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Applied Math II Final Exam

STU Mathematics Department

2. The temperature at a point (x, y) on a metal plate in the xy plane is given by

$$T(x, y) = \frac{xy}{2+x+y}. \text{ Then find}$$

- all first and second order **partial derivatives** of T at $(1, 1)$.
- the **gradient vector**, ∇T at the point $(1, 1)$.
- the **directional derivative** of T at $p(1, 1)$ in the direction of the vector $v = i + j$.
- the **maximum value** of the directional derivative at the point $p(1, 1)$.

a) $\frac{\partial T}{\partial x} = T_x = \frac{y(2+x+y) - xy}{(2+x+y)^2} = \frac{1(2+1+1) - (1)(1)}{(2+1+1)^2} = \frac{2-1}{4^2} = \frac{1}{16}$

$\frac{\partial T}{\partial y} = T_y = \frac{x(2+x+y) - xy}{(2+x+y)^2} = \frac{1(2+1+1) - (1)(1)}{(2+1+1)^2} = \frac{2-1}{4^2} = \frac{1}{16}$

$T_{xx} = \frac{\partial}{\partial x} \left(\frac{\partial T}{\partial x} \right) = \frac{\partial}{\partial x} \left(\frac{y(2+x+y) - xy}{(2+x+y)^2} \right) = \frac{(y-y)(2+x+y)^2 - 2(2+x+y)(y-xy)}{(2+x+y)^4}$

$T_{xx} = \frac{-2y}{(2+x+y)^3} = \frac{-2(1)}{4^3} = \frac{-2}{64} = \frac{-1}{32}$

$T_{yy} = \frac{\partial}{\partial y} \left(\frac{\partial T}{\partial y} \right) = \frac{\partial}{\partial y} \left(\frac{x(2+x+y) - xy}{(2+x+y)^2} \right) = \frac{(x-x)(2+x+y)^2 - 2(2+x+y)(x-xy)}{(2+x+y)^4}$

$T_{yy} = \frac{-2x}{(2+x+y)^3} = \frac{-2(1)}{4^3} = \frac{-2}{64} = \frac{-1}{32}$

b) $\Delta T(1, 1) = T_x(1, 1)i + T_y(1, 1)j$

$\Delta T(1, 1) = \frac{1}{16}i + \frac{1}{16}j$

c) $D_u T(1, 1) = a_1 T_x(1, 1) + a_2 T_y(1, 1)$

$D_u T(1, 1) = \frac{1}{\sqrt{2}} \times \frac{1}{16} + \frac{1}{\sqrt{2}} \times \frac{1}{16}$

$= \frac{\sqrt{2} \times 1}{2} \times \frac{1}{16} + \frac{\sqrt{2} \times 1}{2} \times \frac{1}{16}$

$= \frac{3\sqrt{2}}{32} + \frac{3\sqrt{2}}{32}$

$= \frac{6\sqrt{2}}{32} = \frac{3\sqrt{2}}{16}$

magnitude of the unit vector is

$$\sqrt{1+1} = \sqrt{2}$$

the unit vector will be

$$\vec{u} = \frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}}j$$

d) The maximum value of directional derivative is

$D_u = \|\Delta T(1, 1)\|$

$= \sqrt{\left(\frac{1}{16}\right)^2 + \left(\frac{1}{16}\right)^2}$

$= \sqrt{\frac{1}{256} + \frac{1}{256}} = \sqrt{\frac{2}{256}} = \frac{\sqrt{2}}{16}$

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5. Evaluate the integral $\int_0^2 \int_0^{\sqrt{4-x^2}} e^{-x^2-y^2} dy dx$ by using polar coordinates.

$$y^2 = 4 - x^2$$

$$(y)^2 = (\sqrt{4-x^2})^2$$

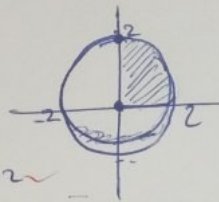
$$y^2 = 4 - x^2$$

$$y^2 + x^2 = 4$$

$$r = 2 \Rightarrow 0 \leq r \leq 2$$

$$\theta = \frac{\pi}{2}$$

$$0 \leq \theta \leq \frac{\pi}{2}$$



$$\int_0^{\pi/2} \int_0^2 e^{-r^2} r dr d\theta \Rightarrow \int_0^{\pi/2} \left[-\frac{e^{-r^2}}{2} \right]_0^2 d\theta$$

using substitution more

$$u = -r^2$$

$$du = -2r dr$$

$$\int_0^{\pi/2} \int_0^2 e^{-r^2} r dr d\theta = \int_0^{\pi/2} \left[-\frac{e^{-r^2}}{2} \right]_0^2 d\theta$$

Integration Using Substitution more

$$u = -r^2$$

$$du = -2r dr$$

$$\frac{1}{2} du = -r dr$$

$$\int_0^2 -du e^u du$$

$$\left(-\frac{e^u}{2} \right) \Big|_0^2 = -\frac{e^2}{2} - \left(-\frac{1}{2} \right)$$

$$\int_0^{\pi/2} \left(-\frac{e^{-r^2}}{2} \right) d\theta$$

$$\int_0^{\pi/2} \frac{-e^u}{2} du d\theta$$

$$\int_0^{\pi/2} \frac{-e^u}{2} du$$

$$\int_0^{\pi/2} \left(\frac{1}{2} - \frac{e^{-4}}{2} \right) d\theta$$

$$\int_0^{\pi/2} \left(\frac{1}{2} - \frac{e^{-4}}{2} \right) d\theta = \left(\frac{\theta}{2} - \frac{\theta e^{-4}}{2} \right) \Big|_0^{\pi/2}$$

$$= \frac{\pi}{4} - \frac{\pi e^{-4}}{4} - (0 - 0)$$

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$$\int_0^{\pi/2} \frac{1-e^{-4}}{2} d\theta$$

$$\left(\frac{\theta}{2} - \frac{\theta e^{-4}}{2} \right) \Big|_0^{\pi/2}$$

$$\left(\frac{\pi}{4} - \frac{\pi e^{-4}}{4} \right) - (0 - 0)$$

$$= \frac{\pi}{4} [1 - e^{-4}]$$

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ASTU Mathematics Department
Applied Math II Final Exam

PART I: For each of the following questions, fill the blank space with the most simplified answers. (each question worth 2 points)

1. If 2 is the period of $f(x)$ and n is any integer, then the period of $f(x + 2n)$ is 2 ✓
2. Assume $f(x)$ is a periodic function with period $2L$, and has a Fourier series on the interval $-L \leq x \leq L$, then its Fourier series expansion formula is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$
3. The domain of the function $f(x, y, z) = x \ln\left(\frac{1}{2y-z}\right)$ is $D = \{(x, y, z) : y > 0, z < 2y\}$ ✓ and $\frac{1}{2y-z} > 0$
4. $\lim_{(x,y) \rightarrow (0,0)} \left(\frac{-y+x}{y^2-x^2}\right) =$ doesn't exist
5. Find the value of c such that the function $f(x, y) = \begin{cases} \frac{e^{x^2+y^2}-1}{x^2+y^2} & \text{for } (x, y) \neq (0,0) \\ c, & \text{for } (x, y) = (0,0) \end{cases}$ is continuous at $(0,0)$. Answer: $c =$ 1 ✓
6. Given $z = x^2 \ln y$, then $\frac{\partial^2 z}{\partial y \partial x}$ at $(2,2)$ is 2 ✓
7. Let $f(x, y) = x e^{xy}$, $x = 3t$, and $y = 1$.
 i. $\frac{df}{dt} =$ $3e^{2t}(1+y^2) = 3e^{3t}(1+3t)$ ✓
 ii. The equation of the tangent plane to the graph of f at a point $(0,1,0)$ is $z = 0$ ✓
8. For $y^2(x^2 - 1) = zx^2$, by implicit differentiation $\frac{\partial z}{\partial y}$ at $(1,1,0) =$ 0 ✓
9. If $R = \{(x, y) : 0 \leq x \leq y, 0 \leq y \leq 1\}$, then $\iint_R e^{y^3} dA =$ $\frac{e}{2} - \frac{1}{2}$ ✓
10. $\int_0^\pi \int_0^{\sin \theta} r dr d\theta =$ $\frac{\pi}{4}$ ✓

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PART II: Workout the following problems by showing all the necessary steps. (30%)

1. Given $f(x) = \begin{cases} 0, & -5 < x < 0 \\ 1, & 0 < x < 5 \end{cases}$; $f(x+10) = f(x)$, then find the Fourier series of $f(x)$.

$$a_0 = \frac{1}{5} \int_{-5}^0 0 dx + \frac{1}{5} \int_0^5 1 dx = \frac{1}{5} \left[0 + x \right]_0^5 = \frac{1}{5} (5) = 1$$

$$a_n = \frac{1}{5} \int_{-5}^0 0 \cos \frac{n\pi x}{5} dx + \frac{1}{5} \int_0^5 1 \cos \frac{n\pi x}{5} dx$$

$$= \frac{1}{5} \int_0^5 \cos \frac{n\pi x}{5} dx$$

$$= \frac{1}{5} \left[\left(\sin \frac{n\pi x}{5} \right) \frac{5}{n\pi} \right]_0^5$$

$$= \frac{1}{5} \left[\left(\sin n\pi \right) \frac{5}{n\pi} - \left(\sin 0 \right) \frac{5}{n\pi} \right]$$

$$a_n = 0$$

$$b_n = \frac{1}{5} \int_{-5}^0 0 \sin \frac{n\pi x}{5} dx + \frac{1}{5} \int_0^5 1 \sin \frac{n\pi x}{5} dx$$

$$= \frac{1}{5} \int_0^5 \sin \left(\frac{n\pi x}{5} \right) dx$$

$$= \frac{1}{5} \left[-\cos \left(\frac{n\pi x}{5} \right) \right]_0^5$$

$$= -\frac{1}{5} \left[\frac{5}{n\pi} \left[\cos \left(\frac{n\pi x}{5} \right) \right]_0^5 \right] = -\frac{1}{5} \left[\cos n\pi \right] - \left[-\frac{5}{5n\pi} \left(\cos 0 \right) \right]$$

$$= \frac{-1}{n\pi} (-1)^n + \frac{1}{n\pi}$$

$$= \frac{1}{n\pi} - \frac{1}{n\pi} (-1)^n = \frac{1}{n\pi} [1 - (-1)^n]$$

$$b_n = \frac{1}{n\pi} [1 - (-1)^n]$$

Therefore the Fourier series will be $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{5} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{5}$

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{n\pi} [1 - (-1)^n] \sin \frac{n\pi x}{5}$$

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4. By using Lagrange's multipliers method, find the extreme values of $f(x, y) = xy$ on the ellipse $2x^2 + y^2 = 1$.

$$\begin{aligned} \frac{\partial}{\partial x}(xy) &= g_x(x, y) \lambda, & y &= 4x\lambda, & \lambda &= \frac{y}{4x} \\ \frac{\partial}{\partial y}(xy) &= g_y(x, y) \lambda, & x &= 2y\lambda, & \lambda &= \frac{x}{2y} \\ \frac{y}{4x} &= \frac{x}{2y} \Rightarrow 2y^2 = 4x^2 \end{aligned}$$

The points are

$$\left(-\frac{1}{2}, \frac{\sqrt{2}}{2}\right), \left(-\frac{1}{2}, -\frac{\sqrt{2}}{2}\right), \left(\frac{1}{2}, \frac{\sqrt{2}}{2}\right), \left(\frac{1}{2}, -\frac{\sqrt{2}}{2}\right)$$

$$f\left(-\frac{1}{2}, \frac{\sqrt{2}}{2}\right) = -\frac{1}{2} \times \frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{4}$$

$$f\left(-\frac{1}{2}, -\frac{\sqrt{2}}{2}\right) = -\frac{1}{2} \times -\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4}$$

$$f\left(\frac{1}{2}, \frac{\sqrt{2}}{2}\right) = \frac{1}{2} \times \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4}$$

$$f\left(\frac{1}{2}, -\frac{\sqrt{2}}{2}\right) = \frac{1}{2} \times -\frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{4}$$

The maximum value is found on $f\left(-\frac{1}{2}, -\frac{\sqrt{2}}{2}\right)$ and $f\left(\frac{1}{2}, \frac{\sqrt{2}}{2}\right)$

Therefore the Max. Value is $\frac{\sqrt{2}}{4}$

The minimum value is found on $f\left(-\frac{1}{2}, \frac{\sqrt{2}}{2}\right)$ and $f\left(\frac{1}{2}, -\frac{\sqrt{2}}{2}\right)$

Therefore the min. Value is $-\frac{\sqrt{2}}{4}$

$$\begin{aligned} 2x^2 + 2x^2 &= 1 \\ 4x^2 &= 1 \\ x^2 &= \frac{1}{4} \end{aligned}$$

$$x = \pm \frac{1}{2}$$

$$2\left(\frac{1}{4}\right) + y^2 = 1$$

$$y^2 = 1 - \frac{1}{2}$$

$$y^2 = \frac{1}{2}$$

$$y = \pm \frac{\sqrt{2}}{2}$$

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3. Let $f(x, y) = x^2y - y^3 - 2x^2 + 2y^2 + 10$, then by using second partial test, find
- all critical points and classify them as points of relative maximum, relative minimum and saddle points.
 - relative maximum and relative minimum values of f .

critical points, $\partial_x = 0$, $2xy - 4x = 0$, $x = 0, y = 4$
 $\partial_y = 0$, $x^2 - 3y^2 + 4y = 0$, put $x = 0$ on equation

$$-3y^2 + 4y = 0$$

$$4y - 3y^2 = 0$$

$$y(4 - 3y) = 0$$

$$y = 0, \frac{4}{3}$$

put $y = 4$ on the same equation

$$x^2 - 48 + 16 = 0$$

$$x^2 = 0$$

$$x = 0$$

* There are critical points at $(0, 0)$, $(0, \frac{4}{3})$, $(0, 4)$

$$D = \partial_{xx}\partial_{yy} - (\partial_{xy})^2$$

$$D = \begin{vmatrix} \partial_{xx} & \partial_{xy} \\ \partial_{yx} & \partial_{yy} \end{vmatrix}$$

$$= \partial_{xx} = 2y - 4$$

$$\partial_{yy} = -6y + 4$$

$$\partial_{xy} = 2x$$

$$\partial_{yx} = 2x$$

$$D = \partial_{xx}(0, 0) \partial_{yy}(0, 0) = \begin{vmatrix} -4 & 0 \\ 0 & 4 \end{vmatrix} = D < 0, \text{ saddle point at } (0, 0)$$

$$D = \partial_{xx}(0, \frac{4}{3}) \partial_{yy}(0, \frac{4}{3}) = \begin{vmatrix} -\frac{4}{3} & 0 \\ 0 & -\frac{4}{3} \end{vmatrix} = D > 0, \text{ maximum value at } (0, \frac{4}{3})$$

$$D = \partial_{xx}(0, 4) \partial_{yy}(0, 4) = \begin{vmatrix} 4 & 0 \\ 0 & -20 \end{vmatrix} = D < 0, \text{ saddle point at } (0, 4)$$

(b) Relative maximum at $(0, \frac{4}{3}) \Rightarrow f(0, \frac{4}{3}) = \frac{-64 + 32 + 10}{27} = \frac{-64 + 96 + 270}{27}$

$$= \frac{-64 + 96 + 270}{27}$$

$$= \frac{302}{27}$$

$$= \frac{302}{27}$$

$$= \frac{302}{27} \text{ is max value}$$

~~$f(0, 0)$ is saddle point~~
 ~~$f(0, 4)$ is saddle point~~
 ~~$f(0, \frac{4}{3})$ is saddle point~~

The function has no relative minimum value.

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3. Let $f(x, y) = x^2y - y^3 - 2x^2 + 2y^2 + 10$, then by using second partial test, find
- all critical points and classify them as points of relative maximum, relative minimum and saddle points.
 - relative maximum and relative minimum values of f .

critical points, $f_x = 0$, $2xy - 4x = 0$, $x = 0$ or $y = 4$
 $f_y = 0$, $x^2 - 3y^2 + 4y = 0$, put $x = 0$ on equation

$$-3y^2 + 4y = 0$$

$$4y - 3y^2 = 0$$

$$y(4 - 3y) = 0$$

$$y = 0, \frac{4}{3}$$

put $y = 4$ on the same equation

$$x^2 - 48 + 16 = 0$$

$$x^2 = 0$$

$$x = 0$$

* Therefore critical points are $(0, 0)$, $(0, \frac{4}{3})$, $(0, 4)$

$$D = f_{xx}f_{yy} - (f_{xy})^2$$

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

$$= f_{xx} = 2y - 4$$

$$f_{yy} = -6y + 4$$

$$f_{xy} = 2x$$

$$f_{yx} = 2x$$

$$D = f_{xx}(0, 0) f_{yy}(0, 0) = \begin{vmatrix} -4 & 0 \\ 0 & 4 \end{vmatrix} = D < 0, \text{ saddle point at } (0, 0)$$

$$D = f_{xx}(0, \frac{4}{3}) f_{yy}(0, \frac{4}{3}) = \begin{vmatrix} -\frac{4}{3} & 0 \\ 0 & -\frac{4}{3} \end{vmatrix} = D > 0, \text{ maximum value at } (0, \frac{4}{3})$$

$$D = f_{xx}(0, 4) f_{yy}(0, 4) = \begin{vmatrix} 4 & 0 \\ 0 & -20 \end{vmatrix} = D < 0, \text{ saddle point at } (0, 4)$$

$$(b) \text{ Relative maximum at } (0, \frac{4}{3}) \Rightarrow f(0, \frac{4}{3}) = \frac{-64 + 32 + 10}{27} = \frac{-64 + 96 + 270}{27}$$

$$= \frac{32 + 270}{27}$$

$$= \frac{302}{27} \Rightarrow \text{max value}$$

$$= \frac{302}{27}$$

$$= \frac{302}{27} \Rightarrow \text{max value}$$

~~$f(0, 0)$ is saddle point~~
 ~~$f(0, 4) = 10$ is saddle point~~
 ~~$f(0, \frac{4}{3})$ is saddle point~~

The function has no relative minimum value.

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