
Ethio Exams

A) (2, 3-5) +1c

B) (2+w) +1a

D) = (7,3) +1b

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FACULTY OF COMPUTER AND MATHEMATICAL SCIENCES
DEPARTMENT OF MATHEMATICS

Applied Mathematics III (Math 331) Final Examination Jan. 14, 2012

Answer Key

1. Let C be a curve parameterized by $r(t) = -2\cos t\,\hat{i} + 2\sin t\,\hat{j} + \sqrt{5}t\hat{k}$; where $0 \le t \le 2\pi$. Find

a) the length of the curve C. [3 pts.]

Solution:

$r'(t) = 2\sin t\,\hat{i} + 2\cos t\,\hat{j} + \sqrt{5}\hat{k}$

$r'(t).r'(t) = 4\sin^2 t + 4\cos^2 t + 5 = 4(\sin^2 t + \cos^2 t) + 5 = 9$

$l = \int_0^{2\pi} \sqrt{r'(t).r'(t)}\,dt$ or $\ell(c) = \int_0^{2\pi} \|r'(t)\|\,dt$ } (2)

$= \int_0^{2\pi} \sqrt{9}\,dt$

$= 3\int_0^{2\pi} dt$ } (1)

$= 3t\Big|_0^{2\pi}$

$= 6\pi$ *units*

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b) the normal vector at a point t. [3 pts.]

Solution:

$r'(t) = 2\sin t\,\hat{i} + 2\cos t\,\hat{j} + \sqrt{5}\hat{k}$

$\|r'(t)\| = \sqrt{4\sin^2 t + 4\cos^2 t + 5} = \sqrt{9} = 3$

$T(t) = \dfrac{r'(t)}{\|r'(t)\|} = \dfrac{1}{3}(2\sin t\,\hat{i} + 2\cos t\,\hat{j} + \sqrt{5}\hat{k})$ } ... (2)

$T'(t) = \dfrac{2}{3}(\cos t\,\hat{i} - \sin t\,\hat{j})$

$\|T'(t)\| = \sqrt{\dfrac{4}{9}(\cos^2 t + \sin^2 t)} = \dfrac{2}{3}$

$\therefore N = \dfrac{T'(t)}{\|T'(t)\|} = \dfrac{3}{2}.\dfrac{2}{3}(\cos t\,\hat{i} - \sin t\,\hat{j}) = \cos t\,\hat{i} - \sin t\,\hat{j}$ } ... (1)

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c) the curvature of C.

Solution:

[3 pts.]

$$\kappa = \kappa(t) = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3} = \frac{\|T'(t)\|}{\|r'(t)\|} = \frac{2/3}{3} = \frac{2}{9} \quad \ldots (3)$$

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2. Let $F(x,y,z) = (axy + z^3)\hat{i} + x^2\hat{j} + bxz^2\hat{k}$. Find the values of the constants a and b such that F is a conservative field.

Solution:

[6 pts.]

F is conservative field if $curl\, F = 0$.

$$curl\, F = \nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ axy + z^3 & x^2 & bxz^2 \end{vmatrix} = 0 \quad \ldots (2)$$

$$\left(\frac{\partial}{\partial y}(bxz^2) - \frac{\partial}{\partial z}(x^2)\right)\hat{i} - \left(\frac{\partial}{\partial x}(bxz^2) - \frac{\partial}{\partial z}(axy + z^3)\right)\hat{j} + \left(\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(axy + z^3)\right)\hat{k} = 0 \quad \ldots (2)$$

$$(0-0)\hat{i} - (bz^2 - 3z^2)\hat{j} + (2x - ax)\hat{k} = 0$$

$$0\hat{i} + (b-3)z^2\hat{j} + (2-a)x\hat{k} = 0$$

$\therefore b = 3$ and $a = 2$ $\quad \ldots (2)$

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3. Let $F(x,y,z) = (yze^{xyz})\hat{i} + (xze^{xyz} + z + \cos y)\hat{j} + (xye^{xyz} + y)\hat{k}$. Find a potential function of F. [6 pts.]

Let f be a potential function of F. Then

$\nabla f = F$

$\Rightarrow \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = F(x,y,z)$

$\Rightarrow \frac{\partial f}{\partial x} = yze^{xyz}, \frac{\partial f}{\partial y} = xze^{xyz} + z + \cos y, \frac{\partial f}{\partial z} = xye^{xyz} + y$ } (1)

$\Rightarrow f(x,y,z) = \int yze^{xyz}\,dx + g(y,z)$

$= e^{xyz} + g(y,z)$ } ... (2) or in other ways

$\frac{\partial f}{\partial y} = xze^{xyz} + g_y(y,z) = xze^{xyz} + z + \cos y$

$\Rightarrow g_y(y,z) = z + \cos y$

$\Rightarrow g(y,z) = \int (z + \cos y)\,dy + h(z)$

$= yz + \sin y + h(z)$ ... (1)

$f(x,y,z) = e^{xyz} + yz + \sin y + h(z)$

$\frac{\partial f}{\partial z} = xye^{xyz} + y + h'(z) = xye^{xyz} + y$

$\Rightarrow h'(z) = 0$

$\Rightarrow h(z) = C$ ... (1)

$\therefore f(x,y,z) = e^{xyz} + yz + \sin y + C$

is a potential function. ... (1)

(3)

4. Find the work done $w = \oint_C F.dr$ by the force field $F(x,y) = (y^3 + \cos 2x)\hat{i} + (e^{3y} - x^3)\hat{j}$, where C is the circle $x^2 + y^2 = 4$ in the counterclockwise direction. [7 pts.]

* If done using parametr. see the steps.

A circle will satisfy the conditions of Green's Theorem since it is closed and simple.

$P(x,y) = y^3 + \cos 2x$ and $Q(x,y) = e^{3y} - x^3$

$\Rightarrow \frac{\partial P}{\partial y} = 3y^2$ and $\frac{\partial Q}{\partial x} = -3x^2$

Now, using Green's theorem on the line integral gives, ...(2)

$$W = \oint_C F.dr = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

where R is a disk of radius 2 centered at the origin. We can use polar coordinates ...(2)

$$W = \oint_C F.dr = \iint_R (-3x^2 - 3y^2)\,dA$$

$$= -3 \iint_R (x^2 + y^2)\,dA$$

$$= -3 \int_0^{2\pi} \int_0^2 r^3\,dr\,d\theta \quad \text{...(2)}$$

$$= -3 \int_0^{2\pi} \frac{1}{4} r^4 \Big|_0^2 d\theta$$

$$= -3 \int_0^{2\pi} 4\,d\theta$$

$$= -12\theta \Big|_0^{2\pi}$$

$$= \underline{\underline{-24\pi}} \quad \text{...(1)}$$

5. Evaluate $\iint_S F \cdot dS$ when $F(x,y,z) = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}$ and S is the part of the paraboloid $z = x^2 + y^2 - 4$ that lies below the square $\{(x,y) : 0 \le x \le 1,\ 0 \le y \le 1\}$ and has downward orientation. [7 pts.]

Define $f(x,y,z) = -z + (x^2+y^2-4) = -z + g(x,y)$

$\nabla f = \langle 2x, 2y, -1 \rangle$

$\vec{n} = \dfrac{\nabla f}{\|\nabla f\|} = \dfrac{\langle 2x, 2y, -1 \rangle}{\|\nabla f\|}$ ...(2)

$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\, dS$

$= \iint_R (xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}) \cdot (2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k})\, dA$ ...(2)

$= \iint_R (2x^2y + 2y^2z - xz)\, dA, \quad z = x^2 + y^2 - 4$

$= \int_0^1 \int_0^1 [2x^2y + 2y^2(x^2+y^2-4) - x(x^2+y^2-4)]\, dy\, dx$

$= \int_0^1 \int_0^1 [2x^2y + 2x^2y^2 + 2y^4 - 8y^2 - x^3 - xy + 4x]\, dy\, dx$ ...(1.5)

$= \int_0^1 \left[x^2y^2 + \frac{2}{3}x^2y^3 + \frac{2}{5}y^5 - \frac{8}{3}y^3 - x^3y - \frac{1}{2}xy^2 + 4xy\right]_0^1 dx$

$= \int_0^1 \left[x^2 + \frac{2}{3}x^2 + \frac{2}{5} - \frac{8}{3} - x^3 - \frac{1}{2}x + 4x\right] dx$

$= \int_0^1 \left[\frac{5}{3}x^2 + \frac{5}{2}x - \frac{34}{15}\right] dx$

$= \left[\frac{5}{9}x^3 + \frac{5}{4}x^2 - \frac{34}{15}x\right]\Big|_0^1$

$= \frac{5}{9} + \frac{5}{4} - \frac{34}{15}$ ...(1.5)

$= \frac{100 + 225 - 418}{180} = \frac{93}{180} \approx 0.52$

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(5)

6. Evaluate the surface integral $\iint_S F.ndS$ where $F(x,y,z) = x\vec{i} + 2y\vec{j} + z\vec{k}$ and S is the boundary of the solid enclosed by the cylinder $x^2 + y^2 = 3$ between the planes $z = 0$ and $z = 4$. [7 pts.]

By Divergence Theorem, we have

$$\iint_S \vec{F}.\vec{n}\,dS = \iiint_D \text{div}F\,dV \quad \ldots (2)$$

Now, $F = xi + 2yj + zk$

$$\Rightarrow \text{div}F = P_x + Q_y + R_z = 1 + 2 + 1 = 5$$

Thus $$\iint_S \vec{F}.\vec{n}\,ds = \iiint_D 5\,dV \quad \ldots (2)$$

D is described in cylinderical coordinates as $D: 0 \le \theta \le 2\pi,\ 0 \le r \le 1,\ 0 \le z \le 4$

Hence

$$\iint_S \vec{F}.\vec{n}\,ds = 5\int_0^{2\pi}\int_0^1\int_0^4 r\,dz\,dr\,d\theta \quad \ldots (1.5)$$

$$= 20\int_0^{2\pi}\int_0^1 r\,dr\,d\theta$$

$$= 10\int_0^{2\pi} d\theta$$

$$= 10(2\pi)$$

$$= 20\pi \quad \ldots (1.5)$$

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attempted directly
the whole work.

7. Evaluate the line integral $\int_C xe^{yz}\,ds$, where C is the line segment from $A(0,0,0)$ to $B(1,2,3)$. [6 pts.]

Solution: The parameterization of the circle C is given by

$r(t) = (0,0,0) + t(1,2,3), \quad 0 \le t \le 1$

$= t\hat{i} + 2t\hat{j} + 3t\hat{k}$ .....(*)

$\Rightarrow r'(t) = \hat{i} + 2\hat{j} + 3\hat{k} \neq 0 \Rightarrow C$ is a smooth curve.

$\|r'(t)\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$ } ... (1)

$f(x,y,z) = xe^{yz} \Rightarrow f(x(t), y(t), z(t)) = te^{(2t)(3t)}$

$= te^{6t^2}$ [Since $x(t) = t$, $y(t) = 2t$, $z(t) = 3t$]

$\int_C xe^{yz}\,ds = \int_0^1 te^{6t^2}\sqrt{14}\,dt$ Since $ds = \|r'(t)\|dt$ .... (3)

$= \sqrt{14}\int_0^1 te^{6t^2}\,dt$

Let $u = 6t^2$

$du = 12t\,dt$

$0 \le t \le 1 \Rightarrow 0 \le u \le 6$

$\int_C xe^{yz}\,ds = \sqrt{14}\int_0^6 e^u \frac{du}{12} = \frac{\sqrt{14}}{12}(e^6 - 1)$ .... (2)

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8. a) Find all points at which $f(z) = 3x^2 + iy^2$ is differentiable. What is the corresponding derivative? [4 pts.]

Solution: For $f(z) = 3x^2 + iy^2$ the real and the imaginary parts are respectively $u(x,y) = 3x^2$ and $v(x,y) = y^2$. ... (1)

$\Rightarrow u_x = 6x$, $u_y = 0$, $v_x = 0$, and $v_y = 2y$.

Here u, v and the partial derivatives are all continuous. So f to be differentiable it must satisfy the Cauchy-Riemann equations, i.e., } ... (1)

$u_x = v_y$ and $u_y = -v_x$

$\Leftrightarrow \quad 6x = 2y$

$\Leftrightarrow \quad y = 3x$

Hence f is differentiable on the line $y = 3x$. ... (1)

$f'(z) = u_x + iv_x = v_y - iu_y = 6x + i0 = 6x$ on $y = 3x$. ... (1)

b) Is there an analytic function with real part $u(x,y)=3x^2-6y$? Justify your answer. [4 pts.]

Solution;

$u(x,y)=3x^2-6y$

$u_x(x,y)=6x,\ u_y(x,y)=-6 \Rightarrow u_{xx}(x,y)=6,$ and $u_{yy}(x,y)=0$ } . . . (1)

$u_{xx}+u_{yy}=6+0=6\neq 0$ . . . . (2)

$\Rightarrow u(x,y)=3x^2-6y$ is not harmonic and hence it can't be a real part of analytic function. . . . (1)

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c) Find the real and the imaginary part of the logarithmic function $\log(1-i\sqrt{3})$.

Solution; $z=x+iy=1-i\sqrt{3}$ [4 pts.]

$r=|z|=\sqrt{1^2+(\sqrt{3})^2}=2$ and $\theta=\arg z=\tan^{-1}(\frac{-\sqrt{3}}{1})=-\frac{\pi}{3}$ } . . (1)

$z=re^{i\theta}=2e^{-i\frac{\pi}{3}}$

$\log(1-i\sqrt{3})=\log(2e^{-i\frac{\pi}{3}})$

$=\ln|z|+i\arg z$

$=\ln 2+i\left(\frac{-\pi}{3}+2k\pi\right),\ k\in Z$ } . . . (2)

So $\mathrm{Re}(\log(1-i\sqrt{3}))=\ln 2$ and $\mathrm{Im}(\log(1-i\sqrt{3}))=-\frac{\pi}{3}+2k\pi,\ k=0,\pm 1,\pm 2,\cdots$ . . . (1)

THE END

ADDIS ABABA UNIVERSITY
DEPARTMENT OF MATHEMATICS
Applied mathematics III (Math331) final Examination

Answer Key

a. Describe and sketch the curve parameterized by the vector valued function $\vec{r}(t) = \sin t\,i + 3j + \cos t\,k$ for $t \in [0, 2\pi]$ (3pts)

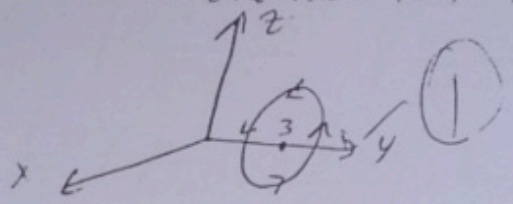
Soln. Let C be the curve parameterized by $\vec{r}(t) = \sin t\,i + 3j + \cos t\,k$.

the parametric eqns of C

$x = \sin t,\ y = 3,\ z = \cos t$

Now $x^2 + z^2 = \sin^2 t + \cos^2 t = 1$ which is a unit circle

$\therefore$ C is a unit circle oriented +ve in the plane $y = 3$



b. Let C be the curve parameterized by $\vec{r}(t) = (\frac{1}{3}t^3 - t)i + t^2 j$ for $t \in [-1, 1]$. Then find:

i. the length of the curve C. (3pts)

The length $L(c)$ of the curve is given by

$$L(c) = \int_{-1}^{1} \|r'(t)\|\,dt$$

Now $r'(t) = (t^2 - 1)i + 2t\,j$

$$\Rightarrow \|r'(t)\| = \sqrt{(t^2-1)^2 + 4t^2} = \sqrt{t^4 + 2t^2 + 1} = t^2 + 1$$

$$\therefore L(c) = \int_{-1}^{1} (t^2 + 1)\,dt = \left(\frac{t^3}{3} + t\right)\Big|_{-1}^{1} = \left(\frac{1}{3} + 1\right) - \left(-\frac{1}{3} - 1\right) = 8/3.$$

Thus $L(c) = 8/3$ Units long.

value of c so that the vector $F(x,y,z) = (x+2y)i + (\quad 3z)j + (y+cz)k$ is solenoidal. (4pts)

is solenoidal iff $\operatorname{div} F = 0$ ✓ (1)

$\operatorname{div} F = \nabla \cdot F = \frac{\partial}{\partial x}(x+2y) + \frac{\partial}{\partial y}(y+3z) + \frac{\partial}{\partial z}(y+cz) = 1 + 1 + c = 2 + c.$

but then $\operatorname{div} F = 0 \Leftrightarrow 2 + c = 0 \Leftrightarrow c = -2.$

Thus. F is solenoidal iff $c = -2$ ✓ (1)

b. Show that any vector field of the form $F(x,y,z) = f(x)i + g(y)j + h(z)k$ where f, g, and h are differentiable functions is irrotational. (4pts)

Soln. F is irrotational iff $\operatorname{curl} F = 0$ ✓ (1)

Now $\operatorname{curl} F = \nabla \times F$ (1)

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f(x) & g(y) & h(z) \end{vmatrix} = \left(\frac{\partial}{\partial y}h(z) - \frac{\partial}{\partial z}g(y)\right)i - \left(\frac{\partial}{\partial x}h(z) - \frac{\partial}{\partial z}f(x)\right)j + \left(\frac{\partial}{\partial x}g(y) - \frac{\partial}{\partial y}f(x)\right)k$$

$$= 0$$

$\therefore$ F is irrotational.

(3)

$F(x,y,z) = e^y i + (xe^y + e^z) j + ye^z k$

Show that F is conservative and find its potential function f. (5pts)

$\text{curl } F = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^y & xe^y + e^z & ye^z \end{vmatrix} = (e^z - e^z) i + (0-0) j + (e^y - e^y) k = 0$

$\therefore$ F is conservative.

$\Rightarrow$ $\exists\ f(x,y,z)$ such that $\nabla f = F(x,y,z)$.

i.e $\frac{\partial f}{\partial x} = e^y$, $\frac{\partial f}{\partial y} = xe^y + e^z$, $\frac{\partial f}{\partial z} = ye^z$.

Given $\frac{\partial f}{\partial x} = e^y$, we have $f(x,y,z) = \int e^y dx + h(y,z) = xe^y + h(y,z)$

$\therefore$ $f(x,y,z) = xe^y + h(y,z)$

$\Rightarrow$ $\frac{\partial f}{\partial y} = xe^y + h_y(y,z)$

Now from $\frac{\partial f}{\partial y} = xe^y + e^z$ and $\frac{\partial f}{\partial y} = xe^y + h_y(y,z)$ we have

$h_y(y,z) = e^z \Rightarrow h(y,z) = \int e^z dy + k(z) \Rightarrow h(y,z) = ye^z + k(z)$

$\therefore$ $f(x,y,z) = xe^y + ye^z + k(z)$.

$\Rightarrow$ $\frac{\partial f}{\partial z} = ye^z + k'(z)$

Now from $\frac{\partial f}{\partial z} = ye^z$ and $\frac{\partial f}{\partial z} = ye^z + k'(z)$ we have

$k'(z) = 0 \Rightarrow k(z) = c$.

$\therefore$ $f(x,y,z) = xe^y + ye^z + c$

Thus The potential fun. f is. $f(x,y,z) = xe^y + ye^z + c$ ✓

b. Find $\int_C F \cdot dr$ where C is the line segment from $P(0,2,0)$ to $Q(4,0,3)$. (3pts)

Since F is conservative by fundamental thm of line integral we have.

$\int_C F \cdot dr = \int_C \nabla f \cdot dr = f(4,0,3) - f(0,2,0)$

$= 4e^0 + 0e^3 + c - (0 + 2e^0 + c)$

$= (4 + c) - (2 + c) = 2$.

$\therefore$ $\int_C F \cdot dr = 2$.

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