KITWE DISTRICT EDUCATION BOARD



ADDITIONAL MATHEMATICS PAMPHLET

(AMIE)
ALL MATHEMATICS IS EASY

DEBS- KITWE
FIRST EDITION(2017)

COORDINATE GEOMETRY

Coordinate geometry is the study geometrical properties of points, straight lines and curves using algebraic methods.

FINDING THE LENGTH OF A LINE GIVEN TWO POINTS:

DISTANCE BETWEEN TWO POINTS

The length of a line joining two points $A(x_1, y_1)$ *and* $B(x_2, y_2)$ is given by

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

EXAMPLE:

1. Find the length of the line joining $P(2,3)$ *and* $Q(4,7)$

$$\begin{aligned} PQ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(4-2)^2 + (7-3)^2} \\ &= \sqrt{2^2 + 4^2} \\ &= \sqrt{4+16} \\ PQ &= \sqrt{20}\ units \end{aligned}$$

FINDING THE MID POINT OF TWO POINTS

The midpoint of the line joining two points $A(x_1, y_1)$ *and* $B(x_2, y_2)$, has coordinates

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

EXAMPLE:

1. Find the coordinates of the midpoint of the straight line joining the points $A(3,7)$ *and* $B(5,9)$.

$$\text{Midpoint is } \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

$$\left(\frac{3+5}{2},\frac{7+9}{2}\right)$$

$$\left(\frac{8}{2},\frac{16}{2}\right)$$

$$(4,8)$$

FINDING THE GRADIENT AND EQUATION OF A STRAGHT LINE

GRADIENT OF A STRAIGHT LINE:

$$m=\frac{y-step}{x-step}$$

$$m=\frac{y_2-y_1}{x_2-x_1}$$

EXAMPLE

1. Find the gradient of the line joining the points $(4,1)$ *and* $(7,3)$.

$$m=\frac{y_2-y_1}{x_2-x_1}$$

$$=\frac{3-1}{7-4}$$

$$m=\frac{2}{3}$$

FINDING THE EQUATION OF A STRAIGHT LINE:

The following can be used to find the equation of a straight line.

1. $y=mx+c$ Where m is the gradient and c is the point where the line meets y axis. (Y-intercept).

2. $y-y_1=m(x-x_1)$ Where m is known and a point (x_1,y_1)

3. $\frac{y-y_2}{x-x_2}=\frac{y_2-y_1}{x_2-x_1}$ Where two points are known (x_1,y_1) *and* (x_2,y_2)

EXAMPLE

1. Find the equation of the lie with gradient 2 passing through the point (5, 3)

We use $y - y_1 = m(x - x_1)$

$$y - (-3) = 2(x - 5)$$

$$y + 3 = 2x - 10$$
$$y = 2x - 10 - 3$$
$$\underline{y = 2x - 13 \text{ or } y - 2x = -13}$$

PARALLEL AND PERPENDICULAR LINES:

1. If two lines AB and PQ are parallel then there gradients are equal.

$$m_1 = m_2$$

2. If two lines AB and PQ are perpendicular then the product of their gradients always equal to -1

$$\therefore m_1 \times m_2 = -1$$
$$m_1 m_2 = -1$$

EXAMPLE

3. Find the equation of the line which is

(a) parallel to $2x + y = 3$ and passes through (0, 1)
(b) Perpendicular to $3x + y = 5$ and passes through (-2,-1)

SOLUTION

(a) $2x + y = 3$

$y = -2x + 3$

m = -2 and point (0, 1)

$y - y_1 = m(x - x_1)$

$y - 1 = -2(x - 0)$

$y = -2x + 1$

-for any two Parallel lines their gradients are the same

(b) $3x + y = 5$

$y = -3x + 5$

$m_1 = -3$

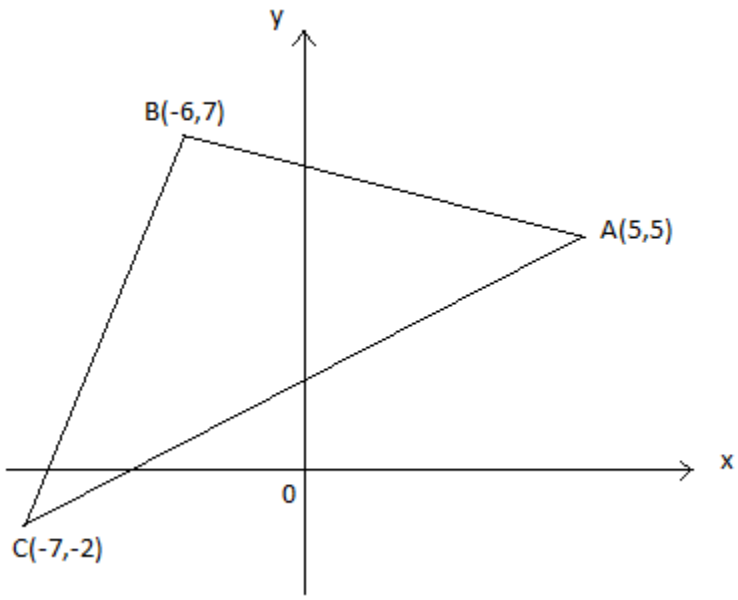
$$m_1 m_2 = -1$$
$$-3m_2 = -1$$
$$m_2 = \frac{-1}{-3}$$
$$m_2 = \frac{1}{3}$$

$$y - y_2 = m(x - x_2)$$
$$y - (-1) = \frac{1}{3}[x - (-2)]$$
$$y + 1 = \frac{1}{3}x + \frac{2}{3}$$
$$y = \frac{1}{3}x - \frac{1}{3}$$

-For Perpendicular lines $m_1 m_2 = -1$

AREA

1. Given that triangle ABC has vertices A(5, 5), B(-6, 7) and C(-7, -2) Find its area



AREA OF RECTILINEAR FIGURES

Area of triangle ABC is

A(x_1, y_1), B(X_2, y_2) and C(x_3,y_3)

Then Area $\frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - x_1y_3 - x_3y_2 - x_2y_1)$

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_1 & y_1 \end{vmatrix} or \begin{vmatrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{vmatrix}$$

The points should be taken in the anticlockwise direction.

Starting point is repeated at the end in order to complete the figure

Area of ΔABC = $= \frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_3 - x_1y_3 - x_3y_2 - x_2y_1)$

$= \frac{1}{2}[5(7) + (-6)(-2) + (-7)(5) - 5(-2) - (-7)(7) - (-6)(5)]$

$= \frac{1}{2}[35 + 12 - 35 + 10 + 49 + 30]$

$= 50.5 \, Squreunits$

SYSTEMS OF EQUATIONS

1. Solve the following simultaneous equations

   $3x + 4y = 2$

   $x^2 + 8xy + 12 = 0$

   Solutions

   $3x + 4y = 2$..... (1)

   $x^2 + 8xy + 12 = 0$..... (2)

   Step (1) from equation (1) $3x + 4y = 2$

   $\Rightarrow 3x = 2 - 4y$

   $\Rightarrow x = \frac{2-4y}{3}$ ..... (3)

   Step (2) substitute (3) in equation (2)

   $x^2 + 8xy + 12 = 0$

   $\Rightarrow (\frac{2-4y}{3})^2 + 8(\frac{2-4y}{3})y + 12 = 0$

$$\Rightarrow \frac{4-16y+16y2}{9} + \frac{16-32y2}{3} + 12 = 0$$

$$\Rightarrow 9\,[\,4 - 16y + 16y^2 + 3(16 - 32y^2) + (9)(12) = 0]$$

$$\Rightarrow 4 - 16y + 16y^2 + 48 - 96y^2 + 108 = 0$$

$$-80y^2 + 32y + 112 = 0$$

$$5y^2 - 2y - 7 = 0$$

$$(5y - 7)(y + 1) = 0$$

$\Rightarrow 5y - 7 = 0$ or $y + 1 = 0$

$y = \frac{7}{5}$ $y = -1$

Step (3) substitute $y = \frac{7}{5}$ into (3): $x = \frac{2-4(\frac{7}{5})}{3}$

$$= -\frac{6}{5}$$

$$= -1\frac{1}{5}$$

Substitute $y = -1$ into (3): $x = \frac{2-4(-1)}{3}$

$$= \frac{6}{3}$$

$$= 2$$

Therefore, $x = -1\frac{1}{2}$, $y = 1\frac{2}{5}$ or $x = 2$, y -1 (ans).

2. Given that (a ,7) is a solution of the simultaneous equations $3x - y = 8$ and $bx^2 - xy + 9 = y^2$. Find
   a) The value of a and b.
   b) The coordinates of the other solution.

a) Step (1) substituting $x = a$ and $y = 7$ in equations (1) and (2)

$\Rightarrow 3a - 7 = 8$..... (3)

$\Rightarrow ba2 - 7a + 9 = 7^2 \Rightarrow ba^2 - 7a + 9 = 49$

$$ba^2 - 7a = 40 \text{..... (4)}$$

Step (2) Solving equations (3) and (4) simultaneously

From equation (3) $3a - 7 = 8$

$3a = 8 + 7$

$\frac{3a}{3} = \frac{15}{3}$

$a = 5$..... (5)

Step (3) substituting (5) into (4)

$ba^2 - 7a = 40$

=> $b(5)^2 - 7(5) = 40$

$25b - 35 = 40$

$25b = 75$

$b = 3$..... (6) Therefore, $a = 5$, $b = 3$ (ans)

b) Step (1) substituting (6) into (2) and solving equations (1) and (2) simultaneously

$3x - y = 8$..... (1)

$3x^2 - xy + 9 = y^2$..... (2)

Step (2) from equation (1) $y = 3x - 8$..... (3)

Step (3) substituting equation (3) into (2)

=> $3x^2 - x(3x - 8) + 9 = (3x - 8)^2$

=> $3x^2 - 3x^2 - 8x + 9 = 9x^2 - 48x + 64$

=> $9x^2 - 56x + 55 = 0$

=> $(9x - 11)(x - 5) = 0$

=> $9x - 11 = 0$ or $x - 5 = 0$

$9x = 11$ $x = 5$

$x = \frac{11}{9}$

Substituting the value of x in equation (3)

$y = 3(\frac{11}{9}) - 8$

$= \frac{11}{9} - 8$

$= \frac{-13}{3}$

$= -4\frac{1}{3}$

Therefore, the coordinates of the other solution are $(1\frac{2}{9}, -4\frac{1}{3})$

3. Find the coordinates of the point of intersection of the line $x + y = 3$ and the curve $x^2 - 2x + 2y^2 = 3$.

Solution

Step (1) to find the coordinates of the point of intersection

Solve the two equations simultaneously

$x + y = 3$..... (1)

$x^2 - 2x + 2y^2 = 3$

From equation (1) $x = 3 - y$

Step (2) substitute equation (3) into (2)

$\Rightarrow (3 - y)^2 - 2(3 - y) + 2y^2 = 3$

$9 - 6y + y^2 - 6 + 2y + 2y^2 = 3$

$3y^2 - 4y + 9 - 6 - 3 = 0$

$3y^2 - 4y = 0$

$Y(3y - 4) = 0$

$Y = 0$ or $3y - 4 = 0$

$\frac{3y}{3} = \frac{4}{3}$

$y = \frac{4}{3}$

Step (3) substitute the values of y in equation (3)

When $y = \frac{4}{3}$, $x = 3 - \frac{4}{3}$ when $y = 0$, $x = 3 - 0$

$x = 1\frac{2}{3}$ $x = 3$

Therefore, the coordinates of the point of intersection are $(1\frac{2}{3}, 1\frac{1}{3})$ and (3, 0).

4. Solve the following equations

$2x + y + 3z = 11$..... (1)

$x + 2y – 2z = 3$..... (2)

$4x + 3y + z = 15$..... (3)

Step (1) eliminating z from equation (1) and (2)

$2(2x + y + 3z = 11)$, $3(x + 2y – 2z = 3)$

$\Rightarrow \underline{4x + 2y + 6z = 22} \qquad \underline{3x + 6y – 6z = 9}$

Adding the two equations gives

$7x + 8y = 31$..... (4)

Step (2) eliminating z from equation (2) and (3)

$X + 2y - 2z = 3$, $2(4x + 3y + z = 15)$

$\underline{8x + 6y + 2z = 30}$

Adding the two equations gives

$9x + 8y = 33$..... (5)

Step (3) solving equations (4) and (5) simultaneously

$7x + 8y = 31$

$\underline{(-)\ 9x + 8y = 33}$

$\frac{-2x}{-2} = \frac{-2}{-2}$

$\underline{x = 1}$

Replacing x = 1 in (4)

$7(1) + 8y = 31$

$7 + 8y = 31$

$8y = 31 - 7$

$8y = 24$

$\underline{y = 3}$

Replacing the values of x and y in (1)

$$2(1) + 3 + 3z = 11$$

$$5 + 3z = 11$$

$$3z = 6$$

$$\underline{z = 2}$$

Therefore, x = 1, y = 3 and z = 2

5. Find the solution of the linear system 2x + 3y + 4z = -4, 4x + 2y + 3z = -11 and 3x + 4y + 2z = -3, using the crammer's rule.

SOLUTIONS

$$2x + 3y + 4z = -4 \quad \ldots\ldots (1)$$

$$4x + 2y + 3z = -11 \quad \ldots\ldots (2)$$

$$3x + 4y + 2z = -3 \quad \ldots\ldots (3)$$

Step (1) forming matrices

$$\begin{pmatrix} 2 & 3 & 4 \\ 4 & 2 & 3 \\ 3 & 4 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 \\ -11 \\ -3 \end{pmatrix}$$

Step (2) let A = $\begin{pmatrix} 2 & 3 & 4 \\ 4 & 2 & 3 \\ 3 & 4 & 2 \end{pmatrix}$, B = $\begin{pmatrix} -4 & 3 & 4 \\ -11 & 2 & 3 \\ -3 & 4 & 2 \end{pmatrix}$, C = $\begin{pmatrix} 2 & -4 & 4 \\ 4 & -11 & 3 \\ 3 & -3 & 2 \end{pmatrix}$, and D =

$$\begin{pmatrix} 2 & 3 & -4 \\ 4 & 2 & -11 \\ 3 & 4 & -3 \end{pmatrix}$$

Step (3) finding determinants

$$|A| = 2\begin{vmatrix} 2 & 3 \\ 4 & 2 \end{vmatrix} - 3\begin{vmatrix} 4 & 3 \\ 3 & 2 \end{vmatrix} + 4\begin{vmatrix} 4 & 2 \\ 3 & 4 \end{vmatrix}$$

$$= 2(4 - 12) - 3(8 - 9) + 4(16 - 6)$$

$$= -16 + 3 + 40$$

$$= 27$$

$$|B| = -4\begin{vmatrix} 2 & 3 \\ 4 & 2 \end{vmatrix} - 3\begin{vmatrix} -11 & 3 \\ -3 & 2 \end{vmatrix} + 4\begin{vmatrix} -11 & 2 \\ -3 & 4 \end{vmatrix}$$

$$= -4(4 - 12) - 3(-22 + 9) + 4(-44 + 6)$$

= -81

$|C| = 2\begin{vmatrix} -11 & 3 \\ -3 & 2 \end{vmatrix} - (-4)\begin{vmatrix} 4 & 3 \\ 3 & 2 \end{vmatrix} + 4\begin{vmatrix} 4 & -11 \\ 3 & -3 \end{vmatrix}$

= 2(-22 + 9) + 4(8 - 9) + 4(-12 + 33)

= 54

$|D| = 2\begin{vmatrix} 2 & -11 \\ 4 & -3 \end{vmatrix} - 3\begin{vmatrix} 4 & -11 \\ 3 & -3 \end{vmatrix} - 4\begin{vmatrix} 4 & 2 \\ 3 & 4 \end{vmatrix}$

= 2(-6 + 44) – 3(-12+ 33) – 4(16 - 6)

= -27

Therefore, $x = \frac{|B|}{|A|}$, $y = \frac{|C|}{|A|}$, $z = \frac{|D|}{|A|}$

$= \frac{-81}{27}$ $= \frac{54}{27}$ $= \frac{-27}{27}$

= -3 = 2 = -1

FUNCTIONS

- When two members of the two sets are connected, it is called a relationship
- A relation is a collection of ordered pairs. A function is a special type of relation
- Functions and relations can be represented by:

(i) a mapping (ii) a table (iii) an ordered pair (iv) an algebraic sentence (v) a graph

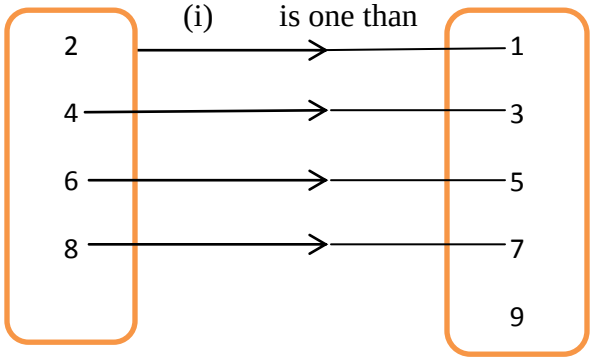
QUESTIONS

1. A relation from set A = {2, 4, 6, 8} to set B = {1, 3, 5, 7, 9} is given as "is one more than"

   (a)Draw an arrow diagram to show the relation

Answers

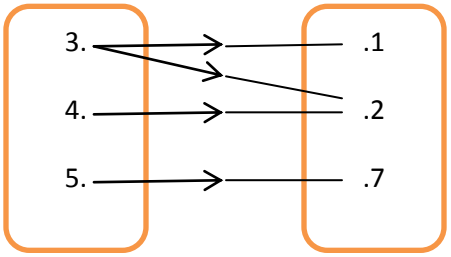
A



(b) the type of relationship

one – to – one relationship

2. Study the mapping below



(a) Complete the following

(i) 3 is mapped into ….. and ….

Answer 1 and 2

(ii) 4 is mapped into……

Answer 2

(iii) 5 is mapped into…….

Answer 7

(b) List the set of:

(i) The domain

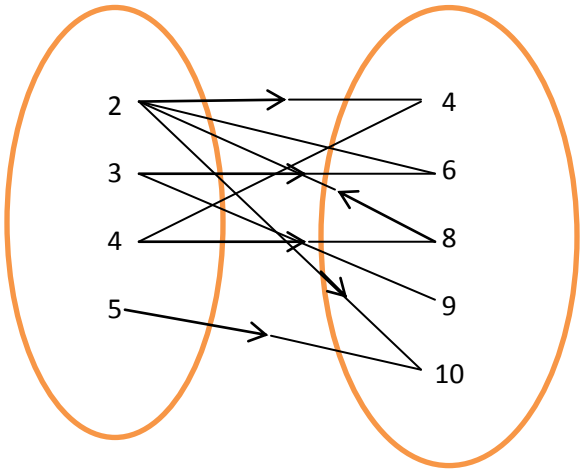
Answer - Domain = $\{3, 4, 5\}$

(ii) The range

Answer - Range = $\{1, 2, 7\}$

3. Set D = $\{(2,4), (2, 6), (2,8), (2,10), (3, 6), (3, 9), (4,4), (4,8), (5, 10)\}$

(a) Illustrate this information on an arrow diagram.



4. Complete the following table

	Input	f: x ⟶ 3x + 1	Output	Ordered pair
i.	1	f:x ⟶ 3 (1) + 1	4	1,4)
ii.	0	f:x ⟶ 3 (0) + 1		
iii.	2			
iv.				

Answer

	input	f: x⟶ 3x + 1	Output	Ordered pair
i.	1	f: x⟶ 3(1) + 1	4	(1,4)
ii.	0	f: x⟶ 3(0) + 1	1	(0,1)
iii.	-2	f: x⟶ 3 (-2) + 1	-5	(-2, -5)
iv.	-3	f: x⟶ 3(-3) + 1	-8	(-3, -8)

5. A function f is such that f)t) = 2t – 2. Find:

(a) f (0) (b) f (2) (c) $(\frac{-1}{2})$

(a) f (t) = 2t – 2
f (0) = 2 (0) -2
= 0 – 2
= -2

(b) f (t) = 2t – 2
f (2) = 2 (2) -2
= 4 – 2
= 2

(c) f (t) = 2 (t) -2
= 2 $(\frac{-1}{2})$ -2
= -1-2
= -3

6. Given that the ordered pairs (m, 25) and (n, -10) belong to the mapping h:x ⟶ x+4 Find the values of M and n.

Answer

(M, 25)
h : x ⟶ x + 4
h (x) = x + 4
h(m) = M = 4
25 = M + 4
25 – 4 = M
21 = M

(n, -10)
h : x ⟶ x + 4
h(x) ⟶ x + 4
h(n) = n + 4
-10 = n + 4
-10 – 4 = n

-14 = n

7. In the first year, Grace made K800 selling cellphones. She increased her earnings by K50 each year for the next four years

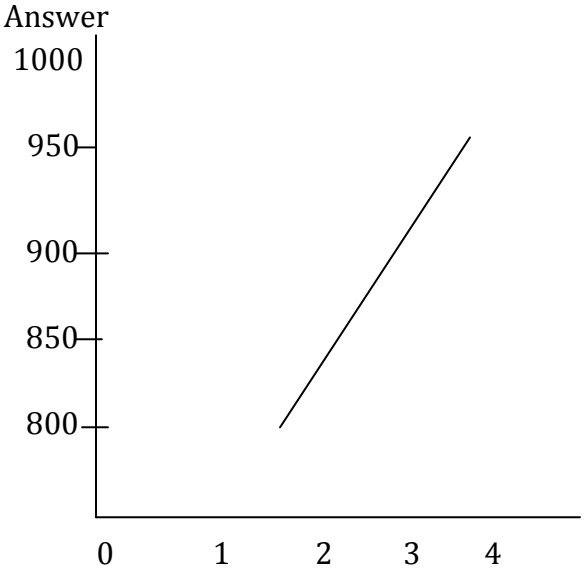
(a) Draw up a table

Answer

Year	Earnings
1	K800
2	K850
3	K900
4	K900

(b) Draw up a linear graph

Answer



(c) Write the co-ordinate pairs

Answer

(x, y) ⟶ (1, K800)
(2, K850)
(3, K900)
(4, K950)

(d) write an algebraic sentence

Algebraic Sentence
Y = x + K50

(a) Give the domain and range

Domain = {1, 2, 3, 4}

Range = {K800, K850, K900, K950}

1. If f : x ⟶ $\frac{2}{3x}$ + 5, find
   a) f(2)
   b) x when f(x) =7
   c) $f^{-1}(x)$
2. A function h is defined as h(x) = $\frac{1}{2}$x – 5, find
   a) h(-4)
   b) the value of x for which h(x)= 3
   c) $h^{-1}(x)$
3. Given that f(x) = $\frac{3x-5}{2}$ and g(x) = $\frac{x-4}{6}$, find
   a) f(-9)
   b) $f^{-1}(x)$

c) the value of x for f(x) = 3g(x)

4. If h(x) = 3x – 5, find
   a) h(3)
   b) h(x) = 10
   c) $h^{-1}(x)$

EXPECTED ANSWERS

1. a) $5\frac{2}{6}$ or $5\frac{1}{2}$
   b) X = $\frac{1}{3}$
   c) $f^{-1}(x) = \frac{2}{3x-15}$

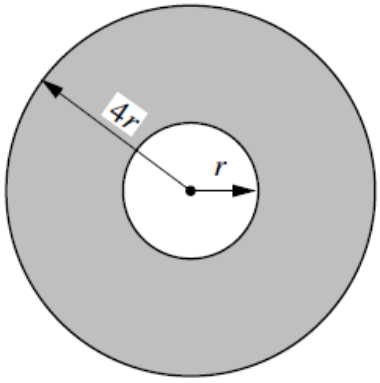
2. a) h (-4) = -7
   b) X = 16
   c) $h^{-1}(x) = 2x + 10$

3. a) f(-9) = -16
   b) $f^{-1}(x) = \frac{2x+5}{3}$
   c) X = $\frac{1}{2}$

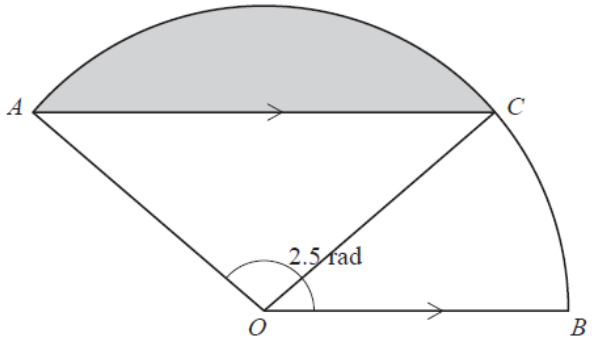
4. a) h(3) = 4
   b) X = 5
   c) h-1(x) = $\frac{x+5}{3}$

CIRCULAR/RADIAN MEASURE

1. Two circles have radii *r* cm and 4*r* cm.
   Find, in terms of π and *r*.
   (a) the area of the circle with radius 4*r* cm, [2]
   (b) given that the area of the circle with radius 4*r* cm is 201.06cm find the area of the shaded ring, [4]
   (c) The total length of the inner and outer edges of the shaded ring. [2]



2. The diagram shows a sector *OACB* of a circle, centre *O*, in which angle *AOB* = 2.5 radians. The line *AC* is parallel to *OB*.



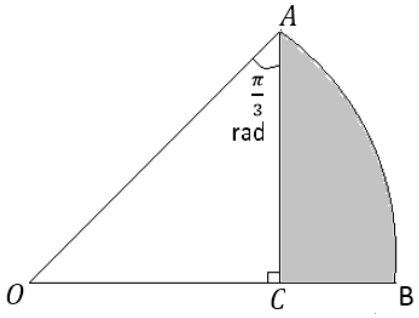
(i) Show that angle *AOC* = (5 – π) radians. [4]

Given that the radius of the circle is 12 cm, find

(ii) the area of the shaded region, [3]

(iii) the perimeter of the shaded region. [3]

3. AOB is a sector of a circle, centre O with $\angle OAC = \frac{\pi}{3}$ radians and $\angle OCA$ a right angle. Given that the arc AB has length 5 cm,



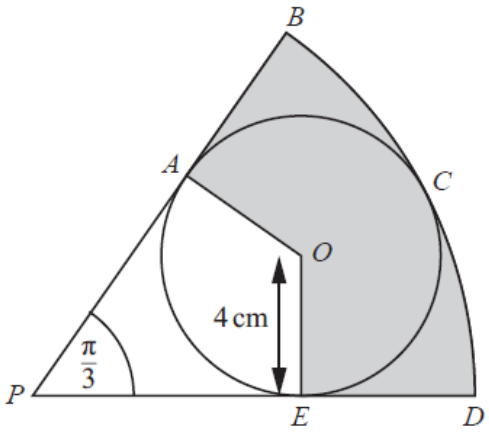
(a) Show that OA = 9.55cm, [3]

(b) Calculate the perimeter of the shaded region [3]

(c) Calculate the area of the shaded region [4]

4.

The diagram shows a circle, centre O, radius 4 cm, enclosed within a sector $PBCDP$ of a circle, centre P. The circle centre O touches the sector at points A, C and E. Angle BPD is $\frac{\pi}{3}$ radians.



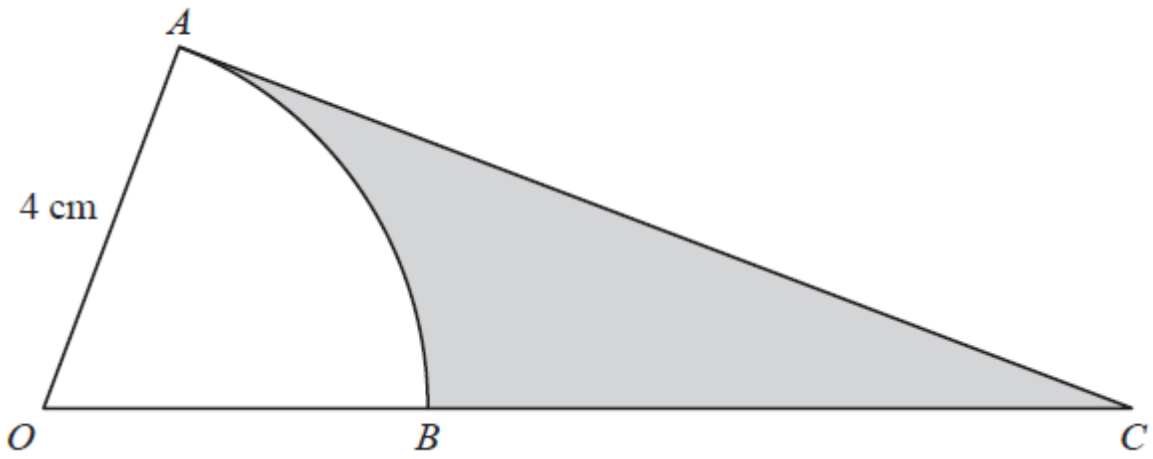
(i) Show that $PA = 4\sqrt{3}\ cm$ and $PB = 12$ cm. [2]

Find, to 1 decimal place,

(ii) The area of the shaded region, [4]

(iii) The perimeter of the shaded region. [4]

5. The diagram shows a sector OAB of a circle, centre O, radius 4 cm. The tangent to the circle at A meets the line OB extended at C. Given that the area of the sector OAB is 10 cm^2, calculate



(i) the angle *AOB* in radians, [2]

(ii) the perimeter of the shaded region. [4]

EXPECTED ANSWERS

Q1. (a) $A = \frac{1}{2} r^2\theta$

$= \frac{1}{2} (4r)^2 2\pi$

$A = 16\pi r^2$ B2

(b) $A = 16\pi r^2$

$\Longrightarrow \quad 201.06 = 16\pi r^2$ M2

$r^2 = \frac{201.06}{16\pi}$

$r = 2$

Shaded Area = Area of Outer circle - area of Inner circle

$A = \frac{1}{2} {r_1}^2\theta - \frac{1}{2} {r_2}^2\theta$ M1

$A = \frac{1}{2}(16r^2)2\pi - \frac{1}{2} r^2(2\pi)$

$A = 60\pi\ cm^2$ A1

(c) Length = Outer circumference – Inner circumference

$= r_1\theta + r_2\theta$ M1

$= 8(2\pi) + 2(2\pi)$

$= 16\pi + 4\pi$

$= 20\pi\ cm$ A1

2. (i) $B\hat{O}C = A\hat{C}O = C\hat{A}O$ M1

$Let\ B\hat{O}C = \theta$

$then\ A\hat{O}C = (2.5 - \theta) rad$ M1

$and\ from\ isosceles\ \Delta AOC,$

$A\hat{O}C + O\hat{C}A + C\hat{A}O = \pi$

$\Longrightarrow (2.5 - \theta) + \theta + \theta = \pi$ M1

$\Longrightarrow \theta = \pi - 2.5$

$\therefore A\hat{O}C = 2.5 - (\pi - 2.5)$

$A\hat{O}C = 5 - \pi$ A1

(ii) Shaded Area = Area of sector AOC - Area of triangle AOC

$Shaded\ area = \frac{1}{2}r^2\theta - \frac{1}{2}r^2 Sin\ \theta$ M1

$= \frac{1}{2}r^2(\theta - \sin\theta)$

$= \frac{1}{2}(12)^2(5 - \pi - \sin(5 - \pi)$ M1

$= 133.8 \sim 133.9$ A1

(iii) Perimeter = line AC + arc length AC

$= r\theta + AC$ where $(AC)^2 = r^2 + r^2 - 2r^2\cos\theta$ M1

$AC = \sqrt{(288 - 288\cos(5 - \pi))}$

Perimeter $= 12(5 - \pi) + \sqrt{(288 - 288\cos(5 - \pi))}$ M1

Perimeter = 41.5 ~ 41.6 A1

3.

(a) $Arc\ Length\ AB = r\theta$

$where\ OA = r\ and\ \theta = \frac{\pi}{2} - \frac{\pi}{3}$

$Thus\ 5 = OA \times \frac{\pi}{6}$

$\Longrightarrow 0A = \frac{30}{\pi} = 9.55$ to 2 decimal places

(a) Perimeter $= \ arc\ AB\ +\ AC\ + CB$

$= \ 12(\frac{\pi}{6}) + 9.55\sin\frac{\pi}{6} + (\ 9.55 - 9.55\cos\frac{\pi}{6}\ )$

Perimeter$= \ 12.34cm$

(b) Shaded area = Area of sector – area of triangle AOC

$= \frac{1}{2}r^2\theta - \frac{1}{2}OC \times CA$

$= \frac{1}{2}(12)^2\left(\frac{\pi}{6}\right) - \frac{1}{2}\left(9.55cos\ \frac{\pi}{6}\right)\left(9.55sin\frac{\pi}{6}\right)$

$A\ = 17.95 \sim 17.96$

4.

(i) $\tan\frac{\pi}{6} = \frac{4}{PA}$

$PA\ = \frac{4}{tan\frac{\pi}{6}}$, by special angles $\tan\frac{\pi}{6}\ =\ \frac{1}{\sqrt{3}}$

$\therefore \quad PA = 4\sqrt{3}$ QED

And PB = PC = radius of circle centre P

$\Longrightarrow PB = PO + OC$

$= \sqrt{(4\sqrt{3}\ )^2 + 4^2} + \ 4$

$= \ \sqrt{64}\ + 4$

$= 12\ cm$

(ii) Shaded Area = Area of sector PBD – area of quadrilateral PAOE

$= \frac{1}{2}r^2\theta - 2(\frac{1}{2}PA \times AO)$

$= \frac{1}{2}(12)^2\ \frac{\pi}{6} - 2(\frac{1}{2}(4\sqrt{3}) \times 4)$

$= 12\pi - 16\sqrt{3}$

≈ 10.0

(iii) Perimeter $= \ 2(AO) + 2(AB) + \ Arc\ length\ BCD$

$= 2(4) + \ 2\left(12 - 4\sqrt{3}\right) + \ 12\left(\frac{\pi}{3}\right)$

$= 30.71$ to 2 decimal places

5. (i) Area of Sector = $\frac{1}{2}r^2\theta$

$$10 = \frac{1}{2}(4^2)\theta$$

$$\theta = \frac{10 \times 2}{16}$$

$$\theta = \frac{5}{4}\ radians$$

(ii) Perimeter = Arc AB + BC + AC

$$= r\theta + (OC - OB) + 4\tan\frac{5}{4}$$

$$= 4\left(\frac{5}{4}\right) + \left[\left(\frac{4}{\cos\frac{5}{4}}\right) - 4\right] + 4\tan\frac{5}{4}$$

TOPIC: PERMUTATIONS AND COMBINATIONS

SPECIFIC OUTCOMES

- Describe permutations and combinations
- Calculate permutations and combinations of 'n' items
- Calculate 'n factorial' (n!).
- Solve problems on linear arrangement and selection

1.1 Describe Permutation and Combination

1.1.1 Permutation

A permutation is simply an arrangement of items; consider three letters A, B and C arranged in a linear manner, we have

ABC
ACB
BAC
BCA
CAB
CBA

All these are possible arrangements or permutations of the three letters

In short in permutation attention is paid to the order in which items are arranged.

1.1.2 Combinations

A combination is about the number of possible choices or selections. Note that a combination does not depend on order but the content of the group, while a permutation does.

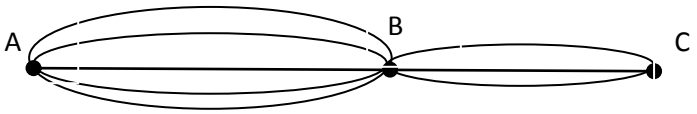
It is extremely important to distinguish between permutation case and a combination case. This will help you determine which formula or approach to use.

2.1 Calculate Permutations and combinations

2.1.1 Permutations of Objects which are all different

Example 1

How many ways are there of getting from A to C, passing through each point at most once?



How many **arrangements** are there of the letter

(i) DOG
(ii) COMPUTER
(iii) DUCKLING

Solutions

In general number of arrangements, or permutations, of n different objects = n(n -1)(n – 2)……..3 x 2 x 1= n!

(i) Since the word DOG has three different characters we have n = 3
Implying that the number of different permutations = 3!
= 3 x 2 x 1
= 6

(ii) Here, n = 8
Therefore number of arrangements = 8!
= 8 x 7 x 6 x 5 x 4 x 3 x 2 x 1
= 40,320

(iii) Here, n = 9
There number of arrangements = 9!
= 362,880----------use the calculator to evaluate

2.1.2 Arranging r objects from n different objects

Example 2

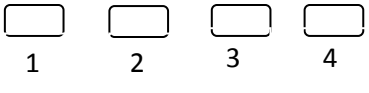
How many arrangements are there of

(a) The four letters from the word MICROWAVE
(b) Five letters from the word GARNISH?

Solutions

(a)

- This can be done in two ways; first consider the four letters to be arranged from the word MICROWAVE as slots which are to be filled up



Note that for each first each of the 9 choices for first slot there 8 choices for the second , 7 for the third and 6 for the fourth

Therefore number of possible arrangements = 9 x 8 x 7 x 6
= 3, 024

- The second method involves the use of the formula;
In general if r objects are arranged from n different items then the number of permutations is given by
${}^{n}P_{r}$ = n(n – 1)(n – 2)…….(n – r + 1)
$= \frac{n!}{(n-r)!}$
From the example r = 4 and n = 9

Therefore number of permutations of four letters from the word MICROWAVE is

$= \frac{9!}{(9-4)!} = 3,024$

(b) Check if your answer is 2, 520

Demonstrate the use of a calculator in solving these problems by applying the function ${}^{n}P_{r}$ in shift mode.

2.1.3 Arrangement of like and unlike things

The number of arrangement of n things in which p are of one kind, q of another and r of another is given by $\frac{n!}{p!q!r!}$

Example

In how many ways can 4 red, 3 yellow and 2 green discs be arranged in a row if discs of the same color are indistinguishable?

Solutions

Here n = 4+3+2 = 9

Therefore the number of arrangements = $\frac{9!}{4!3!2!}$

= 1260 ways

Example 2

Find(a) in how many ways the letters of the word ALGEBRA can be arranged in a row

(b)in how many of these arrangements are the two A's together,(c) in how many arrangements the two A's are not together.

Solutions

(a) There are 7 letters including the two A's

Therefore total arrangements = $\frac{7!}{2!}$

= 2520

(b) If the two A's are together there are effectively 6 letters to arrange, hence the number of arrangements = 6! = 720

(c) If the two A's are not together = 2520 – 720

= 1800

TRY

1. In how many ways can the letters of the following word be arranged in a row

(a) PARALLELOGRAM

(b)PHOTOGRAPHY

2. In how many ways can the letters of the word GEOMETRY

(a)Be arranged in a row

(b) In how many of these arrangements are the two E's together?

(c) In how many of these arrangements are the two E's not together?

2.0 Combinations

2.0 Def: A combination is selection. In making a selection from a number of items only the contents of the group selected are important and not the order in which the items are selected. Consider the following AD and DA has the same combination but different permutations

2.2 Formula

The number of possible combinations of n different objects, taking r at a time, is given by ${}^{n}C_{r}$ also written as $\binom{n}{r}$ where ${}^{n}C_{r} = \frac{n!}{(n-r)!r!}$

Example

How many selections of 4 letters can be made from the 6 letters a,b,c,d,e and f?

SOLUTION

Here n = 6 and r= 4

${}^{n}C_{r} = \frac{n!}{(n-r)!r!} = {}^{6}C_{4} = \frac{6!}{(6-4)!4!}$

= 15

Example 2

How many different committees, each consisting of 3 boys and 2 girls, can be chosen from 7 boys and 5 girls?

SOLUTIONS

Number of ways of choosing 3 boys from 7 = $\frac{7!}{(7-3)!3!} = 35$

Number of ways of choosing 2 girls from 5 = $\frac{5!}{(5-2)!2!} = 10$

Number of committees that can be chosen = 35 x 10 = 350

Note that 35 is multiplied by 10 since the choice of the boys and the choice of the girls are independent operations

Example 3

A group consists of 4 boys and 7 girls. In how many ways can a team be selected if it is to contain

(a) No boys
(b) At least one of each sex
(c) 2 boys and 3 girls
(d) At least 3 boys

Solutions

The solution of this kind of a problem is best solved using a table indicating the possible combinations as shown below.

Boys 4	Girls 7
0	5
1	4
2	3
3	2
4	1
5	0

(a) No boys selected, so the team is chosen from the 7 girls

${}^{7}C_{5} = 21$

(b) At least one of each sex

Total number of possible selections without restrictions = ${}^{11}C_{5} = 462$

Thus, number of possible selections = 462 -21

= 441

(c) Try to work out this one and check your answer

PAST EXAM QUESTIONS

1.1 A family consists of a father, mother and six children. Find the number of ways of arranging them in a straight line if

(a) There are no restrictions

(b) The arrangements start and end with a parent

[4030/2/2014]

2.1 A chess team consisting of 8 boys and 5 girls is to be chosen from 10 boys and 7 girls. In how many ways can this be done

[4030/2/2014]

3.1

(a) Calculate the number of arrangements of the letters in the word PREVIOUSLY

(b) A School has three grade 10 classes. In how many can three out of 8 Mathematics teachers be allocated to the classes?

(c) A group of 7 students is to be chosen from 11 boys and 9 girls. Find the number of ways of choosing 5 boys and 2 girls.

[4030/2/2013]

4.1

(a) In how many ways can 7 red marbles and 3 green marbles be put in a straight line, if

(i) There are no restrictions

(ii) Green marbles should not be next to each other?

(b) A group of 6 pupils is to be chosen from 10 boys and 8 girls. Find the of ways of choosing at least 4 boys.

[4030/2/2012

(a) Calculate the number arrangements of the word ‘’DISCOVERY’.

(b) A group of 7 pupils is to be chosen from 11 boys and 9 girls. Find the number of ways of choosing

(i) 7 pupils

(ii) At least 5 boys

5.1

(a) Calculate the number of arrangements of the name GROSJEAN

(b) Seven runners are hoping to take part in a race, but the truck has only five lanes. In how many ways can five of the seven runners be assigned to the lanes.

(c) A team of five persons is chosen from 7 women and 8 men. How many different teams can be selected, if the team has to contain at least 3 women?

[4030/2/2008]

6.1

(a) In how many ways can 5 boys and 3 girls stand in a straight line, if

(i) There are no restrictions

(ii) The boys stand next to each other

(b) A collection of 16 books contain one of Harry Potter’s book, Mary is going to choose 6 of these books to take on holiday.

(i) In how many ways can she choose 6 books

(ii) How many of these choices will include the Harry Potter's books?

[4030/2/2006]

ADDITIONAL MATHEMATICS

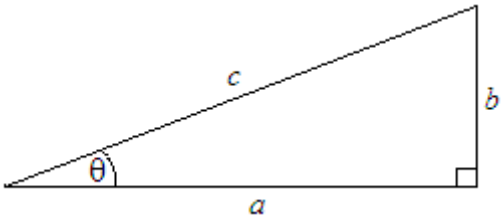
1. TRIGONOMETRIC FUNCTIONS

1.1 Trigonometrical ratios

The six trigonometrical ratios are sine(sin), cosine(cos), tangent(tan), cosecant(cosec), secant(sec) and cotangent(cot).

Definitions

The six trigonometrical ratios are derived from the right-angled triangle.

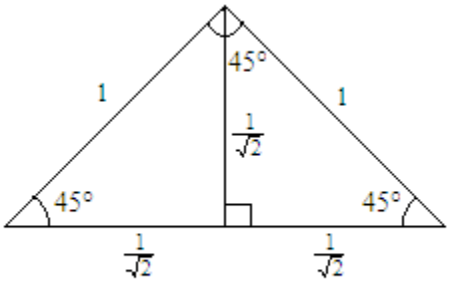


$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{b}{c}$, $\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{a}{c}$ and $\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{b}{a}$.

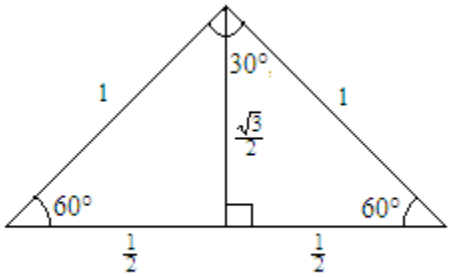
The ratios $\sec\theta, \operatorname{cosec}\theta$ and $\cot\theta$ are reciprocals of $\cos\theta, \sin\theta$ and $\tan\theta$ respectively.

This means $\sec\theta = \frac{1}{\cos\theta} = \frac{c}{a}$, $\operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{c}{b}$ and $\cot\theta = \frac{1}{\tan\theta} = \frac{a}{b}$.

1.2 Special angles



Isosceles triangle



Equilateral triangle

The angles 0°, 30°, 45°, 60° and 90° (in radians 0, $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$) are known as special angles. The sine, cosine and tangent of 30° and 60° are derived from an equilateral triangle of unit length and the sine, cosine and tangent of 45° are derived from a right angled isosceles triangle.

The table below shows the values of the sine, cosine and tangent of special angles.

θ (degs)	0°	30°	45°	60°	90°
θ (rads)	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1

$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

Examples

Find the exact value of each of the following:

(a) $3\tan 45^0 + 4\sin 60^0$ (b) $7\sin 45^0 - 4\cos^2 60^0$

Solutions

If the question asks you to find 'the exact value' leave your answer in surd form. Do not use your calculator.

(a) $3\tan 45^0 + 4\sin 60^0 = 3(1) + 4(\frac{\sqrt{3}}{2})$

$= 3 + 2\sqrt{3}$

(b) $7\sin 45^0 - 4\cos^2 60^0 = 7(\frac{1}{\sqrt{2}}) - 8(\frac{1}{2})^2$

$= \frac{7}{\sqrt{2}} - 8(\frac{1}{4})$

$= \frac{7}{\sqrt{2}} - 2$

If you want you can rationalize the denominator of the first term so that your final answer becomes $\frac{7}{\sqrt{2}} - 2 = \frac{7}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} - 2$

$= \frac{7\sqrt{2}}{2} - 2$ or $\frac{1}{2}(7\sqrt{2} - 4)$

Exercise

Find the exact value of each of the following:

(a) $2\sin^2 60^0 + 5\tan^2 60^0 - \cos ec\, 30^0$ (b) $4(1 - \sin 45^0 \cos ec\, 45^0) - \cot 60^0$

(c) $6\sqrt{3}\cos 30^0 - 9\sqrt{2}\sin 60^0 + (\sqrt{5})^2$ (d) $\dfrac{1 + 2\tan 30^0 + \sec 60^0}{1 - 2\tan 30^0 + \sec 60^0}$

(e) $10\sin\frac{\pi}{6} + 42\cos\frac{\pi}{3} - \sqrt{3}\cos ec\frac{\pi}{3}$

Answers

(a) $\frac{25}{2}$ (b) $-\frac{1}{\sqrt{3}}$ (c) $14 - \frac{9\sqrt{6}}{2}$ (d) $\dfrac{31+12\sqrt{3}}{23}$ (e) 23

1.3 Trigonometrical Identities

At this level the following trigonometrical identities should be mastered. If you master them you will have little or no difficulties at all when trying to answer questions that will ask you to prove given identities.

Pythagorean identities

(i) $\sin^2\theta + \cos^2\theta = 1$ (ii) $\sec^2\theta = 1 + \tan^2\theta$ (iii) $\cos ec^2\theta = 1 + \cot^2\theta$

(iv) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ (v) $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

Examples

(i) Prove that $(\sec x + \sec x \ \text{cosec}\, x)(1-\sin x) = \cot x$.

(ii) Show that $\dfrac{1+\cos\theta}{1-\cos\theta} - \dfrac{1-\cos\theta}{1+\cos\theta} = 4\cot\theta\,\text{cosec}\,\theta$

Solutions

(i) $\text{LHS} = (\sec x + \sec x \ \text{cosec}\, x)(1-\sin x) = \left(\dfrac{1}{\cos x} + \dfrac{1}{\cos x \sin x}\right)(1-\sin x)$

$$= \left(\frac{1+\sin x}{\cos x \sin x}\right)(1-\sin x)$$

$$= \frac{(1+\sin x)(1-\sin x)}{\cos x \sin x}$$

$$= \frac{1-\sin^2 x}{\cos x \sin x}$$

$$= \frac{\cos^2 x}{\cos x \sin x}$$

$$= \frac{\cos x}{\sin x}$$

$$= \cot x$$

(ii) $\text{LHS} = \dfrac{1+\cos\theta}{1-\cos\theta} - \dfrac{1-\cos\theta}{1+\cos\theta} = \dfrac{(1+\cos\theta)^2 - (1-\cos\theta)^2}{(1-\cos\theta)(1+\cos\theta)}$

$$= \frac{1+2\cos\theta+\cos^2\theta - (1-2\cos\theta+\cos^2\theta)}{(1-\cos\theta)(1+\cos\theta)}$$

$$= \frac{1+2\cos\theta+\cos^2\theta - 1+2\cos\theta-\cos^2\theta}{1-\cos^2\theta}$$

$$= \frac{4\cos\theta}{\sin^2\theta}$$

$$= \frac{4\cos\theta}{\sin\theta} \times \frac{1}{\sin\theta}$$

$$= 4\cot\theta\,\text{cosec}\,\theta$$

Exercise

(i) Prove that $(\sec\theta-\tan\theta)^2=\dfrac{1-\sin\theta}{1+\sin\theta}$.

(ii) Prove the identity $\dfrac{\tan^2 x-1}{1-\cot^2 x}=\tan^2 x$.

(iii) Show that $\dfrac{\cos ec\, A}{\cot A+\tan A}=\cos A$

(iv) Prove that $\cos^2\beta-\dfrac{1}{\sec^2-1}=\cot^2\beta\cos^2\beta$.

Compound Angles and Multiple Angles

(i) $\sin(A+B)=\sin A\cos B+\cos A\sin A$ (ii) $\sin(A+B)=\sin A\cos B+\cos A\sin A$

(iii) $\cos(A+B)=\cos A\cos B-\sin A\cos A$ (iv) $\cos(A-B)=\cos A\cos B+\sin A\cos A$

(v) $\sin 2A=2\sin A\cos A$ (vi) $\tan A=\dfrac{2\tan A}{1-\tan^2 A}$

(vii) $\cos 2A=\cos^2 A-\sin^2 A$
$=2\cos^2 A-1$
$=1-2\sin^2 A$

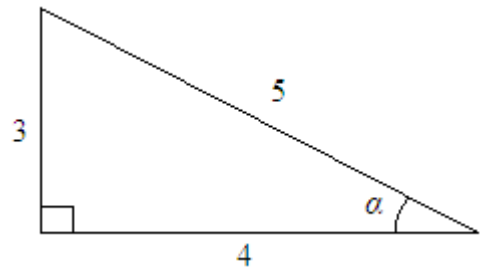
Examples

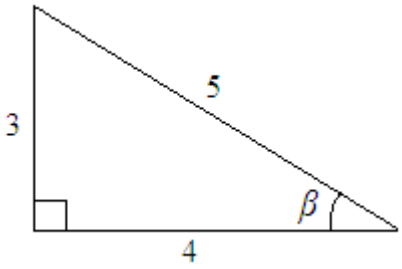
1. It is given that $\sin\beta=\frac{3}{5}$ and $\cos\alpha=\frac{4}{5}$, where α is acute and β is a third quadrant angle. Find the exact value of each of the following.

(i) $\sin\alpha$, $\tan\beta$ and $\tan\alpha$ (ii) $\sin(\beta+\alpha)$ (iii) $\tan(\beta-\alpha)$

Solutions

(i) Draw two right angled triangles with the angles α and β as shown below. Use Pythagoras theorem to find all the sides of the triangles.





From the triangles, $\sin\alpha=\frac{3}{5}$, $\tan\beta=\frac{3}{4}$ and $\tan\alpha=\frac{3}{4}$.

Alternatively:

$\sin^2\alpha=1-\cos^2\alpha$

$\sin^2\alpha=1-\frac{16}{25}$

$\sin^2\alpha=\frac{9}{25}$

$\therefore\ \sin\alpha=\frac{3}{5}$

$\cos^2\beta=1-\sin^2\beta$

$\cos^2\beta=1-\frac{9}{25}$

$\cos^2\beta=\frac{16}{25}$

$\therefore\ \cos\beta=-\frac{4}{5}$ (3rd Quadrant)

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \tfrac{3}{5} \div \tfrac{4}{5} \qquad \tan\beta = \frac{-\sin\beta}{-\cos\beta} = -\tfrac{3}{5} \div -\tfrac{4}{5}$$

$$\therefore \tan\alpha = \tfrac{3}{4} \qquad \therefore \tan\beta = \tfrac{3}{4}$$

(ii) $$\sin(\beta+\alpha) = \sin\beta\cos\alpha + \cos\beta\sin\alpha$$
$$= -\tfrac{3}{5}\times\tfrac{4}{5} + -\tfrac{4}{5}\times\tfrac{3}{5}$$
$$= -\tfrac{24}{25}$$

(iii) $$\tan(\beta-\alpha) = \frac{\sin(\beta-\alpha)}{\cos(\beta-\alpha)} = \frac{\sin\beta\cos\alpha - \cos\beta\sin\alpha}{\cos\beta\sin\alpha + \sin\beta\sin\alpha}$$
$$= \frac{-\tfrac{3}{5}\times\tfrac{4}{5} - (-\tfrac{4}{5})\times\tfrac{3}{5}}{-\tfrac{4}{5}\times\tfrac{3}{5} + (-\tfrac{3}{5})\times\tfrac{3}{5}}$$
$$= 0$$

Exercise

(a) Find the exact value of each of the following:

(i) $\sin 105^0$ (ii) $\cos 75^0$ (iii) $\tan 15^0$

(b) Show that $\cos 2\theta = \cos^2\theta - \sin^2\theta$.

(c) Prove that $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$.

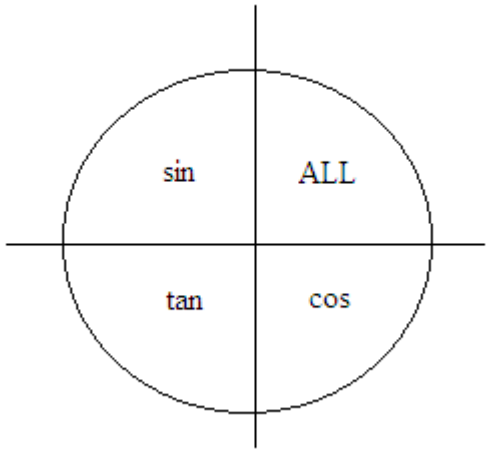
(d) It is given that $\sin\theta = -\tfrac{3}{5}$ and $\tan\alpha = \tfrac{3}{4}$, where α is acute and θ is a third quadrant angle. Find the exact value of each of the following.

(i) $\sin\alpha$, $\cos\theta$ and $\cos\alpha$ (ii) $\sin(\theta+\alpha)$ (iii) $\tan(\theta-\alpha)$

1.4 Trigonometrical equations

The Four Quadrants

The sign (+ve or −ve) of a trigonometrical ratio depends on the size of the angle
The sign is thus determined by the quadrant where it is located.



The diagram above shows the quadrants in which sine, cosine and tangent is positive.

The principle angle α is the acute angle in the first quadrant, where all the trigonometrical ratios are positive. A trigonometrical equation has an infinite number of solutions. This is why a range of values of solutions is normally given. To find all the solutions in a given range, the principle angle α should be determined.

$$\text{If } \theta \text{ is in the } 1^{st} \text{ quadrant : } \theta = \alpha$$
$$\text{If } \theta \text{ is in the } 2^{nd} \text{ quadrant : } \theta = 180 - \alpha$$
$$\text{If } \theta \text{ is in the } 3^{rd} \text{ quadrant : } \theta = 180 + \alpha$$
$$\text{If } \theta \text{ is in the } 4^{th} \text{ quadrant : } \theta = 360 - \alpha$$

Examples

Solve the following equations for angles between 0° and 360°.

(a) $\sqrt{2}\sin\theta = 1$ (b) $2\cos 2x = -\sqrt{3}$

Solutions

(a)
$$\sqrt{2}\sin\theta = 1$$
$$\sin\theta = \tfrac{1}{\sqrt{2}}$$
$$\alpha = \sin^{-1}(\tfrac{1}{\sqrt{2}}) = 45^0$$

Sine is positive in the 1st and 2nd quadrant.

1st quadrant : $\theta = \alpha = 45^0$, 2nd quadrant : $\theta = 180^0 - \alpha = 180^0 - 45^0 = 135^0$

(b)
$$2\cos 2x = -\sqrt{3}$$
$$\cos 2x = -\tfrac{\sqrt{3}}{2}$$
$$\alpha = \cos^{-1}(\tfrac{\sqrt{3}}{2}) = 30^0$$

Cosine is negative in the 2nd and 3rd quadrant.

2nd quadrant : $2x = 180^0 - \alpha = 180^0 - 30^0 = 150^0$ $\therefore\ x = 75^0$

The coefficient of x in 2x, that is 2, implies you can make another complete turn (360°). This means add 360° to 150°.

$$2x = 360^0 + 150^0 \Rightarrow x = 255^0$$

3rd quadrant : $2x = 180^0 + \alpha = 180^0 + 30^0 = 210^0$ $\therefore\ x = 105^0$

$$2x = 360^0 + 210^0 \Rightarrow x = 285^0$$

Equations of the form $a\cos\theta + b\sin\theta = c$

(i) When $c = 0$, the equation can be simplified by converting it to one involving $\tan\theta$ only..

Example:

$$5\cos\theta + 12\sin\theta = 0 \Rightarrow\ 12\sin\theta = -5\cos\theta$$

$$\frac{12\sin\theta}{\cos\theta}=\frac{-5\cos\theta}{\cos\theta},\quad \text{dividing both sides by } \cos\theta$$

$$\tan\theta=\frac{-5}{12}$$

$$\alpha=\tan^{-1}(\tfrac{5}{12})=22.6^0$$

2nd quadrant : $\theta=180^0-22.6=157.4^0$

3rd quadrant : $\theta=180^0+22.6=202.6^0$

(ii) When $c\neq 0$, $a\cos\theta+b\sin\theta=c$ should then be written in the form

$R\cos(\theta-\alpha)=\sqrt{a^2+b^2}\cos(\theta-\alpha)$, where α is acute and $R>0$.

Example: Solve the equation $5\cos\theta+12\sin\theta=10$, for the range $0°\leq\theta\leq 360°$.

$$5\cos\theta+12\sin\theta=R\cos(\theta-\alpha)$$
$$5\cos\theta+12\sin\theta=R\cos\theta\cos\alpha+R\sin\theta\sin\alpha$$

Equating coefficients of $\cos\theta$ and $\sin\theta$: $R\cos\alpha=5$ --- (I) and $R\sin\alpha=12$ --- (II)

Dividing (II) by (I) : $\frac{R\sin\alpha}{R\cos\alpha}=\frac{12}{5}\Rightarrow \tan\alpha=\frac{12}{5}$ and $\therefore\ \alpha=\tan^{-1}(\frac{12}{5})=67.4^0$

Squaring and adding: $R^2\cos^2\alpha+R^2\sin^2\alpha=5^2+12^2$

$$R^2(\cos^2\alpha+\sin^2\alpha)=169$$
$$R=13\ ,\ \text{since}\ \ \cos^2\alpha+\sin^2\alpha=1.$$

Hence $5\cos\theta+12\sin\theta=13\cos(\theta-67.4^0)$

$$10=13\cos(\theta-67.4^0)$$
$$\cos(\theta-67.4^0)=\tfrac{10}{13}$$
$$\theta-67.4^0=\cos^{-1}(\tfrac{10}{13})$$
$$\theta-67.4^0=39.7^0\quad (\text{1st quadrant})$$
$$\theta=107.1^0$$

and $\quad\theta-67.4^0=360-39.7^0\quad (\text{4th quadrant})$

$$\theta=387.7^0$$

Although $\theta=387.7^0$ is a solution, it is inadmissible since it is outside the given range.
Therefore the only solution is $\theta=107.1^0$.

Exercise

(a) $4\cos\theta+3\sin\theta=2$ (b) $\sqrt{3}\cos 2\theta-\sin 2\theta=\sqrt{2}$ (c) $\cos\theta+3\sin\theta=2$

(d) $2\sqrt{2}\sin\theta-12\cos\theta=2$

(e) (i) Express $\sqrt{2}\sin\theta+\sqrt{7}\cos\theta$ in the form $R\sin(x+\alpha)$, where α is acute and $R>0$.

(ii) Hence solve the equation $\sqrt{2}\sin\theta+\sqrt{7}\cos\theta=2$, for the domain $0° \leq \theta \leq 360°$.

Answers

(a) 36.8° (b) 7.5°, 142.5°, 187.5°, 322.5° (c) 20.8°, 122.3°

(d) 61.3°, 157.7° (e)(i) $3\sin(\theta-28.1^0)$ (ii) 69.9°, 166.3°

THE BINOMIAL THEOREM

A **binomial** is an algebraic expression containing 2 terms. For example, $(x + y)$ is a binomial.

We sometimes need to expand binomials as follows:

$(a + b)^0 = 1$

$(a + b)^1 = a + b$

$(a + b)^2 = a^2 + 2ab + b^2$

$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

Clearly, doing this by direct multiplication gets quite tedious and can be rather difficult for larger powers or more complicated expressions

The Binomial Theorem

We use the **binomial theorem** to help us expand binomials to any given power without direct multiplication. As we have seen, multiplication can be time-consuming or even not possible in some cases.

Binomial Theorem Formula

Based on the binomial properties, the binomial theorem states that the following **binomial formula** is valid for all positive integer values of n:

This can be written more simply as:

$$(a + b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + {}^nC_3a^{n-3}b^3 + ... + {}^nC_nb^n$$

1. **Expand $(x^2 + 3)^6$**

$$(x^2 + 3)^6 = {}_6C_0\,(x^2)^6(3)^0 + {}_6C_1(x^2)^5(3)^1 + {}_6C_2\,(x^2)^4(3)^2 + {}_6C_3\,(x^2)^3(3)^3$$

$$+ {}_6C_4\,(x^2)^2(3)^4 + {}_6C_5\,(x^2)^1(3)^5 + {}_6C_6\,(x^2)^0(3)^6$$

$$(1)(x^{12})(1) + (6)(x^{10})(3) + (15)(x^8)(9) + (20)(x^6)(27)$$

$$+ (15)(x^4)(81) + (6)(x^2)(243) + (1)(1)(729)$$

$$= \mathbf{x^{12} + 18x^{10} + 135x^8 + 540x^6 + 1215x^4 + 1458x^2 + 729}$$

2. Expand $(2x - 5y)^7$

I'll plug "$2x$", "$–5y$", and "7" into the Binomial Theorem, counting up from zero to seven to get each term. (I mustn't forget the "minus" sign that goes with the second term in the binomial.)

$$(2x - 5y)^7 = {}_7C_0\,(2x)^7(–5y)^0 + {}_7C_1\,(2x)^6(–5y)^1 + {}_7C_2\,(2x)^5(–5y)^2$$
$$+ {}_7C_3\,(2x)^4(–5y)^3 + {}_7C_4\,(2x)^3(–5y)^4 + {}_7C_5\,(2x)^2(–5y)^5$$
$$+ {}_7C_6\,(2x)^1(–5y)^6 + {}_7C_7\,(2x)^0(–5y)^7$$

$$(1)(128x^7)(1) + (7)(64x^6)(–5y) + (21)(32x^5)(25y^2) + (35)(16x^4)(–125y^3)$$
$$+ (35)(8x^3)(625y^4) + (21)(4x^2)(–3125y^5) + (7)(2x)(15625y^6)$$
$$+ (1)(1)(–78125y^7)$$
$$= \mathbf{128x^7 – 2240x^6y + 16800x^5y^2 – 70000x^4y^3 + 175000x^3y^4 – 262500x^2y^5}$$
$$\mathbf{+ 218750xy^6 – 78125y^7}$$

3. What is the fourth term in the expansion of $(3x – 2)^{10}$?

$$(3x – 2)^{10} = {}_{10}C_0\,(3x)^{10–0}(–2)^0 + {}_{10}C_1\,(3x)^{10–1}(–2)^1 + {}_{10}C_2\,(3x)^{10–2}(–2)^2$$
$$+ \mathbf{{}_{10}C_3\,(3x)^{10–3}(–2)^3} + {}_{10}C_4\,(3x)^{10–4}(–2)^4 + {}_{10}C_5\,(3x)^{10–5}(–2)^5$$
$$+ {}_{10}C_6\,(3x)^{10–6}(–2)^6 + {}_{10}C_7\,(3x)^{10–7}(–2)^7 + {}_{10}C_8\,(3x)^{10–8}(–2)^8$$
$$+ {}_{10}C_9\,(3x)^{10–9}(–2)^9 + {}_{10}C_{10}\,(3x)^{10–10}(–2)^{10}$$

$(x + y)^2 = x^2 + 2xy + y^2$ (second power: three terms)

$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$ (third power: four terms)

$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$ (fourth power: five terms)

The expansion in this exercise, $(3x – 2)^{10}$, has power of $n = 10$, so the expansion will have eleven terms, and the terms will count up, not from 1 to 10 or from 1 to 11, but from 0 to 10. This is why the fourth term will not the one where I'm using "4" as my counter, but will be the one where I'm using "3".

$${}_{10}C_3\,(3x)^{10–3}(–2)^3 = (120)(2187)(x^7)(–8) = \mathbf{–2099520x}$$

4. Find the tenth term in the expansion of $(x + 3)^{12}$.

To find the tenth term, I plug *x*, 3, and 12 into the Binomial Theorem, using the number 10 – 1 = 9 as my counter:

$${}_{12}C_9\,(x)^{12-9}(3)^9 = (220)x^3(19683) = \mathbf{4330260}\boldsymbol{x^3}$$

5. Find the middle term in the expansion of $(4x - y)^8$.

Since this binomial is to the power 8, there will be nine terms in the expansion, which makes the fifth term the middle one. So I'll plug 4*x*, –*y*, and 8 into the Binomial Theorem, using the number 5 – 1 = 4 as my counter.

$${}_8C_4\,(4x)^{8-4}(-y)^4 = (70)(256x^4)(y^4) = \mathbf{17920}\boldsymbol{x^4y^4}$$

REMAINDER AND FACTOR THEOREM

1. Find the remainder when $3x^3 - x^2 - 5x + 2$ is divided by $3x + 2$

   If f (x) is divided by (3x + 2)

   $3x + 2 = 0$

   $x = \frac{-2}{3}$

   $f(x) = 3x^3 - x^2 - 5x + 2$

   $f(\frac{-2}{3}) = 3(\frac{-2}{3})^3 - (\frac{-2}{3})^2 - 5(\frac{-2}{3}) + 2$

   $= 3(\frac{-8}{27}) - \frac{4}{9} + \frac{10}{3} + 2$

   $= 4$

2. Factorise completely the expression

   $2x^3 - 11x^2 - 20x - 7$

   By trial let $p(x) = 2x^3 - 11x^2 - 20x - 7$

   If x = -1

   $P(-1) = 2(-1)^3 - 11(-1)^2 - 20(-1_ -7$

   $= -11 + 20 - 7$

   $= 0 \quad \therefore$ (x + 1) is a factor

   $2x^3 - 11x^2 - 20x - 7 = (x + 1)(2x^2 - 13x - 7)$

   $= (x + 1)(2x^2 - 14x + x - 7)$

   $= (x + 1)[2x(x - 7) + 1(x - 7)]$

   $= (x + 1)(2x + 1)(x - 7)$

3. A curve has equation $y = 3x^3 - 2x^2 + 2x$ show that the equation of the tangent to the curve at the point where x = 1 is

$y = 7x - 4$

$y = 3x^3 - 2x^2 = 2x$

When x = 1 $y = 3\,(1)^3 - 2\,(1)^2 = 2\,(1)$

$= 3 - 2 + 2$

$= 3 + 0 = 3$ $\therefore$ (1, 3) is the point on the curve

$\frac{dy}{dx} = 9x^2 - 4x + 2$

When x = 1 $\frac{dy}{dx} = 9\,(1)^2 - 4\,(1) + 2$

$= 9 - 4 + 2$

$= 7$

Gradient of tanget at (1,3) = 7

$y - 3 = 7\,(x - 1)$

$y - 3 = 7x - 7$

$y = 7x - 7 + 3$

$y = 7x - 4$ shown

4. Give that $f(x) = 2x^3 - 7x^2 + 7ax + 16$ is divisible by x – a, find

(i) the value of the constant a

(ii) the remainder when f (x) is divided by 2x + 1

(i) $f(x) = 2x^3 - 7x^2 + 7ax + 16$

f (x) is divisible by x – a

$f(a) = 0$

$2a^3 - 7a^2 + 7a\,(a) + 16 = 0$

$2a^3 - 7a^2 + 7a^2 + 16 = 0$

$\frac{2}{2}a^3 + \frac{16}{2} = 0$

$a^3 + 8 = 0$

$a^3 = -8$

$a^3 = (-2)^3$ $\therefore a = -2$

(ii) Since a = -2

$f(x) = 2x^3 - 7x^2 - 14x + 16$

$f(\frac{-1}{2}) = 2\,(\frac{-1}{2})^3 - 7\,(\frac{-1}{2})^2 - 14\,(\frac{-1}{2}) + 16$

$= 2\left(\frac{-1}{8}\right) - 7\left(\frac{1}{4}\right) + \frac{14}{2} = 16$

$= -\frac{1}{4} - \frac{7}{4} + 7 + 16$

$= 21$

5. Given that: $2x^3 + 5x2 - 6x - 5 = (Ax - 3)(x = B)(x + 1) + C$ for all the values of x, find the values of each of A, B and C.

$2x3 = 5x^2 - 6x - 5 = (Ax - 3)(x + B)(x + 1) + C$

$= (Ax^2 + ABx - 3x - 3B)(x + 1) + C$

$= [Ax^3 + (AB - 3)x2 - 3Bx + Ax^2 + (AB - 3)x - 3B] + C$

Comparing coefficients of x^2

$Ax^3 = 2x^3$

$A = 2$

$5 = A - 3 + AB$

$5 = 2 - 3 + 2B$

$-1 + 2B = 5$

$B = \frac{6}{2}$

$B = 3$

$-5 = -3b + C$

$-3B + C = -5$

$-9 + C = -5$

$C = -5 + 9$

$= +4 \quad \therefore A = 2, B = 3, C = 3$

6. Given that $x^2 + 2x - 3$ is a factor of f (x), where $f(x) = x^4 + 6x^3 + 2ax + bx - 3a$, find

(i) the value of a and the value of b

(ii) the other quadratic factor of f (x)

(i) $x^2 + 2x - 3 = (x - 1)(x + 3)$ f (1), f (-3)

$f(x) = x^4 + 6x^3 + 2ax^2 + bx - 3a$

$f(1) = 1^4 + 6(1)^3 + 2a(1)^2 + b(1) - 3a$

$1 + 6 + 2 + b - 3a = 0$

$b - a = -7 \quad$ (i)

$f(-3) = (-3)^4 + 6(-3)^3 + 2a(-3)^2 + b(-3) - 3a$

$81 - 162 + 18a - 3b - 3a = 0$

$15a - 3b = 81$

$5a - b = 27$ (ii)

Solve simultaneously

$-a + 5a = -7 + 27$ $\quad b - 5 + -7$

$\frac{4a}{4} = \frac{20}{4}$ $\quad b = -7 + 5$

$a = 5$ $\quad = -2$ $\quad \therefore$ <u>$a = 5, b = -2$</u>

(ii) $x^4 + 6x^3 + 2ax^2 + bx - 3a$

$= x^4 + 6x^3 + 2(5)x^2 = -2x - 3(5)$

$= x^4 = 6x^3 + 10x^2 - 2x - 15$

$= (x^2 + 2x - 3)(x^2 + 4x = 5)$

$\therefore$ other factor = <u>$x^2 + 4x + 5$</u>

7. Find the remainder when $2x^4 + 5x^2 - 7$ is divided by $x + 3$

$$
\begin{array}{r|l}
 & 2x^3 - 6x^2 + 23x - 69 \\
\hline
x + 3 & 2x^4 + 0x3 + 5x^2 + 0x - 7 \\
 & -\ 2x^4 + 6x^3 \\
\hline
 & \quad -\ 6x^3 + 5x^2 \\
 & \quad -\ 6x^3 - 18x^2 \\
\hline
 & \quad\quad -\ 23x^2 + 0x \\
 & \quad\quad 23x^2 + 69x \\
\hline
 & \quad\quad\quad -69x - 7 \\
 & \quad\quad\quad -69x - 207 \\
\hline
 & \quad\quad\quad 200 \text{ Remainder}
\end{array}
$$

ARITHMETIC AND GEOMETRIC PROGRESSIONS

SEQUENCE

Sequence is a set of numbers listed in a well defined order with a specific rule that can be used to state the next numbers in that set.

1. Write the next three terms of each of the sequences below

(a) 1, 2,4,8,…….

Answers

16, 32,64

(b)-4,-1,2,5,8,11

Answers

14,17,20

Series

A series is the sum of all the terms of a sequence e.g

(i) 1+2+4+8+16+,…….

Answers

14,17,20

1. . For the AP, 2+5+8+…… find

(i) The 10^{th} term

(ii)

Answers

a=2, first term

d = 5 - 2

D = 3

T_{10} = a+ (n-1)d

= 2+ (10-1)3

= 29

(iii) The 51 st term

Answers

$T_{51} = 2+ (51 - 1)3$
- $= 2+ 50 \times 3$
$= 152$

(iv) The n^{th} term
(v)

Answers

$T_n = a + (n - 1) d$
$= 2+ (n - 1)3$
$= 2 + 3n - 3$
$= 3n - 1$

3. find the number of terms in the AP 3 + (-1) + (-5) +….+ (-53).

Answers

$a = 3, d = -4, \quad T_n = -53$
$T_n = a + (n - 1)d$
$-53 = 3 + (n - 1) -4$
$-53 = 3 - 4n + 4$
$4n = 53 + 7$
$\frac{1}{4} \times 4n = 60 \times \frac{1}{4}$
$n = 15$

4. The 10^{th} term of an AP is 37 and the 16^{th} term is 61, for this AP find:

(i) The common difference

Answers

$T_n = a + (n - 1)d$
$T_{10} = a + 9d$
$37 = a + 9d$……eqn 1
and
$T_{16} = a + (16 - 1)d$
$16 = a+15d$……..eqn 2 and solve the equations simultaneously.
$a + 9d = 37$
$-(a+15d = 61)$

$-\frac{6}{4} = \frac{-24}{4}$

$d = 4$

(ii) The first term

Answers

First term

$a + 9d = 37$

$a + 9(4) = 37$

$a + 36 = 37$

$a = 37 - 36$

$a = 1$

(iii) the 30^{th} term

Answers

$Tn = a + (n - 1)$

$T_{30} = 1 + (30 - 1)4$

$T_{30} = \underline{117}$

5. The n^{th} term (Tn) of an AP is given by $Tn = 1/2(4n - 3)$.

(a) State (i) the 5^{th} term (ii) the 10^{th} term (iii) the 6^{th} term

Answers
The 5^{th} term
$T_5 = 1/2(4n - 3)$
$T_5 = 1/2(4 \times 5n - 3$
$T_5 = 1/2(17)$
$T_5 = 8.5$

Answers
$T_{10} = 1/2(4 \times 10 - 30)$
$T_{10} = 1/2(40 - 3)$
$T_{10} = 18.5$

$T_6 = 1/2(4 \times 6 - 3)$
$T_6 = 1/2(24 - 3)$
$T_6 = 1/2(21)$
$T_6 = 10.5$

(b) the common difference

Answers
$d = T_6 - T_5$
$d = 10.5 - 8.5$
$d = \underline{2}$

Therefore, the common difference is 2.

6. If $x + 1$, $2x - 1$ and $x + 5$ are three consecutive terms, find the value of x.

Answers

X + 1 , 2x - 1 , x + 5
T_1 T_2 T_3

For an AP,

Common difference, d = $T_2 - T_1 = T_3 - T_1$
(2x - 1) - (x + 1) = (x + 5) - (2x - 1)
2x - 1 - x - 1 = x + 5 - 2x
2x - x - 1 - 1 = x - 2x + 5 + 1
X - 2 = -x + 6
X + x = 2 + 6
2x = 8
X=4

7. (i) if the numbers 3,m,n and 8 are three consecutive terms of an AP, find the values of m and n.

Answers

M - 3 = n - m
M + m = n + 3
2m = n + 3
m= $\frac{n+3}{2}$ ……..eq1

and n - m = 18 - n
n + n = 18 + m
2n = 18 + m
m = 2n - 18…….eq2

Equate m = m

n + $\frac{3}{1}$ = 2n - 18/1

n + 3 = 2(2n - 18)
n + 3 = 4n - 36
n-4n = -36 -3
-3n = -39
$\frac{-3n}{-3}$ = $\frac{-39}{-3}$
n = 13

for m

m = $\frac{n+3}{2}$

m = $\frac{13+3}{2}$

m = $\frac{16}{2}$

m = 8

Therefore, m = 8 and n = 13

(ii) The numbers m - 1, 4m + 1 and 5m - 1 are three consecutive terms of an AP, find the numbers.

(iii)

Answers

m - 1, 4m + 1, 5m - 1 and 4m + 1 is an arithmetic mean between m - 1 and 5m - 1

$b = \frac{a + c}{2}$

$4m + 1 = \frac{m - 1 + 5m - 1}{2}$

$2(4m + 1) = m + 5m - 1 - 1$

$8m + 2 = 6m - 2$

$8m - 6m = -2 - 2$

$2m = -4$

$M = -2$

Substitute for m = 1 in the series we get, -3, -7 and -11

8.(i) Find the arithmetic mean of the first 6 terms of 3 + 8 +……

Answers

First term
a = 3
Common difference d = 8-3
d = 5

Therefore, the 6 terms are 3,8,13,18,23,28.

Arithmetic mean = $\frac{3 + 8 + 13 + 18 + 23 + 28}{6}$

Arithmetic mean = $\frac{93}{6}$

Arithmetic mean = 15.5

Or

Arithmetic mean = median

$$= \frac{13 + 18}{2}$$

= <u>15.5</u>

(ii) Find the arithmetic mean and the geometric mean of 4 and 64.

Answers

Given 4 and 64

Arithmetic mean = $\frac{4 + 64}{2}$

$$= \frac{68}{2}$$

= <u>34</u>

Geometric mean = square root of 4 and 64

= 2 x 8

= <u>16</u>

9. An arithmetic progression has a 1st term to be 2 and common difference of 2, show that the sum of the first nth terms of the AP is given by $S_n = n^2 + n$. hence find the sum of the 21st terms of an AP.

Answers

A = 2, d = 2

$S_n = \frac{n}{2}(2a + (n-1)d)$

$= \frac{n}{2}(2x2 + (n-1)2)$

$= \frac{n}{2}(4 + 2n - 2)$

$\frac{n}{2}(2 + 2n) =$

$= n + n^2$

$S_n = \underline{n^2 + n}$ is required

The sum of the first 21st terms

$S_{21} = 21^2 + 21$

$S_{21} = 441 + 21$

= 462

10. The sum S_n of the first n terms of an AP is given by $S_n = n^2 + n$, find (i) the first term (ii) common difference (iii) the formula for the sum of the first n - 1 terms

Answers

(i) $S_n = n^2 + 2n$
To find the first term we put n = 1 in the given sum
$S_1 = 1^2 + 2(1)$
$a = 3$

(ii) The common difference
$d = S_2 - 2S_1$
$= 8 - 6$
$= 2$

(iii) $S_n = n^2 + 2n$
$S_{n-1} = (n - 1)^2 + 2(n - 1)$
$= n^2 - n - n + 1 + 2n - 2$
$= n^2 - 2n + 2n + 1 - 2$
$= n^2 - 1$

Geometric Progression (GP)

A geometric progression (GP) is a sequence in which each term is formed by multiplying the previous term by a constant amount.

n^{th} term of a Geometric Progression

The n^{th} term of a GP with first term a and common ratio r is: $T_n = ar^{n-1}$

1. For a GP, 2 + 6 + 18 + ……, find (i) the tenth term (ii) the 17^{th} term

Solution

First term (a) = 2

Common ratio r = $\frac{T_2}{T_1} = \frac{6}{2}$

$r = 3$

n = 10

$T_n = ar^{n-1}$

$T_{10} = 2 \times 3^{10-1}$

$T_{10} = 39366$ So the 10^{th} term is 39 366

(ii) $T_{17} = 2 \text{ X } 3^{17 - 1}$

$= 2 \text{ x } 3^{16}$

$= 86\ 093\ 442$

2. The third term of a GP is 9 and the tenth term is 19 683, find;

(i) the common ratio

(ii) the 8^{th} term

Solutions

(i) $T_3 = ar^2$

$ar^2 = 9$……eqn (i)

$T_{10} = ar^9$

$Ar^9 = 19\ 683$……..eqn (ii)

Dividing equation (i) by equation (ii)

$$\frac{ar^9}{ar^2} = \frac{19683}{9}$$

$\sqrt[7]{r^7} = \sqrt[7]{2187}$ Therefore the common ratio is 3

$r = 3$

(ii) $ar^2 = 9$

$A \text{ x } (3)^2 = 9$

$$\frac{9a}{9} = \frac{9}{9}$$

$a = 1$ The first term is 1

(iii) $Tn = ar^{n-1}$

$T_8 = 1 \text{ x } 3^{8 - 1}$

$= 3^7$

$T_8 = 2187$ the 8^{th} term is 2 187

3. Given that x +2, x + 3 and x + 6 are the first three terms of a GP, find

(a) the value of x

(ii) the 5^{th} term of the GP.

Solutions

(i) Common ratio (r) $\frac{T_2}{T_1} = \frac{T_3}{T_2}$ =

$$\frac{x+3}{x+2} = \frac{x+6}{x+3}$$

$(x + 3)(x + 3) = (x + 2)(x + 6)$

$X^2 + 3x + 3x + 9 = X^2 + 6x + 2x + 12$

$6x + 9 = 8x + 12$

$8x - 6x = 9 - 12$

$2x = -3$

$x = -1\frac{1}{2}$

(ii) First term (a) = x + 2

$$= \frac{-3}{2} + 2$$

$$= \frac{-3+4}{2}$$

$$a = \frac{1}{2}$$

Common ratio (r)= $\dfrac{x+3}{x+2}$

$$= \frac{\frac{-3}{2} + \frac{3}{1}}{\frac{-3}{2} + \frac{2}{1}}$$

$$= \left(\frac{-3}{2} + \frac{3}{1}\right) \div \left(\frac{-3}{2} + \frac{2}{1}\right)$$

$$= \frac{3}{2} \div \frac{1}{2}$$

$$= \frac{3}{2} \times 2$$

$$\underline{\underline{r = 3}}$$

$$T_5 = ar^{n-1}$$

$$= \frac{1}{2} \times 3^4$$

$$= \frac{81}{2} or 40.5$$

The n^{th} term (T_n) of a GP is given by $T_n = 2^{9-n}$. Find

(i) the first term

(ii) the common ratio

(iii) the sum of the first 9 terms.

Solutions

(i) $T_n = 2^{9-n}$

$T_1 = 2^{9-1}$

$T_1 = 2^8$

$T_1 = 256$

$a = 256$

(ii) To find the common ratio, first calculate the second term (T_2)

$T_2 = 2^{9-2}$

$= 2^7$

$= 128$

Common ratio (r) $= \frac{T_2}{T_1}$

$= \frac{128}{256}$

$= \frac{1}{2}$

(iii) Sum

$= \frac{a(1-r^n)}{1-r}$

$= \frac{256[1-(\frac{1}{2})^9]}{1-\frac{1}{2}}$

$= \frac{256(0.998046875)}{\frac{1}{2}}$

$= \frac{255.5}{0.5}$

$Sum = 511$

Sum of a GP

6. Calculate, correct to three significant figures, the sum of the first 8 terms of the GP 12, 8, $5\frac{1}{3}$.....

Solutions

First term a = 12

$r = \frac{8}{12}$ or 0.75

$S_8 = \frac{a(1-r^n)}{1-r}$

$= \frac{12[1-(\frac{3}{4})^8]}{1-\frac{3}{4}}$

$= \frac{12(0.899837085)}{1-\frac{3}{4}}$

$= \frac{10.79864502}{0.25}$

$= 43.19458008$

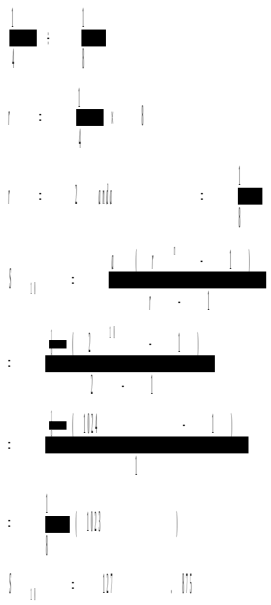
= 43.2 correct to 3 significant figures

$\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \ldots\ldots\ldots$ ..

7. Work out the sum of the first 10 terms of

Solution

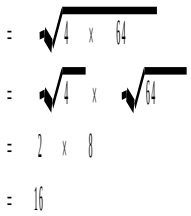
Common ratio (r) =



GEOMETRIC MEAN

8. Find the geometric Mean of 4 and 64.

SOLUTION



9. The sum of infinity of a certain GP is 28. if the first term is 37, find r

Sum to infinity

$$S\infty = \frac{a}{1-r} = 28$$

$$a = 37$$

$$\frac{a}{1-r} = 28$$

$$\frac{37}{1-r} = 28$$

$$28(1-r) = 37$$

$$28 - 28r = 37$$

$$-28r = 37 - 28$$

$$\frac{28r}{28} = \frac{9}{-28} or -032$$

10. Write down the number of terms in the following GP_s

(i) 2 + 4 + 8 + ……..+512

(ii) 81 + 27 + 9 + ……. +

Solution

First term a = 2, r =2
Last term = 512

$$L = ar^{n-1}$$
$$\frac{512}{2} = \frac{2 \times 2^{n-1}}{2}$$
$$256 = 2^{n-1}$$

Factorising 256

2 | 128
2 | 64
2 | 32
2 | 16
2 | 8
2 | 4
2 | 2
2 | 1

$2^8 = 2^{n-1}$
$8 = n - 1$
$n = 8 + 1$
$n = 9$

The GP has 9 terms

(ii) 1 + 27 + 9 + ……. +

$$a = 81, r = \frac{1}{3}, last = \frac{1}{27}$$
$$n = 1 + \frac{\log \frac{L}{a}}{\log r}$$
$$= 1 + \frac{\log \frac{\frac{1}{27}}{81}}{\log \frac{1}{3}}$$
$$= 1 + \frac{\log(\frac{1}{27} \times \frac{1}{81})}{\log \frac{1}{3}}$$
$$n = 1 + \frac{\log(\frac{1}{3})^3 \times (\frac{1}{3})^4}{\log(\frac{1}{3})}$$
$$n = 1 + \frac{\log(\frac{1}{3})^7}{\log(\frac{1}{3})}$$
$$n = 1 + \frac{7\log(\frac{1}{3})}{\log(\frac{1}{3})}$$
$$n = 1 + 7$$
$$n = 8$$

The GP has 8 terms.

DIFFERENTIATION

3.1 Differentiation

Sub - Topic

- The derived function
- Application of the derived function.

3.2 Specific out comes

- Find the derivative of a polynomial $f'(x) = y' = \frac{dy}{dx}$
- Find the derivative of a sum of functions or of composite functions.
- Find derivative of gradient,
- Tangents, normal and stationary points.
- Calculate maxima and minima
- Differentiate exponential functions
- Differentiate trigonometric functions

REVISION OF A DERIVATIVE BY FIRST PRINCIPLE

1. DIFFERENTIATING FUNCTIONS FROM FIRST PRINCIPLES.

EXAMPLES:

1. If $f(x) = 2x - 5$, $f'(x)$ from first principle.

   SOLUTION

$$f'(x) = \lim_{h \to o} \frac{f(x+h) - f(x)}{h}$$

   DATA:

$f(x) = 2x - 5$

$f(x+h) = 2(x+h) - 5$ [Plug in these functions in the formula above]

$$f'(x) = \lim_{h \to o} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to o} \frac{2(x+h) - 5 - (2x-5)}{h}$$

$$= \lim_{h \to o} \frac{2x + 2h - 5 - 2x + 5}{h}$$

$$= \lim_{h \to o} \frac{2h}{h}$$

$$= \lim_{h \to o} 2$$

$$f'(x) = 2$$

2. Find $\frac{dy}{dx}$ from first principle for the function $y = 2x^2$.

SOLUTION:

$$\frac{dy}{dx} = \lim_{h \to o} \frac{f(x+h) - f(x)}{h}$$

DATA.

$$f(x) = 2x^2$$

$$f(x+h) = 2(x+h)^2$$

[plug in these in the formula above]

$$f'(x) = \lim_{h \to o} \frac{f(x+h) - f(x)}{h}$$

$$\frac{dy}{dx} = \lim_{h \to o} \frac{2(x+h)^2 - 2x^2}{h}$$

$$= \lim_{h \to o} + \frac{2(x^2 + 2xh + h^2) - 2x^2}{h}$$

$$= \lim_{h \to o} \frac{2x^2 + 4xh + 2h^2 - 2x^2}{h}$$

$$= \lim_{h \to o} \frac{4xh + 2h^2}{h}$$

$$= \lim_{h \to o} 4x + 2h$$

$$\frac{dy}{dx} = 4x \quad (note\, that\, as\, h \to o, 2h = 0)$$

EXERCISE:

1. Find $f'(x)$ for each of the following functions by first principle.

$(a)\ f(x) = 5x + 4$

$(b)\ f(x) = x^2 - 1$

$(c)\ f(x) = 20x^2 - 6x + 7$

Expected Answers:

$(a)\ f'(x) = 5$

$(b)\ f'(x) = 2x$

$(c)\ f'(x) = 40x - 6$

3.2.1 Finding the Derivative of a polynomial

Even though the syllabus does not mention of differentiating from first principles the student should have profound knowledge in this concept and the concept of limits as outlined above. This revision kit will simply focus on differentiating by rule.

Notation:

In general given a y = f(x)

y' = f'(x) = $\frac{dy}{dx}$

1. **Rule if $y = ax^n$ where a and n are real numbers $f'(x) = a\,nx^{n-1}$**

   Example

   Find the derivatives of the following functions

   (a) $Y = 3x^3 - 4x^2 + 5x + 1$

   (b) $Y = \sqrt{x} + 3x^3$

   Solutions

   (a) $\frac{dy}{dx} = 9x^2 - 8x + 5$

   (b) $\frac{dy}{dx} = ½ \text{ x } ^{½ - 1} + 9x^2$

   $= \frac{1}{2\sqrt{x}} + 9x^2$

2. **Chain – rule Composite functions**

   Let y = u where u is function of x

   It can be shown that $y' = \frac{dy}{dx} = \frac{dy}{du} \text{ x } \frac{du}{dx}$

   Example

   Differentiate each of the following with respect to x

   (a) $Y = \frac{3}{\sqrt{2x+5}}$

   (b) $Y = (3x^4 + 4x^2 - 10)^7$

SOLUTIONS

(a) $Y = \frac{3}{\sqrt{2x+5}}$

$Y = 3(2x + 5)^{-½}$

Let $u = 2x + 5$

Thus $y = 3u^{-½}$

$\frac{dy}{du} = 3 .(- ½ U^{-2/3})$

$\frac{du}{dx} = 2$

BY CHAIN RULE

(a)

$\frac{dy}{dx} = 2 . 3. (- ½ U^{-2/3})$

$= \frac{-3}{\sqrt[3]{(2x+5)^2}}$

(b) $Y = (3x^4 + 4x^2 – 10)^7$

Let $u = 3x^4 + 4x^2 – 10$

Thus $y = U^7$

$U' = 12x^3 + 8x$

$Y' = 7U^6$

By chain rule we have

$\frac{dy}{dx} = U'.y'$

$= (12x^3 + 8x) . 7U^6$

$= 7((3x^4 + 4x^2 – 10)^6.(12x^3 + 8x)$

3. PRODUCT RULE

Given u and v which are both functions of x such that

$Y = uv$

$Y' = u'v + v'u$……product rule

Example

Given that $Y = \frac{3x-4}{1+x^2}$, find the derivative of y with respect to x

Solution

$Y = (3x – 4) (1 + x^2)^{-1}$

Let $u = 3x – 4$ and $v = (1 + x^2)^{-1}$

$U' = 3$ $v' = - 2x (1 +x^2)^{-2}$

By product rule we have $y' = u'v + v'u$

$= 3(1 + x^2)^{-1} + [- 2x (1 +x^2)^{-2}]. (3x – 4)$

$= \frac{3}{1 + x^2} - \frac{2x(3x-4)}{(1+x^2)^2}$

4. QUOTIENT RULE

Given u and v both functions of x such that

Y = $\frac{u}{v}$

Y' = $\frac{vu'-v'u}{v^2}$

Example

Given that y = $\frac{x-2}{2x+3}$, where x ≠ - 1.5 , find derivative of y wrt x

Solution

Let u = x – 2 and v = 2x + 3

U' = 1 v' = 2

By quotient rule we have

Y' = $\frac{(2x+3).1+2(x-2)}{(2x+3)^2}$+

= $\frac{2x+3+2x-4}{(2x+3)^2}$ = $\frac{4x-1}{(2x+3)^2}$

3.2.2 Tangents, Normals and Stationary Points

2.1 Tangents and Normals

- In general the syllabus is concerned with finding the equations of the tangents and the normals at a particular point.
- These concepts are highly related to coordinated geometry so it here by advised that the teacher should emphasize the need to apply the straight line equation in solving these problems.
- The Normal to a curve at one point is the line which passes through the point and is perpendicular to the tangent at that point.
- The equation of the normal at $P_0(x_0, y_0)$ is given by

  $y - y_0 = \frac{-1}{\left(\frac{dy}{dx}\right)} (x - x_0)$

  Example

  Find the equation of the tangent and the normal to $y = x^3 – 2x^2 + 4$ at (2,4)

  Solutions

  $Y' = 3x^2 – 4x$; the slope of the tangent at x =2 is y' (2) = 4

  The equation of the tangent is y – 4 = 4 (x – 2)

  Y = 4x – 4

  The equation of the normal is y – 4 = - ¼ (x – 2)

  x + 4y = 18

  Example 2

  Given that y = $\frac{2x-3}{x-2}$, x ≠ 2, calculate

  (a) The value of k for which $\frac{dy}{dx} = \frac{k}{(x-2)^2}$

  (b) The equation of the normal to the curve at the point where x = 1

  Solutions

  (a) Apply quotient rule to get the derivative of y wrt x

$Y' = \frac{(x-2).2-(2x-3).1}{(x-2)^2}$

$= \frac{2x-4-2x+3}{(x-2)^2}$

$= \frac{-1}{(x-2)^2} = \frac{k}{(x-2)^2}$, as given

Therefore k = - 1

(b) To find the equation of the normal we need to find the gradient of the curve at x = 1 by substituting this value in the expression for $\frac{dy}{dx}$

$\frac{dy}{dx} = -1$, $y_0 = 1$

Equation of normal is given by

$$y - y_0 = \frac{-1}{\left(\frac{dy}{dx}\right)}(x - x_0)$$

$$y - 1 = 1(x - 1)$$

$$y = x - 1 + 1$$

$$y = x$$

Example

Two variables, x and y are related by the equation

$Y = x^2 + \frac{k}{x}$, where k is a constant.

(i) obtain an expression for $\frac{dy}{dx}$

(ii) Find the value of k for which the line y - 4x = 3 is a tangent to the curve where x=3.

(iii) Find the equation of the normal to the curve at the point where x = 3.

Solutions

(i) $\frac{dy}{dx} = 2x - \frac{k}{x^2}$……by rule

(ii) If y – 4x = 3 is tangent to the curve then it gradient is equal to the gradient of the curve at x =3

y = 4x + 3 implying that the gradient of the line is 4

Thus $\frac{dy}{dx} = 4$

Substituting this value in (i) we have

$4 = 2(3) - \frac{k}{3^2}$……..since x = 3

Solving for k we have

K = 18……..verify

(iii) To find the equation of the normal we need to the gradient of the curve at x = 3, which has already been given since $\frac{dy}{dx} = 4$

Thus gradient of the normal is m = – ¼ and the point is (3, 15)

$y - y_0 = m(x - x_0)$

y – 3 = – ¼(x – 15)

4y – 12 = - x + 15

x + 4y = 15 + 12

x + 4y = 27 is the equation of the Normal at x = 3

3.2.3 Calculate maxima and minima

Example

(a) Given that y = $\frac{8}{x^2} + \frac{x^2}{6}$, find the stationary value of y and determine whether it is maximum or minimum.

(b) A cuboid has a total surface area of 150cm^2 and is such that its base s a square of side x cm.

(i) Show that the height h, of the cuboid is h =$\frac{75-x^2}{2x}$.

(ii) Express the volume, v, of cuboid in terms of x.

(iii) Given that that x can varies, find the value of x for which v has a stationary value. Find this value of v and determine whether it is a maximum or minimum.

Solutions

(a) Y = $8x^{-2} + \frac{x^3}{6}$,

Y' = $-16x^{-3} + \frac{3x^2}{6}$,

= $\frac{-16}{x^3} + \frac{x^2}{2}$,

The stationary value is found at $\frac{dy}{dx} = 0$

$\frac{-16}{x^3} + \frac{x^2}{2} = 0$

$-32 + x^5 = 0$

$X^5 = 32$

$X = 2$

Therefore the stationary value is found by substituting the value of x when $\frac{dy}{dx} = 0$, into the original function , x= 2

Y = $8x^{-2} + \frac{x^3}{6}$,

= $8\ (2^{-2}) + \frac{2^3}{6}$,

= 2 + 8/6

= $3\frac{1}{3}$ is the stationary value

To determine whether it is maximum or minimum we use the second derivative test

If $\frac{d^2y}{dx^2} < 0$ maximum value

If $\frac{d^2y}{dx^2} > 0$ minimum value

Since $y' = -16x^{-3} + \frac{3x^2}{6}$,

$y'' = -48x^{-4} + x$

At x = 2

$y''(2) = -48(2)^{-4} + 4$

$= -3 + 4 = 1 > 0$

Therefore the stationary value is minimum.

(b) Try to work out this one.

3.2.3 Differentiating of Exponential functions

Exponential functions occur in various forms. $Y = a^x$ where a is a real number and x is a variable is one such example. However, if $y = e^x$ where e is a natural number is one such a function that the syllabus focuses on.

In general

If $y = e^u$ where u is a function of x then $\frac{dy}{dx} = u'e^u$ ……chain rule

Example

Differentiate the following wrt x

(a) $Y = e^{2x}$

(b) $Y = 4e^{5x + 7}$

(c) $Y = (3x^2 + 4x + 2)e^{3x}$

SOLUTION

Apply the chain rule stated above

3.2.4 Differentiate trigonometric functions

In general if x is angle in radians it is possible to find the derivative of the basic six trigonometric functions. In general if u is function of x

1. y = Sin u ……y' = u Cos u
2. y = Cos u……y' = -u Sin u
3. y = tan u ……....y'' = u Sec^2 u…….try to derive this using quotient rule

EXAMPLE

Differentiate wrt x each of the following

(a) $y = Sin3x^2$

(b) $y = Cos \sqrt{2x + 5}$

(c) $y = \tan e^{5x}$

INTEGRATION

Integration is the reverse of differentiation.

Since integration is the reverse process of differentiation, the **standard integrals** listed in table 1 may be deduced and readily checked by differentiation.

Table 1. STANDARD INTEGRALS

(i) $\int ax^n dx = \frac{ax^{n+1}}{n+1} + c \qquad except\ where\ n = -1$

(ii) $\int cosax\ dx = \frac{1}{a} sin\ \ ax + c$

(iii) $\int sinax\ dx = -\frac{1}{a} cos\ \ ax + c$

(iv) $\int sec^2 ax\ dx = \frac{1}{a} tan\ ax + c$

(v) $\int cosec^2 ax\ dx = \ -\frac{1}{a} cot\ \ ax\ + c$

(vi) $\int e^{ax} dx = \frac{1}{a} e^{ax} + c$

(vii) $\int \frac{1}{x} dx = In\ x + c$

QUESTIONS

1. Determine (a) $\int 5x^2\ dx$ (b) $\int 2t^3 dx$

SOLUTIONS

(a) $\int 5x^2\ dx\ = \frac{5x^{2+1}}{2+1} + c$

$= \mathbf{\frac{5x^3}{3} + c}$ **ans.**

(b) $\int 2t^3 dx = \frac{2t^{3+1}}{3+1} + c$

$= \mathbf{\frac{2t^4}{4} + c}$ ***ans***.

2. Determine $(a) \int \frac{3}{x^2} dx$ $(b) \int 3\sqrt{x}\ dx$

SOLUTIONS

(a) $\int \frac{3}{x^2} dx\ = \int 3x^{-2} dx = \frac{3x^{-2+1}}{-2+1} + c$

$= \ -3x^{-1} + c$

(b) $\int 3\sqrt{x}\ dx = \int 3x^{\frac{1}{2}}\ dx = \frac{3x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$

$= \frac{3x^{\frac{3}{2}}}{\frac{3}{2}} + c$

$= \frac{-3}{x} + c$ ***ans.***

$= 2x^{\frac{3}{2}+c}$

$= \mathbf{2\sqrt{x^3}} +$ $\boldsymbol{c}$

3. Determine (a) $\int 4cos\ 3x\ dx$ (b) $\int 5sin2\theta d\theta$

SOLUTIONS

(a) $\int 4cos\ 3x\ dx = (4)\left(\frac{1}{3}\right) sin3x + c$

$= \frac{4}{3}\boldsymbol{sin3x} + \boldsymbol{c}$

(b) $\int 5sin2\theta d\theta = 5\left(-\frac{1}{2}\right) cos2\theta + c$

$= -\frac{5}{2}\boldsymbol{cos2\theta} + \boldsymbol{c}$

4. Determine (a) $\int 7sec^2\ 4t\ dt$ (b) $\int 3cosec^2\ 2x\ dx$

SOLUTIONS

(a) $\int 7sec^2\ 4t\ dt = (7)\left(\frac{1}{4}\right) \tan 4t + c$

$= \frac{7}{4}\mathbf{tan\ 4}\boldsymbol{t} + \boldsymbol{c}$

(b) $\int 3cosec^2\ 2x\ dx = (3)\left(-\frac{1}{2}\right) cot2x + c$

$= -\frac{3}{2}\boldsymbol{cot2x} + \boldsymbol{c}$

5. Determine (a) $\int 5e^{3x}dx$ (b) $\int \frac{2}{3e^{4t}}dt$

SOLUTIONS

(a) $\int 5e^{3x}dx$ $=5\left(\frac{1}{3}\right)e^{3x} + c$

$= \frac{5}{3}\boldsymbol{e^{3x}} + \boldsymbol{c}$

(b) $\int \frac{2}{3e^{4t}}dt = \int \frac{2}{3}e^{-4t}dt = \left(\frac{2}{3}\right)\left(-\frac{1}{4}\right)e^{-4t} + c$

$= -\frac{1}{6}\boldsymbol{e^{-4t}} + \boldsymbol{c}$

$= -\frac{1}{\boldsymbol{6e^{4t}}} + \boldsymbol{c}$

6. Determine $\int \frac{3}{5x}dx$

SOLUTION

$$\int \frac{3}{5x}dx = \int \left(\frac{3}{5}\right)\left(\frac{1}{x}\right)dx$$

$= \frac{3}{5}\boldsymbol{Inx} + \boldsymbol{c}$ **ans.**

APPLICATION OF INTEGRATION

7. Find y given that $\frac{dy}{dx} = 2x - 3$ and that $y = -4$ *when* $x = 1$.

SOLUTION

$$If\ \frac{dy}{dx} = 2x - 3, \text{then} \quad y = \int (2x - 3)dx = x^2 - 3x + c$$
$$when\ x = 1\ ,\ y = 1 - 3 + c = -4\ \ so\ c = -2$$
$$Hence\ y = x^2 - 3x - 2$$

8. The gradient of the tangent at a point on a curve is given by $x^2 + x - 2$. Find the equation of the curve if it passes through (2, 1).

SOLUTION

Gradient $= \frac{dy}{dx} = x^2 + x - 2$

Then $y = \int (x^2 + x - 2)dx = \frac{x^3}{3} + \frac{x^2}{2} - 2x + c$

$$when\ x = 2, \qquad y = \frac{8}{3} + \frac{4}{2} - 4 + c = 1$$
$$Hence\ c = \frac{1}{3}.$$

$\boldsymbol{The\ equation\ of\ the\ curve\ is\ y = \frac{x^3}{3} + \frac{x^2}{2} - 2x + \frac{1}{3}\ \ or\ \ 6y = 2x^3 + 3x^2 - 12 + 2}$.

STATISTICS

1. (a) The number of people living in six houses is 3, 8, 4, x, y and z, if

The median is 7.5.

The mode is 8.

The mean is 7.

Find the value of x, y and z. [5]

(b) Calculate the standard deviation [3]

(c) The grouped frequency table below shows the amount (KA) spent on travel by a number of students.

Cost of travel (KA)	$0 < A \leqslant 10$	$10 < A \leqslant 20$	$20 < A \leqslant 40$
Frequency	15	m	n

(i) Write down an estimate for the total amount in terms of m and n. [2]

(ii) The calculated estimate of the mean amount is K13 exactly. Write down an equation containing m and n, and show that it simplifies to $2m + 17n = 120$. [3]

(iii) A student drew a histogram to represent this data. The area of the rectangle representing the $0 < A \leq 10$ group was equal to the sum of the areas of the other two rectangles. Write an equations in m and n for this relationship [1]

(iv) Find the values of m and n by solving the simultaneous equations

$2m + 17n = 120$,

$m + n = 15$. [3]

2. Answer the whole of this question on a sheet of graph paper.

In a survey, 200 shoppers were asked how much they had just spent in a supermarket. The results are shown in the table.

Amount(Kx)	$0 < x \leqslant 20$	$20 < x \leqslant 40$	$40 < x \leqslant 60$	$60 < x \leqslant 80$	$80 < x \leqslant 100$	$100 < x \leqslant 140$
Number of shoppers	10	32	48	54	36	20

(c) (i) Write down the modal class. [1]

(ii) Calculate an estimate of the mean amount, giving your answer correct to 2 decimal places. [4]

(b) (i) Make a cumulative frequency table for these 200 shoppers. [2]

(ii) Using a scale of 2 cm to represent K20 on the horizontal axis and 2 cm to represent 20 shoppers on the vertical axis, draw a cumulative frequency diagram for this data. [4]

(c) Use your cumulative frequency diagram to find

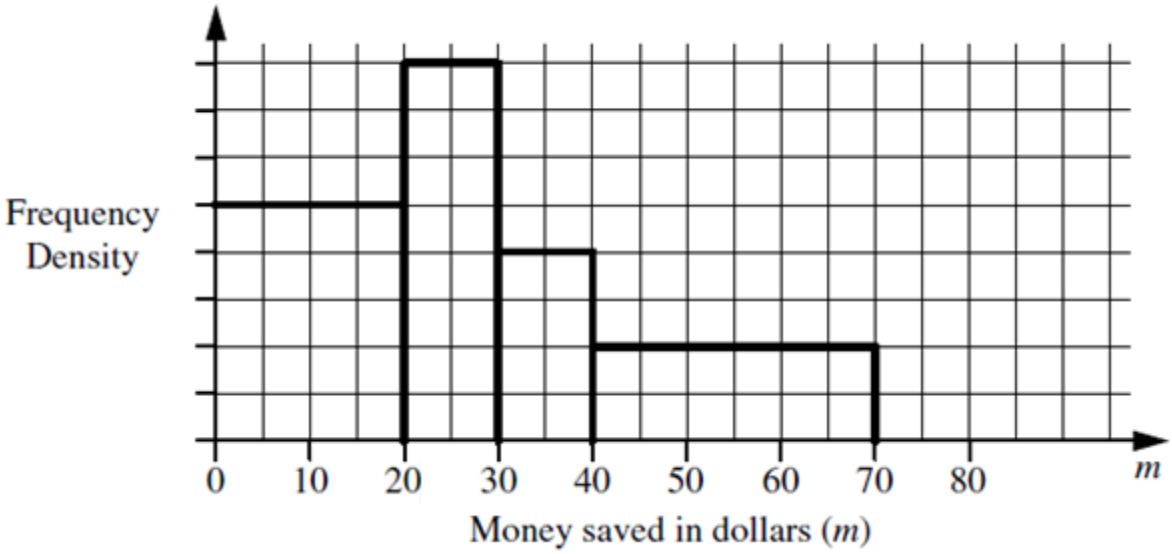
(i) The median amount, [1]

(ii) The upper quartile, [1]

(iii) the inter-quartile range, [1]

(iv) How many shoppers spent at least K75? [2]

3. A group of children were asked how much money they had saved. The histogram and table show the results.



Money saved ($\$m$)	$0 < m \leqslant 20$	$20 < m \leqslant 30$	$30 < m \leqslant 40$	$40 < m \leqslant 70$
Frequency	25	p	q	r

Use the histogram to calculate the values of p, q and r. [4]

4. Answer the whole of this question on a sheet of graph paper.

120 passengers on an aircraft had their baggage weighed. The results are shown in the table.

Mass of baggage (M kg)	$0 < M \leqslant 10$	$10 < M \leqslant 15$	$15 < M \leqslant 20$	$20 < M \leqslant 25$	$25 < M \leqslant 40$
Number of passengers	12	32	28	24	24

(a) **(i)** Write down the modal class. [1]

(ii) Calculate an estimate of the mean mass of baggage for the 120 passengers. Show all your working. [4]

(iii) Sophia draws a pie chart to show the data. What angle should she have in the $0 < M \leq 10$ sector? [1]

(b) Calculate the Standard deviation for the data in the table above. [7]

ANSWERS

1. (a) For median to be 7.5, the numbers must be arranged such as

$3, 4, x, 8, y \text{ and } z$

$\therefore \quad 7.5 = \frac{x + 8}{2}$

$x = 15 - 8$

$x = 7$

If the mode is 8, y can be 8 as well

The mean is given as 7

$\Rightarrow \quad 3 + 4 + 7 + 8 + 8 + z = 6 \times 7$

$z = 42 - 30$

$x = 7, \quad y = 8 \quad \text{and} \quad z = 12$

(b) SD = $\sqrt{\frac{\sum x^2}{n} - \bar{x}^2}$

Where $\sum x^2 = 3^2 + 4^2 + 7^2 + 8^2 + 8^2 + 12^2$

$= 9 + 16 + 49 + 64 + 64 + 144$

$\sum x^2 = 346$

$$SD = \sqrt{\frac{346}{6} - 49}$$

$$SD = 2.94$$

(i) $15(5) + 15m + 30n$

$KA = 75 + 15m + 30n$

(ii) $\frac{75+15m+30n}{15+m+n} = 13$

$2m + 17n = 120 \quad QED$

(iii) $m + n = 15$

2. (a)

(i) Modal Class is $60 < x \leq 80$

(ii) $mean = \frac{10(10)+32(30)+48(50)+54(70)+36(90)+20(120)}{200}$

$\bar{x} = \frac{12880}{200} = K64.80$

(b) (i)

Amount (KA)	≤ 20	≤ 40	≤ 60	≤ 80	≤ 100	≤ 140
Number of shoppers	10	42	90	144	180	200

(d) (i) $Median\ K63 - K64$

(ii) $Q_3 = K82 - K84$

(iii) $Q_3 - Q_1 = K38 - K41$

(iv) $Using\ \$75\ reading\ on\ Cum.Freq.Graph$ – 67 *or* 68 *or* 69 *or* 70 *or* 71 *or* 72