Introduction to Ordinary and Partial Differential Equations

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Chapter 1

Introduction

1.1 Classification of Differential Equations

Definition: A *differential equation* is an equation which contains derivatives of the unknown. (Usually it is a mathematical model of some physical phenomenon.)

Two classes of differential equations:

- O.D.E. (ordinary differential equations): linear and non-linear;
- P.D.E. (partial differential equations). (not covered in math250, but in math251)

Some concepts related to differential equations:

- system: a collection of several equations with several unknowns.
- order of the equation: the highest order of derivatives.
- linear or non-linear equations: Let $y(t)$ be the unknown. Then,

$$a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y = g(t), \qquad (*)$$

  is a linear equations. If the equation can not be written as $(*)$, then it's non-linear.

Two things you must know: identify the linearity and the order of an equation.

Example 1. Let $y(t)$ be the unknown. Identify the order and linearity of the following equations.

(a). $(y+t)y' + y = 1$,
(b). $3y' + (t+4)y = t^2 + y''$,
(c). $y''' = \cos(2ty)$,
(d). $y^{(4)} + \sqrt{t}y''' + \cos t = e^y$.

Answer.

Problem	order	linear?
(a). $(y+t)y' + y = 1$	1	No
(b). $3y' + (t+4)y = t^2 + y''$	2	Yes
(c). $y''' = \cos(2ty)$	3	No
(d). $y^{(4)} + \sqrt{t}y''' + \cos t = e^y$	4	No

What is a solution? A solution is a function that satisfies the equation, the boundary conditions (if any), the initial conditions (if any), and whose derivatives exist.

Example 2. Verify that $y(t) = e^{at}$ is a solution of the IVP (initial value problem)

$$y' = ay, \qquad y(0) = 1.$$

Here $y(0) = 1$ is called the initial condition.

Answer. Let's check if $y(t)$ satisfies the equation and the initial condition:

$$y' = ae^{at} = ay, \qquad y(0) = e^0 = 1.$$

They are both OK. So it is a solution.

Example 3. Verify that $y(t) = 10 - ce^{-t}$ with c a constant, is a solution to $y' + y = 10$.

Answer.

$$y' = -(-ce^{-t}) = ce^{-t}, \qquad y' + y = ce^{-t} + 10 - ce^{-t} = 10. \qquad \text{OK.}$$

Let's try to solve one equation.

Example 4. Consider the equation

$$(t+1)y' = t^2$$

We can rewrite it as (for $t \neq -1$)

$$y' = \frac{t^2}{t+1} = \frac{t^2 - 1 + 1}{t+1} = \frac{(t+1)(t-1)+1}{t+1} = (t-1) + \frac{1}{t+1}$$

To find y, we need to integrate y':

$$y = \int y'(t)dt = \int \left[(t-1) + \frac{1}{t+1}\right] dt = \frac{t^2}{2} - t + \ln|t+1| + c$$

where c is an integration constant which is arbitrary. This means there are infinitely many solutions.

Additional condition: initial condition $y(0) = 1$. (meaning: $y = 1$ when $t = 0$) Then

$$y(0) = 0 + \ln|1| + c = c = 1, \qquad \text{so} \qquad y(t) = \frac{t^2}{2} - t + \ln|t+1| + 1.$$

So for equation like $y' = f(t)$, we can solve it by integration: $y = \int f(t)dt$.

Review on integration:

$$\begin{aligned}
\int x^n\,dx &= \frac{1}{n+1}x^{n+1}+c, \quad (n\neq -1)\\
\int \frac{1}{x}\,dx &= \ln|x|+c\\
\int \sin x\,dx &= -\cos x+c\\
\int \cos x\,dx &= \sin x+c\\
\int e^x\,dx &= e^x+c\\
\int a^x dx &= \frac{a^x}{\ln a}+c
\end{aligned}$$

Integration by parts:

$$\int u\,dv = uv - \int v\,du, \qquad \int_a^b u(x)v'(x)\,dx = u(x)v(x)\Big|_a^b - \int_a^b v(x)u'(x)\,dx$$

Chain rule:

$$\frac{d}{dt}(f(g(t)) = f'(g(t))\cdot g'(t)$$

1.2 Directional Fields

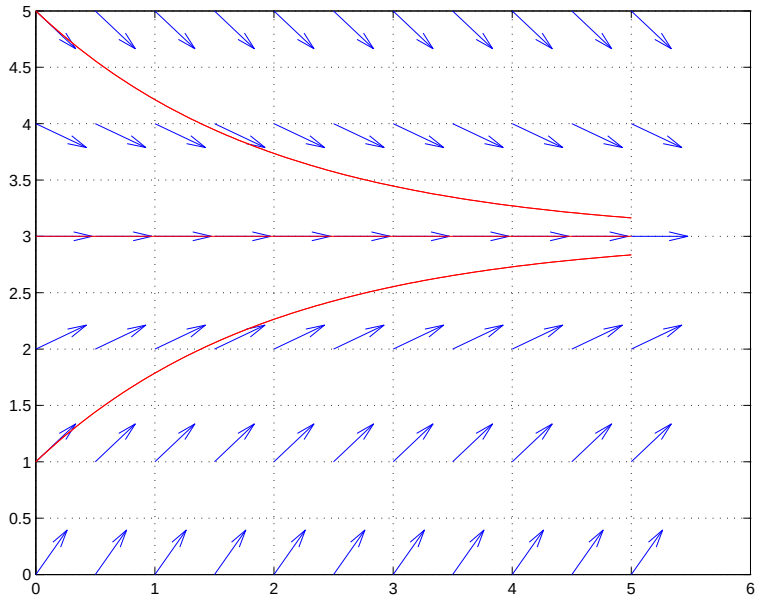
Directional field: for first order equations $y' = f(t,y)$.
Interpret y' as the slope of the tangent to the solution $y(t)$ at point (t,y) in the $y-t$ plane.

- If $y'=0$, the tangent line is horizontal;
- If $y'>0$, the tangent line goes up;
- If $y'<0$, the tangent line goes down;
- The value of $|y'|$ determines the steepness.

Example 5. Consider the equation $y' = \frac{1}{2}(3-y)$. We know the following:

- If $y=3$, then $y'=0$, flat slope,
- If $y>3$, then $y'<0$, down slope,
- If $y<3$, then $y'>0$, up slope.

See the directional field below (with some solutions sketched in red):



We note that, if $y(0) = 3$, then $y(t) = 3$ is the solution.
Asymptotic behavior: As $t \to \infty$, we have $y \to 3$.

Remarks:

(1). For equation $y'(t) = a(b - y)$ with $a > 0$, it will have similar behavior to **Example** 5, where $b = 3$ and $a = \frac{1}{2}$. Solution will approach $y = b$ as $t \to +\infty$.

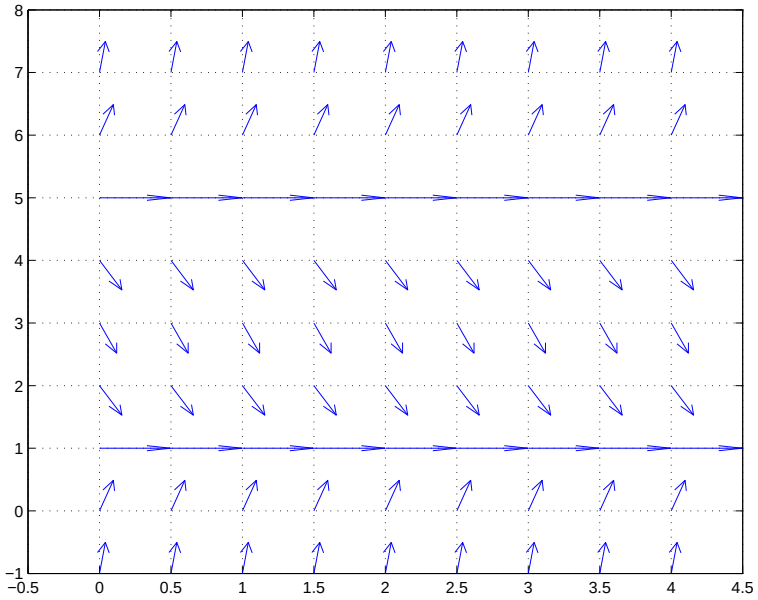
(2). Now consider $y'(t) = a(b-y)$, but with $a < 0$. This changes the sign of y'. We now have

- If $y(0) = b$, then $y(t) = b$;
- If $y(0) > b$, then $y \to +\infty$ as $t \to +\infty$;
- If $y(0) < b$, then $y \to -\infty$ as $t \to +\infty$.

Example 6: Let $y'(t) = (y - 1)(y - 5)$. Then,

- If $y = 1$ or $y = 5$, then $y' = 0$.
- If $y < 1$, then $y' > 0$;
- If $1 < y < 5$, then $y' < 0$;
- If $y > 5$, then $y' < 0$.

Directional field looks like:

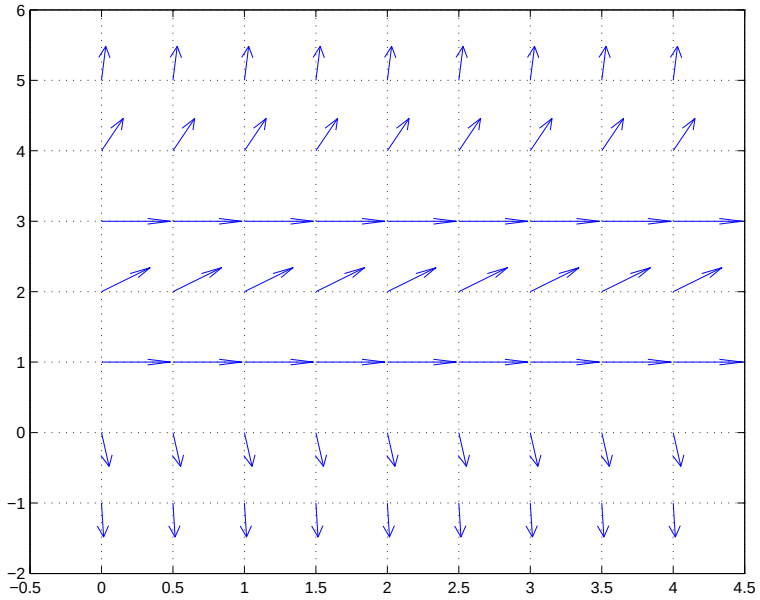


What can we say about the solutions?

- If $y(0) = 1$, then $y(t) = 1$;
- If $y(0) = 5$, then $y(t) = 5$;
- If $y(0) < 1$, then $y \to 1$ as $t \to +\infty$;
- If $1 < y(0) < 5$, then $y \to 1$ as $t \to +\infty$;
- If $y(0) > 5$, then $y \to +\infty$ as $t \to +\infty$.

Remark: If we have $y'(t) = f(y)$, and for some y_0 we have $f(y_0) = 0$, then, $y(t) = y_0$ is a solution.

Example 7: Given the plot of a directional field,



which of the following ODE could have generated it?

(a). $y'(t) = (y-2)(y-4)$

(b). $y'(t) = (y-1)^2(y-3)$

(c). $y'(t) = (y-1)(y-3)^2$

(d). $y'(t) = -(y-1)(y-3)^2$

We first check the constant solution, $y = 1$ and $y = 3$. Then (a) can not be. Then, we check the sign of y' on the intervals: $y < 1$, $1 < y < 3$, and $y > 3$, to match the directional field. We found that (c) could be the equation.

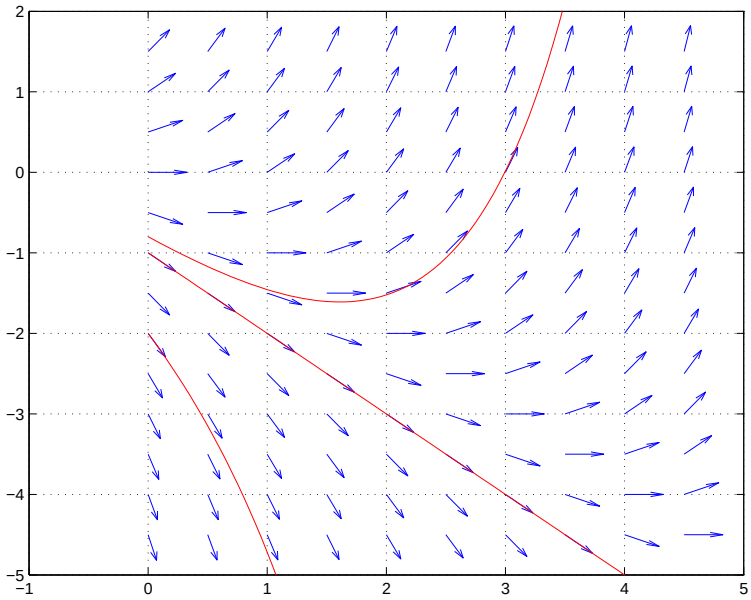
Example 8. Consider a more complicated situation where y' depends on both t and y. Consider $y' = t + y$. To generate the directional field, we see that:

- We have $y' = 0$ when $y = -t$,
- We have $y' > 0$ when $y > -t$,
- We have $y' < 0$ when $y < -t$.

One can sketch the directional field along lines of $y = -t + c$ for various values of c.

- If $y = -t$, then $y' = 0$;
- If $y = -t - 1$, then $y' = -1$;
- If $y = -t - 2$, then $y' = -2$;
- If $y = -t + 1$, then $y' = 1$;
- If $y = -t + 2$, then $y' = 2$;

Below is the graph of the directional field, with some solutions plotted in red.



What can we say about the solutions?
The solution depends on the initial condition $y(0)$.

- We see first that if $y(0) = -1$, the solution is $y(t) = -t - 1$;
- If $y(0) > -1$, then $y \to +\infty$ as $t \to +\infty$;
- If $y(0) < -1$, then $y \to \mp\infty$ as $t \to +\infty$.

We can also discuss the asymptotic behavior as $t \to -\infty$:

- If $y(0) > -1$, then $y \to +\infty$ as $t \to -\infty$.
- If $y(0) < -1$, then $y \to +\infty$ as $t \to -\infty$.

We see that, for general $y' = f(t, y)$ where the function f depends on t, the directional fields could be rather tedious to generate by hand. There are computer softwares that will do this for you, as I did with these plots. They are generated in `Matlab`, a powerful software that can solve many math problems.

Chapter 2

First Order Differential Equations

We consider the equation

$$\frac{dy}{dt} = f(t, y)$$

Overview:

- Two special types of equations: linear, and separable;
- Linear vs. nonlinear;
- modeling;
- autonomous equations and population model;
- a little bit on Euler's method.

2.1 Linear equations; Method of integrating factors

The function $f(t, y)$ is a linear function in y, i.e, we can write

$$f(t, y) = -p(t)y + g(t).$$

So we will study the equation

$$y' + p(t)y = g(t). \tag{A}$$

We introduce the method of integrating factors (due to Leibniz): We multiply equation (A) by a function $\mu(t)$ on both sides

$$\mu(t)y' + \mu(t)p(t)y = \mu(t)g(t)$$

The function μ is chosen such that the equation is integrable, meaning the LHS (Left Hand Side) is the derivative of something. In particular, we require:

$$\mu(t)y' + \mu(t)p(t)y = (\mu(t)y)', \quad \Rightarrow \quad \mu(t)y' + \mu(t)p(t)y = \mu(t)y' + \mu'(t)y$$

which requires

$$\mu'(t) = \frac{d\mu}{dt} = \mu(t)p(t), \quad \Rightarrow \quad \frac{d\mu}{\mu} = p(t)\,dt$$

Integrating both sides

$$\ln \mu(t) = \int p(t)\,dt$$

which gives a formula to compute μ

$$\mu(t) = \exp\left(\int p(t)\,dy\right).$$

Therefore, this μ is called the *integrating factor*.

Note that μ is not unique. In fact, adding an integration constant, we will get a different μ. But we don't need to be bothered, since any such μ will work. We can simply choose one that is convenient.

Putting back into equation (A), we get

$$\frac{d}{dt}(\mu(t)y) = \mu(t)g(t), \quad \mu(t)y = \int \mu(t)g(t)\,dt + c$$

which gives the formula for the solution

$$y(t) = \frac{1}{\mu(t)}\left[\int \mu(t)g(t)\,dt + c\right], \quad \text{where} \quad \mu(t) = \exp\left(\int p(t)\,dt\right).$$

Example 1. Solve $y' + ay = b$ $(a \neq 0)$.

Answer. We have $p(t) = a$ and $g(t) = b$. So

$$\mu = \exp\left(\int a\,dt\right) = e^{at}$$

so

$$y = e^{-at}\int e^{at}b\,dt = e^{-at}\left(\frac{b}{a}e^{at} + c\right) = \frac{b}{a} + ce^{-at},$$

where c is an arbitrary constant. Pay attention to where one adds this integration constant!

Example 2. Find the general solution of $y' + y = e^{2t}$.

Answer. We have $p(t) = 1$ and $g(t) = e^{2t}$. So

$$\mu(t) = \exp\left(\int 1\,dt\right) = e^t$$

and

$$y(t) = e^{-t}\int e^t e^{2t}\,dt = e^t\int e^{3t}dt = e^{-t}\left(\frac{1}{3}e^{3t} + c\right) = \frac{1}{3}e^{2t} + ce^{-t}.$$

Can you discuss the behavior of this solution, as $t \to \pm\infty$?

Example 3. Solve the IVP

$$(1+t^2)y' + 4ty = \left(1+t^2\right)^{-2}, \quad y(0) = 1.$$

Answer. First, let's rewrite the equation into the normal form

$$y' + \frac{4t}{1+t^2}y = \left(1+t^2\right)^{-3},$$

so

$$p(t) = \frac{4t}{1+t^2}, \qquad g(t) = \left(1+t^2\right)^{-3}.$$

Then

$$\begin{aligned} \mu(t) &= \exp\left(\int p(t)\,dt\right) = \exp\left(\int \frac{4t}{1+t^2}dt\right) = \exp\left(\int \frac{2}{1+t^2}d(t^2)\right) \\ &= \exp(2\ln(1+t^2)) = \exp(\ln(1+t^2)^2) = \left(1+t^2\right)^2. \end{aligned}$$

Then

$$y = \frac{\int (1+t^2)^2(1+t^2)^{-3}dt}{(1+t^2)^2} = \frac{\int (1+t^2)^{-1}dt}{(1+t^2)^2} = \frac{\arctan t + c}{(1+t^2)^2}.$$

By the IC: $y(0) = 1$:

$$y(0) = \frac{0+c}{1} = c = 1, \quad \Rightarrow \quad y(t) = \frac{\arctan t + 1}{(1+t^2)^2}.$$

Example 4. Find the general solution for $ty' - y = t^2e^{-t}$, $(t > 0)$.

Answer. Rewrite it into normal form

$$y' - \frac{1}{t}y = te^{-t}$$

so

$$p(t) = -1/t, \qquad g(t) = te^{-t}.$$

We have

$$\mu(t) = \exp\left(\int (-1/t)dt\right) = \exp\left(-\ln t\right) = \frac{1}{t}$$

and

$$y(t) = t\int \frac{1}{t}te^{-t}dt = t\int e^{-t}dt = t\left(-e^{-t} + c\right) = -te^{-t} + ct.$$

Example 5. Solve $y - \frac{1}{3}y = e^{-t}$, with $y(0) = a$, and discuss the behavior of y as $t \to \infty$, as one chooses different initial values for $y(0) = a$.

Answer. Let's solve it first. We have

$$\mu(t) = e^{-\frac{1}{3}t}$$

so

$$y(t) = e^{\frac{1}{3}t} \int e^{-\frac{1}{3}t} e^{-t} dt = e^{\frac{1}{3}t} \int e^{-\frac{4}{3}t} dt = e^{\frac{1}{3}t} \left(-\frac{3}{4} e^{-\frac{4}{3}t} + c \right).$$

Plug in the IC to find c

$$y(0) = e^0(-\frac{3}{4} + c) = a, \quad c = a + \frac{3}{4}$$

so

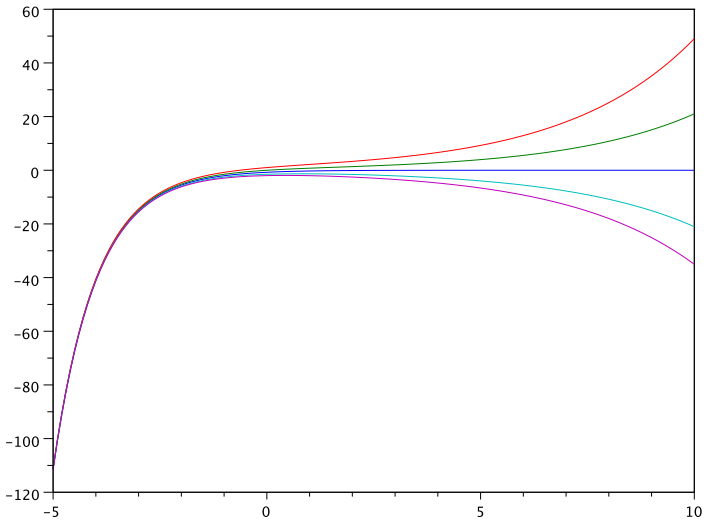
$$y(t) = e^{\frac{1}{3}t} \left(-\frac{3}{4} e^{-\frac{4}{3}t} + a + \frac{3}{4} \right) = -\frac{3}{4} e^{-t} + \left(a + \frac{3}{4} \right) e^{t/3}.$$

To see the behavior of the solution, we see that it contains two terms. The first term e^{-t} goes to 0 as t grows. The second term $e^{t/3}$ goes to ∞ as t grows, but the constant $a + \frac{3}{4}$ is multiplied on it. So we have

- If $a + \frac{3}{4} = 0$, i.e., if $a = -\frac{3}{4}$, we have $y \to 0$ as $t \to \infty$;
- If $a + \frac{3}{4} > 0$, i.e., if $a > -\frac{3}{4}$, we have $y \to \infty$ as $t \to \infty$;
- If $a + \frac{3}{4} < 0$, i.e., if $a < -\frac{3}{4}$, we have $y \to -\infty$ as $t \to \infty$;

On the other hand, as $t \to -\infty$, the term e^{-t} will blow up to $-\infty$, and will dominate. Therefore, $y \to -\infty$ as $t \to -\infty$ for any values of a.

See plot below:



Example 6. Solve $ty' + 2y = 4t^2$, $y(1) = 2$.

Answer. Rewrite the equation first

$$y' + \frac{2}{t} y = 4t, \quad (t \neq 0)$$

So $p(t) = 2/t$ and $g(t) = 4t$. We have

$$\mu(t) = \exp\left(\int 2/t\, dt\right) = \exp(2\ln t) = t^2$$

and

$$y(t) = t^{-2}\int 4t \cdot t^2 dy = t^{-2}\left(t^4 + c\right) = t^2 + ct^{-2}.$$

We see that the solution has 2 terms, t^2 and t^{-2}. With different initial value, we will get different values of c, and the solution will be very different.

By our IC $y(1) = 2$, we get

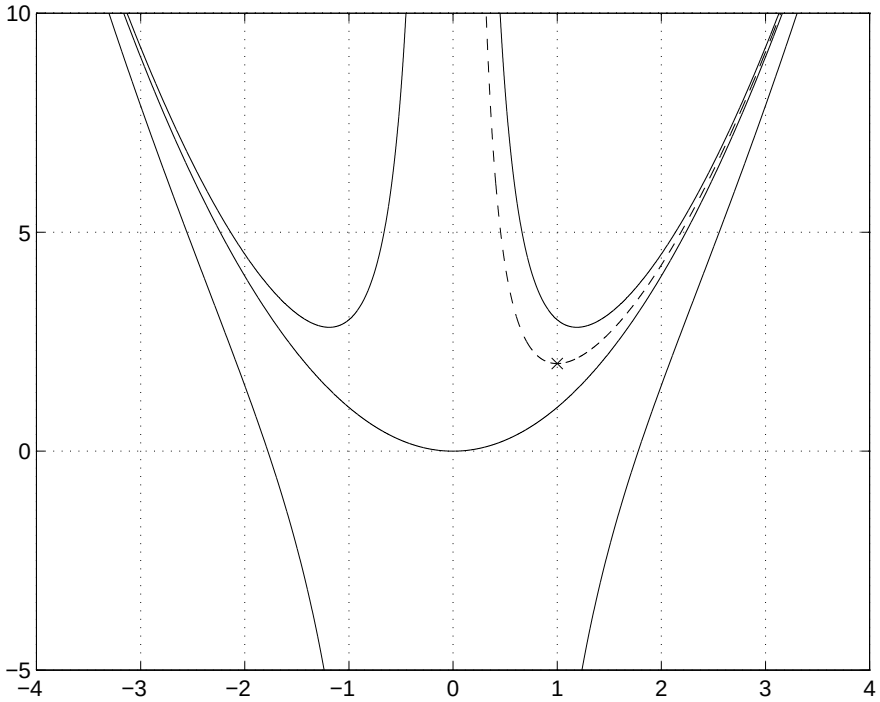
$$y(1) = 1 + c = 2, \quad c = 1$$

we get the solution:

$$y(t) = t^2 + \frac{1}{t^2}, \quad t > 0.$$

Note the condition $t > 0$ comes from the fact that the initial condition is given at $t = 1$, and we require $t \neq 0$.

In the graph below we plot several solutions in the $t - y$ plan, depending on initial data. The one for our solution is plotted with a dashed line where the initial point is marked with a '$\times$'.



Bernoulli Equations*. This is an example of solving nonlinear first order ODE by a variable change and turn it into a linear equation. Consider the **Bernoulli differential equations**

$$y'(t) + p(t)y = q(t)y^n$$

where n is an integer.

If $n = 0$ or $n = 1$, this equation is linear and we can solve it by the method integrating factor. Otherwise, the equation is non-linear. Such equations arises in applications such as population models and models of one-dimensional motion influenced by drag forces.

We now make a variable change, and call $v(t) = y(t)^m$, for some m to be determined. Then, $y = v^{1/m}$. By the chain rule, we have

$$y' = \frac{1}{m} v^{1/m-1} v'.$$

Put these into the differential equation, we get

$$\frac{1}{m} v^{1/m-1} v' + p(t) v^{1/m} = q(t) v^{n/m}.$$

Multiply both sides by $mv^{1-1/m}$, we get

$$v' + mp(t)v = mq(t)v^{(m+n-1)/m}.$$

This may look more complicated than the original equation, but remember we can choose the value m in a smart way. Now let $m = 1 - n$, we get

$$v' + (1-n)p(t)v = (1-n)q(t),$$

which is linear, and we can solve it! Once we get $v(t)$, we can then easily recover $y(t)$.

Example 7. Solve the IVP

$$y' + y = ty^3, \qquad y(0) = 2.$$

Answer. We make the variable change and let $v = y^{1-n} = y^{1-3} = y^{-2}$. By the derivation we did, we have

$$v' - 2v = -2t, \qquad v(0) = y(0)^{-2} = 1/4.$$

The general solution is

$$v(t) = Ce^{2t} + (t + \frac{1}{2}).$$

Imposing the initial condition, we get

$$C = -\frac{1}{4}, \qquad v(t) = -\frac{1}{4}e^{2t} + (t + \frac{1}{2}).$$

Finally, we go back to $y(t) = v^{-1/2}$, and get

$$y(t) = \left[-\frac{1}{4}e^{2t} + (t + \frac{1}{2}) \right]^{-1/2}.$$

Remarks on the definition of linear and nonlinear equations:

(1) For any linear equation

$$y' + p(t)y = g(t)$$

if we make a non-linear variable change, say $y = f(v)$ for some nonlinear function f (for example, $f(v) = v^2$), we turn the linear equation for y into a nonlinear equation for v.

(2) However, the other way around is not always possible. Only for some special nonlinear equations, a suitable (carefully chosen) variable change could turn it into a linear equation in the new variable. Therefore, such special equations are worth mentioning.

2.2 Separable Equations

Let $y(x)$ denote the unknown. We study first order equations that can be written as

$$\frac{dy}{dx} = f(x, y) = \frac{M(x)}{N(y)}$$

where $M(x)$ and $N(y)$ are suitable functions of x and y only. Then we have

$$N(y)\,dy = M(x)\,dx, \qquad \Rightarrow \quad \int N(y)\,dy = \int M(x)\,dx$$

and we get implicitly defined solutions of $y(x)$.

Example 1. Consider

$$\frac{dy}{dx} = \frac{\sin x}{1 - y^2}.$$

We can separate the variables:

$$\int (1 - y^2)\,dy = \int \sin x\,dx, \quad \Rightarrow \quad y - \frac{1}{3}y^3 = -\cos x + c.$$

If one has IC as $y(\pi) = 2$, then

$$2 - \frac{1}{3} \cdot 2^3 = -\cos \pi + c, \qquad \Rightarrow \quad c = -\frac{5}{3},$$

so the solution $y(x)$ is implicitly given as

$$y - \frac{1}{3}y^3 + \cos x + \frac{5}{3} = 0.$$

Example 2. Find the solution in explicit form for the equation

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y - 1)}, \qquad y(0) = -1.$$

Answer. Separate the variables

$$\int 2(y - 1)\,dy = \int (3x^2 + 4x + 2)\,dx\,, \quad \Rightarrow \quad (y - 1)^2 = x^3 + 2x^2 + 2x + c$$

Set in the IC $y(0) = -1$, i.e., $y = -1$ when $x = 0$, we get

$$(-1-1)^2 = 0 + c, \qquad c = 4, \qquad (y-1)^2 = x^3 + 2x^2 + 2x + 4.$$

In explicitly form, one has two choices:

$$y(t) = 1 \pm \sqrt{x^3 + 2x^2 + 2x + 4}.$$

To determine which sign is the correct one, we check again by the initial condition:

$$y(0) = 1 \pm \sqrt{4} = 1 \pm 2, \qquad \text{must have} \quad y(0) = -1.$$

We see we must choose the '-' sign. The solution in explicitly form is:

$$y(x) = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}.$$

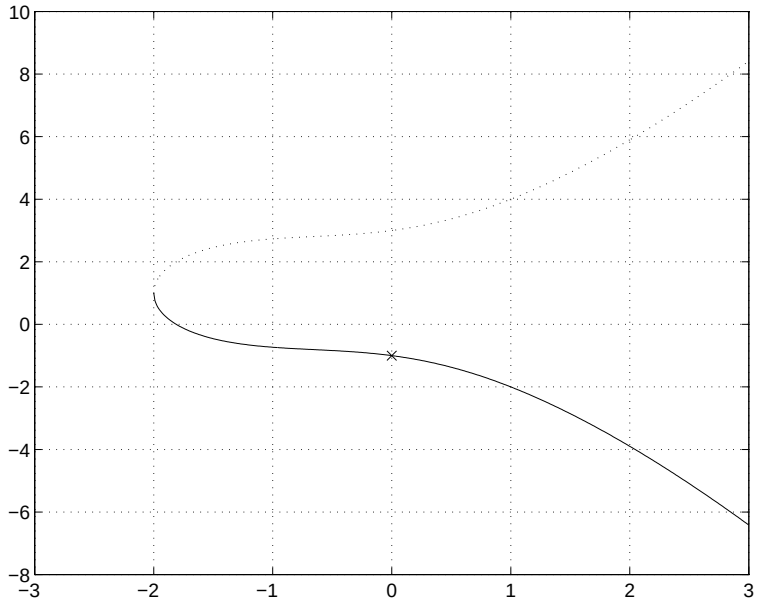
On which interval will this solution be defined?

$$x^3 + 2x^2 + 2x + 4 \geq 0, \quad \Rightarrow \quad x^2(x+2) + 2(x+2) \geq 0$$

$$\Rightarrow \quad (x^2+2)(x+2) \geq 0, \quad \Rightarrow \quad x \geq -2.$$

We can also argue that when $x = -2$, we have $y = 1$. At this point $|dy/dx| \to \infty$, therefore solution can not be defined at this point.

The plot of the solution is given below, where the initial data is marked with 'x'. We also include the solution with the '+' sign, using dotted line.



Example 3. Solve $y' = 3x^2 + 3x^2y^2$, $y(0) = 0$, and find the interval where the solution is defined.

Answer. Let's first separate the variables.

$$\frac{dy}{dx} = 3x^2(1+y^2), \quad \Rightarrow \quad \int \frac{1}{1+y^2} dy = \int 3x^2\, dx, \quad \Rightarrow \quad \arctan y = x^3 + c.$$

Set in the IC:

$$\arctan 0 = 0 + c, \quad \Rightarrow \quad c = 0$$

we get the solution

$$\arctan y = x^3, \quad \Rightarrow \quad y = \tan(x^3).$$

Since the initial data is given at $x = 0$, i.e., $x^3 = 0$, and tan is defined on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, we have

$$-\frac{\pi}{2} < x^3 < \frac{\pi}{2}, \quad \Rightarrow \quad -\left[\frac{\pi}{2}\right]^{1/3} < x < \left[\frac{\pi}{2}\right]^{1/3}.$$

Example 4. Solve

$$y' = \frac{1 + 3x^2}{3y^2 - 6y}, \qquad y(0) = 1$$

and identify the interval where solution is valid.

Answer. Separate the variables

$$\int (3y^2 - 6y)dy = \int (1 + 3x^2)dx \quad y^3 - 3y^2 = x + x^3 + c.$$

Set in the IC: $x = 0, y = 1$, we get

$$1 - 3 = c, \quad \Rightarrow \quad c = -2,$$

Then,

$$y^3 - 3y^2 = x^3 - x - 2.$$

Note that solution is given in implicitly form.

To find the valid interval of this solution, we note that y' is not defined if $3y^2 - 6y = 0$, i.e., when $y = 0$ or $y = 2$. These are the two so-called "bad points" where you can not define the solution. To find the corresponding values of x, we use the solution expression:

$$y = 0 : \quad x^3 + x - 2 = 0,$$

$$\Rightarrow \quad (x^2 + x + 2)(x - 1) = 0, \quad \Rightarrow \quad x = 1$$

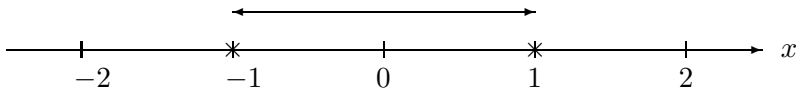
and

$$y = 2 : \quad x^3 + x - 2 = -4, \quad \Rightarrow \quad x^3 + x + 2 = 0,$$

$$\Rightarrow \quad (x^2 - x + 2)(x + 1) = 0, \quad \Rightarrow \quad x = -1$$

(Note that we used the facts $x^2 + x + 2 \neq 0$ and $x^2 - x + 2 \neq 0$ for all x.)

Draw the real line and work on it as following:



Therefore the interval is $-1 < x < 1$.

2.3 Differences between linear and nonlinear equations; existence and uniqueness of solutions

We discuss here some fundamental differences between linear and nonlinear equations regarding existence and uniqueness of solutions.

For a linear equation

$$y' + p(t)y = g(t), \qquad y(\bar{t}) = \bar{y},$$

if we require the solution $y(t)$ to be differentiable functions, then we have the following existence and uniqueness theorem.

Theorem . If $p(t)$ and $g(t)$ are continuous and bounded on an open interval containing t_0, then it has an unique solution on that interval.

A brief proof. The existence of a solution is obvious, since we can write it out using the method of integrating factors. For the uniqueness, let y_1 and y_2 be two solutions of the problem, i.e.,

$$y_1' + p(t)y_1 = g(t), \qquad y_1(\bar{t}) = \bar{y}, \qquad y_2' + p(t)y_2 = g(t), \qquad y_2(\bar{t}) = \bar{y}.$$

Define the error $e(t) = y_1 - y_2$. Then, $e(t)$ solves the equation

$$e'(t) + p(t)e(t) = 0, \qquad e(\bar{t}) = 0.$$

From the directional field, we see that $e(t) \equiv 0$, proving $y_1(t) = y_2(t)$, which implies uniqueness.

One can use this Theorem to identify intervals where the solution of the IVP could be defined.

Example 1. Find the largest interval where the solution can be defined for the following problems.

(A). $ty' + y = t^3$, $y(-1) = 3$.

(B). $ty' + y = t^3$, $y(1) = -3$.

(C). $(t-3)y' + (\ln t)y = 2t$, $y(1) = 2$

(D). $y' + (\tan t)y = \sin t$, $y(\pi) = 100$.

Answer. (A). Rewrite: $y' + \frac{1}{t}y = t^2$, so $t \neq 0$. Since $t_0 = -1$, the interval is $t < 0$.

(B). The equation is same as (A), so $t \neq 0$. $t_0 = 1$, the interval is $t > 0$.

(C). Rewrite: $y' + \frac{\ln t}{t-3}y = \frac{2t}{t-3}$, so $t \neq 3$ and $t > 0$ for the ln function. Since $t_0 = 1$, the interval is then $0 < t < 3$.

(D). Since $t_0 = \pi$, and for $\tan t$ to be defined we must have $t \neq \frac{2k+1}{2}\pi$, $k = \pm 1, \pm 2, \cdots$. So the interval is $\frac{\pi}{2} < t < \frac{3\pi}{2}$.

Remark: The conditions in the Theorem guarantees the uniqueness. If the conditions fail, the uniqueness might still hold, but not granted. Also, in many cases one will have discontinuous functions of $p(t)$ and $g(t)$. One can relax the restriction on the solution, and require $y(t)$ to be only continuous. Then, the conditions reduces to: p, q are integrable functions.

Example (With discontinuous coefficient functions). Consider

$$y' - y = g(t), \qquad y(0) = 0, \qquad g(t) = \begin{cases} 1, & 0 \le t < 1 \\ -2, & 1 \le t \le 2. \end{cases}$$

Find a solution on the interval $0 \le t \le 2$.

Answer. The function $g(t)$ has a jump at $t = 1$. We can solve the equation using 2 steps.
(1) For $0 \le t < 1$, we have the IVP

$$y' - y = 1, \qquad y(0) = 0.$$

Solving this, we obtain

$$y(t) = e^t - 1.$$

At $t = 1$, we have $y(1) = e - 1$.
(2) For $1 \le t \le 2$, we have the IVP

$$y' - y = -2, \qquad y(1) = e - 1.$$

Note that the initial condition is given at $t = 1$, and we use the solution in step 1 and evaluate it at $t = 1$.

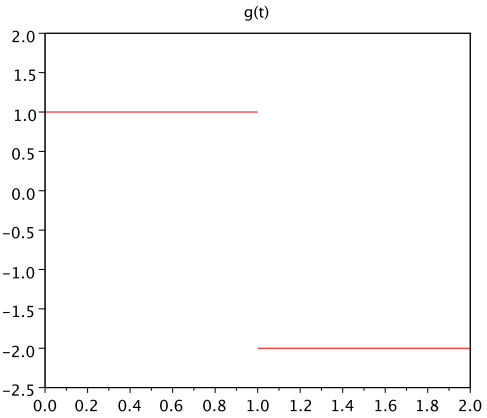
This can be easily solved, and we get

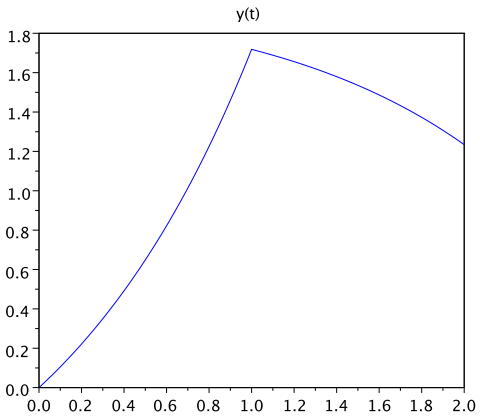
$$y(t) = (1 - 3/e)e^t + 2, \qquad 1 \le t \le 2.$$

Put them together, we obtain the solution, piecewise defined

$$y(t) = \begin{cases} e^t - 1, & 0 \le t < 1 \\ y(t) = (1 - 3/e)e^t + 2, & 1 \le t \le 2. \end{cases}$$

See graphs of the functions g and y:





We see that $y(t)$ is continuous at $t = 1$, but the graph has a kink, since $y'(t)$ is discontinuous at $t = 1$.

In general, ODEs with discontinuous coefficients have their own sets of Theorems.

For non-linear equation

$$y' = f(t, y), \quad y(\bar{t}) = \bar{y},$$

we have a much weaker theorem, for differentiable functions $y(t)$.

Theorem . If $f(t, y)$, $\frac{\partial f}{\partial y}(t, y)$ are continuous and bounded on an rectangle $(\alpha < t < \beta, a < y < b)$ containing $(\bar{t}, \bar{y})$, then there exists an open interval around $\bar{t}$, contained in (α, β), where the solution exists and is unique.

We note that the statement of this theorem is not as strong as the one for linear equation.

The proof uses the Picard iteration, and is much more complicated. It is outlined in the textbook, you may read it if you are curious.

Some remarks*:

(1). If $f(t, y)$, $\frac{\partial f}{\partial y}(t, y)$ are continuous at (t_0, y_0), then in a small neighborhood around the point (t_0, y_0), one can linearize $f(t, y)$, and $f(t, y) \approx a+by$ will be a good approximation. Then by the Theorem for linear equation, solution exits and is unique. This is a rather standard technique for nonlinear problems, studying the local linearized problem.

(2). A rough (not rigorous) argument to see how the bound on $\frac{\partial f}{\partial y}(t, y)$ helps the uniqueness: Let y_1, y_2 be two solutions, then

$$(y_1 - y_2)' = f(t, y_1) - f(t, y_2), \qquad y_1(\bar{t}) - y_2(\bar{t}) = 0.$$

This implies

$$y_1(t) - y_2(t) = \int_{\bar{t}}^{t} \Big(f(s, y_1(s)) - f(s, y_2(s)) \Big)\, ds$$

The bound on $\frac{\partial f}{\partial y}$ implies that there exist a constant M, such that

$$|f(t, y_1) - f(t, y_2)| \le M\, |y_1 - y_2|\,.$$

We now have

$$|y_1(t) - y_2(t)| \le \int_{\bar{t}}^{t} |f(s, y_1(s)) - f(s, y_2(s))|\, ds \le M \int_{\bar{t}}^{t} |y_1(s) - y_2(s)|\, ds.$$

Write now $E(t) = |y_1(t) - y_2(t)| > 0$. We have

$$E(t) \le M \int_{\bar{t}}^{t} E(t)\, ds, \qquad E(\bar{t}) = 0.$$

We now use a comparison argument. Let $w(t)$ be the solution for

$$w(t) = M \int_{\bar{t}}^{t} w(t)\, ds, \qquad w(\bar{t}) = E(\bar{t}) = 0.$$

By comparison we have $E(t) \le w(t)$. For $w(t)$, is satisfies the equation $w' = Mw$ with $w(\bar{t}) = 0$, which implies $w(t) \equiv 0$. Therefore $E(t) \equiv 0$, leading to uniqueness.

Below we give several counter examples.

Example 1. Loss of uniqueness. Consider

$$\frac{dy}{dt} = f(t, y), \qquad f(x, y) = -\frac{t}{y}, \qquad y(-2) = 0.$$

We first note that at $y = 0$, which is the initial value of y, we have $y' = f(t, y) \to \infty$. So the conditions of the Theorem are not satisfied, and we expect something to go wrong.

Solve the equation as an separable equation, we get

$$\int y\, dy = -\int t\, dt, \qquad y^2 + t^2 = c,$$

and by IC we get $c = (-2)^2 + 0 = 4$, so $y^2 + t^2 = 4$. In the $y - t$ plan, this is the equation for a circle, centered at the origin, with radius 2. The initial condition is given at $t_0 = -2, y_0 = 0$, where the tangent line is vertical (i.e., with infinite slope). We have two solutions: $y = \sqrt{4 - t^2}$ and $y = -\sqrt{4 - t^2}$. We lose uniqueness of solutions.

Example 2. Loss of uniqueness; another example. Consider the IVP

$$y' = y^{1/3}, \qquad y(0) = 0.$$

We see a trivial solution is $y(t) \equiv 0$. Now consider the following function:

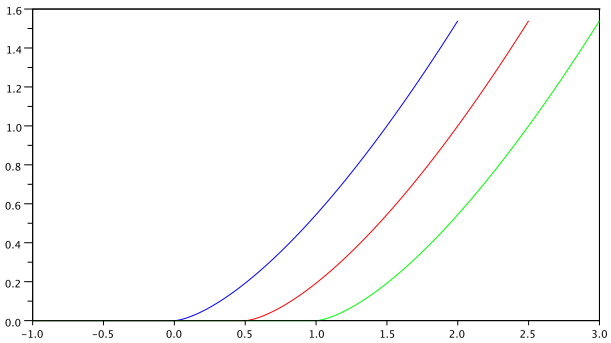
$$y_c(t) = \begin{cases} (2(t - c)/3)^{3/2}, & t > c \\ 0, & t \le c \end{cases}$$

where $c \ge 0$ is any constant. We can easily check that $y_c(t)$ is a solution to the IVP, for any $c \ge 0$. Indeed, we have, for $t \ge c$,

$$y' = \frac{3}{2} (2(t - c)/3)^{1/2} \frac{2}{3} = (2(t - c)/3)^{1/2} = y^{1/3},$$

and for $t < c$, $y' = 0 = y^{1/3}$. The initial condition is also satisfied.

Therefore, there exist infinitely many solutions! See graphs of these solutions for $c = 0, 0.5, 1$.



This is no surprise, since $f(t, y) = y^{1/3}$, then $f_y = \frac{1}{3} y^{-2/3}$ is not bounded at $y = 0$, which is where the initial condition is given.

Example 3. Blow-up of solution. Consider a simple non-linear equation:

$$y' = y^2, \qquad y(0) = 1.$$

Note that $f(t,y) = y^2$, which is defined for all t and y. But, due to the non-linearity of f, solution can not be defined for all t.

This equation can be easily solved as a separable equation.

$$\int \frac{1}{y^2} dy = \int dt, \qquad -\frac{1}{y} = t + c, \quad y(t) = \frac{-1}{t+c}.$$

By IC $y(0) = 1$, we get $1 = -1/(0+c)$, and so $c = -1$, and

$$y(t) = \frac{-1}{t-1}.$$

We see that the solution *blows up* as $t \to 1$, and can not be defined beyond that point.

This kind of blow-up phenomenon is well-known for nonlinear equations.

2.4 Modeling with first order equations

General modeling concept: derivatives describe "rates of change".

Model I: Exponential growth/decay.

$Q(t)$ = amount of quantity at time t

Assume the rate of change of $Q(t)$ is proportional to the quantity at time t. We can write

$$\frac{dQ}{dt}(t) = r \cdot Q(t), \qquad r: \text{ rate of growth/decay}$$

If $r > 0$: exponential growth
If $r < 0$: exponential decay

Differential equation:

$$Q' = rQ, \quad Q(0) = Q_0\,.$$

Solve it: separable equation.

$$\int \frac{1}{Q} dQ = \int r\, dt, \quad \Rightarrow \quad \ln Q = rt + c, \quad \Rightarrow \quad Q(t) = e^{rt+c} = ce^{rt}$$

Here r is called the *growth rate*. By IC, we get $Q(0) = C = Q_0$. The solution is

$$Q(t) = Q_0 e^{rt}.$$

Two concepts:

- For $r > 0$, we define Doubling time T_D, as the time such that $Q(T_D) = 2Q_0$.

$$Q(T_D) = Q_0 e^{rT_D} = 2Q_0, \qquad e^{rT_D} = 2, \quad rT_D = \ln 2, \quad T_D = \frac{\ln 2}{r}.$$

- For $r < 0$, we define Half life (or half time) T_H, as the time such that $Q(T_H) = \frac{1}{2}Q_0$.

$$Q(T_H) = Q_0 e^{rT_H} = \frac{1}{2}Q_0, \quad e^{rT_H} = \frac{1}{2}, \quad rT_H = \ln\frac{1}{2} = -\ln 2, \quad T_H = \frac{\ln 2}{-r}.$$

Note here that $T_H > 0$ since $r < 0$.

NB! T_D, T_H do not depend on Q_0. They only depend on r.

Example 1. If interest rate is 8%, compounded continuously, find doubling time.

Answer. Since $r = 0.08$, we have $T_D = \frac{\ln 2}{0.08}$.

Example 2. A radio active material is reduced to 1/3 after 10 years. Find its half life.

Answer. Model: $\frac{dQ}{dt} = rQ$, r is rate which is unknown. We have the solution $Q(t) = Q_0 e^{rt}$. So

$$Q(10) = \frac{1}{3}Q_0, \qquad Q_0 e^{10r} = \frac{1}{3}Q_0, \qquad r = \frac{-\ln 3}{10}.$$

To find the half life, we only need the rate r

$$T_H = -\frac{\ln 2}{r} = -\ln 2 \frac{10}{-\ln 3} = 10\frac{\ln 2}{\ln 3}.$$

Model II: Interest rate/mortgage problems.

Example 3. Start an IRA account at age 25. Suppose deposit \$2000 at the beginning and \$2000 each year after. Interest rate 8% annually, but assume compounded continuously. Find total amount after 40 years.

Answer. Set up the model: Let $S(t)$ be the amount of money after t years

$$\frac{dS}{dt} = 0.08S + 2000, \qquad S(0) = 2000.$$

This is a first order linear equation. Solve it by integrating factor

$$S' - 0.08S = 2000, \qquad \mu = e^{-0.08t}$$

$$S(t) = e^{0.08t} \int 2000 \cdot e^{-0.08t} dt = e^{0.08t} \left[2000 \frac{e^{-0.08t}}{-0.08} + c \right] = \frac{2000}{-0.08} + ce^{0.08t} = -25000 + ce^{0.08t}.$$

By IC,

$$S(0) = -25000 + c = 2000, \qquad C = 27000,$$

we get

$$S(t) = 27000e^{0.08t} - 25000.$$

When $t = 40$, we have

$$S(40) = 27000 \cdot e^{3.2} - 25000 \approx 637,378.$$

Compare this to the total amount invested: $2000 + 2000 * 40 = 82,000$.

Example 4: A home-buyer can pay \$800 per month on mortgage payment. Interest rate is r annually, (but compounded continuously), mortgage term is 20 years. Determine maximum amount this buyer can afford to borrow. Calculate this amount for $r = 5\%$ and $r = 9\%$ and observe the difference.

Answer. Set up the model: Let $Q(t)$ be the amount borrowed (principle) after t years

$$\frac{dQ}{dt} = rQ(t) - 800 * 12$$

The terminal condition is given $Q(20) = 0$. We must find $Q(0)$.

Solve the differential equation:

$$Q' - rQ = -9600, \qquad \mu = e^{-rt}$$

$$Q(t) = e^{rt} \int (-9600)e^{-rt}dt = e^{rt}\left[-9600\frac{e^{-rt}}{-r} + c\right] = \frac{9600}{r} + ce^{rt}$$

By terminal condition

$$Q(20) = \frac{9600}{r} + ce^{20r} = 0, \qquad c = -\frac{9600}{r \cdot e^{20r}}$$

so we get

$$Q(t) = \frac{9600}{r} - \frac{9600}{r \cdot e^{20r}}e^{rt}.$$

Now we can get the initial amount

$$Q(0) = \frac{9600}{r} - \frac{9600}{r \cdot e^{20r}} = \frac{9600}{r}(1 - e^{-20r}).$$

If $r = 5\%$, then

$$Q(0) = \frac{9600}{0.05}(1 - e^{-1}) \approx \$121,367.$$

If $r = 9\%$, then

$$Q(0) = \frac{9600}{0.09}(1 - e^{-1.8}) \approx \$89,034.$$

We observe that with higher interest rate, one could borrow less.

A few words on compounding of interests rate. Let r be the annual interest rate, and Q_0 be the amount of deposit initially. If the interest is compounded continuously, we see that it gives exponential growth. After t years and we have

$$Q(t) = Q_0e^{rt}.$$

If the interest is compounded quarterly, we get

$$Q_4(t) = Q_0(1 + r/4)^{4t}.$$

If the interest is compounded monthly, we get

$$Q_{12}(t) = Q_0(1 + r/12)^{12t}.$$

If the interest is compounded daily, we get

$$Q_{365}(t) = Q_0(1 + r/365)^{365t}.$$

One can show that

$$Q(T) > Q_{365}(t) > Q_{12}(t) > Q_4(t).$$

Indeed, consider the function

$$Q_n(t) = (1 + r/n)^{nt}.$$

Write it out by binomial formula (Taylor expansion)

$$\begin{aligned} Q_n(t) &= 1 + nt(r/n) + \frac{1}{2}nt(nt-1)(r/n)^2 + \frac{1}{6}nt(nt-1)(nt-2)(r/n)^3 \cdots \\ &= 1 + rt + \frac{1}{2}\frac{nt-1}{nt}(rt)^2 + \frac{1}{6}\frac{nt-1}{nt}\frac{nt-2}{nt}(rt)^2 + \cdots \end{aligned}$$

The Taylor expansion for the exponential function is

$$Q(t) = 1 + rt + \frac{1}{2}(rt)^2 + \frac{1}{6}(rt)^3 + \cdots$$

Comparing term-by-term, we have

$$Q(t) > Q_n(t), \qquad Q_n(t) > Q_m(t) \quad (n > m), \qquad \lim_{n\to+\infty} Q_n(t) = Q(t).$$

Model III: Mixing Problem.

Example 5. At $t = 0$, a tank contains Q_0 lb of salt dissolved in 100 gal of water. Assume that water containing 1/4 lb of salt per gal is entering the tank at a rate of r gal/min. At the same time, the well-mixed mixture is draining from the tank at the same rate.

(1). Find the amount of salt in the tank at any time $t \geq 0$.

(2). When $t \to \infty$, meaning after a long time, what is the limit amount Q_L?

Answer. Set up the model:
$Q(t)$ = amount (lb) of salt in the tank at time t (min)
Then, $Q'(t)$ = [in-rate] − [out-rate].
In-rate: r gal/min × 1/4 lb/gal = $\dfrac{r}{4}$ lb/min
concentration of salt in the tank at time $t = \dfrac{Q(t)}{100}$
Out-rate: r gal/min × $\dfrac{Q(t)}{100}$ lb/gal = $\dfrac{r}{100}Q(t)$ lb/min

$$Q'(t) = [\text{In-rate}] - [\text{Out-rate}] = \frac{r}{4} - \frac{r}{100}Q(t), \qquad \text{I.C.} \quad Q(0) = Q_0.$$

(1). Solve the equation

$$Q' + \frac{r}{100}Q = \frac{r}{4}, \qquad \mu = e^{(r/100)t}.$$

$$Q(t) = e^{-(r/100)t}\int \frac{r}{4}e^{(r/100)t}dt = e^{-(r/100)t}\left[\frac{r}{4}e^{(r/100)t}\frac{100}{r} + c\right] = 25 + ce^{-(r/100)t}.$$

By IC

$$Q(0) = 25 + c = Q_0, \qquad c = Q_0 - 25,$$

we get

$$Q(t) = 25 + (Q_0 - 25)e^{-(r/100)t}.$$

(2). As $t \to \infty$, the exponential term goes to 0, and we have

$$Q_L = \lim_{t\to\infty} Q(t) = 25\text{lb}.$$

We can also observed intuitively that, as time goes on for long, the concentration of salt in the tank must approach the concentration of the salt in the inflow mixture, which is 1/4. Then, the amount of salt in the tank would be $1/4 \times 100 = 25$ lb, as $t \to +\infty$.

Example 6. Tank contains 50 lb of salt dissolved in 100 gal of water. Tank capacity is 400 gal. From $t = 0$, 1/4 lb of salt/gal is entering at a rate of 4 gal/min, and the well-mixed mixture is drained at 2 gal/min. Find:

(1) time t when it overflows;

(2) amount of salt before overflow;

(3) the concentration of salt at overflow.

Answer. (1). Since the inflow rate 4 gal/min is larger than the outflow rate 2 gal/min, the tank will be filled up at t_f:

$$t_f = \frac{400 - 100}{4 - 2} = 150\text{min}.$$

(2). Let $Q(t)$ be the amount of salt at t min.
In-rate: 1/4 lb/gal $\times$ 4 gal/min = 1 lb/min
Out-rate: 2 gal/min $\times \dfrac{Q(t)}{100 + 2t}$ lb/gal $= \dfrac{Q(t)}{50 + t}$ lb/min

$$Q'(t) = 1 - \frac{Q(t)}{50 + t}, \qquad Q' + \frac{1}{50 + t}Q = 1, \qquad Q(0) = 50$$

$$\mu = \exp\left(\int \frac{1}{50 + t}dt\right) = \exp\left(\ln(50 + t)\right) = 50 + t$$

$$Q(t) = \frac{1}{50 + t}\int (50 + t)dt = \frac{1}{50 + t}\left[50t + \frac{1}{2}t^2 + c\right]$$

By IC:

$$Q(0) = c/50 = 50, \qquad c = 2500,$$

We get

$$Q(t) = \frac{50t + t^2/2 + 2500}{50 + t}.$$

(3). The concentration of salt at overflow time $t = 150$ is

$$\frac{Q(150)}{400} = \frac{50 \cdot 150 + 150^2/2 + 2500}{400(50 + 150)} = \frac{17}{64}\text{lb/gal}.$$

Model IV: Air resistance

Example 7. A ball with mass 0.5 kg is thrown upward with initial velocity 10 m/sec from the roof of a building 30 meter high. Assume air resistance is $|v|/20$. Find the max height above ground the ball reaches.

Answer. Let $S(t)$ be the position (m) of the ball at time t sec. Then, the velocity is $v(t) = dS/dt$, and the acceleration is $a = dv/dt$. Let upward be the positive direction. We have by Newton's Law:

$$F = ma = -mg - \frac{v}{20}, \qquad a = -g - \frac{v}{20m} = \frac{dv}{dt}$$

Here $g = 9.8$ is the gravity, and $m = 0.5$ is the mass. We have an equation for v:

$$\frac{dv}{dt} = -\frac{1}{10}v - 9.8 = -0.1(v + 98),$$

so

$$\int \frac{1}{v+98}dv = \int (-0.1)dt, \quad \Rightarrow \quad \ln|v+98| = -0.1t + c$$

which gives

$$v + 98 = \bar{c}e^{-0.1t}, \quad \Rightarrow \quad v = -98 + \bar{c}e^{-0.1t}.$$

By IC:

$$v(0) = -98 + \bar{c} = 10, \quad \bar{c} = 108, \quad \Rightarrow \quad v = -98 + 108e^{-0.1t}.$$

To find the position S, we use $S' = v$ and integrate

$$S(t) = \int v(t)\,dt = \int (-98 + 108e^{-0.1t})dt = -98t + 108e^{-0.1t}/(-0.1) + c$$

By IC for S,

$$S(0) = -1080 + c = 30, \qquad c = 1110, \qquad S(t) = -98t - 1080e^{-0.1t} + 1110.$$

At the maximum height, we have $v = 0$. Let's find out the time T when max height is reached.

$$v(T) = 0, \quad -98 + 108e^{-0.1T} = 0, \quad 98 = 108e^{-0.1T}, \quad e^{-0.1T} = 98/108,$$

$$-0.1T = \ln(98/108), \quad T = -10\ln(98/108) = 10\ln(108/98).$$

So the max height S_M is

$$\begin{aligned} S_M &= S(T) = -980\ln\frac{108}{98} - 1080e^{-0.1(10)\ln(108/98)} + 1110 \\ &= -980\ln\frac{108}{98} - 1080(98/108) + 1110 \approx 34.78 \text{ m}. \end{aligned}$$

Other possible questions:

- Find the time when the ball hit the ground.
  Solution: Find the time $t = t_H$ for $S(t_H) = 0$.

- Find the speed when the ball hit the ground.
  Solution: Compute $|v(t_H)|$.
- Find the total distance traveled by the ball when it hits the ground.
  Solution: Add up twice the max height S_M with the height of the building.

Example 8. (Nonlinear air resistance). A rocket sled with initial speed 100 m/s is slowed by a channel of water. We assume that the acceleration satisfies $a = -0.01v^2$ where v is the velocity.

(1) Find the velocity $v(t)$ at any time $t > 0$.
(2) Find the distance traveled $s(t)$ at any time $t > 0$.
(3) Find the distance traveled when the speed is reduced to 10 m/s.

Answer. (1). We set up the equation

$$v' = -0.01v^2, \qquad v(0) = 100.$$

This is a separable equation:

$$\int -\frac{1}{v^2}dv = \int 0.01\,dt, \quad \to \quad \frac{1}{v} = 0.01t + c, \quad \to \quad v(t) = \frac{1}{0.01t + c}.$$

Use the initial condition:

$$v(0) = \frac{1}{c} = 100, \quad \to \quad c = 0.01.$$

So the solution is

$$v(t) = \frac{1}{0.01t + 0.01} = \frac{100}{t + 1}.$$

(2). We have $s'(t) = v(t)$ and $s(0) = 0$. Then

$$s(t) = \int_0^t v(\tau)\,d\tau = \int_0^t \frac{100}{\tau + 1}d\tau = 100\ln(\tau + 1)\Big|_{\tau=0}^{t} = 100\ln(t + 1).$$

(3). Let $\bar{t}$ be the time such that $v(\bar{t}) = 10$. Then

$$\frac{100}{\bar{t} + 1} = 10, \quad \to \quad \bar{t} = 9\,\text{s}.$$

Thus

$$s(\bar{t}) = 100 \cdot \ln(9 + 1) \approx 230.3\,\text{m}$$

2.5 Autonomous equations and population dynamics

Definition: An autonomous equation is of the form $y' = f(y)$, where the function f for the derivative depends only on y, not on t.

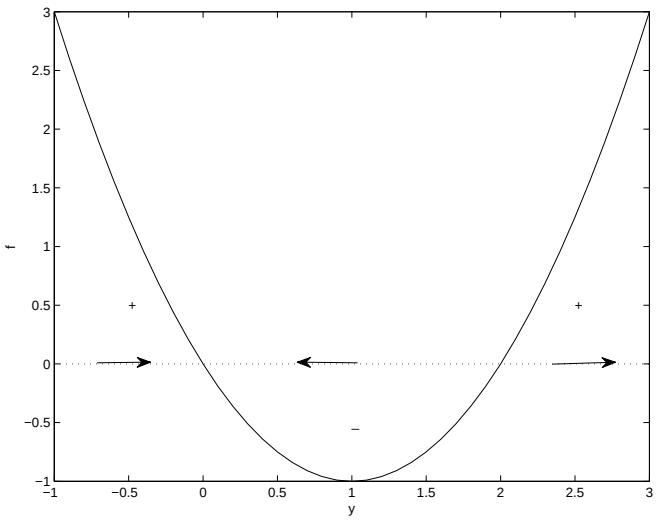
Simplest example: $y' = ry$, exponential growth/decay, where solution is $y = y_0e^{rt}$.

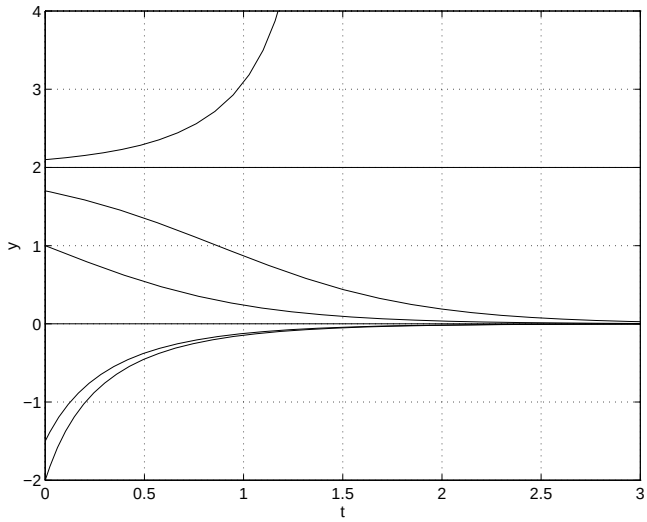
Definition: Zeros of f where $f(y) = 0$ are called critical points or equilibrium points, or equilibrium solutions.

Why? Because if $f(y_0) = 0$, then $y(t) = y_0$ is a constant solution. It is called an equilibrium.

Question: Is an equilibrium stable or unstable?

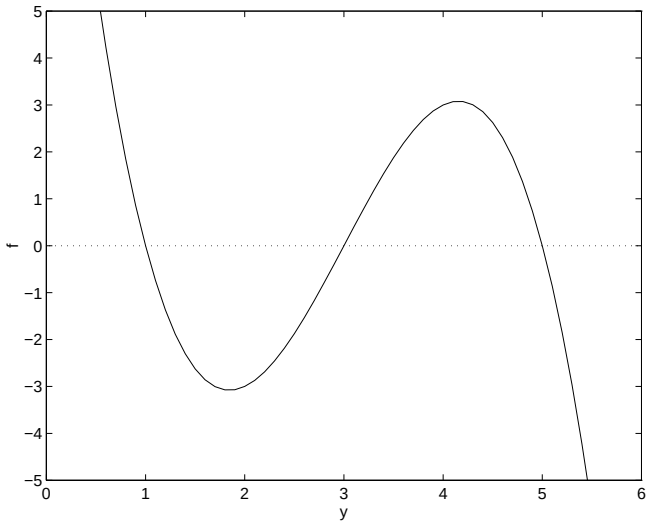
Example 1. $y' = y(y-2)$. We have two critical points: $y_1 = 0, \quad y_2 = 2$.





We see that $y_1 = 0$ is stable, and $y_2 = 2$ is unstable.

Example 2. For the equation $y' = f(y)$ where $f(y)$ is given in the following plot:

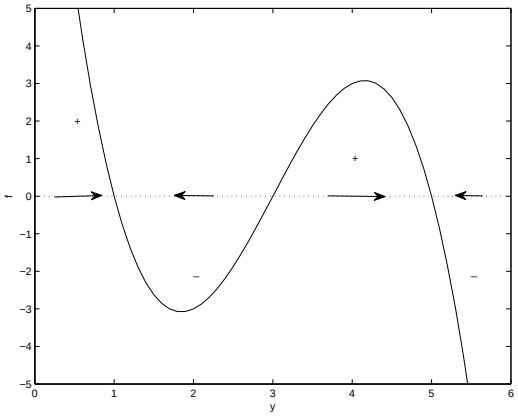


(A). What are the critical points?

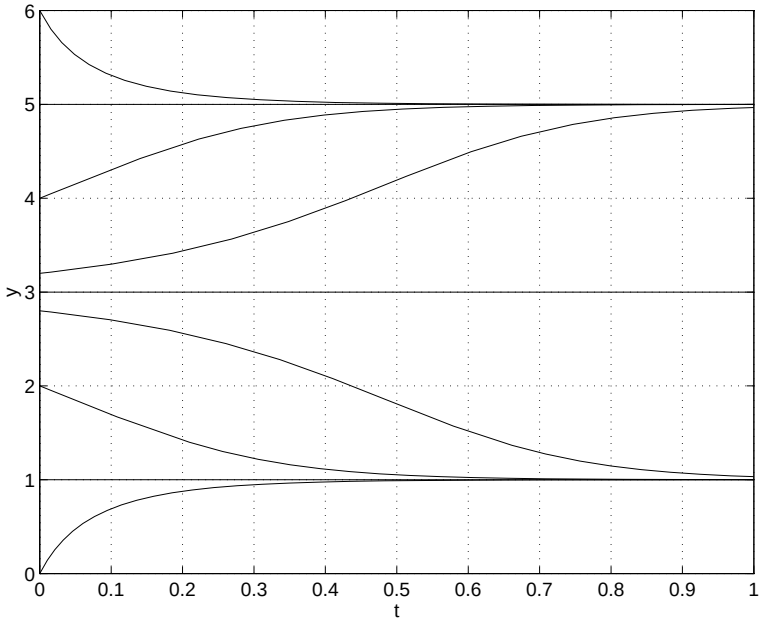
(B). Are they stable or unstable?

(C). Sketch the solutions in the $t - y$ plan, and describe the behavior of y as $t \to \infty$ (as it depends on the initial value $y(0)$.)

Answer. (A). There are three critical points: $y_1 = 1,\ y_2 = 3,\ y_3 = 5$.
(B). To see the stability, we add arrows on the y-axis:



We see that $y_1 = 1$ is stable, $y_2 = 3$ is unstable, and $y_3 = 5$ is stable.
(C). The sketch is given below:



Asymptotic behavior for y as $t \to \infty$ depends on the initial value of y:

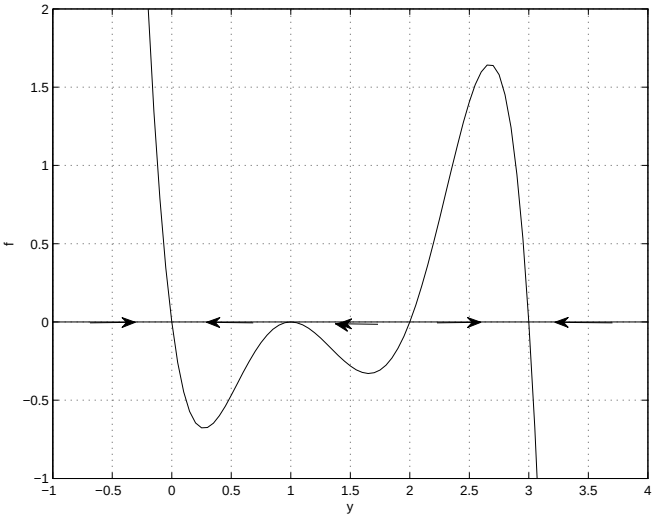
- If $y(0) < 1$, then $y(t) \to 1$,
- If $y(0) = 1$, then $y(t) = 1$;
- If $1 < y(0) < 3$, then $y(t) \to 1$;
- If $y(0) = 3$, then $y(t) = 3$;
- If $3 < y(0) < 5$, then $y(t) \to 5$;
- If $y(0) = 5$, then $y(t) = 5$;

- if $y(t) > 5$, then $y(t) \to 5$.

Stability: is not only stable or unstable.

Example 3. For $y' = y^2$, we have only one critical point $y_1 = 0$. For $y < 0$, we have $y' > 0$, and for $y > 0$ we also have $y' > 0$. So solution is increasing on both intervals. So on the interval $y < 0$, solution approaches $y = 0$ as t grows, so it is stable. But on the interval $y > 0$, solution grows and leaves $y = 0$, and it is unstable. This type of critical point is called *semi-stable*. This happens when one has a double root for $f(y) = 0$.

Example 4. For equation $y' = f(y)$ where $f(y)$ is given in the plot



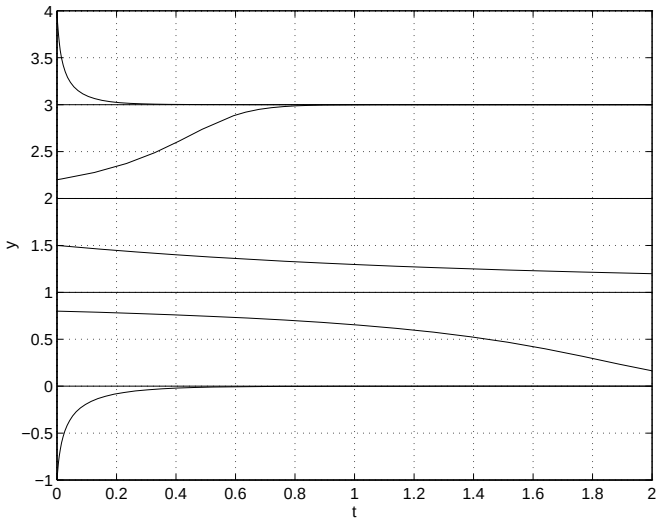
(A). Identify equilibrium points;

(B). Discuss their stabilities;

(C). Sketch solution in $y - t$ plan;

(D). Discuss asymptotic behavior as $t \to \infty$.

Answer. (A). $y = 0, y = 1, y = 2, y = 3$ are the critical points.
(B). $y = 0$ is stable, $y = 1$ is semi-stable, $y = 2$ is unstable, and $y = 3$ is stable.
(C). The Sketch is given in the plot:



(D). The asymptotic behavior as $t \to \infty$ depends on the initial data.

- If $y(0) < 1$, then $y \to 0$;
- If $1 \le y(0) < 2$, then $y \to 1$;
- If $y(0) = 2$, then $y(t) = 2$;
- If $y(0) > 2$, then $y \to 3$.

Application in population dynamics: let $y(t)$ be the population of a species.

Typically, the rate of change in the population depends on the population y, at any time t. This means $y'(t)$ typically does NOT depend on t, and we end up with autonomous equations.

Model 1. The simplest model is the exponential growth, with growth rate r:

$$y'(t) = ry, \qquad y(0) = y_0.$$

The solution can be written explicitly as

$$y(t) = y_0 e^{rt}.$$

If initially there is no life, i.e., $y_0 = 0$, then it remains that way and $y(t) \equiv 0$. Otherwise, if only a very small amount of population exists, i.e., $y_0 > 0$, then $y(t)$ will grow exponentially in time.

Of course, this model is not realistic. In nature there is limited natural resource that can support only limited amount of population.

Model 2. The more realistic model is the "logistic equation":

$$\frac{dy}{dt} = (r - ay)y.$$

or

$$\frac{dy}{dt} = r\left(1 - \frac{y}{k}\right)y, \qquad k = \frac{r}{a},$$

r=intrinsic growth rate,
k=environmental carrying capacity.

critical points: $y = 0,\ y = k$. Here $y = 0$ is unstable, and $y = k$ is stable.

If $0 < y(0) < k$, then $y \to k$ as t grows.

If $y(0) > k$, then $y \to k$ as t grows.

Im summary, if $y(0) > 0$, then

$$\lim_{t\to+\infty} y(t) = k.$$

Remark*. Since this autonomous equation is separable, it is possible to solve it and obtain an explicit expression for the solution. Consider now the IVP

$$\frac{dy}{dt} = r\left(1 - \frac{y}{k}\right)y, \qquad y(0) = y_0,$$

after separating the variables, we have

$$\int \frac{1}{\left(1 - \frac{y}{k}\right)y} dy = \int r dt$$

The tricky part is the integration of the left hand side. We apply a technique called partial fraction, and get

$$\frac{1}{\left(1 - \frac{y}{k}\right)y} = \frac{k}{(k-y)\,y} = \frac{1}{y} + \frac{1}{k-y}.$$

Then

$$\int \left[\frac{1}{y} + \frac{1}{k-y}\right] dy = \ln|y| - \ln|k-y| = \ln\left|\frac{y}{k-y}\right|.$$

This yields

$$\ln\left|\frac{y}{k-y}\right| = rt + c, \quad \to \quad \left|\frac{y}{k-y}\right| = ce^{rt}, \quad \text{where} \quad c = \left|\frac{y_0}{k-y_0}\right|.$$

If $0 < y_0 < k$, so does $0 < y(t) < k$, and we can remove the absolute value signs in the solution, and after some manipulation we get

$$y(t) = \frac{cke^{rt}}{1 + ce^{rt}} = \frac{ky_0e^{rt}}{(k-y_0) + y_0e^{rt}} = \frac{ky_0}{y_0 + (k-y_0)e^{-rt}}.$$

We see that $y \to k$ as $t \to \infty$.

On the other hand, if $y_0 > k$, we have

$$\frac{y}{y-k} = ce^{rt}, \qquad c = \frac{y_0}{y_0 - k}.$$

After some manipulation we get

$$y(t) = \frac{kce^{rt}}{ce^{rt} - 1} = \frac{ky_0e^{rt}}{y_0e^{rt} - (y_0 - k)} = \frac{ky_0}{y_0 - (y_0 - k)e^{-rt}} = \frac{ky_0}{y_0 + (k-y_0)e^{-rt}}$$

We see in both cases the solution takes the same form!

Model 3. An even more detailed model is the logistic growth with a threshold:

$$y'(t) = -r\left(1 - \frac{y}{T}\right) \cdot \left(1 - \frac{y}{K}\right) y, \qquad r > 0, \quad 0 < T < K.$$

We see that there are 3 critical points: $y = 0, y = T, y = K$, where $y = T$ is unstable, and $y = 0, y = K$ are stable.

Let $y(0) = y_0$ be the initial population. We discuss the asymptotic behavior as $t \to +\infty$.

- If $0 < y_0 < T$, then $y \to 0$.
- If $T < y_0 < K$, then $y \to K$.
- If $y_0 > K$, then $y \to K$.

We see that $y_0 = T$ work as a threshold. We have

$$\lim_{t \to +\infty} y(t) = 0 \qquad \text{if} \quad y(0) < T$$
$$\lim_{t \to +\infty} y(t) = K \qquad \text{if} \quad y(0) > T.$$

Some populations follow this model, for example, the some fish population in the ocean. If we over-fishing and make the population below certain threshold, then the fish will go extinct. That's too sad.

Remark*. One can also solve this equation and obtain an explicit expression, but the computation is lengthy!

2.6 Exact Equations

We will review: partial derivatives and Chain Rules for functions of 2 variables.

Consider a function $f(x, y)$. We use these notations for the partial derivatives:

$$f_x = \frac{\partial f}{\partial x}, \qquad f_y = \frac{\partial f}{\partial y}$$

and correspondingly the higher derivatives:

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}, \qquad f_{yy} = \frac{\partial^2 f}{\partial y^2}$$

and the cross derivatives

$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right), \qquad f_{yx} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right),$$

where we have the identity

$$f_{xy} = f_{yx}.$$

Now, consider $x = x(t)$ and $y = y(t)$, and we form a composite function as $f(x(t), y(t))$. We see that f now depends only on t.

Chain Rule:

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} = f_x x' + f_y y'.$$

Example 1. Let $f(x, y) = x^2y^2 + e^x$, and $x(t) = t^2$, $y(t) = e^t$, and consider the composite function $f(x(t), y(t))$. Compute $\frac{df}{dt}$.

Answer. We first compute the derivatives

$$f_x = 2xy^2 + e^x, \quad f_y = 2x^2y, \qquad x'(t) = 2t, \quad y'(t) = e^t$$

By the Chain Rule, we compute

$$\begin{aligned} \frac{df}{dt} &= (2xy^2 + e^x)2t + 2x^2ye^t \\ &= 2t\left(2t^2e^{2t} + e^{t^2}\right) + 2t^4e^te^t \end{aligned}$$

Special case: If $y = y(x)$, then the composite function $f(x, y(x))$ will follow this form of Chain Rule

$$\frac{df}{dx} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = f_x + f_y y'(x).$$

Exact Equations. We will start with an Example.

Example 2. Let $y(x)$ be the unknown. Consider the equation

$$6x + e^xy^2 + 2e^xyy' = 0$$

We see that the equation is NOT linear. It is NOT separable either. None of the methods we know can solve it.

However, define the function

$$\psi(x, y) = 3x^2 + e^xy^2$$

We notice that

$$\psi_x = 6x + e^xy^2, \qquad \psi_y = 2e^xy$$

and the equation can be written as

$$\psi_x(x, y) + \psi_y(x, y)y' = 0.$$

Since $y = y(x)$, we apply the Chain Rule to the composite function $\psi(x, y(x))$ and get

$$\frac{d\psi}{dx} = \psi_x + \psi_y y'(x)$$

which is the left-hand side of the equation. By the differential equation, we now have

$$\frac{d\psi}{dx} = 0, \qquad \rightarrow \quad \psi(x, y) = C$$

where C is an arbitrary constant, to be determined by initial condition. We have found the solution in an *implicit form*:

$$3x^2 + e^x y^2 = C.$$

In this example, we are even able to write out the solution in an explicit form by algebraic manipulation

$$y^2 = e^{-x}(C - 3x^2), \qquad y = \pm\sqrt{e^{-x}(C - 3x^2)}.$$

Here, the choice of $+$ or $-$ sign should be determined by initial condition.

Definition of an exact equation. En equation in the form

$$M(x, y) + N(x, y)y' = 0$$

is called exact if there exists a function $\psi(x, y)$ such that $\psi_x = M$ and $\psi_y = N$.

How to check if an equation is exact? By the identity $\psi_{xy} = \psi_{yx}$, we must have

$$M_y(x, y) = N_x(x, y).$$

How to solve it? Need to find the function ψ, then we get implicit solution

$$\psi(x, y) = C.$$

How to find ψ? By using the facts that

$$\psi_x = M(x, y), \qquad \psi_y = N(x, y)$$

and integrate. We will see this through an example.

Example 3. Check if the following equation is exact

$$(2x + 3y) + (x - 2y)y' = 0.$$

Answer. Here we have

$$M(x, y) = 2x + 3y, \qquad N(x, y) = x - 2y.$$

Then

$$M_y = 3, \quad N_x = 1, \qquad M_y \neq N_x,$$

so the equation is not exact.

Remark 1. Consider a separable equation:

$$y' = \frac{f(x)}{g(y)}.$$

Multiply both sides by $g(y)$ and re-arranging terms, we get

$$f(x) - g(y)y' = 0.$$

Now we check if this equation is exact. Clearly, we have $f_y = 0$ and $g_x = 0$, so it is exact. We may conclude that all separable equations can be rewritten into an exact equation.

Remark 2. However, the exactness of an equation is up to manipulation. Consider the separable equation in Remark 1, and rewrite the equation now into

$$\frac{f(x)}{g(y)} - y' = 0.$$

So now we have

$$M(x, y) = \frac{f(x)}{g(y)}, \quad N(x, y) = 1.$$

Since $N_x = 0$ and $M_y \neq 0$ in general, the equation is not exact.

So, be careful. When you say an equation is exact, you must specify in which form you present your equation.

Example 4. Consider the equation

$$(2x + y) + (x + 2y)y' = 0$$

(1) Is it exact? (2) If yes, find the solution with the initial condition $y(1) = 1$.

Answer. (1). We have $M = 2x + y$ and $N = x + 2y$, so $M_y = 1$ and $N_x = 1$, so the equation is exact.

(2). To solve it, we need to find the function ψ. We have

$$\psi_x = 2x + y, \qquad \psi_y = x + 2y. \tag{A}$$

Integrating the first equation in x:

$$\psi(x, y) = \int \psi_x \, dx = \int (2x + y) \, dx + h(y) + x^2 + xy + h(y).$$

(Review: To perform a partial integration in x, we treat y as a constant. Therefore, the integration constant could depend on y since it is a constant. That's why we add $h(y)$, a function of y, to the anti-derivative.)

To determine $h(y)$, we use the second equation in (A).

$$\psi_y = x + h'(y) = x + 2y, \qquad h'(y) = 2y, \quad h(y) = y^2.$$

Therefore

$$\psi = x^2 + xy + y^2$$

and the implicit solution is

$$x^2 + xy + y^2 = C.$$

Finally, we determine the constant C by initial condition. Plug in $x = 1, y = 1$, we get $C = 3$, so the implicit solution is

$$x^2 + xy + y^2 = 3.$$

Remark. This procedure is rather lengthy, and could easily cause many mistakes. An alternative (shorter) method is described below. We integrate the two equations in (A), with

respect to the partial derivatives. This means, for the first equation we integrate in x and the second equation in y. We do not take the integration constants, we only find terms in the anti-derivatives. This gives two expressions of ψ:

$$\psi = \int_x (2x + y)\, dx = x^2 + xy, \qquad \psi = \int_y (x + 2y)\, dy = xy + y^2.$$

Now we comb through the two expressions, and collect all *different* terms. We get

$$\psi = x^2 + xy + y^2.$$

Note that we get the same answer.

Example 5. Given equation

$$(xy^2 + bx^2y) + (x + y)x^2y' = 0.$$

(1) Find the values of b such that the equation is exact. (2) Solve it with that value of b.

Answer. (1). We have

$$M(x, y) = xy^2 + bx^2y, \qquad N(x, y) = x^3 + x^2y$$

so

$$M_y = 2xy + bx^2, \qquad N_x = 3x^2 + 2xy$$

We see that we must have $b = 3$ to ensure $M_y = N_x$ which would make the equation exact.

(2). We now set $b = 3$. To solve the equation, we need to find the function ψ. We have

$$\psi_x = xy^2 + 3x^2y, \qquad \psi_y = x^3 + x^2y.$$

Integrating the first equation in x:

$$\psi(x, y) = \int (xy^2 + 3x^2y)dx + h(y) = \frac{1}{2}x^2y^2 + x^3y + h(y)$$

To find $h(y)$, we use ψ_y:

$$\psi_y = x^2y + x^3 + h'(y) = N = x^3 + x^2y$$

We must have $h'(y) = 0$, so we can use $h(y) = 0$.

The implicit solution is

$$\frac{1}{2}x^2y^2 + x^3y = C,$$

where C is arbitrary, to be determined by initial condition.

(We remark that, if we use the alternative method in the previous Remark, we would get the same answer. Try this yourself.)

We make some observations.

(1). The exactness of an equation is up to manipulation. Consider an exact equation $M(x,y)+N(x,y)y'=0$ where $M_y=N_x$. If we multiply the equation by some function $f(x,y)$ on both sides, we get

$$f(x,y)M(x,y)+f(x,y)N(x,y)y'=0$$

which in general is not exact for an arbitrary choice of f.

(2). Then, the other way around is also possible. Consider an equation $\hat{M}(x,y)+\hat{N}(x,y)y'=0$, which is not exact as they appear. If the conditions are favorable, it might be possible to find some function $\mu(x,y)$, such that

$$\mu(x,y)\hat{M}(x,y)+\mu(x,y)\hat{N}(x,y)y'=0$$

becomes exact. If this is true, then $\mu(x,y)$ is called an *integrating factor*.

(3). Consider now an separable equation $y'=f(x)/g(y)$. Write it as

$$\frac{f(x)}{g(y)}-y'=0$$

the equation is not exact. But if we multiply both sides by $g(y)$, we get

$$f(x)-g(y)y'=0$$

which is exact. This shows that: *any separable equation could be written into an exact equation.*

2.7 Euler's method*

In general, for an equation

$$y'(t)=f(t,y), \qquad y(\bar{t})=\bar{y},$$

it might not be possible to find an analytic expression for the solution $y(t)$. If one wants to know how the solution behaves, one way would be using directional field and sketch the solutions. Another way is to use numerical methods to compute some approximate solution.

We choose a grid size, i.e., a small value $h>0$, and make the grid, i.e., a set of points for t, such that

$$t_0=\bar{t}, \qquad t_k=t_0+kh, \quad k=1,2,3,\cdots$$

We seek approximate solution only at these grid points, i.e.,

$$y_k\approx y(x_k), \qquad k=1,2,3,\cdots$$

The values of y_k are computed through iteration. The main idea here is, at $t=t_k$, one checks the value for the gradient $y'(t_k)=f(t_k,y_k)$, and use this as the slope and take a step forward in time. This gives the following algorithm: Given the starting point $t_0=\bar{t}, y+0=\bar{y}$, perform the following computation:

$$y_{k+1}=y_k+h\cdot f(t_k,y_k), \qquad \text{for} \quad k=0,1,2,\cdots$$

Then, one can plot the graph $y(t)$ using these grid points, in a programing language.

One can illustrate this idea through a graph, together with directional field.

Numerical methods for ODEs are a well-studied topic. There are many methods developed, for various models. Better methods (better than Euler) are available , would would give higher order accuracy. Here we don't have time to go into details. If you are interested in this topic, you could take the course CMPSC/Math 451, or an extended version of it, CMPSC/Math 455 + 456.

Chapter 3

Second Order Linear Equations

In this chapter we study linear 2nd order ODEs. The general form of these equations is

$$a_2(t)y'' + a_1(t)y' + a_0(t)y = b(t),$$

where

$$a_2(t) \neq 0, \qquad y(t_0) = y_0, \quad y'(t_0) = \bar{y}_0.$$

If $b(t) \equiv 0$, we call it *homogeneous*. Otherwise, it is called *non-homogeneous*.

3.1 Homogeneous equations with constant coefficients

This is the simplest case: a_2, a_1, a_0 are all constants, and $g = 0$. Let's write:

$$a_2y'' + a_1y' + a_0y = 0.$$

We start with an example.

Example 1. Solve $y'' - y = 0$, (we have here $a_2 = 1, a_1 = 0, a_0 = 1$).

Answer. Let's guess an answer of the form $y_1(t) = e^t$.
Check to see if it satisfies the equation: $y'' = e^t$, so $y'' - y = e^t - e^t = 0$. So it is a solution.

Guess another function: $y_2(t) = e^{-t}$.
Check: $y' = -e^{-t}$, so $y'' = e^{-t}$, so $y'' - y = e^t - e^t = 0$. So it is also a solution.

Claim: Another function $y = c_1y_1 + c_2y_2$ for any arbitrary constants c_1, c_2 (this is called "a linear combination of y_1, y_2") is also a solution.

Check if this claim is true:

$$y(t) = c_1e^t + c_2e^{-t},$$

then

$$y' = c_1e^t - c_2e^{-t}, \quad y'' = c_1e^t + c_2e^{-t}, \quad \Rightarrow \quad y'' - y = 0.$$

Actually this claim is a general property. It is called *the principle of superposition*.

Theorem (The Principle of Superposition) *Let $y_1(t)$ and $y_2(t)$ be solutions of*

$$a_2(t)y'' + a_1(t)y' + a_0(t)y = 0$$

Then, $y = c_1y_1 + c_2y_2$ for any constants c_1, c_2 is also a solution.

Proof: If y_1 solves the equation, then

$$a_2(t)y_1'' + a_1(t)y_1' + a_0(t)y_1 = 0. \qquad (I)$$

If y_2 solves the equation, then

$$a_2(t)y_2'' + a_1(t)y_2' + a_0(t)y_2 = 0. \qquad (II)$$

Multiple (I) by c_1 and (II) by c_2, and add them up:

$$a_2(t)(c_1y_1 + c_2y_2)'' + a_1(t)(c_1y_1 + c_2y_2)' + a_0(t)(c_1y_1 + c_2y_2) = 0.$$

Let $y = c_1y_1 + c_2y_2$, we have

$$a_2(t)y'' + a_1(t)y' + a_0(t)y = 0$$

therefore y is also a solution to the equation.

How to find the solutions of $a_2y'' + a_1y' + a_0y = 0$?
We seek solutions in the form $y(t) = e^{rt}$. Find r.

$$y' = re^{rt} = ry, \qquad y'' = r^2e^{rt} = r^2y$$

$$a_2r^2y + a_1ry + a_0y = 0$$

Since $y \neq 0$, we get

$$a_2r^2 + a_1r + a_0 = 0$$

This is called the *characteristic equation*.

Conclusion: If r is a root of the characteristic equation, then $y = e^{rt}$ is a solution.

If there are two real and distinct roots $r_1 \neq r_2$, then the **general solution** is $y(t) = c_1e^{r_1t} + c_2e^{r_2t}$ where c_1, c_2 are two arbitrary constants to be determined by initial conditions (ICs).

Example 2. Consider $y'' - 5y' + 6y = 0$.

(a). Find the general solution.

(b). If ICs are given as: $y(0) = -1, y'(0) = 5$, find the solution.

(c). What happens to $y(t)$ when $t \to \infty$?

Answer. (a). The characteristic equation is: $r^2 - 5r + 6 =$, so $(r-2)(r-3) = 0$, two roots: $r_1 = 2, r_2 = 3$. General solution is:

$$y(t) = c_1e^{2t} + c_2e^{3t}.$$

(b). $y(0) = -1$ gives: $c_1 + c_2 = -1$.
$y'(0) = 5$: we have $y' = 2c_1e^{2t} + 3c_2e^{3t}$, so $y'(0) = 2c_1 + 3c_2 = 5$.

Solve these two equations for c_1, c_2: Plug in $c_2 = -1 - c_1$ into the second equation, we get $2c_1 + 3(-1 - c_1) = 5$, so $c_1 = -8$. Then $c_2 = 7$. The solution is

$$y(t) = -8e^{2t} + 7e^{3t}.$$

(c). We see that $y(t) = e^{2t} \cdot (-8 + 7e^t)$, and both terms in the product go to infinity as t grows. So $y \to +\infty$ as $t \to +\infty$.

Example 3. Find the solution for $2y'' + y' - y = 0$, with initial conditions $y(1) = 0,\ y'(1) = 3$.

Answer. Characteristic equation:

$$2r^2 + r - 1 = 0, \quad \Rightarrow \quad (2r-1)(r+1) = 0, \quad \Rightarrow \quad r_1 = \frac{1}{2},\ r_2 = -1.$$

General solution is:

$$y(t) = c_1 e^{\frac{t}{2}} + c_2 e^{-t}.$$

The ICs give

$$y(1) = 0: \quad c_1 e^{\frac{1}{2}} + c_2 e^{-1} = 0. \tag{A}$$

$$y'(1) = 3: \quad y'(t) = \frac{1}{2} c_1 e^{\frac{1}{2}t} - c_2 e^{-t}, \quad \frac{1}{2} c_1 e^{\frac{1}{2}} - c_2 e^{-1} = 3. \tag{B}$$

(A)+(B) gives

$$\frac{3}{2} c_1 e^{\frac{1}{2}} = 3, \quad c_1 = 2e^{-\frac{1}{2}}.$$

Plug this in (A):

$$c_2 = -e c_1 e^{\frac{1}{2}} = -e 2 e^{\frac{1}{2}} e^{\frac{1}{2}} = -2e.$$

The solution is

$$y(t) = 2e^{-\frac{1}{2}} e^{\frac{1}{2}t} - 2ee^{-t} = 2e^{\frac{1}{2}(t-1)} - 2e^{-(t-1)},$$

and as $t \to \infty$ we have $y \to \infty$.

Remark. Note that the initial data is given at $t = 1$, not $t = 0$. It appears that this causes difficulties in computing the constants c_1, c_2. Note that we write the final answer in the form such that it becomes a function of $t - 1$. Indeed, this suggests a better form for the general solution that would lead to simpler computation when one plugs in the initial data. We take the general solution as

$$y(t) = \bar{c}_1 e^{(t-1)/2} + \bar{c}_2 e^{-(t-1)}.$$

Note that we replaced t with $t - t_0$, where $t_0 = 1$ is the time when the initial conditions are given. We have

$$y'(t) = \frac{1}{2} \bar{c}_1 e^{(t-1)/2} - \bar{c}_2 e^{-(t-1)}$$

and the initial conditions give us

$$y(1) = \bar{c}_1 + \bar{c}_2 = 0, \qquad y'(1) = \frac{1}{2} \bar{c}_1 - \bar{c}_2 = 3.$$

This leads to easy computation to find $\bar{c}_1 = 2, \bar{c}_2 = -2$, and the solution is the same

$$y(t) = 2e^{\frac{1}{2}(t-1)} - 2e^{-(t-1)}\,.$$

Summary of receipt:

1. Write the characteristic equation; $y'' \to r^2,\ y' \to r,\ y \to 1$.
2. Find the roots;
3. Write the general solution;
4. Set in ICs to get the arbitrary constants c_1, c_2.

Example 4. Consider the equation $y'' - 4y = 0$.

(a). Find the general solution.

(b). If the initial conditions are given as $y(0) = 1$ and $y'(0) = a$, then, for what values of a would y remain bounded as $t \to +\infty$?

Answer. (a). Characteristic equation

$$r^2 - 4 = 0, \quad \Rightarrow \quad r_1 = -2, \ \ r_2 = 2.$$

General solution is

$$y(t) = c_1 e^{-2t} + c_2 e^{2t}.$$

(b). If $y(t)$ remains bounded as $t \to \infty$, then the term e^{2t} must vanish, which means we must have $c_2 = 0$. This means $y(t) = c_1 e^{-2t}$. If $y(0) = 1$, then $y(0) = c_1 = 1$, so $y(t) = e^{-2t}$. This gives $y'(t) = -2e^{-2t}$ which means $a = y'(0) = -2$.

Example 5. Consider the equation $2y'' + 3y' = 0$. The characteristic equation is

$$2r^2 + 3r = 0, \quad \Rightarrow \quad r(2r+3) = 0, \quad \Rightarrow \quad r_1 = -\frac{3}{2},\ r_2 = 0$$

The general solution is

$$y(t) = c_1 e^{-\frac{3}{2}t} + c_2 e^{0t} = c_1 e^{-\frac{3}{2}t} + c_2.$$

As $t \to \infty$, the first term in y vanishes, and we have $y \to c_2$.

Example 6. Find a 2nd order equation such that $c_1 e^{3t} + c_2 e^{-t}$ is its general solution.

Answer. From the form of the general solution, we see the two roots are $r_1 = 3, r_2 = -1$. The characteristic equation could be $(r-3)(r+1) = 0$, or this equation multiplied by any non-zero constant. So $r^2 - 2r - 3 = 0$, which gives us the equation

$$y'' - 2y' - 3y = 0.$$

3.2 Solutions of Linear Homogeneous Equations; the Wronskian

We consider some theoretical aspects of the solutions to a general 2nd order linear equations.

Theorem . (Existence and Uniqueness Theorem) *Consider the initial value problem*

$$y'' + p(t)y' + q(t)y = g(t), \qquad y(t_0) = y_0, \quad y'(t_0) = \bar{y}_0.$$

If $p(t), q(t)$ and $g(t)$ are continuous and bounded on an open interval I containing t_0, then there exists exactly one solution $y(t)$ of this equation, valid on I.

Example 1. Given the equation

$$(t^2 - 3t)y'' + ty' - (t+3)y = e^t, \qquad y(1) = 2, \quad y'(1) = 1.$$

Find the largest interval where solution is valid.

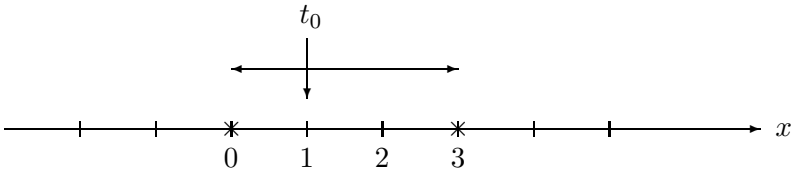
Answer. Rewrite the equation into the proper form:

$$y'' + \frac{t}{t(t-3)}y' - \frac{t+3}{t(t-3)}y = \frac{e^t}{t(t-3)},$$

so we have

$$p(t) = \frac{t}{t(t-3)}, \quad q(t) = -\frac{t+3}{t(t-3)}, \quad g(t) = \frac{e^t}{t(t-3)}.$$

We see that we must have $t \neq 0$ and $t \neq 3$. Since $t_0 = 1$, then the largest interval is $I = (0, 3)$, or $0 < t < 3$. See the figure below.



Definition. Given two functions $f(t), g(t)$, the **Wronskian** is defined as

$$W(f, g)(t) \doteq fg' - f'g.$$

Remark: One way to remember this definition could be using the determinant of a 2×2 matrix,

$$W(f, g)(t) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}.$$

Main property of the Wronskian:

- If $W(f,g) \equiv 0$, then f and g are *linearly dependent*.
- Otherwise, they are *linearly independent*.

If two functions are linearly dependent, they are essentially the same function. In fact, one function could be written as a scalar multiple of the other. To be precise, $W(f,g) \equiv 0$ (linearly dependent) if and only if $f(t) = k \cdot g(t)$ for some constant k. To see this, assume $W(f,g) \equiv 0$, i.e., $fg' - f'g \equiv 0$, then

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} = \frac{-W(f,g)}{g^2} = 0, \quad \to \quad \frac{f}{g} = k(\text{constant}).$$

On the other hand, if $f(t) = kg(t)$ for some constant k, then

$$W(f,g) = kg(t) \cdot g'(t) - kg'(t)g(t) \equiv 0.$$

Example 2. Compute the Wronskian of the following function pairs.
(a). $f = e^t$, $g = e^{-t}$.

Answer. We have

$$W(f,g) = e^t(-e^{-t}) - e^t e^{-t} = -2 \neq 0$$

so they are linearly independent.

(b). $f(t) = \sin t$, $g(t) = \cos t$.

Answer. We have

$$W(f,g) = \sin t(\sin t) - \cos t \cos t = -1 \neq 0$$

and they are linearly independent.

(c). $f(t) = t + 1$, $g(t) = 4t + 4$.

Answer. We have

$$W(f,g) = (t+1)4 - (4t+4) = 0$$

so they are linearly dependent. (In fact, we have $g(t) = 4 \cdot f(t)$.)
(d*). $f(t) = 2t$, $g(t) = |t|$.

Answer. Note that $g'(t) = \text{sign}(t)$ where sign is the sign function. So

$$W(f,g) = 2t \cdot \text{sign}(t) - 2|t| = 0$$

(we used $t \cdot \text{sign}(t) = |t|$). So they are linearly dependent for all $t \neq 0$.

Theorem . *Suppose $y_1(t), y_2(t)$ are two solutions of*

$$y'' + p(t)y' + q(t)y = 0.$$

Then

(I) We have either $W(y_1, y_2) \equiv 0$ or $W(y_1, y_2)$ never zero;

(II) If $W(y_1, y_2) \neq 0$, the $y = c_1 y_1 + c_2 y_2$ is the general solution. They are also called to form a fundamental set of solutions. As a consequence, for any ICs $y(t_0) = y_0, y'(t_0) = \bar{y}_0$, there is a unique set of (c_1, c_2) that gives a unique solution.

Proof of part (II): Plug in the initial conditions, we get

$$\begin{aligned} c_1 y_1(t_0) + c_2 y_2(t_0) &= y_0 \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) &= \bar{y}_0. \end{aligned}$$

Writing this into matrix-vector form:

$$\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ \bar{y}_0 \end{pmatrix}.$$

We see that there exist a uniqueness solution for c_1, c_2 if the determinant of the coefficient matrix is not zero. This determinant is precisely the Wronskian.

Example . Show that $y_1(t) = \sqrt{t}$ and $y_2(t) = 1/t$ form a fundamental set of solution for

$$2t^2 y'' + 3ty' - y = 0, \quad (t > 0).$$

Answer. One needs to check two things: (1) plug in y_1, y_2 to see that they are solutions. (skip details). (2) Compute the Wronskian $W(y_1, y_2)$ to see if it is not 0. In fact, one get $W(y_1, y_2) = -\frac{3}{2} t^{-3/2} < 0$ for $t > 0$.

The next Theorem is probably the most important one in this chapter.

Theorem (Abel's Theorem) *Let y_1, y_2 be two (linearly independent) solutions to*

$$y'' + p(t)y' + q(t)y = 0$$

on an open interval I. Then, the Wronskian $W(y_1, y_2)$ on I is given by

$$W(y_1, y_2)(t) = C \cdot \exp(\int -p(t)\, dt),$$

for some constant C depending on y_1, y_2, but independent on t or on I.

Proof. Since y_1, y_2 are solutions, we have

$$\begin{aligned} y_1'' + p(t)y_1' + q(t)y_1 &= 0 \\ y_2'' + p(t)y_2' + q(t)y_2 &= 0 \end{aligned}$$

Multiply the first equation by $-y_2$ and the second one by y_1, and add them up, we get

$$(y_1 y_2'' - y_1'' y_2) + p(t)(y_1 y_2' - y_1' y_2) = 0.$$

Note that $W(y_1, y_2) = y_1 y_2' - y_1' y_2$ and $W' = y_1 y_2'' - y_1'' y_2$, we get

$$W' + p(t)W = 0$$

which is a first order linear equation which we can solve. We get

$$W(y_1, y_2)(t) = C \exp\left[- \int p(t) dt \right]$$

Completing the proof. Note that this also proves part (I) of the previous Theorem. The Wronskian $W(y_1, y_2)$ is either identically 0 (when $C = 0$) or never 0 (when $C \neq 0$).

Example 3. Given

$$t^2 y'' - t(t+2)y' + (t+2)y = 0.$$

Find $W(y_1, y_2)$ without solving the equation.

Answer. We first find the $p(t)$

$$p(t) = -\frac{t+2}{t}$$

which is valid for $t \neq 0$. By Abel's Theorem, we have

$$W(y_1, y_2) = C \cdot \exp(\int -p(t)\, dt) = C \cdot \exp(\int \frac{t+2}{t}\, dt) = C e^{t + 2\ln|t|} = C t^2 e^t.$$

NB! The solutions are defined on either $(0, \infty)$ or $(-\infty, 0)$, depending on t_0.

From now on, when we say two solutions y_1, y_2 of the equation, we mean two linearly independent solutions that can form a fundamental set of solutions.

Example 4. If y_1, y_2 are two solutions of

$$ty'' + 2y' + te^t y = 0,$$

and $W(y_1, y_2)(1) = 2$, find $W(y_1, y_2)(5)$.

Answer. First we find that $p(t) = 2/t$. By Abel's Theorem we have

$$W(y_1, y_2)(t) = C \cdot \exp\left\{ - \int \frac{2}{t}\, dt \right\} = C \cdot e^{-\ln t} = C t^{-2}.$$

If $W(y_1, y_2)(1) = 2$, then $C = 2$. So we have

$$W(y_1, y_2)(5) = (2)5^{-2} = \frac{2}{25}.$$

Example 5. If $W(f, g) = 3e^{4t}$, and $f = e^{2t}$, find g.

Answer. By definition of the Wronskian, we have

$$W(f,g) = fg' - f'g = e^{2t}g' - 2e^{2t}g = 3e^{4t},$$

which gives a 1st order equation for g:

$$g' - 2g = 3e^{2t}.$$

Solve it for g, by method of integrating factor:

$$\mu(t) = e^{-2t}, \quad g(t) = e^{2t}\int e^{-2t}3e^{2t}\,dy = e^{2t}(3t+c).$$

We can choose $c = 0$, and get $g(t) = 3te^{2t}$.

Next example shows how Abel's Theorem can be used to solve 2nd order differential equations.

Example 6. Consider the equation $y'' + 2y' + y = 0$. Find the general solution.

Answer. The characteristic equation is $r^2 + 2r + 1 = 0$, which given double roots $r_1 = r_2 = -1$. So we know that $y_1 = e^{-t}$ is a solutions. How can we find another solution y_2 that's linearly independent?

By Abel's Theorem, we have

$$W(y_1, y_2) = C\exp\left\{\int -2\,dt\right\} = Ce^{-2t},$$

and we can choose $C = 1$ and get $W(y_1, y_2) = e^{-2t}$. By the definition of the Wronskian, we have

$$W(y_1, y_2) = y_1y_2' - y_1'y_2 = e^{-t}y_2' - (-e^{-t}y_2) = e^{-t}(y_2' + y_2).$$

These two computation must have the same answer, so

$$e^{-t}(y_2' + y_2) = e^{-2t}, \qquad y_2' + y_2 = e^{-t}.$$

This is a 1st order equation for y_2. Solve it:

$$\mu(t) = e^t, \qquad y_2(t) = e^{-t}\int e^te^{-t}\,dt = e^{-t}(t+c).$$

Choosing $c = 0$, we get $y_2 = te^t$. The general solution is

$$y(t) = c_1y_1 + c_2y_2 = c_1e^{-t} + c_2te^{-t}.$$

This is called *the method of reduction of order*. We will study it more later in chapter 3.4.

3.3 Complex Roots

We start with an example.

Example Consider the equation $y'' + y = 0$, find the general solution.

Answer. By inspection, we need to find a function such that $y'' = -y$. We see that $y_1 = \cos t$ and $y_2 = \sin t$ both work. By the Wronskian $W(y_1, y_2) = -2 \neq 0$, we see that these two solutions are linearly independent. Therefore, the general solution is

$$y(t) = c_1 y_1 + c_2 y_2 = c_1 \cos t + c_2 \sin t.$$

Let' try to connect this with the characteristics equation:

$$r^2 + 1 = 0, \quad r^2 = -1, \quad r_1 = i, \quad r_2 = -i.$$

The roots are complex. In fact, they are complex-conjugate pair. We see that the imaginary part seems to give sin and cos functions.

In general, the roots of the characteristic equation can be complex numbers. Consider the equation

$$ay'' + by' + cy = 0, \qquad \to \quad ar^2 + br + c = 0.$$

The two roots are

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If $b^2 - 4ac < 0$, the root are complex, i.e., a pair of complex conjugate numbers. We will write $r_{1,2} = \lambda \pm i\mu$. There are two solutions:

$$y_1 = e^{(\lambda + i\mu)t} = e^{\lambda t} e^{i\mu t}, \qquad y_2 = y_1 = e^{(\lambda - i\mu)t} = e^{\lambda t} e^{-i\mu t}.$$

To deal with exponential function with pure imaginary exponent, we need the Euler's Formula:

$$e^{i\beta} = \cos\beta + i \sin\beta, \qquad e^{-i\beta} = \cos\beta - i \sin\beta.$$

Then

$$y_1 = e^{\lambda t}(\cos \mu t + i \sin \mu t), \qquad y_2 = e^{\lambda t}(\cos \mu t - i \sin \mu t).$$

But these solutions are complex-valued. We want real-valued solutions! To achieve this, we use the Principle of Superposition. If y_1, y_2 are two solutions, then $c_1 y_1 + c_2 y_2$ is also a solution for any constants c_1, c_2. In particular, the functions $\frac{1}{2}(y_1 + y_2)$, $\frac{1}{2i}(y_1 - y_2)$ are also solutions. Write

$$\tilde{y}_1 \doteq \frac{1}{2}(y_1 + y_2) = e^{\lambda t} \cos \mu t, \qquad \tilde{y}_2 \doteq \frac{1}{2i}(y_1 - y_2) = e^{\lambda t} \sin \mu t.$$

We need to make sure that they are linearly independent. We can check the Wronskian,

$$W(\tilde{y}_1, \tilde{y}_2) = \mu e^{2\lambda t} \neq 0. \quad \text{(home work problem)}.$$

So y_1, y_2 are linearly independent, and we have the general solution

$$y(t) = c_1 e^{\lambda t} \cos \mu t + c_2 e^{\lambda t} \sin \mu t = e^{\lambda t}(c_1 \cos \mu t + c_2 \sin \mu t).$$

Remark: We note here that if a complex valued function is a solution, then the real part and the imaginary part are each a solution. This is a more general result. Now let $y(t) = u(t) + iv(t)$ be a solution of

$$y'' + p(t)y' + q(t)y = 0$$

Then, using $y' = u' + iv'$ and $y'' = u'' + iv''$, we get

$$(u'' + iv'') + p(t)(u' + iv') + q(t)(u + iv) = 0$$

Collect the real part and the imaginary part, we have

$$\Big(u'' + p(t)u' + q(t)u\Big) + i\Big(v'' + p(t)v' + q(t)\Big) = 0$$

which implies

$$u'' + p(t)u' + q(t)u = 0, \qquad v'' + p(t)v' + q(t) = 0,$$

proving that u and v are both solutions.

Example 1. (Perfect Oscillation: Simple harmonic motion.) Solve the initial value problem

$$y'' + 4y = 0, \qquad y(\frac{\pi}{6}) = 0, \quad y'(\frac{\pi}{6}) = 1.$$

Answer. The characteristic equation is

$$r^2 + 4 = 0, \quad \Rightarrow \quad r = \pm 2i, \quad \Rightarrow \quad \lambda = 0,\ \mu = 2.$$

The general solution is

$$y(t) = c_1 \cos 2t + c_2 \sin 2t.$$

Find c_1, c_2 by initial conditions: since $y' = -2c_1 \sin 2t + 2c_2 \cos 2t$, we have

$$y(\frac{\pi}{6}) = 0: \quad c_1 \cos\frac{\pi}{3} + c_2 \sin\frac{\pi}{3} = \frac{1}{2}c_1 + \frac{\sqrt{3}}{2}c_2 = 0,$$

$$y'(\frac{\pi}{6}) = 1: \quad -2c_1 \sin\frac{\pi}{3} + 2c_2 \cos\frac{\pi}{3} = -2c_1\frac{\sqrt{3}}{2} + 2c_2\frac{1}{2} = 1.$$

Solve these two equations, we get $c_1 = -\frac{\sqrt{3}}{4}$ and $c_2 = \frac{1}{4}$. So the solution is

$$y(t) = -\frac{\sqrt{3}}{4}\cos 2t + \frac{1}{4}\sin 2t,$$

which is a periodic oscillation. This is also called perfect oscillation or simple harmonic motion.

Example 2. (Decaying oscillation.) Find the solution to the IVP (Initial Value Problem)

$$y'' + 2y' + 101y = 0, \qquad y(0) = 1, \quad y'(0) = 0.$$

Answer. The characteristic equation is

$$r^2 + 2r + 101 = 0, \quad \Rightarrow \quad r_{1,2} = -1 \pm 10i, \quad \Rightarrow \quad \lambda = -1,\ \mu = 10.$$

So the general solution is

$$y(t) = e^{-t}(c_1 \cos 10t + c_2 \sin 10t),$$

so

$$y'(t) = -e^{-t}(c_1 \cos t + c_2 \sin t) + e^{-t}(-10c_1 \sin t + 10c_2 \cos t)$$

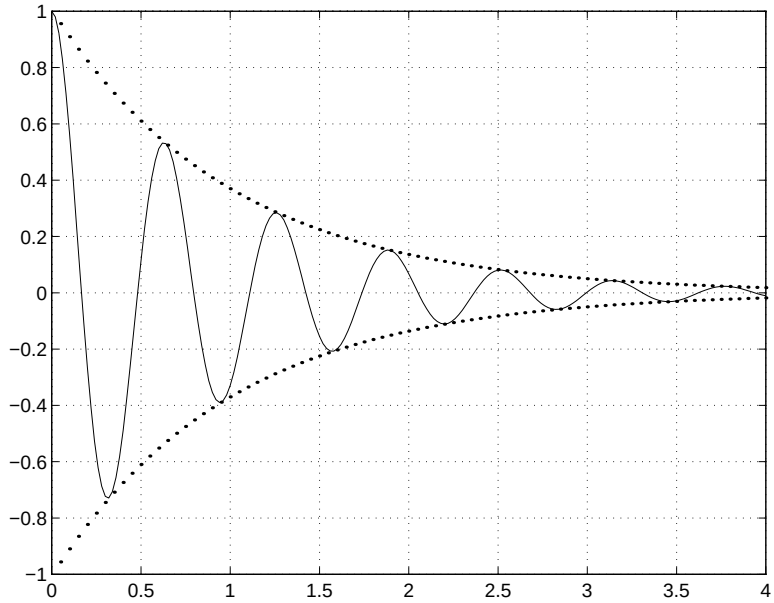
Fit in the ICs:

$$y(0) = 1: \qquad y(0) = e^0(c_1 + 0) = c_1 = 1,$$
$$y'(0) = 0: \qquad y'(0) = -1 + 10c_2 = 0, \quad c_2 = 0.1.$$

Solution is

$$y(t) = e^{-t}(\cos t + 0.1 \sin t).$$

The graph is given below:



We see it is a decaying oscillation. The sin and cos part gives the oscillation, and the e^{-t} part gives the decaying amplitude. As $t \to \infty$, we have $y \to 0$.

Example 3. (Growing oscillation) Find the general solution of $y'' - y' + 81.25y = 0$.

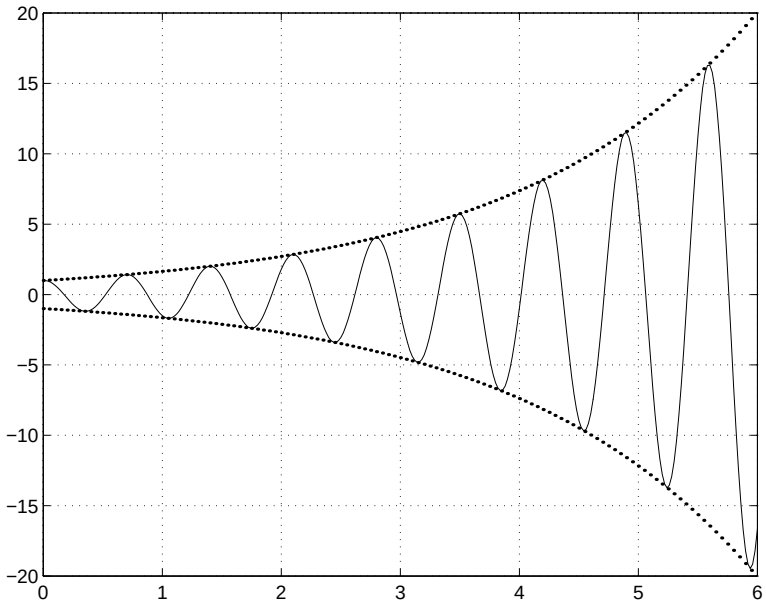
Answer.

$$r^2 - r + 81.25 = 0, \qquad \Rightarrow \quad r = 0.5 \pm 9i, \quad \Rightarrow \quad \lambda = 0.5, \quad \mu = 9.$$

The general solution is

$$y(t) = e^{0.5t}(c_1 \cos 9t + c_2 \sin 9t).$$

A typical graph of the solution looks like:



We see that y oscillate with growing amplitude as t grows. In the limit when $t \to \infty$, y oscillates between $-\infty$ and $+\infty$.

Conclusion: Sign of λ, the real part of the complex roots, decides the type of oscillation:

- $\lambda = 0$: perfect oscillation;
- $\lambda < 0$: decaying oscillation;
- $\lambda > 0$: growing oscillation.

We note that since $\lambda = \frac{-b}{2a}$, so the sign of λ follows the sign of $-b/a$.

3.4 Repeated roots; reduction of order

For the characteristic equation $ar^2 + br + c = 0$, if $b^2 = 4ac$, we will have two repeated roots

$$r_1 = r_2 = \bar{r} = -\frac{b}{2a}.$$

We have one solution $y_1 = e^{\bar{r}t}$. How can we find the second solution which is linearly independent of y_1?

From experience in an earlier example, we claim that $y_2 = te^{\bar{r}t}$ is a solution. To prove this claim, we plug it back into the equation. If $\bar{r}$ is the double root, then, the characteristic equation can be written

$$r^2 - 2\bar{r}r + \bar{r}^2 = 0$$

which gives the equation

$$y'' - 2\bar{r}y' + \bar{r}^2 y = 0.$$

We can check if y_2 satisfies this equation. We have

$$y' = e^{\bar{r}t} + \bar{r}te^{\bar{r}t}, \quad y'' = 2\bar{r}e^{\bar{r}t} + \bar{r}^2te^{\bar{r}t}.$$

Put into the equation, we get

$$2\bar{r}e^{\bar{r}t} + \bar{r}^2te^{\bar{r}t} - 2\bar{r}(e^{\bar{r}t} + \bar{r}te^{\bar{r}t}) + \bar{r}^2te^{\bar{r}t} = 0.$$

Finally, we must make sure that y_1, y_2 are linearly independent. We compute their Wronskian

$$W(y_1, y_2) = y_1y_2' - y_1'y_2 = e^{\bar{r}t}(e^{\bar{r}t} + \bar{r}te^{\bar{r}t}) - \bar{r}e^{\bar{r}t}te^{\bar{r}t} = e^{2\bar{r}t} \neq 0.$$

We conclude now, the general solution is

$$y(t) = c_1y_1 + c_2y_2 = c_1e^{\bar{r}t} + c_2te^{\bar{r}t} = e^{\bar{r}t}(c_1 + c_2t).$$

Example 1. (not covered in class) Consider the equation $y'' + 4y' + 4y = 0$. We have $r^2 + 4r + 4 = 0$, and $r_1 = r_2 = r = -2$. So one solution is $y_1 = e^{-2t}$. What is y_2?

Method 1. Use Wronskian and Abel's Theorem. By Abel's Theorem we have

$$W(y_1, y_2) = c\exp(-\int 4\,dt) = ce^{-4t} = e^{-4t}, \quad (\text{let } c = 1).$$

By the definition of Wronskian we have

$$W(y_1, y_2) = y_1y_2' - y_1'y_2 = e^{-2t}y_2' - (-2)e^{-2t}y_2 = e^{-2t}(y_2' + 2y_2).$$

They must equal to each other:

$$e^{-2t}(y_2' + 2y_2) = e^{-4t}, \quad y_2' + 2y_2 = e^{-2t}.$$

Solve this for y_2,

$$\mu = e^{2t}, \quad y_2 = e^{-2t}\int e^{2t}e^{-2t}\,dt = e^{-2t}(t + C)$$

Let $C = 0$, we get $y_2 = te^{-2t}$, and the general solution is

$$y(t) = c_1y_1 + c_2y_2 = c_1e^{-2t} + c_2te^{-2t}.$$

Method 2. This is the textbook's version. We guess a solution of the form $y_2 = v(t)y_1 = v(t)e^{-2t}$, and try to find the function $v(t)$. We have

$$y_2' = v'e^{-2t} + v(-2e^{-2t}) = e^{-2t}(v' - 2v), \qquad y_2'' = e^{-2t}(v'' - 4v' + 4v).$$

Put them in the equation

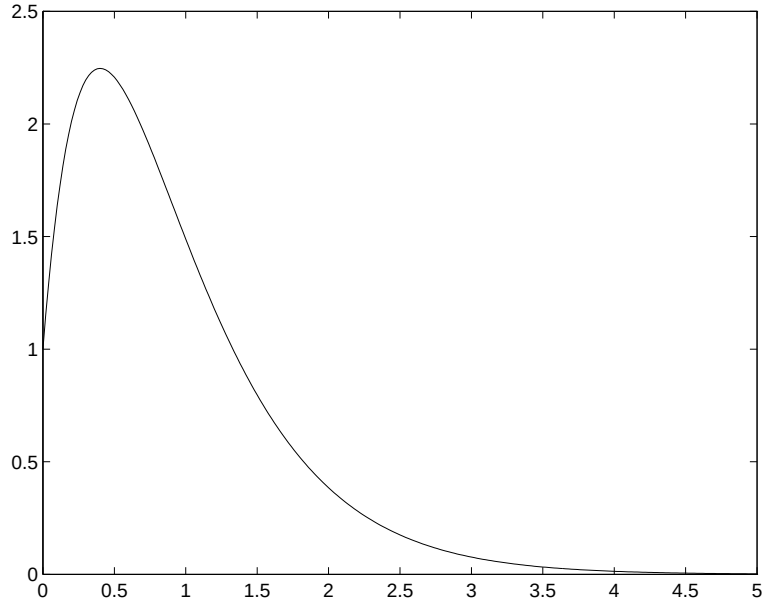
$$e^{-2t}(v'' - 4v' + 4v) + 4e^{-2t}(v' - 2v) + 4v(t)e^{-2t} = 0.$$

Cancel the term e^{-2t}, and we get $v'' = 0$, which gives $v(t) = c_1t + c_2$. So

$$y_2(t) = vy_1 = (c_1t + c_2)e^{-2t} = c_1te^{-2t} + c_2e^{-2t}.$$

Note that the term c_2e^{-2t} is already contained in cy_1. Therefore we can choose $c_1 = 1, c_2 = 0$, and get $y_2 = te^{-2t}$, which gives the same general solution as Method 1. We observe that this method involves more computation than Method 1.

A typical solution graph is included below:



We see if $c_2 > 0$, y increases for small t. But as t grows, the exponential (decay) function dominates, and solution will go to 0 as $t \to \infty$.

One can show that in general if one has repeated roots $r_1 = r_2 = r$, then $y_1 = e^{rt}$ and $y_2 = te^{rt}$, and the general solution is

$$y = c_1e^{rt} + c_2te^{rt} = e^{rt}(c_1 + c_2t).$$

Example 2. Solve the IVP

$$y'' + 2y' + y = 0, \qquad y(0) = 2, \quad y'(0) = 1.$$

Answer. This follows easily now

$$r^2 + 2r + 1 = 0, \quad \Rightarrow \quad r_1 = r_2 = -1, \quad \Rightarrow \quad y(t) = (c_1 + c_2t)e^{-t}.$$

The ICs give

$$y(0) = 2: \quad c_1 + 0 = 2, \quad \Rightarrow \quad c_1 = 2.$$

$$y'(t) = (-c_1 + c_2t)e^{-t} + c_2e^{-t}, \quad y'(0) = -c_1 + c_2 = 1, \quad \Rightarrow \quad c_2 = 1 + c_1 = 3.$$

So the solution is $y(t) = (2 + 3t)e^t$. How would the graph of $y(t)$ look like? We have $y \to 0$ as $t \to \infty$.

Example 2b. If the initial conditions in Example 2 are changed into

$$y(2) = 2, \quad y'(2) = 1,$$

it would be more convenient to write the general solution in the form

$$y(t) = c_1 e^{-(t-2)} + c_2(t-2)e^{-(t-2)}.$$

Note that we substitute t with $t-2$ in the general solution form in Example 2. Now we have

$$y'(t) = -c_1 e^{-(t-2)} + c_2 e^{-(t-2)} - c_2(t-2)e^{-(t-2)}.$$

The initial conditions now give:

$$y(2) = c_1 + 0 = 2, \qquad y'(2) = -c_1 + c_2 = 1, \qquad \to \quad c_1 = 2, \quad c_2 = 3.$$

Note that the computation remains the same! The solution is

$$y(t) = [2 + 3(t-2)]e^{-(t-2)}.$$

Of course you can write the same general solution as in Example 2 and work out the detail. Try that, and you will be convinced that it takes much more work!

Example 2c. Let's look at even another way of solving Example 2b. Note that the equation is autonomous, i.e., no t appears in the equation. Let the initial data be given at $t_0 = 2$. Introduce a new time variable $\tau = t - t_0 = t - 2$. Note this is just shift the time by 2 units into the future. We have $d\tau = dt$, and the IVP becomes

$$y''(\tau) + 2y'(\tau) + y(\tau) = 0, \qquad y(0) = 2, \quad y'(0) = 1.$$

which is same as Example 2. We can therefore write out the solution

$$y(\tau) = (2 + 3\tau)e^{-\tau}.$$

Plug back in $\tau = t - 2$, we have the solution

$$y(t) = [2 + 3(t-2)]e^{-(t-2)},$$

which is of course the same as in Example 2b.

Summary: For $ay'' + by' + cy = 0$, and $ar^2 + br + c = 0$ has two roots r_1, r_2, we have

- If $r_1 \neq r_2$ (real): $\quad y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$;
- If $r_1 = r_2 = \bar{r}$ (real): $\quad y(t) = (c_1 + c_2 t)e^{\bar{r}t}$;
- If $r_{1,2} = \lambda \pm i\mu$ complex: $\quad y(t) = e^{\lambda t}(c_1 \cos \mu t + c_2 \sin \mu t)$.

On reduction of order: This method can be used to find a second solution y_2 if the first solution y_1 is given for a second order linear equation.

Example 3. For the equation

$$2t^2y'' + 3ty' - y = 0, \quad t > 0,$$

given one solution $y_1 = \frac{1}{t}$, find a second linearly independent solution.

Answer. Method 1: Use Abel's Theorem and Wronskian. By Abel's Theorem, and choose $C = 1$, we have

$$W(y_1, y_2) = \exp\left\{-\int p(t)\,dt\right\} = \exp\left\{-\int \frac{3t}{2t^2}dt\right\} = \exp\left\{-\frac{3}{2}\ln t\right\} = t^{-3/2}.$$

By definition of the Wronskian,

$$W(y_1, y_2) = y_1y_2' - y_1'y_2 = \frac{1}{t}y_2' - (-\frac{1}{t^2})y_2 = t^{-3/2}.$$

Solve this for y_2:

$$\mu = \exp\left(\int \frac{1}{t}\,dt\right) = \exp(\ln t) = t, \quad \Rightarrow \quad y_2 = \frac{1}{t}\int t \cdot t^{-\frac{3}{2}}dt = \frac{1}{t}\frac{2}{3}t^{\frac{3}{2}} = \frac{2}{3}\sqrt{t}.$$

Drop the constant $\frac{2}{3}$, we get $y_2 = \sqrt{t}$.

Method 2: This is the textbook's version. Guess solution to be $y_2(t) = y_1(t)v(t)$, and find $v(t)$ by solving a first order equation!

Comment 1: This method is more complicated in computation than the 1st method. However, it is a more general method. In the case one does not have something like the Abel's Theorem, such a method will always work, and will always reduce the order by 1.

Comment 2: A similar situation occurs in finding roots (or factorizing) a polynomial. Let $P_n(t)$ be a polynomial of degree n. If we found that r_1 is a root, then it has a factor $t - r_1$, so $P_n(t) = (t - r_1)Q(t)$, where $Q(t)$ is a polynomial of degree $n - 1$.

Let's introduce another method that combines the ideas from Method 1 and Method 2.

Method 3. We will use Abel's Theorem, and at the same time we will seek a solution of the form $y_1 = vy_1$.

By Abel's Theorem, we have (worked out in M1) $W(y_1, y_2) = t^{-\frac{3}{2}}$. Now, seek $y_2 = vy_1$. By the definition of the Wronskian, we have

$$W(y_1, y_2) = y_1y_2' - y_1'y_2 = y_1(vy_1)' - y_1'(vy_1) = y_1(v'y_1 + vy_1'') - vy_1y_1' = v'y_1^2.$$

Note that this is a general formula:

$$W(y_1, y_2) = v'y_1^2, \qquad \text{if} \quad y_2 = vy_1.$$

Now putting $y_1 = 1/t$, we get

$$v'\frac{1}{t^2} = t^{-\frac{3}{2}}, \qquad v' = t^{\frac{1}{2}}, \qquad v = \int t^{\frac{1}{2}}dt = \frac{2}{3}t^{\frac{3}{2}}.$$

Drop the constant $\frac{2}{3}$, we get

$$y_2 = vy_1 = t^{\frac{3}{2}}\frac{1}{t} = t^{\frac{1}{2}}.$$

We see that Method 3 is the most efficient one among all three methods. We will focus on this method from now on.

Example 4. Consider the equation

$$t^2y'' - t(t+2)y' + (t+2)y = 0, \quad (t > 0).$$

Given $y_1 = t$, find the general solution.

Answer. We have

$$p(t) = -\frac{t(t+2)}{t^2} = -\frac{t+2}{t} = -1 - \frac{2}{t}.$$

Let y_2 be the second solution. By Abel's Theorem, choosing $c = 1$, we have

$$W(y_1, y_2) = \exp\left\{\int \left(1 + \frac{2}{t}\right) dt\right\} = \exp\{t + 2\ln t\} = t^2e^t.$$

Let $y_2 = vy_1$, the $W(y_1, y_2) = v'y_1^2 = t^2v'$. Then we must have

$$t^2v' = t^2e^t, \qquad v' = e^t, \qquad v = e^t, \qquad y_2 = te^t.$$

(A cheap trick to double check your solution y_2 would be: plug it back into the equation and see if it satisfies it.) The general solution is

$$y(t) = c_1y_2 + c_2y_2 = c_1t + c_2te^t.$$

We observe here that Method 3 is very efficient.

Example 5. Given the equation $t^2y'' - (t - \frac{3}{16})y = 0, \quad (t > 0)$, and $y_1 = t^{(1/4)}e^{2\sqrt{t}}$, find y_2.

Answer. We will always use method 3. We see that $p = 0$. By Abel's Theorem, setting $c = 1$, we have

$$W(y_1, y_2) = \exp\left(\int 0dt\right) = 1.$$

Seek $y_2 = vy_1$. Then, $W(y_1, y_2) = y_1^2v' = t^{\frac{1}{2}}e^{4\sqrt{t}}v'$. So we must have

$$t^{\frac{1}{2}}e^{4\sqrt{t}}v' = 1, \quad \Rightarrow \quad v' = t^{-\frac{1}{2}}e^{-4\sqrt{t}}, \quad \Rightarrow \quad v = \int t^{-\frac{1}{2}}e^{-4\sqrt{t}}dt.$$

Let $u = -4\sqrt{t}$, so $du = -2t^{-\frac{1}{2}}dt$, we have

$$v = \int -\frac{1}{2}e^u\, du = -\frac{1}{2}e^u = -\frac{1}{2}e^{-4\sqrt{t}}.$$

So drop the constant $-\frac{1}{2}$, we get

$$y_2 = vy_1 = e^{-4\sqrt{t}}t^{\frac{1}{4}}e^{2\sqrt{t}} = t^{\frac{1}{4}}e^{-2\sqrt{t}}.$$

The general solution is

$$y(t) = c_1y_1 + c_2y_2 = t^{\frac{1}{4}}\left(c_1e^{2\sqrt{t}} + c_2e^{-2\sqrt{t}}\right).$$

3.5 Non-homogeneous equations; method of undetermined coefficients

Want to solve the non-homogeneous equation

$$y'' + p(t)y' + q(t)y = g(t), \tag{N}$$

Steps:

1. First solve the homogeneous equation

$$y'' + p(t)y' + q(t)y = 0, \tag{H}$$

   i.e., find y_1, y_2, linearly independent of each other, and form the general solution

$$y_H = c_1 y_1 + c_2 y_2.$$

2. Find a particular/specific solution Y for (N), by MUC (method of undetermined coefficients);

3. The general solution for (N) is then

$$y = y_H + Y = c_1 y_1 + c_2 y_2 + Y.$$

   Find c_1, c_2 by initial conditions, if given.

Key step: step 2.

Why $y = y_H + Y$?

A quick proof: If y_H solves (H), then

$$y_H'' + p(t)y_H' + q(t)y_H = 0, \tag{A}$$

and since Y solves (N), we have

$$Y'' + p(t)Y' + q(t)Y = g(t), \tag{B}$$

Adding up (A) and (B), and write $y = y_H + Y$, we get $y'' + p(t)y' + q(t)y = g(t)$.

Main focus: constant coefficient case, i.e.,

$$ay'' + by' + cy = g(t).$$

Example 1. Find the general solution for $y'' - 3y' - 4y = 3e^{2t}$.

Answer. Step 1: Find y_H.

$$r^2 - 3r - 4 = (r+1)(r-4) = 0, \quad \Rightarrow \quad r_1 = -1,\ r_2 = 4,$$

so

$$y_H = c_1 e^{-t} + c_2 e^{4t}.$$

Step 2: Find Y. We guess/seek solution of the same form as the source term $Y = Ae^{2t}$, and will determine the coefficient A.

$$Y' = 2Ae^{2t}, \quad Y'' = 4Ae^{2t}.$$

Plug these into the equation:

$$4Ae^{2t} - 3 \cdot 2Ae^{2t} - 4Ae^{2t} = 3e^{2t}, \quad \Rightarrow \quad -6A = 3, \quad \Rightarrow \quad A = -\frac{1}{2}.$$

So $Y = -\frac{1}{2}e^{2t}$.

Step 3. The general solution to the non-homogeneous solution is

$$y(t) = y_H + Y = c_1e^{-t} + c_2e^{4t} - \frac{1}{2}e^{2t}.$$

Observation: The particular solution Y take the same *form* as the source term $g(t)$.

But this is not always true.

Example 2. Find general solution for $y'' - 3y' - 4y = 2e^{-t}$.

Answer. The homogeneous solution is the same as Example 1: $y_H = c_1e^{-t} + c_2e^{4t}$. For the particular solution Y, let's first try the same form as g, i.e., $Y = Ae^{-t}$. So $Y' = -Ae^{-t}, Y'' = Ae^{-t}$. Plug them back in to the equation, we get

$$\text{LHS} = Ae^{-t} - 3(-Ae^{-t}) - 4Ae^{-t} = 0 \neq 2e^{-et} = \text{RHS}.$$

So it doesn't work. Why?

We see $r_1 = -1$ and $y_1 = e^{-t}$, which means our guess $Y = Ae^{-t}$ is a solution to the homogeneous equation. It will never work.

Second try: $Y = Ate^{-t}$. So

$$Y' = Ae^{-t} - Ate^{-t}, \quad Y'' = -Ae^{-t} - Ae^{-t} + Ate^{-t} = -2Ae^{-t} + Ate^{-t}.$$

Plug them in the equation

$$(-2Ae^{-t} + Ate^{-t}) - 3(Ae^{-t} - Ate^{-t}) - 4Ate^{-t} = -5Ae^{-t} = 2e^{-t},$$

we get

$$-5A = 2, \quad \Rightarrow \quad A = -\frac{2}{5},$$

so we have $Y = -\frac{2}{5}te^{-t}$.

Summary 1. If $g(t) = ae^{\alpha t}$, then the form of the particular solution Y depends on r_1, r_2 (the roots of the characteristic equation).

case	form of the particular solution Y
$r_1 \neq \alpha$ and $r_2 \neq \alpha$	$Y = Ae^{\alpha t}$
$r_1 = \alpha$ or $r_2 = \alpha$, but $r_1 \neq r_2$	$Y = Ate^{\alpha t}$
$r_1 = r_2 = \alpha$	$Y = At^2e^{\alpha t}$

Example 3. Find the general solution for

$$y'' - 3y' - 4y = 3t^2 + 2.$$

Answer. The y_H is the same $y_H = c_1 e^{-t} + c_2 e^{4t}$.

Note that $g(t)$ is a polynomial of degree 2. We will try to guess/seek a particular solution of the same form:

$$Y = At^2 + Bt + C, \qquad Y' = 2At + B, \qquad Y'' = 2A$$

Plug back into the equation

$$2A - 3(2At + b) - 4(At^2 + Bt + C) = -4At^2 - (6A + 4B)t + (2A - 3B - 4C) = 3t^2 + 2.$$

Compare the coefficient, we get three equations for the three coefficients A, B, C:

$$\begin{aligned} -4A = 3 \qquad &\to \quad A = -\frac{3}{4} \\ -(6A + 4B) = 0, \quad &\to \quad B = \frac{9}{8} \\ 2A - 3B - 4C = 2, \quad &\to \quad C = \frac{1}{4}(2A - 3B - 2) = -\frac{55}{32} \end{aligned}$$

So we get

$$Y(t) = -\frac{3}{4}t^2 + \frac{9}{8}t - \frac{55}{32}.$$

But sometimes this guess won't work.

Example 4. Find the particular solution for $y'' - 3y' = 3t^2 + 2$.

Answer. We see that the form we used in the previous example $Y = At^2 + Bt + C$ won't work because $Y'' - 3Y'$ will not have the term t^2.

New try: multiply by a t. So we guess $Y = t(At^2 + Bt + C) = At^3 + Bt^2 + Ct$. Then

$$Y' = 3At^2 + 2Bt + C, \quad Y'' = 6At + 2B.$$

Plug them into the equation

$$(6At + 2B) - 3(3At^2 + 2Bt + C) = -9At^2 + (6A - 6B)t + (2B - 3C) = 3t^2 + 2.$$

Compare the coefficient, we get three equations for the three coefficients A, B, C:

$$\begin{aligned} -9A = 3 \qquad &\to \quad A = -\frac{1}{3} \\ (6A - 6B) = 0, \quad &\to \quad B = A = -\frac{1}{3} \\ 2B - 3C = 2, \quad &\to \quad C = \frac{1}{3}(2B - 2) = -\frac{8}{9} \end{aligned}$$

So $Y = t(-\frac{1}{3}t^2 - \frac{1}{3}t - \frac{8}{9})$.

Summary 2. If $g(t)$ is a polynomial of degree n, i.e.,

$$g(t) = \alpha_n t^n + \cdots + \alpha_1 t + \alpha_0$$

the particular solution for

$$ay'' + by' + cy = g(t)$$

(where $a \neq 0$) depends on b, c:

case	form of the particular solution Y
$c \neq 0$	$Y = P_n(t) = A_n t^n + \cdots + A_1 t + A_0$
$c = 0$ but $b \neq 0$	$Y = tP_n(t) = t(A_n t^n + \cdots + A_1 t + A_0)$
$c = 0$ and $b = 0$	$Y = t^2 P_n(t) = t^2(A_n t^n + \cdots + A_1 t + A_0)$

Example 5. Find a particular solution for

$$y'' - 3y' - 4y = \sin t.$$

Answer. Since $g(t) = \sin t$, we will try the same form. Note that $(\sin t)' = \cos t$, so we must have the $\cos t$ term as well. So the form of the particular solution is

$$Y = A\sin t + B\cos t.$$

Then

$$Y' = A\cos t - B\sin t, \qquad Y'' = -A\sin t - B\cos t.$$

Plug back into the equation, we get

$$\begin{aligned} &(-A\sin t - B\cos t) - 3(A\cos t - B\sin t) - 4(A\sin t + b\cos t) \\ &= \quad (-5A + 3B)\sin t + (-3A - 5B)\cos t = \sin t. \end{aligned}$$

So we must have

$$-5A + 3B = 1, \quad -3A - 5B = 0, \qquad \to \quad A = \frac{5}{34}, \quad B = \frac{3}{34}.$$

So we get

$$Y(t) = -\frac{5}{34}\sin t + \frac{3}{34}\cos t.$$

We observe that: (1). If the right-hand side is $g(t) = a\cos t$, then the same form would work; (2). More generally, if $g(t) = a\sin t + b\sin t$ for some a, b, then the same form still work.

However, this form won't work if it is a solution to the homogeneous equation.

Example 6. Find a general solution for $y'' + y = \sin t$.

Answer. Let's first find y_H. We have $r^2+1=0$, so $r_{1,2}=\pm i$, and $y_H = c_1\cos t + c_2 \sin t$.

For the particular solution Y: We see that the form $Y = A\sin t + B\cos t$ won't work because it solves the homogeneous equation.

Our new guess: multiply it by t, so

$$Y(t) = t(A\sin t + B\cos t).$$

Then

$$Y' = (A\sin t + B\cos t) + t(A\cos t + B\sin t),$$
$$Y'' = (-2B - At)\sin t + (2A - Bt)\cos t.$$

Plug into the equation

$$Y'' + Y = -2B\sin t + 2A\cos t = \sin t, \quad \Rightarrow \quad A = 0, \quad B = -\frac{1}{2}$$

So

$$Y(y) = -\frac{1}{2}t\cos t.$$

The general solution is

$$y(t) = y_H + Y = c_1\cos t + c_2\sin t - \frac{1}{2}t\cos t.$$

Summary 3. If $g(t) = a\sin\alpha t + b\cos\alpha t$, the form of the particular solution depends on the roots r_1, r_2.

case	form of the particular solution Y
(1). $r_{1,2} \neq \pm\alpha i$	$Y = A\sin\alpha t + B\cos\alpha t$
(2). $r_{1,2} = \pm\alpha i$	$Y = t(A\sin\alpha t + B\cos\alpha t)$

Note that case (2) occurs when the equation is $y'' + \alpha^2 y = a\sin\alpha t + b\cos\alpha t$.

We now have discovered some general rules to obtain the form of the particular solution for the non-homogeneous equation $ay'' + by' + cy = g(t)$.

- Rule (1). Usually, Y take the same form as $g(t)$;
- Rule (2). Except, if the form of $g(t)$ provides a solution to the homogeneous equation. Then, one can multiply it by t.
- Rule (3). If the resulting form in Rule (2) is still a solution to the homogeneous equation, then, multiply it by another t.

Next we study a couple of more complicated forms of g.

Example 7. Find a particular solution for

$$y'' - 3y' - 4y = te^t.$$

Answer. We see that $g = P_1(t)e^{at}$, where P_1 is a polynomial of degree 1. Also we see $r_1 = -1, r_2 = 4$, so $r_1 \neq a$ and $r_2 \neq a$. For a particular solution we will try the same form as g, i.e., $Y = (At + B)e^t$. So

$$Y' = Ae^t + (At + b)e^t = (A + b)e^t + Ate^t,$$

$$Y'' = \cdots = (2A + B)e^t + Ate^t.$$

Plug them into the equation,

$$[(2A + B)e^t + Ate^t] - 3[(A + b)e^t + Ate^t] - 4(At + B)e^t = (-6At - A - 6B)e^t = te^t.$$

We must have $-6At - A - 6B = t$, i.e.,

$$-6A = 1, \quad -A - 6B = 0, \quad \Rightarrow \quad A = -\frac{1}{6}, B = \frac{1}{36}, \quad \Rightarrow \quad Y = (-\frac{1}{6}t + \frac{1}{36})e^t.$$

However, if the form of g is a solution to the homogeneous equation, it won't work for a particular solution. We must multiply it by t in that case.

Example 8. Find a particular solution of

$$y'' - 3y' - 4y = te^{-t}.$$

Answer. Since $a = -1 = r_1$, so the form we used in Example 7 won't work here. (Can you intuitively explain why?) Try a new form now

$$Y = t(At + B)e^{-t} = (At^2 + Bt)e^{-t}.$$

Then

$$Y' = \cdots = [-At^2 + (2A - B)t + B]e^{-t},$$

$$Y'' = \cdots = [At^2 + (B - 4A)t + 2A - 2B]e^{-t}.$$

Plug into the equation

$$\begin{aligned}&[At^2 + (B - 4A)t + 2A - 2B]e^{-t} - 3[-At^2 + (2A - B)t + B]e^{-t} - 4(At^2 + Bt)e^{-t}\\ &= \quad [-10At + 2A - 5B]e^{-t} \;=\; te^t.\end{aligned}$$

So we must have $-10At + 2A - 5B = t$, which means

$$-10A = 1, \quad 2A - 5B = 0, \qquad \Rightarrow \qquad A = -\frac{1}{10}, \quad B = -\frac{1}{25}.$$

Then

$$Y = \left(-\frac{1}{10}t^2 - \frac{1}{25}t\right)e^{-t}.$$

Summary 4. If $g(t) = P_n(t)e^{at}$ where $P_n(t) = \alpha_n t^n + \cdots + \alpha_1 t + \alpha_0$ is a polynomial of degree n, then the form of a particular solution depends on the roots r_1, r_2.

case	form of the particular solution Y
$r_1 \neq a$ and $r_2 \neq a$	$Y = \tilde{P}_n(t)e^{at} = (A_n t^n + \cdots + A_1 t + A_0)e^{at}$
$r_1 = a$ or $r_2 = a$ but $r_1 \neq r_2$	$Y = t\tilde{P}_n(t)e^{at} = t(A_n t^n + \cdots + A_1 t + A_0)e^{at}$
$r_1 = r_2 = a$	$Y = t^2\tilde{P}_n(t)e^{at} = t^2(A_n t^n + \cdots + A_1 t + A_0)e^{at}$

Other cases of g are treated in a similar way: Check if the form of g is a solution to the homogeneous equation. If not, then use it as the form of a particular solution. If yes, then multiply it by t or t^2.

We summarize a few cases below.

Summary 5. If $g(t) = e^{\alpha t}(a\cos\beta t + b\sin\beta t)$, and r_1, r_2 are the roots of the characteristic equation. Then

case	form of the particular solution Y
$r_{1,2} \neq \alpha \pm i\beta$	$Y = e^{\alpha t}(A\cos\beta t + B\sin\beta t)$
$r_{1,2} = \alpha \pm i\beta$	$Y = t \cdot e^{\alpha t}(A\cos\beta t + B\sin\beta t)$

Summary 6. If $g(t) = P_n(t)\cos\beta t + \tilde{P}_n(t)\sin\beta t)$ where $P_n(t)$ and $\tilde{P}_n(t)$ are polynomials of degree n, and r_1, r_2 are the roots of the characteristic equation. Then

case	form of the particular solution Y
$r_{1,2} \neq \pm i\beta$	$Y = (A_n t^n + \cdots + A_0)\cos\beta t + (B_n t^n + \cdots + B_0)\sin\beta t$
$r_{1,2} = \alpha \pm i\beta$	$Y = t \cdot [(A_n t^n + \cdots + A_0)\cos\beta t + (B_n t^n + \cdots + B_0)\sin\beta t]$

Summary 7. If $g(t) = P_n(t)e^{\alpha t}(a\cos\beta t + b\sin\beta t)$ where $P_n(t)$ is a polynomial of degree n, and r_1, r_2 are the roots of the characteristic equation. Then

case	form of the particular solution Y
$r_{1,2} \neq \alpha \pm i\beta$	$Y = e^{\alpha t}[(A_n t^n + \cdots + A_0)\cos\beta t + (B_n t^n + \cdots + B_0)\sin\beta t]$
$r_{1,2} = \alpha \pm i\beta$	$Y = t \cdot e^{\alpha t}[(A_n t^n + \cdots + A_0)\cos\beta t + (B_n t^n + \cdots + B_0)\sin\beta t]$

More terms in the source. If the source $g(t)$ has several terms, we treat each separately and add up later. Let $g(t) = g_1(t) + g_2(t) + \cdots g_n(t)$, then, find a particular solution Y_i for each $g_i(t)$ term as if it were the only term in g, then $Y = Y_1 + Y_2 + \cdots Y_n$. This claim follows from the principle of superposition. (Can you provide a brief proof?)

In the examples below, we want to write the form of a particular solution.

Example 9. $y'' - 3y' - 4y = \sin 4t + 2e^{4t} + e^{5t} - t$.

Answer. Since $r_1 = -1, r_2 = 2$, we treat each term in g separately and the add up:

$$Y(t) = A\sin 4t + B\cos 4t + Cte^{4t} + De^{5t} + (Et + F).$$

Example 10. $y'' + 16y = \sin 4t + \cos t - 4\cos 4t + 4.$

Answer. The char equation is $r^2 + 16 = 0$, with roots $r_{1,2} = \pm 4i$, and

$$y_H = c_1 \sin 4t + c_2 \cos 4t.$$

We also note that the terms $\sin 4t$ and $-4\cos 4t$ are of the same type, and we must multiply it by t. So

$$Y = t(A\sin 4t + B\cos 4t) + (C\cos t + D\sin t) + E.$$

Example 11. $y'' - 2y' + 2y = e^t \cos t + 8e^t \sin 2t + te^{-t} + 4e^{-t} + t^2 - 3.$

Answer. The char equation is $r^2 - 2r + 2 = 0$ with roots $r_{1,2} = 1 \pm i$. Then, for the term $e^t \cos t$ we must multiply by t.

$$Y = te^t(A_1\cos t + A_2 \sin t) + e^t(B_1 \cos 2t + B_2 \sin 2t) + (C_1 t + C_0)e^{-t} + De^{-t} + (F_2 t^2 + F_1 t + F_0).$$

3.6 Variation of Parameters*

This is a general method to find a particular solution for the non-homogeneous equation, 2nd order and linear. Consider

$$y'' + p(t)y' + q(t) = 0, \tag{H}$$

and

$$y'' + p(t)y' + q(t) = g(t), \tag{N}$$

Assume that we have found the general solution for (H), given by

$$y_H(t) = c_1 y_1(t) + c_2 y_2(t),$$

where c_1, c_2 are arbitrary constants.

We now guess that a particular solution for (N) as

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t).$$

Our goal is to find some functions u_1, u_2 that will make y_p a particular solution.

Note that we have a lot of freedom here. If we plug in y_p in (N), we will get one constraint. But we have 2 functions. This means there is one constraint that is free-choice for us! We should use it wisely, to get 1st order equations for u_1, u_2.

We compute the derivative of y_p

$$y_p' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2'.$$

If we differentiate one more time, the expression gets large. The term with u_1' and u_2' would give u_1'', u_2'', which we should avoid. We see it is a good place to use our free choice and require

$$y_1 u_1' + y_2 u_2' = 0. \tag{A}$$

Then, we have

$$y_p' = y_1' u_1 + y_2' + u_2, \quad y_p'' = y_1'' u_1 + y_2'' u_2 + y_1' u_1' + y_2' u_2'.$$

Plug into (N), we get

$$\Big[y_1''u_1 + y_2''u_2 + y_1'u_1' + y_2'u_2'\Big] + p(t)\Big[y_1'u_1 + y_2' + u_2\Big] + q(t)\Big[y_1u_1 + y_2u_2\Big] = g(t)$$

Cleaning up, we get

$$\Big[y_1'' + p(t)y_1' + q(t)y_1\Big]u_1 + \Big[y_2'' + p(t)y_2' + q(t)y_2\Big]u_2 + [y_1'u_1' + y_2'u_2'] = g(t)$$

Since y_1, y_2 are solutions of (H), the two brackets are 0, and we get

$$y_1'u_1' + y_2'u_2' = g(t). \qquad (B)$$

Note now (A) and (B) are two linear equations for the unknowns u_1', u_2'. Write it into matrix-vector form:

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g(t) \end{pmatrix}$$

Note the determinant of the coefficient matrix is $W(y_1, y_2) \not\equiv 0$, which implies unique solution for u_1', u_2':

$$\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \frac{1}{W}\begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix}\begin{pmatrix} 0 \\ g(t) \end{pmatrix}$$

which gives

$$u_1' = -\frac{1}{W}y_2(t)g(t), \qquad u_2' = \frac{1}{W}y_1(t)g(t). \qquad (C)$$

We recover u_1, u_2 by integrating

$$u_1(t) = -\int \frac{1}{W(t)}y_2(t)g(t)\,dt, \qquad u_2(t) = \int \frac{1}{W(t)}y_1(t)g(t)dt.$$

The particular solution is

$$y_p(t) = u_1y_1 + u_2y_2 = -y_1(t)\int \frac{1}{W(t)}y_2(t)g(t)\,dt + y_2(t)\int \frac{1}{W(t)}y_1(t)g(t)dt.$$

Remark 1: In general, finding the general solution for (H) is not trivial. We don't have a general algorithm yet.

Remark 2: However, if one can find one solution of (H), call it y_1, by reduction of order we can find y_2. Then, by variation of parameter, we can find y_p, and therefore the general solution of (N).

Example Find the general solution for

$$y'' - 2y' + y = e^t \ln t, \quad t > 0.$$

Answer. We first find the two linearly independent solutions y_1, y_2 for the homogeneous equation. Note that it has constant coefficients. The char eqn $r^2 - 2r + 1 = 0$ gives two repeated roots $r_1 = r_2 = 1$, so

$$y_1 = e^t, \quad y_2 = te^t.$$

Then

$$y_1' = e^t, \quad y_2' = (1+t)e^t, \qquad W(y_1, y_2) = (1+t)e^{2t} - te^{2t} = e^{2t}.$$

Write the particular solution as $y_p = u_1 y_1 + u_2 y_2$, by (C) we have

$$u_1' = -\frac{te^t e^t \ln t}{e^{2t}} = -t \ln t, \qquad u_2' = \ln t$$

which gives

$$u_1(t) = -\int t \ln t dt = -\frac{t^2}{2} \ln t + \frac{t^2}{4}, \qquad u_2(t) = \int \ln t dt = t \ln t - t.$$

(by integration-by-parts). We get

$$y_p = \frac{1}{2} t^2 e^t \Big[\ln t - \frac{3}{2} \Big]$$

The general solution is

$$y(t) = c_1 y_1 + c_2 y_2 + y_p(t) = c_1 e^t + c_2 t e^t + \frac{1}{2} t^2 e^t \Big[\ln t - \frac{3}{2} \Big].$$

Example Given that $y_1(t) = t$ is a solution of the homogeneous equation

$$t^2 y'' - t y' + y = 0, \quad t > 0$$

find the solution for the IVP

$$t^2 y'' - t y' + y = t, \quad y(1) = 1, \quad y'(1) = 4.$$

Answer. . Sketch: by reduction of order, we find $y_2 = t\ lnt$ (see previous example).
By variation of parameter, we get $y_p = \frac{1}{2} t (\ln t)^2$ (details...)
General solution

$$y(t) = c_1 t + c_2 t \ln t + \frac{1}{2} t (\ln t)^2.$$

Find the constants c_1, c_2 by initial conditions (details...)

$$c_1 = 1, \quad c_2 = 3,$$

so the solution to the IVP is

$$y(t) = t + 3t \ln t + \frac{1}{2} t (\ln t)^2.$$

3.7 Mechanical Vibrations

In this chapter we study some applications of the IVP

$$ay'' + by' + cy = g(t), \qquad y(0) = y_0, \quad y'(0) = \bar{y}_0.$$

The spring-mass system: See figure below.

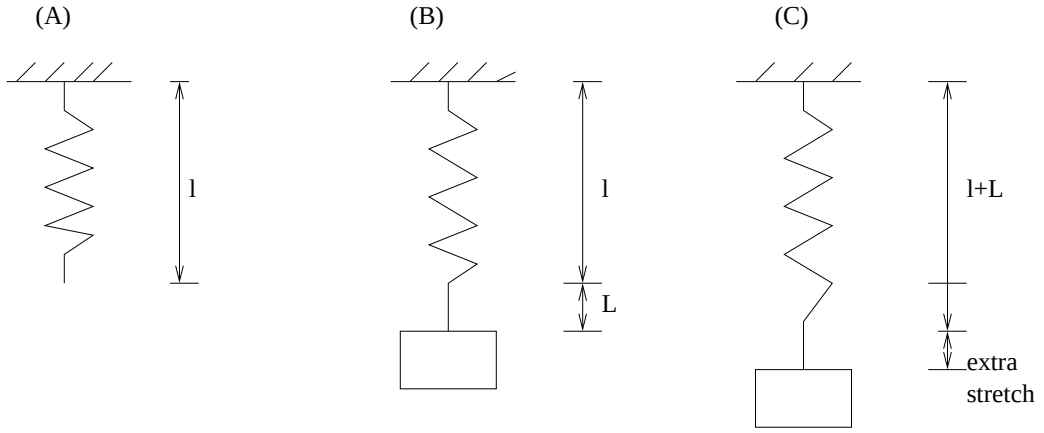
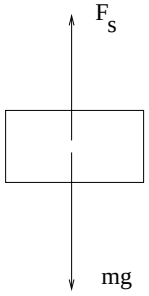


Figure (A): a spring in rest, with length l.

Figure (B): we put a mass m on the spring, and the spring is stretched. We call length L the elongation.

Figure (C): The spring-mass system is set in motion by stretch/squeeze it extra, with initial velocity, or with external force.

Force diagram at equilibrium position: $mg = F_s$.



Hooke's law: Spring force $F_s = -kL$, where L =elongation and k =spring constant.

So: we have $mg = kL$ which give

$$k = \frac{mg}{L}$$

which gives a way to obtain k by experiment: hang a mass m and measure the elongation L.

Model the motion: Let $u(t)$ be the displacement/position of the mass at time t, assuming the origin $u = 0$ is at the equilibrium position, and downward is the positive direction.

Total elongation: $L + u$

Total spring force: $F_s = -k(L + u)$

Other forces:

* damping/resistent force: $F_d(t) = -\gamma v = -\gamma u'(t)$, where γ is the damping constant, and v

is the velocity

* External force applied on the mass: $F(t)$, given function of t

Total force on the mass: $\sum f = mg + F_s + F_d + F$.

Newton's law of motion $ma = \sum f$ gives

$$ma = mu'' = \sum f = mg + F_s + F_d + F, \qquad mu'' = mg - k(L+u) - \gamma u' + F.$$

Since $mg = kL$, by rearranging the terms, we get

$$mu'' + \gamma u' + ku = F$$

where m ia the mass, γ is the damping constant, k is the spring constant, and F is the external force.

Next we study several cases.

Case 1: Undamped free vibration (simple harmonic motion). We assume no damping ($\gamma = 0$) and no external force ($F = 0$). So the equation becomes

$$mu'' + ku = 0.$$

Solve it

$$mr^2 + k = 0, \quad r^2 = -\frac{k}{m}, \quad r_{1,2} = \pm\sqrt{\frac{k}{m}}i = \pm\omega_0 i, \quad \text{where } \omega_0 = \sqrt{\frac{k}{m}}.$$

General solution

$$u(t) = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t.$$

Four terminologies of this motion: **frequency, period, amplitude and phase**, defined below.

Frequency: $\omega_0 = \sqrt{\dfrac{k}{m}}$

Period: $T = \dfrac{2\pi}{\omega_0}$

Amplitude and phase: We need to work on this a bit. We can write

$$u(t) = \sqrt{c_1^2 + c_2^2}\left(\frac{c_1}{\sqrt{c_1^2 + c_2^2}}\cos\omega_0 t + \frac{c_2}{\sqrt{c_1^2 + c_2^2}}\sin\omega_0 t\right).$$

Now, define δ, such that $\tan\delta = c_2/c_1$, then

$$\sin\delta = \frac{c_2}{\sqrt{c_1^2 + c_2^2}}, \qquad \cos\delta = \frac{c_1}{\sqrt{c_1^2 + c_2^2}}$$

so we have

$$u(t) = \sqrt{c_1^2 + c_2^2}(\cos\delta \cdot \cos\omega_0 t + \sin\delta \cdot \sin\omega_0 t) = \sqrt{c_1^2 + c_2^2}\cos(\omega_0 t - \delta).$$

So amplitude is $R = \sqrt{c_1^2 + c_2^2}$ and phase is $\delta = \arctan\dfrac{c_2}{c_1}$.

A trick to memorize the last term formula: Consider a right triangle, with c_1 and c_2 as the sides that form the right angle. Then, the amplitude equals to the length of the hypotenuse, and the phase δ is the angle between side c_1 and the hypotenuse. Draw a graph and you will see it better.

A few words on units:

force (f)	weight (mg)	length (u)	mass (m)	gravity (g)
lb	lb	ft	lb $\cdot$ sec^2/ft	32 ft/sec^2
newton	newton	m	kg	9.8 m/sec^2

Problems in this part often come in the form of word problems. We need to learn the skill of extracting information from the text and put them into mathematical terms.

Example 1. A mass weighing 10 lb stretches a spring 2 in. We neglect damping. If the mass is displaced an additional 2 in, and is then set in motion with initial upward velocity of 1 ft/sec, determine the position, frequency, period, amplitude and phase of the motion.

Answer. We see this is free harmonic oscillation. The equation is

$$mu'' + ku = 0$$

with initial conditions (Pay attention to units!)

$$u(0) = 2\text{in} = \frac{1}{6}\text{ft}, \qquad u'(0) = -1.$$

We need find the values for m and k. We have

$$mg = 10, \quad g = 32, \quad m = \frac{10}{g} = \frac{10}{32} = \frac{5}{16}.$$

To find k, we see that the elongation is $L = 2\text{in} = \frac{1}{6}\text{ft}$ if the mass $m = \frac{5}{16}$. By Hooke's law, we have

$$kL = mg, \qquad k = mg/L = 60.$$

Put in these values, we get the equation

$$u'' + 192u = 0, \qquad u(0) = \frac{1}{6}, \quad u'(0) = -1.$$

So the frequency is $\omega_0 = \sqrt{192}$, and the general solution is

$$u(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$$

We can find c_1, c_2 by the ICs:

$$u(0) = c_1 = \frac{1}{6}, \quad u'(0) = \omega_0 c_2 = -1, \quad c_2 = -\frac{1}{\omega_0} = -\frac{1}{\sqrt{192}}.$$

(Note that $c_1 = u(0)$ and $c_2 = u'(0)/\omega_0$.) Now we have the position at any time t

$$u(t) = \frac{1}{6} \cos \omega_0 t - \frac{1}{\sqrt{192}} \sin \omega_0 t.$$

The four terms of the motion are

$$\omega_0 = \sqrt{192}, \quad T = \frac{2\pi}{\omega_0} = \frac{\pi}{\sqrt{48}}, \quad R = \sqrt{c_1^2 + c_2^2} = \sqrt{\frac{19}{576}} \approx 0.18,$$

and

$$\delta = \arctan\frac{c_2}{c_1} = \arctan -\frac{6}{\sqrt{192}} = -\arctan\frac{\sqrt{3}}{4}.$$

Case II: Damped free vibration. We assume that $\gamma \neq 0 (> 0)$ and $F = 0$.

$$mu'' + \gamma u' + ku = 0$$

then

$$mr^2 + \gamma r + k = 0, \qquad r_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}.$$

We see the type of root depends on the sign of the discriminant $\Delta = \gamma^2 - 4km$.

- If $\Delta > 0$, (i.e., $\gamma > \sqrt{4km}$, large damping,) we have two real roots, and they are both negative. The general solution is $u = c_1 e^{r_1 t} + c_2 e^{r_2 t}$, with $r_1 < 0, r_2 < 0$.

  Due to the large damping force, there will be no vibration in the motion. The mass will simply return to the equilibrium position exponentially. This kind of motion is called **overdamped**.

- If $\Delta = 0$, (i.e., $\gamma = \sqrt{4km}$) we have double roots $r_1 = r_2 = r < 0$. So $u = (c_1 + c_2 t)e^{rt}$.

  Depending on the sign of c_1, c_2 (which is determined by the ICs), the mass may cross the equilibrium point maximum once. This kind of motion is called **critically damped**, and this value of γ is called **critical damping**.

- If $\Delta < 0$, (i.e., $\gamma < \sqrt{4km}$, small damping) we have complex roots

  $$r_{1,2} = -\lambda \pm \mu i, \qquad \lambda = \frac{\gamma}{2m}, \qquad \mu = \frac{\sqrt{4km - \gamma^2}}{2m}.$$

  So the position function is

  $$u(t) = e^{-\lambda t}\left(c_1 \cos \mu t + c_2 \sin \mu t\right).$$

  This motion is called **damped oscillation**. We can re-write it as

  $$u(t) = e^{-\lambda t} R \cdot \cos(\mu t - \delta), \quad R = \sqrt{c_1^2 + c_2^2}, \quad \delta = \arctan\frac{c_2}{c_1}.$$

  Here the term $e^{-\lambda t}R$ is the amplitude, and μ is called the quasi frequency, and the quasi period is $\frac{2\pi}{\mu}$. The graph of the solution looks like the one for complex roots with negative real part.

Summary: For all cases, since the real part of the roots are always negative, u will go to zero as t grow. This means, if there is damping, no matter how big or small, the motion will eventually come to a rest.

Example 2. A mass of 1 kg is hanging on a spring with $k = 1$. The mass is in a medium that exerts a viscous resistance of 6 newton when the mass has a velocity of 48 m/sec. The mass is then further stretched for another 2m, then released from rest. Find the position $u(t)$ of the mass.

Answer. We have $\gamma = \frac{6}{48} = \frac{1}{8}$. So the equation for u is

$$mu'' + \gamma u' + ku = 0, \quad u'' + \frac{1}{8}u' + u = 0, \quad u(0) = 2, \quad u'(0) = 0.$$

Solve it

$$r^2 + \frac{1}{8}r + 1 = 0, \quad r_{1,2} = -\frac{1}{16} \pm \frac{\sqrt{255}}{16}i, \quad \omega_0 = \frac{\sqrt{255}}{16}$$

$$u(t) = e^{-\frac{1}{16}t}\left(c_1 \cos \omega_0 t + c_2 \sin \omega_0 t\right).$$

By ICs, we have $u(0) = c_1 = 2$, and

$$u'(t) = -\frac{1}{16}u(t) + e^{-\frac{1}{16}t}(-\omega_0 c_1 \sin \omega_0 t + \omega_0 c_2 \cos \omega_0 t),$$

$$u'(0) = -\frac{1}{16}u(0) + \omega_0 c_2 = 0, \quad c_2 = \frac{2}{\sqrt{255}}.$$

So the position at any time t is

$$u(t) = e^{-t/16}\left(2\cos \omega_0 t - \frac{2}{\sqrt{255}} \sin \omega_0 t\right).$$

3.8 Forced Vibrations

In this chapter we assume the external force is $F(t) = F_0 \cos \omega t$. (The case where $F(t) = F_0 \sin \omega t$ is totally similar.)

The reason for this particular choice of force will be clear later when we learn Fourier series, i.e., we represent periodic functions with the sum of a family of sin and cos functions.

Case 1: With damping.

$$mu'' + \gamma u' + ku = F_0 \cos \omega t.$$

Solution consists of two parts:

$$u(t) = u_H(t) + U(t),$$

$u_H(t)$: the solution of the homogeneous equation,
$U(t)$: a particular solution for the non-homogeneous equation.

From discussion is the previous chapter, we know that $u_H \to 0$ as $t \to +\infty$ for systems with damping. Therefore, this part of the solution is called the *transient solution.*

The appearance of U is due to the force term F. Therefore it is called *the forced response*. The form of this particular solution is $U(t) = A_1 \cos \omega t + A_2 \sin \omega t$. As we have seen, we can rewrite it as $U(t) = R\cos(\omega t - \delta)$ where R is the amplitude and δ is the phase. We see it is a periodic oscillation for all time t.

As time $t \to \infty$, we have $u(t) \to U(t)$. So $U(t)$ is called the *steady state*.

Case 2: Without damping. The equation now is

$$mu'' + ku = F_0 \cos \omega t$$

Let $w_0 = \sqrt{k/m}$ denote the system frequency (i.e., the frequency for the free oscillation). The homogeneous solution is

$$u_H(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t.$$

The form of the particular solution depends on the value of w. We have two cases.

Case 2A: if $w \neq w_0$. The particular solution is of the form

$$U = A \cos wt.$$

(Note that we did not take the $\sin wt$ term, because there is no u' term in the equation.) And $U'' = -w^2 A \cos wt$. Plug these in the equation

$$m(-w^2 A \cos wt) + kA \cos wt = F_0 \cos wt,$$

$$(k - mw^2)A = F_0, \qquad A = \frac{F_0}{k - mw^2} = \frac{F_0}{m(k/m - w^2)} = \frac{F_0}{m(w_0^2 - w^2)}.$$

Note that if w is close to w_0, then A takes a large value.

General solution

$$u(t) = c_1 \cos w_0 t + c_2 \sin w_0 t + A \cos wt$$

where c_1, c_2 will be determined by ICs.

Now, assume ICs:

$$u(0) = 0, \qquad u'(0) = 0.$$

Let's find c_1, c_2 and the solution:

$$u(0) = 0: \quad c_1 + A = 0, \quad c_1 = -A$$

$$u'(0) = 0: \quad 0 + w_0 c_2 + 0 = 0, \quad c_2 = 0$$

Solution

$$u(t) = -A \cos w_0 t + A \cos wt = A(\cos wt - \cos w_0 t).$$

We see that the solution consists of the sum of two cosine functions, with different frequencies. In order to have a better idea of how the solution looks like, we apply some manipulation. Recall the trig identity:

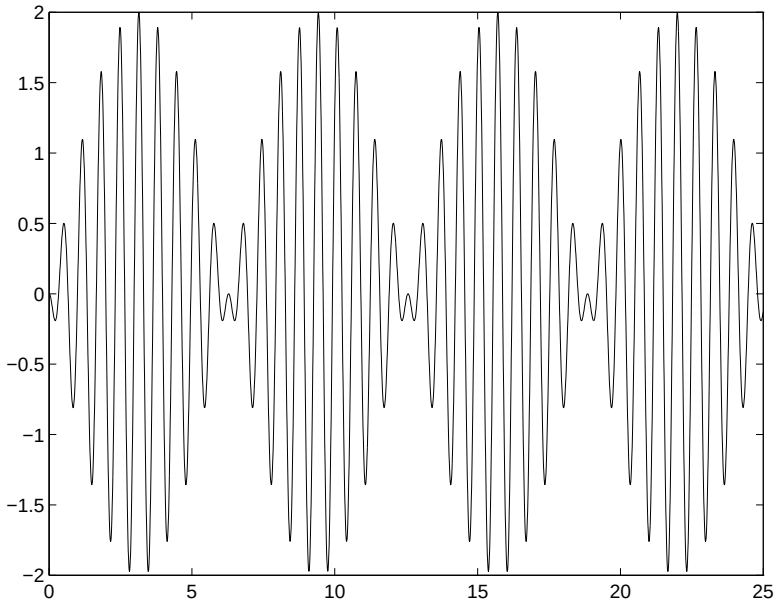
$$\cos a - \cos b = 2 \sin \frac{b-a}{2} \sin \frac{a+b}{2}.$$

We now have

$$u(t) = 2A \sin \frac{w_0 - w}{2} t \cdot \sin \frac{w_0 + w}{2} t.$$

Since both w_0, w are positive, then $w_0 + w$ has larger value than $w_0 - w$. Then, the first term $2A \sin \frac{w_0 - w}{2} t$ can be viewed as the varying amplitude, and the second term $\sin \frac{w_0 + w}{2} t$ is the vibration/oscillation.

One particular situation of interests: if $w_0 \neq w$ but they are very close $w_o \approx w$, then we have $|w_0 - w| << |w_0 + w|$, meaning that $|w_0 - w|$ is much smaller than $|w_0 + w|$. The plot of $u(t)$ looks like (we choose $w_0 = 9, w = 10$)

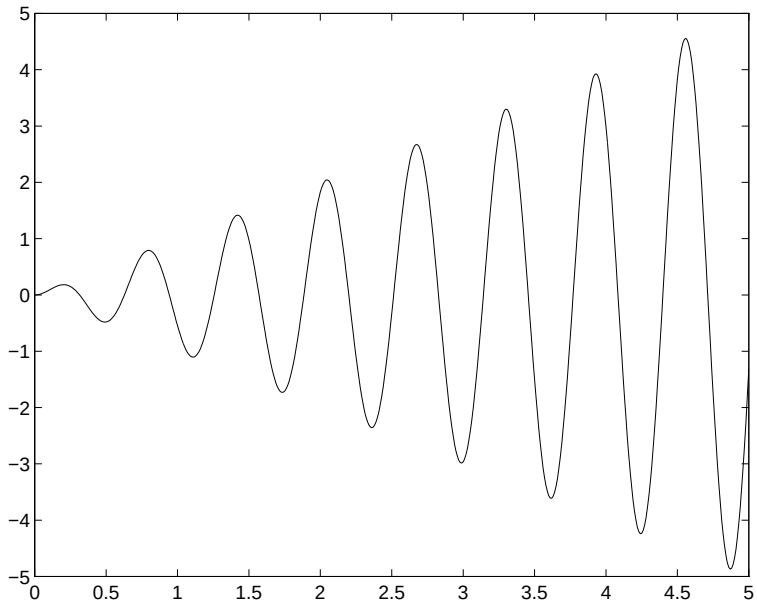


This is called a *beat*. (One observes it by hitting a key on a piano that's not tuned, for example.)

Case 2B: If $w = w_0$. The particular solution is

$$U = At \cos w_0 t + Bt \sin w_0 t$$

A typical plot looks like:



This is called *resonance*. If the frequency of the source term ω equals to the frequency of the system ω_0, then, small source term could make the solution grow very large!

One can bring down a building or bridge by small periodic perturbations.

Historical disasters such as the French troop marching over a bridge and the bridge collapsed. Why? Unfortunate for the French, the system frequency of the bridge matches the frequency of their foot-steps.

Summary:

- With damping:
  Transient solution u_H plus the forced response term $U(t)$ (steady state),
- Without damping:
  if $w = w_0$: resonance.
  if $w \neq w_0$ but $w \approx w_0$: beat.

Chapter 4

Higher Order Linear Equations

4.1 General Theory of n-th Order Linear Equations

The general form of a linear equation of n-th order, with $y(t)$ as the unknown, is

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y = g(t). \qquad (A)$$

Let L denote the linear differential operator

$$L(y) \doteq y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y, \qquad L(y) = g(t).$$

We need to assign n initial/boundary conditions. Let t_0 be the initial time. Normally, the lower derivatives are given at t_0, i.e.,

$$y(t_0) = y_o, y'(t_0) = y'_o, \cdots, y^{(n-1)}(t_0) y_o^{(n-1)}.$$

Theoretical aspects are very similar to those for 2nd order linear equations, with suitable extensions.

Existence and Uniqueness. If the coefficient functions $p_0(t), p_1(t), \cdots, p_{n-1}(t)$ are continuous and bounded on an open interval I containing t_0, then equation (A) has a uniqueness solution on the interval I.

Typical problem types: Find the largest interval where solution is valid.

Linear dependency of n functions: The Wronskian determinant for n functions $(y_1, y_2, \cdots, y_n)$ is defined as

$$W(y_1, y_2, \cdots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y'_1 & y'_2 & \cdots & y'_n \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

The determinant is lengthy to compute for general $n \times n$ matrices. The simpler cases are when $n = 2$ and $n = 3$, which we recall here

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc, \qquad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{array}{l} a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} \\ -a_{31}a_{22}a_{13} - a_{21}a_{12}a_{33} - a_{11}a_{32}a_{23}. \end{array}$$

One can use the Wronskian determinant to check if a set of functions are linearly dependent or not.

Homogeneous equations when $g(t) \equiv 0$: There are n solutions, $y_1, y_2, \cdots y_n$, linearly independent, with $W(y_1, y_2, \cdots, y_n) \neq 0$, that forms a set of fundamental solutions, whose linear combination gives the general solution

$$y_H(t) = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n.$$

Here the constants $c_1, c_2, \cdots, c_n$ are determined by the n initial conditions.

Remark: We observe now how the property $W \neq 0$ leads to unique solutions for the constants $c_1, \cdots, c_n$. Pugging in the n initial conditions, we have

$$\begin{aligned} c_1 y_1(t_0) + c_2 y_2(t_0) + \cdots + c_n y_n(t_0) &= y_o \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) + \cdots + c_n y_n'(t_0) &= y_o' \\ &\vdots \\ c_1 y_1^{(n-1)}(t_0) + c_2 y_2^{(n-1)}(t_0) + \cdots + c_n y_n^{(n-1)}(t_0) &= y_o^{(n-1)} \end{aligned}$$

Writing it into matrix-vector form, we get

$$\begin{pmatrix} y_1(t_0) & y_2(t_0) & \cdots & y_n(t_0) \\ y_1'(t_0) & y_2'(t_0) & \cdots & y_n'(t_0) \\ \vdots & & & \\ y_1^{(n-1)}(t_0) & y_2^{(n-1)}(t_0) & \cdots & y_n^{(n-1)}(t_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} y_o \\ y_o' \\ \vdots \\ y_o^{(n-1)} \end{pmatrix}$$

The system has a uniqueness solution if the determinant of the coefficient matrix is not 0, i.e., $W(y_1, \cdots, y_n) \neq 0$.

Abel's Theorem.* Let $y_1(t), \cdots, y_n(t)$ be n linearly independent solutions to $L(y) = 0$, i.e.,

$$y^{(n)} + p_{n-1}(t) y^{(n-1)} + \cdots + p_1(t) y' + p_0(t) y = 0, \quad a < t < b$$

Let $W(t)$ be the Wronskian of $y_1 \cdot, y_n$. Then, $W(t)$ is a solution of the first order linear equation

$$W' - p_{n-1}(t) W = 0, \qquad W(t) = C \exp\Big\{ - \int p_{n-1}(t)\, dt \Big\}, \qquad a < t < b.$$

The proof could be found in most advanced texts on differential equations. The basic argument is similar to that of the case $n = 2$. The computations are more involved, however, since one needs to compute the derivatives of an $(n \times n)$ determinant of functions. Here is a brief proof.

By product rule, the derivative of the determinant is:

$$W' = \begin{vmatrix} y_1' & y_2' & \cdots & y_n' \\ y_1' & y_2' & \cdots & y_n' \\ y_1'' & y_2'' & \cdots & y_n'' \\ y_1''' & y_2''' & \cdots & y_n''' \\ \vdots & & & \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} + \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1'' & y_2'' & \cdots & y_n'' \\ y_1'' & y_2'' & \cdots & y_n'' \\ y_1''' & y_2''' & \cdots & y_n''' \\ \vdots & & & \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} + \cdots + \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ y_1'' & y_2'' & \cdots & y_n'' \\ \vdots & & & \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{vmatrix}$$

Here in the first matrix we differentiate the first row, in the the second matrix we differentiate the second row and so on. There are totally n terms. We observe now that all the terms, except the last, have two identical rows in the matrix. Then, their determinants are all 0, except the last term. Therefore, we have

$$W' = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ y_1'' & y_2'' & \cdots & y_n'' \\ \vdots & & & \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{vmatrix} . \qquad (A)$$

Now, since y_i's are solutions, we have

$$y_i^{(n)} = -p_{n-1}y_i^{(n-1)} - p_{n-2}y_i^{(n-2)} - \cdots - p_1 y_i' - p_o y_i$$

Now, multiply first row by p_0, the second row by p_i, and so on, and the $(n-1)$-th row by p_{n-2}, and add them all to the last row. Remember that this operation does not change the determinant. Then the last row becomes $-p_{n-1}y_i^{(n-1)}$ for the i-th entry. We now have

$$W' = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ y_1'' & y_2'' & \cdots & y_n'' \\ \vdots & & & \\ p_{n-1}y_1^{(n-1)} & p_{n-1}y_2^{(n-1)} & \cdots & p_{n-1}y_n^{(n-1)} \end{vmatrix} = p_{n-1}(t)W \,,$$

completing the proof.

Solution for the non-homogeneous equation: $y(t) = y_H(t) + Y(t)$, where y_H is the homogeneous solution, and Y is a particular solution.

4.2 Homogeneous Equations with Constant Coefficients.

This follows in the same setup as that for 2nd order equations. Consider the equation

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

Characteristic equation

$$a_n r^n + \cdots a_1 r + a_0 = 0$$

In general, one find n roots, (counting multiplicity)

$$(r - r_1)(r - r_2)\cdots(r - r_n) = 0.$$

From these roots one can find n solutions. It follows the same rules as for 2nd order equations, with some extensions (marked with * in the following table).

	root type	solution(s)
	r is real, un-repeated	e^{rt}
	r is real, double root	e^{rt}, te^{rt}
(*)	r is real, triple root	e^{rt}, te^{rt}, t^2e^{rt}
(*)	r is real, repeated with multiplicity m	e^{rt}, te^{rt}, $\cdots$ $t^{m-1}e^{rt}$
	$r = \lambda \pm i\mu$ complex	$e^{\lambda t}\cos\mu t, e^{\lambda t}\sin\mu t$
(*)	$r = \lambda \pm i\mu$ complex and double roots	$e^{\lambda t}\cos\mu t, e^{\lambda t}\sin\mu t$ and $te^{\lambda t}\cos\mu t, te^{\lambda t}\sin\mu t$
(*)	$r = \lambda \pm i\mu$ complex, repeated m times	similar ...

The hard work is to find the roots, i.e., factorization of polynomials!

Example 1. (a). Find the general solution of $y^{(4)} - 4y'' = 0$.
(b). Find the solution with initial conditions

$$y(0) = 0, \qquad y'(0) = 0, \qquad y''(0) = 8, \qquad y'''(0) = 0.$$

Answer. (a). Write out the characteristic polynomial

$$r^4 - 4r^2 = 0, \qquad r^2(r^2 - 4) = 0, \qquad r^2(r-2)(r+2) = 0$$

We find the roots

$$r_1 = -2, \qquad r_2 = 2, \qquad r_3 = r_4 = 0$$

The corresponding solutions

$$y_1 = e^{-2t}, \qquad y_2 = e^{2t}, \qquad y_3 = e^{0t} = 1, \qquad y_4 = te^{0t} = t$$

General solution

$$y(t) = c_1y_1 + c_2y_2 + c_3y_3 + c_4y_4 = c_1e^{-2t} + c_2e^{2t} + c_3 + c_4t.$$

(b). We now determine the constants by initial data. It's useful to work out the derivatives first:

$$\begin{aligned} y'(t) &= -2c_1e^{-2t} + 2c_2e^{2t} + c_4. \\ y''(t) &= 4c_1e^{-2t} + 4c_2e^{2t} \\ y'''(t) &= -8c_1e^{-2t} + 8c_2e^{2t} \end{aligned}$$

Then, the 4 ICs give

$$\begin{aligned} y(0) = 0: &\quad c_1 + c_2 + c_3 = 0 \\ y'(0) = 0: &\quad -2c_1 + 2c_2 + c_4 = 0 \\ y''(0) = 0: &\quad 4c_1 + 4c_2 = 8, \qquad c_1 + c_2 = 2 \\ y'''(0) = 0: &\quad -8c_1 + 8c_2 = 0, \qquad c_1 = c_2 \end{aligned}$$

From the last two equation, we get $c_1 = c_2 = 1$. Putting these back into the first 2 equations, we get $c_3 = -2$ and $c_4 = 0$.

The solution is

$$y(t) = e^{-2t} + e^{2t} - 2.$$

In the next example, we will focus on finding general solutions.

Example 2. Find the general solution for the following equations:

$$\begin{aligned}
\text{(I)}: &\quad y^{(4)} + 4y'' = 0 \\
\text{(II)}: &\quad y''' - y = 0 \\
\text{(III)}: &\quad y^{(4)} + 8y'' + 16y = 0 \\
\text{(VI)}: &\quad y''' + 3y'' + 3y' + y = 0 \\
\text{(V)*}: &\quad y^{(4)} + 8y = 0
\end{aligned}$$

Answer. (I): Characteristic equation and the roots:

$$r^4 + 4r^2 = 0, \quad r^2(r^2+4) = 0, \quad r_1 = r_2 = 0, \qquad r_{3,4} = \pm 2i$$

General solution

$$y(t) = c_1 + c_2 t + c_3 \cos 2t + c_4 \sin 2t.$$

(II): Characteristic equation and the roots:

$$r^3 - 1 = 0, \qquad (r-1)(r^2+r+1) = 0, \qquad r_1 = 1, \quad r_2 = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}$$

General solution

$$y(t) = c_1 e^t + e^{-\frac{1}{2}t}\left(c_2 \cos \frac{\sqrt{3}}{2}t + c_3 \sin \frac{\sqrt{3}}{2}t\right).$$

(III): Characteristic equation and the roots:

$$r^4 + 8r^2 + 16 = 0, \qquad (r^2+4)^2 = 0, \qquad r_1 = r_2 = 2i, \quad r_3 = 3_4 = -2i.$$

We see that we have double complex roots. General solution is

$$y(t) = c_1 \cos 2t + c_2 \sin 2t + c_3 t \cos 2t + c_4 t \sin 2t.$$

(VI): Characteristic equation and the roots:

$$r^3 + 3r^2 + 3r + 1 = 0, \qquad (r+1)^3 = 0, \qquad r_1 = r_2 = r_3 = -1$$

This is a triple root! General solution is

$$y(t) = c_1 e^{-t} + c_2 t e^{-t} + c_3 t^2 e^{-t} = e^{-t}(c_1 + c_2 t + c_3 t^2).$$

(V)*: Characteristic equation and the roots:

$$r^4 + 8 = 0, \quad r^4 = -8 = 8e^{i\pi}, \quad r_k = 8^{1/4} e^{i(\pi + 2(k-1)\pi)/4}, \quad k = 1, 2, 3, 4.$$

so we have 4 roots

$$r_1 = 8^{1/4}e^{i\pi/4}, \qquad r_2 = 8^{1/4}e^{i3\pi/4}, \qquad r_3 = 8^{1/4}e^{i5\pi/4}, \qquad r_4 = 8^{1/4}e^{i7\pi/4}.$$

We now write out the real and imaginary parts, using Euler's formula. Let $a = 8^{1/4}/\sqrt{2}$, we can now write

$$r_1 = a + ia, \qquad r_2 = -a + ia, \qquad r_3 = -a - ia, \qquad r_4 = a - ia.$$

We see that r_1, r_4 are conjugate pairs, so are r_2, r_3. The general solution is

$$y(t) = e^{at}\Big(c_1\cos(at) + c_2\sin(at)\Big) + e^{-at}\Big(c_3\cos(at) + c_4\sin(at)\Big).$$

NB! Factorizing a polynomial is non-trivial!

4.3 Higher Order Linear Non-homogeneous Differential Equations

We focus on how to find a particular solution.

In the simple case with constant coefficients, and we have found a set of fundamental solutions $y_1, \cdots, y_n$, and the right-hand-side $g(t)$ is a combination of e^{at}, $\sin(\alpha t), \cos(\alpha t)$ and polynomials, we follow the same pattern as for 2nd order equation, to get the correct form of a particular solution.

In the more general case, we will use the method called: variation of parameters. Consider the equation

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y = g(t),$$

and let $y_1, y_2, \cdots, y_n$ be the set of fundamental solutions for the corresponding homogeneous equation.

We seek a particular solution of the form

$$Y = u_1(t)y_1 + u_2(t)y_2 + \cdots + u_n(t)y_n$$

The n functions $u_1, u_2, \cdots, u_n$ are to be determined. We see now we have $(n-1)$ free choices to impose addition constraints these functions. We choose

$$\begin{aligned}
y_1u_1' + y_2u_2' + \cdots + y_nu_n' &= 0\\
y_1'u_1' + y_2'u_2' + \cdots + y_n'u_n' &= 0\\
y_1''u_1' + y_2''u_2' + \cdots + y_n''u_n' &= 0\\
&\vdots\\
y_1^{(n-2)}u_1' + y_2^{(n-2)}u_2' + \cdots + y_n^{(n-2)}u_n' &= 0
\end{aligned}$$

By these constraints, we now have simple forms for the derivatives of Y.

$$\begin{aligned}
Y' &= y_1'u_1 + y_2'u_2 + \cdots + y_n'u_n\\
Y'' &= y_1''u_1 + y_2''u_2 + \cdots + y_n''u_n\\
&\vdots\\
Y^{(n-1)} &= y_1^{(n-1)}u_1 + y_2^{(n-1)}u_2 + \cdots + y_n^{(n-1)}u_n
\end{aligned}$$

The n-th derivative is

$$Y^{(n)} = \left[y_1^{(n)}u_1 + y_2^{(n)}u_2 + \cdots + y_n^{(n)}u_n\right] + \left[y_1^{(n-1)}u_1' + y_2^{(n-1)}u_2' + \cdots + y_n^{(n-1)}u_n'\right]$$

Plug all these into the equation (A), and collect like terms, we get

$$\begin{aligned}\text{LHS} &= u_1\left[y_1^{(n)} + p_{n-1}(t)y_1^{(n-1)} + \cdots + p_1(t)y_1' + p_0(t)y_1\right] \\ &\quad + u_2\left[y_2^{(n)} + p_{n-1}(t)y_2^{(n-1)} + \cdots + p_1(t)y_2' + p_0(t)y_2\right] + \cdots \\ &\quad + u_n\left[y_n^{(n)} + p_{n-1}(t)y_n^{(n-1)} + \cdots + p_1(t)y_n' + p_0(t)y_n\right] \\ &\quad + \left[y_1^{(n-1)}u_1' + y_2^{(n-1)}u_2' + \cdots + y_n^{(n-1)}u_n'\right].\end{aligned}$$

Here all the terms are 0 except the last one. We get

$$y_1^{(n-1)}u_1' + y_2^{(n-1)}u_2' + \cdots + y_n^{(n-1)}u_n' = g(t).$$

Combine the last equation with the $(n-1)$ constraints we imposed earlier, we get a system of n linear equations to solve for u_i''s. Indeed, in matrix-vector form, it can be written as

$$\begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ g(t) \end{pmatrix}. \qquad (*)$$

Note that the coefficient matrix is the Wronskian, and is never 0. There this system has a unique solution for u_i'. One can the recover u_i by integration.

Example Consider the non-homogeneous differential equation

$$t^3y''' - 3t^2y'' + 6ty' - 6y = t, \quad (t > 0)$$

Show that

$$y_1(t) = t, \qquad y_2(t) = t^2 \qquad y_3(t) = t^3$$

form a fundamental set of solution for the corresponding homogeneous equation. Then, find a particular solution for the non-homogeneous equation.

Answer. (1) One can easily plug these 3 function and verify that they are solutions for the homogeneous equation.

(2) To check if they are linearly independent, we compute the Wronskian

$$W(t) = \begin{vmatrix} t & t^2 & t^3 \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = 2t^3.$$

(3). Let the particular solution take the form

$$Y = t\,u_1(t) + t^2u_2(t) + t^3u_3(t).$$

By formula (*), we have

$$\begin{pmatrix} t & t^2 & t^3 \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ t^{-2} \end{pmatrix}$$

We get

$$\begin{pmatrix} u_1' \\ u_2' \\ u_3' \end{pmatrix} = \begin{pmatrix} 1/(2t) \\ -1/t^2 \\ 1/(2t^3) \end{pmatrix}, \qquad \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\ln t \\ 1/t \\ -t^{-2}/4 \end{pmatrix}.$$

A particular solution is

$$Y = t\frac{1}{2}\ln t + t^2(1/t) - t^3 t^{-2}/4 = \frac{t}{2}\ln t + \frac{3}{4}t, \quad (t > 0).$$

We see that the last term is already in y_1, so we can drop it and simply take

$$Y = \frac{t}{2}\ln t, \qquad (t > 0)$$

We can now form the general solution for the non-homogeneous equation

$$Y = c_1 t = c_2 t^2 + c_3 t^3 + \frac{t}{2}\ln t.$$

Remark. In general case, find the solution for the homogeneous equation with variable coefficient is not easy! We can only handle the constant coefficient case so far.

Chapter 6

The Laplace Transform

Laplace transform is mainly used to handle piecewise continuous or impulsive force.

6.1 Definition of the Laplace transform

Topics:

- Definition of Laplace transform,
- Compute Laplace transform by definition, including piecewise continuous functions.

Definition: Given a function $f(t)$, $t \geq 0$, its Laplace transform is defined as

$$F(s) \doteq \mathcal{L}\{f(t)\} \doteq \int_0^\infty e^{-st} f(t)\, dt \doteq \lim_{A\to\infty} \int_0^A e^{-st} f(t)\, dt\,.$$

We say the transform converges if the limit exists, and diverges if not.

Next we will give examples on computing the Laplace transform of given functions by definition.

Example 1. $f(t) = 1$ for $t \geq 0$.

Answer.

$$\begin{aligned} F(s) = \mathcal{L}\{f(t)\} &= \lim_{A\to\infty} \int_0^A e^{-st}\, dt = \lim_{A\to\infty} -\frac{1}{s} e^{-st}\Big|_{t=0}^{A} \\ &= \lim_{A\to\infty} -\frac{1}{s}\left[e^{-sA} - 1\right] = \lim_{A\to\infty} \left\{-\frac{1}{s} e^{-sA}\right\} + \frac{1}{s} = \frac{1}{s}, \quad (s > 0) \end{aligned}$$

Note that the condition $s > 0$ is needed to ensure that the limit exists, and it is 0.

Example 2. $f(t) = e^{at}$.

Answer.

$$\begin{aligned} F(s) &= \mathcal{L}\{f(t)\} = \lim_{A\to\infty} \int_0^A e^{-st} e^{at} dt = \lim_{A\to\infty} \int_0^A e^{-(s-a)t} dt = \lim_{A\to\infty} -\frac{1}{s-a} e^{-(s-a)t}\Big|_0^A \\ &= \lim_{A\to\infty} -\frac{1}{s-a}\left(e^{-(s-a)A} - 1\right) = \frac{1}{s-a}, \quad (s > a) \end{aligned}$$

Note that the condition $s - a > 0$ is needed to ensure that the limit exists.

Example 3. $f(t) = t^n$, for $n \geq 1$ integer.

Answer. Review integration-by-parts:

$$\int u(t)v'(t)\,dt = uv - \int u'(t)v(t)\,dt.$$

For here, we have

$$\begin{aligned} F(s) &= \lim_{A\to\infty} \int_0^A e^{-st}t^n dt = \lim_{A\to\infty} \left\{ t^n \left.\frac{e^{-st}}{-s}\right|_0^A - \int_0^A \frac{nt^{n-1}e^{-st}}{-s} dt \right\} \\ &= 0 + \frac{n}{s} \lim_{A\to\infty} \int_0^A e^{-st}t^{n-1}dt = \frac{n}{s}\mathcal{L}\{t^{n-1}\}. \end{aligned}$$

So we get a recursive relation

$$\mathcal{L}\{t^n\} = \frac{n}{s}\mathcal{L}\{t^{n-1}\}, \qquad \text{for all } n,$$

which means

$$\mathcal{L}\{t^{n-1}\} = \frac{n-1}{s}\mathcal{L}\{t^{n-2}\}, \quad \mathcal{L}\{t^{n-2}\} = \frac{n-2}{s}\mathcal{L}\{t^{n-3}\}, \quad \cdots$$

By induction, we get

$$\begin{aligned} \mathcal{L}\{t^n\} &= \frac{n}{s}\mathcal{L}\{t^{n-1}\} = \frac{n}{s}\frac{(n-1)}{s}\mathcal{L}\{t^{n-2}\} = \frac{n}{s}\frac{(n-1)}{s}\frac{(n-2)}{s}\mathcal{L}\{t^{n-3}\} \\ &= \cdots = \frac{n}{s}\frac{(n-1)}{s}\frac{(n-2)}{s}\cdots\frac{1}{s}\mathcal{L}\{1\} = \frac{n!}{s^n}\frac{1}{s} = \frac{n!}{s^{n+1}}, \quad (s>0) \end{aligned}$$

Example 4. Find the Laplace transform of $\sin at$ and $\cos at$.

Answer. Method 1. Compute by definition, with integration-by-parts, twice. (lots of work...)

Method 2. Use the Euler's formula

$$e^{iat} = \cos at + i \sin at, \qquad \Rightarrow \qquad \mathcal{L}\left\{e^{iat}\right\} = \mathcal{L}\{\cos at\} + i\mathcal{L}\{\sin at\}.$$

By Example 2 we have

$$\mathcal{L}\left\{e^{iat}\right\} = \frac{1}{s-ia} = \frac{1(s+ia)}{(s-ia)(s+ia)} = \frac{s+ia}{s^2+a^2} = \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}.$$

Comparing the real and imaginary parts, we get

$$\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}, \qquad \mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}, \quad (s>0).$$

Remark: Now we will use $\int_0^\infty$ instead of $\lim_{A\to\infty}\int_0^A$, without causing confusion.

For piecewise continuous functions, Laplace transform can be computed by integrating each integral and add up at the end.

Example 5. Find the Laplace transform of

$$f(t) = \begin{cases} 1, & 0 \le t < 2, \\ t-2, & 2 \le t. \end{cases}$$

We do this by definition:

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st} f(t)\,dt = \int_0^2 e^{-st}dt + \int_2^\infty (t-2)e^{-st}dt \\ &= \frac{1}{-s}e^{-st}\Big|_{t=0}^{2} + (t-2)\frac{1}{-s}e^{-st}\Big|_{t=2}^{\infty} - \int_2^A \frac{1}{-s}e^{-st}dt \\ &= \frac{1}{-s}(e^{-2s}-1) + (0-0) + \frac{1}{s}\,\frac{1}{-s}e^{-st}\Big|_{t=2}^{\infty} = \frac{1}{-s}(e^{-2s}-1) + \frac{1}{s^2}e^{-2s} \end{aligned}$$

Remark. Later in Ch 6.3 we will use a different method to deal with discontinuous (piecewise continuous) functions.

6.2 Solution of initial value problems

Topics:

- Properties of Laplace transform, with proofs and examples
- Inverse Laplace transform, with examples, and review of partial fraction,
- Solution of initial value problems, with continuous source terms, with examples covering various cases.

Properties of Laplace transform:

1. Linearity: $\mathcal{L}\{c_1 f(t) + c_2 g(t)\} = c_1\mathcal{L}\{f(t)\} + c_2\mathcal{L}\{g(t)\}$.
2. First derivative: $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$.
   Second derivative: $\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$.
   Higher order derivative:
   $$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0).$$
3. $\mathcal{L}\{-tf(t)\} = F'(s)$ where $F(s) = \mathcal{L}\{f(t)\}$. This also implies $\mathcal{L}\{tf(t)\} = -F'(s)$.
4. Shift Theorem 1: $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$ where $F(s) = \mathcal{L}\{f(t)\}$.
   This implies $e^{at}f(t) = \mathcal{L}^{-1}\{F(s-a)\}$.

Remarks:

- Note properties 2 are useful in differential equations. It shows that each derivative in t caused a multiplication of s in the Laplace transform.
- Property 3 is the counter part for Property 2. It shows that each derivative in s causes a multiplication of $-t$ in the inverse Laplace transform.
- Property 4 is the first Shift Theorem. A counter part of it will come later in chapter 6.3.

Proof:

1. This follows by definition.
2. By definition

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)dt = e^{-st} f(t)\Big|_0^\infty - \int_0^\infty (-s)e^{-st} f(t)dt = -f(0) + s\mathcal{L}\{f(t)\}.$$

   The second derivative formula follows from that of the first derivative. Set f to be f' we get

$$\mathcal{L}\{f''(t)\} = s\mathcal{L}\{f'(t)\} - f'(0) = s(s\mathcal{L}\{f(t)\} - f(0)) - f'(0) = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0).$$

   For high derivatives, it follows by induction.

3. The proof follows from the definition:

$$F'(s) = \frac{d}{ds}\int_0^\infty e^{-st} f(t)dt = \int_0^\infty \frac{\partial}{\partial s}(e^{-st}) f(t)dt = \int_0^\infty (-t)e^{-st} f(t)dt = \mathcal{L}\{-tf(t)\}.$$

4. This proof also follows from definition:

$$\mathcal{L}\{e^{at} f(t)\} = \int_0^\infty e^{-st} e^{at} f(t)dt = \int_0^\infty e^{-(s-a)t} f(t)dt = F(s-a).$$

By using these properties, we could find more easily Laplace transforms of many other functions.

Example 1.

$$\text{From} \qquad \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \qquad \text{we get} \qquad \mathcal{L}\{e^{at}t^n\} = \frac{n!}{(s-a)^{n+1}}.$$

Example 2.

$$\text{From} \qquad \mathcal{L}\{\sin bt\} = \frac{b}{s^2+b^2}, \qquad \text{we get} \qquad \mathcal{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}.$$

Example 3.

$$\text{From} \qquad \mathcal{L}\{\cos bt\} = \frac{s}{s^2+b^2}, \qquad \text{we get} \qquad \mathcal{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2}.$$

Example 4.

$$\mathcal{L}\{t^3+5t-2\} = \mathcal{L}\{t^3\} + 5\mathcal{L}\{t\} - 2\mathcal{L}\{1\} = \frac{3!}{s^4} + 5\frac{1}{s^2} - 2\frac{1}{s}.$$

Example 5.

$$\mathcal{L}\{e^{2t}(t^3+5t-2)\} = \frac{3!}{(s-2)^4} + 5\frac{1}{(s-2)^2} - 2\frac{1}{s-2}.$$

Example 6.

$$\mathcal{L}\{(t^2+4)e^{2t} - e^{-t}\cos t\} = \frac{2}{(s-2)^3} + \frac{4}{s-2} - \frac{s+1}{(s+1)^2+1},$$

because

$$\mathcal{L}\{t^2+4\} = \frac{2}{s^3} + \frac{4}{s}, \qquad \Rightarrow \quad \mathcal{L}\{(t^2+4)e^{2t}\} = \frac{2}{(s-2)^3} + \frac{4}{s-2}.$$

Next are a few examples for Property 3.

Example 7.

$$\text{Given} \quad \mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \qquad \text{we get} \qquad \mathcal{L}\{te^{at}\} = -\left(\frac{1}{s-a}\right)' = \frac{1}{(s-a)^2}$$

Example 8.

$$\mathcal{L}\{t\sin bt\} = -\left(\frac{b}{s^2+b^2}\right)' = \frac{2bs}{(s^2+b^2)^2}.$$

Example 9.

$$\mathcal{L}\{t\cos bt\} = -\left(\frac{s}{s^2+b^2}\right)' = \cdots = \frac{s^2-b^2}{(s^2+b^2)^2}.$$

Inverse Laplace transform. Definition:

$$\mathcal{L}^{-1}\{F(s)\} = f(t), \qquad \text{if} \qquad F(s) = \mathcal{L}\{f(t)\}.$$

Technique: find the way back.

Some simple examples:

Example 10.

$$\mathcal{L}^{-1}\left\{\frac{3}{s^2+4}\right\} = \mathcal{L}^{-1}\left\{\frac{3}{2}\cdot\frac{2}{s^2+2^2}\right\} = \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\} = \frac{3}{2}\sin 2t.$$

Example 11.

$$\mathcal{L}^{-1}\left\{\frac{2}{(s+5)^4}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{3}\cdot\frac{6}{(s+5)^4}\right\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3!}{(s+5)^4}\right\} = \frac{1}{3}e^{-5t}\mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\} = \frac{1}{3}e^{-5t}t^3.$$

Example 12.

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+4}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = \cos 2t + \frac{1}{2}\sin 2t.$$

Example 13.

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2-4}\right\} = \mathcal{L}^{-1}\left\{\frac{s+1}{(s-2)(s+2)}\right\} = \mathcal{L}^{-1}\left\{\frac{3/4}{s-2} + \frac{1/4}{s+2}\right\} = \frac{3}{4}e^{2t} + \frac{1}{4}e^{-2t}.$$

Here we used partial fraction to find out:

$$\frac{s+1}{(s-2)(s+2)} = \frac{A}{s-2} + \frac{B}{s+2}, \qquad A = 3/4, \quad B = 1/4.$$

Solutions of initial value problems.
We will go through one example first.

Example 14. (Two distinct real roots.) Solve the initial value problem by Laplace transform,

$$y'' - 3y' - 10y = 2, \qquad y(0) = 1, \quad y'(0) = 2.$$

Answer. Step 1. Take Laplace transform on both sides: Let $\mathcal{L}\{y(t)\} = Y(s)$, and then

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0) = sY - 1, \qquad \mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0) = s^2Y - s - 2.$$

Note the initial conditions are the first thing to go in!

$$\mathcal{L}\{y''(t)\} - 3\mathcal{L}\{y'(t)\} - 10\mathcal{L}\{y(t)\} = \mathcal{L}\{2\}, \qquad \Rightarrow \qquad s^2Y - s - 2 - 3(sY - 1) - 10Y = \frac{2}{s}.$$

Now we get an algebraic equation for $Y(s)$.

Step 2: Solve it for $Y(s)$:

$$(s^2 - 3s - 10)Y(s) = \frac{2}{s} + s + 2 - 3 = \frac{s^2 - s + 2}{s}, \qquad \Rightarrow \qquad Y(s) = \frac{s^2 - s + 2}{s(s-5)(s+2)}.$$

Step 3: Take inverse Laplace transform to get $y(t) = \mathcal{L}^{-1}\{Y(s)\}$. The main technique here is **partial fraction**.

$$Y(s) = \frac{s^2 - s + 2}{s(s-5)(s+2)} = \frac{A}{s} + \frac{B}{s-5} + \frac{C}{s+2} = \frac{A(s-5)(s+2) + Bs(s+2) + Cs(s-5)}{s(s-5)(s+2)}.$$

Compare the numerators:

$$s^2 - s + 2 = A(s-5)(s+2) + Bs(s+2) + Cs(s-5).$$

The previous equation holds for all values of s. We will now choose selected values of s such that only one of the constants A, B, C will be non-zero, so we can solve for it.

$$\begin{aligned} s = 0: \quad & -10A = 2, \quad \Rightarrow \quad A = -\frac{1}{5} \\ s = 5: \quad & 35B = 22, \quad \Rightarrow \quad B = \frac{22}{35} \\ s = -2: \quad & 14C = 8, \quad \Rightarrow \quad C = \frac{4}{7} \end{aligned}$$

Now, $Y(s)$ is written into sum of terms which we can find the inverse transform:

$$y(t) = A\mathcal{L}^{-1}\{\frac{1}{s}\} + B\mathcal{L}^{-1}\{\frac{1}{s-5}\} + C\mathcal{L}^{-1}\{\frac{1}{s+2}\} = -\frac{1}{5} + \frac{22}{35}e^{5t} + \frac{4}{7}e^{-2t}.$$

NB! Pay attention to the roots of the denominators for $F(s)$. Note that the factors $(s-5)$ and $(s+2)$ come from the characteristic equation, and the term s comes from the source term.

Algorithm for finding solutions:

- Take Laplace transform on both sides. You will get an algebraic equation for $Y(s)$.
- Solve this equation to get $Y(s)$.
- Take inverse transform to get $y(t) = \mathcal{L}^{-1}\{Y\}$.

Example 15. (Distinct real roots, but one matches the source term.) Solve the initial value problem by Laplace transform,

$$y'' - y' - 2y = e^{2t}, \qquad y(0) = 0, \quad y'(0) = 1.$$

Answer. Take Laplace transform on both sides of the equation, we get

$$\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - \mathcal{L}\{2y\} = \mathcal{L}\{e^{2t}\}, \qquad \Rightarrow \qquad s^2Y(s) - 1 - sY(s) - 2Y(s) = \frac{1}{s-2}.$$

Solve it for Y:

$$(s^2-s-2)Y(s) = \frac{1}{s-2}+1 = \frac{s-1}{s-2}, \qquad \Rightarrow \qquad Y(s) = \frac{s-1}{(s-2)(s^2-s-2)} = \frac{s-1}{(s-2)^2(s+1)}.$$

Use partial fraction:

$$\frac{s-1}{(s-2)^2(s+1)} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{(s-2)^2}.$$

Compare the numerators:

$$s - 1 = A(s-2)^2 + B(s+1)(s-2) + C(s+1)$$

Set $s=-1$, we get $A=-\frac{2}{9}$. Set $s=2$, we get $C=\frac{1}{3}$. Set $s=0$ (any convenient values of s can be used in this step), we get

$$-1=4A-2B+C, \quad -1=-\frac{8}{9}-2B+\frac{1}{3}, \quad 2B=\frac{1}{9}+\frac{1}{3}=\frac{4}{9}, \quad B=\frac{2}{9}.$$

So

$$Y(s)=-\frac{2}{9}\frac{1}{s+1}+\frac{2}{9}\frac{1}{s-2}+\frac{1}{3}\frac{1}{(s-2)^2}$$

and

$$y(t)=\mathcal{L}^{-1}\{Y\}=-\frac{2}{9}e^{-t}+\frac{2}{9}e^{2t}+\frac{1}{3}te^{2t}.$$

Discussion: We compare this to the method of undetermined coefficient. General solution of the equation should be $y=y_H+Y$, where y_H is the general solution to the homogeneous equation and Y is a particular solution. The characteristic equation is $r^2-r-2=(r+1)(r-2)=0$, so $r_1=-1, r_2=2$, and $y_H=c_1e^{-t}+c_2e^{2t}$. Since 2 is a root, so the form of the particular solution is $Y=Ate^{2t}$. This discussion concludes that the solution should be of the form

$$y=c_1e^{-t}+c_2e^{2t}+Ate^{2t}$$

for some constants c_1, c_2, A. This fits well with our result.

Example 16. (Complex roots.) Solve

$$y''-2y'+10y=e^{-t}, \quad y(0)=0, \quad y'(0)=1.$$

Answer. Before we solve it, let's use the method of undetermined coefficients to find out which terms will be in the solution.

$$r^2-2r+10=0, \qquad (r-1)^2+9=0, \qquad r_{1,2}=1\pm 3i,$$

$$y_H=c_1e^t\cos 3t+c_2e^t\sin 3t, \qquad Y=Ae^{-t},$$

so the solution should have the form:

$$y=y_H+Y=c_1e^t\cos 3t+c_2e^t\sin 3t+Ae^{-t}.$$

The Laplace transform would be

$$Y(s)=c_1\frac{s-1}{(s-1)^2+3^2}+c_2\frac{3}{(s-1)^2+3^2}+A\frac{1}{s+1}=\frac{c_1(s-1)+c_2(3)}{(s-1)^2+3^2}+\frac{A}{s+1}.$$

This gives us some idea on which terms to look for in partial fraction.

Now let's use the Laplace transform:

$$Y(s)=\mathcal{L}\{y\}, \quad \mathcal{L}\{y'\}=sY-y(0)=sY,$$

$$\mathcal{L}\{y''\}=s^2Y-sy(0)-y(0)=s^2Y-1.$$

$$s^2Y(s)-1-2sY(s)+10Y(s)=\frac{1}{s+1}, \qquad \Rightarrow \qquad (s^2-2s+10)Y(s)=\frac{1}{s+1}+1=\frac{s+2}{s+1}$$

$$Y(s) = \frac{s+2}{(s+1)(s^2-2s+10)} = \frac{s+2}{(s+1)((s-1)^2+3^2)} = \frac{A}{s+1} + \frac{B(s-1)+3C}{(s-1)^2+3^2}$$

Compare the numerators:

$$s+2 = A((s-1)^2+9) + (B(s-1)+3C)(s+1).$$

Set $s=-1$: $13A = 1$, $A = \frac{1}{13}$.
Compare coefficients of s^2-term: $A + B = 0$, $B = -A = -\frac{1}{13}$.
Set any value of s, say $s = 0$, we get

$$2 = 10A - B + 3C, \qquad 3C = 2 - 10A + B = \frac{15}{13}, \qquad C = \frac{5}{13}.$$

Taking inverse transform, we get the solution

$$y(t) = Ae^{-t} + Be^t \cos 3t + Ce^t \sin 3t = \frac{1}{13}e^{-t} - \frac{1}{13}e^t \cos 3t + \frac{5}{13}e^t \sin 3t.$$

We see that this fits our prediction.

Example 17. (Pure imaginary roots.) Solve

$$y'' + y = \cos 2t, \qquad y(0) = 2, \quad y'(0) = 1.$$

Answer. Again, let's first predict the terms in the solution:

$$r^2 + 1 = 0, \qquad r_{1,2} = \pm i, \qquad y_H = c_1 \cos t + c_2 \sin t, \qquad Y = A \cos 2t$$

so

$$y = y_H + Y = c_1 \cos t + c_2 \sin t + A \cos 2t,$$

and the Laplace transform would be

$$Y(s) = c_1 \frac{s}{s^1+1} + c_2 \frac{1}{s^2+1} + A \frac{s}{s^2+4}.$$

Now, let's take Laplace transform on both sides:

$$s^2Y - 2s - 1 + Y = \mathcal{L}\{\cos 2t\} = \frac{s}{s^2+4}$$

$$(s^2+1)Y(s) = \frac{s}{s^2+4} + 2s + 1 = \frac{2s^3+s^2+9s+4}{s^2+4}$$

$$Y(s) = \frac{2s^3+s^2+9s+4}{(s^2+2^2)(s^2+1)} = \frac{As+B}{s^2+1} + \frac{Cs+D(2)}{s^2+2^2}.$$

Comparing numerators, we get

$$2s^3 + s^2 + 9s + 4 = (As+B)(s^2+4) + (Cs+2D)(s^2+1).$$

One may expand the right-hand side and compare terms $s^3, s^2, s, 1$ to find A, B, C, D, but that takes more work.

Let's try by setting s into complex numbers.

Set $s = i$, and remember the facts $i^2 = -1$ and $i^3 = -i$, we have

$$-2i - 1 + 9i + 4 = (Ai + B)(-1 + 4),$$

which gives

$$3 + 7i = 3B + 3Ai, \quad \Rightarrow \quad B = 1, \quad A = \frac{7}{3}.$$

Set now $s = 2i$:

$$-16i - 4 + 18i + 4 = (2Ci + 2D)(-3),$$

then

$$0 + 2i = -6D - 6Ci, \quad \Rightarrow \quad D = 0, \quad C = -\frac{1}{3}.$$

So

$$Y(s) = \frac{7}{3}\frac{s}{s^2+1} + \frac{1}{s^2+1} - \frac{1}{3}\frac{s}{s^2+4}$$

and

$$y(t) = \frac{7}{3}\cos t + \sin t - \frac{1}{3}\cos 2t.$$

We see that the method can be applied to higher order equations with constant coefficients. The hard part in the computation is to factorize the characteristic polynomial.

Example 18. Solve the initial value problem

$$y^{(4)} - 16y = 0, \qquad y(0) = 0, \quad y'(0) = 4, \quad y''(0) = 0, \quad y'''(0) = 0.$$

Answer. We use

$$\mathcal{L}\{y^{(4)}\} = s^4Y - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) = s^4Y - 4s^2.$$

Taking Laplace transform of the equation, we get

$$s^4Y - 4s^2 - 16Y = 0, \qquad Y(s) = \frac{4s^2}{s^4 - 16} = \frac{4s^2}{(s^2-4)(s^2+4)}.$$

By Partial fraction, we fit in

$$Y(s) = \frac{as + 2b}{s^2 - 2^2} + \frac{cs + 2d}{s^2 + 2^2} \qquad (s^2+4)(as+2b) + (s^2-4)(cs+2d) = 4s^2.$$

Set $s = 2$ and $s = -2$, we get

$$8(2a + 2b) = 4(4), \qquad 8(-2a + 2b) = 4(4), \qquad a = 0, \quad b = 1.$$

By checking the s^3-term, we get $c = -a = 0$. By checking the constant term, we get $d = b = 1$. So

$$Y(s) = \frac{2}{s^2 - 2^2} + \frac{2}{s^2 + 2^2}.$$

Take the inverse transform, we get the solution

$$y(t) = \sinh(2t) + \sin(2t).$$

A very brief review on partial fraction, targeted towards inverse Laplace transform.

Goal: rewrite a fractional form $\frac{P_n(s)}{P_m(s)}$ (where P_n is a polynomial of degree n) into sum of "simpler" terms. We assume $n < m$.

The type of terms appeared in the partial fraction is solely determined by the denominator $P_m(s)$. First, we factorize $P_m(s)$ and write it into product of terms of

$$(s-a), \qquad (s^2+a^2), \qquad (s-\lambda)^2+\mu^2.$$

The following table gives the terms in the partial fraction and their corresponding inverse Laplace transform.

term in $P_M(s)$	from where?	term in partial fraction	inverse L.T.
$s-a$	real root, or $g(t)=e^{at}$	$\frac{A}{s-a}$	Ae^{at}
$(s-a)^2$	double roots, or $r=a$ and $g(t)=e^{at}$	$\frac{A}{s-a}+\frac{B}{(s-a)^2}$	$Ae^{at}+Bte^{at}$
$(s-a)^3$	double roots, and $g(t)=e^{at}$	$\frac{A}{s-a}+\frac{B}{(s-a)^2}+\frac{C}{(s-a)^3}$	$Ae^{at}+Bte^{at}+\frac{C}{2}t^2e^{at}$
$s^2+\mu^2$	imaginary roots or $g(t)=\cos\mu t$ or $\sin\mu t$	$\frac{As+B\mu}{s^2+\mu^2}$	$A\cos\mu t+B\sin\mu t$
$(s-\lambda)^2+\mu^2$	complex roots, or $g(t)=e^{\lambda t}\cos\mu t$ or $e^{\lambda t}\sin\mu t$	$\frac{A(s-\lambda)+B\mu}{(s-\lambda)^2+\mu^2}$	$e^{\lambda t}(A\cos\mu t+B\sin\mu t)$

In summary, this table can be written as

$$\frac{P_n(s)}{(s-a)(s-b)^2(s-c)^3((s-\lambda)^2+\mu^2)}$$
$$= \frac{A}{s-a}+\frac{B_1}{s-b}+\frac{B_2}{(s-b)^2}+\frac{C_1}{s-c}+\frac{C_2}{(s-c)^2}+\frac{C_3}{(s-c)^3}+\frac{D_1(s-\lambda)+D_2\mu}{(s-\lambda)^2+\mu^2}.$$

Remark. As we have mentioned before, Laplace transform is basically used to deal with discontinuous (piecewise continuous) functions. The examples we have seen so far are all with continuous functions. These example serve as a way for us to get familiar with the method and the basic techniques involved. Next, we will study discontinuous functions.

6.3 Step functions

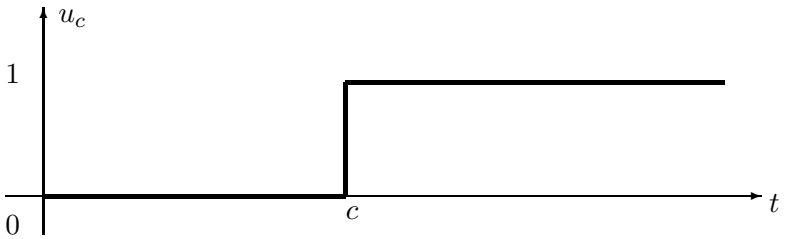
Topics:

- Definition and basic application of unit step (Heaviside) function,
- Laplace transform of step functions and functions involving step functions (piecewise continuous functions),
- Inverse transform involving step functions.

We use steps functions to form piecewise continuous functions.
Unit step function(Heaviside function):

$$u_c(t) = \begin{cases} 0, & 0 \le t < c, \\ 1, & c \le t. \end{cases}$$

for $c \ge 0$. A plot of $u_c(t)$ is below:



Note: It is common to write $u(t) = u_0(t)$ where the step occurs at $t = 0$, and $u_c(t)$ is then shifted by c units in t axis, i.e., $u_c(t) = u(t - c)$.

For a given function $f(t)$, if it is multiplied with $u_c(t)$, then

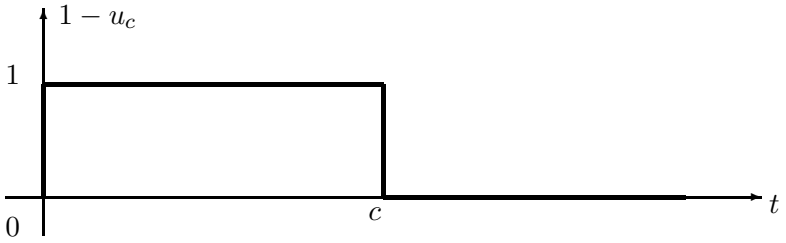
$$u_c(t)f(t) = \begin{cases} 0, & 0 < t < c, \\ f(t), & c \le t. \end{cases}$$

This says that $u_c(t)f(t)$ equals to $f(t)$ on the interval $[c, \infty)$, and is 0 everywhere else.
We say $u_c(t)$ *picks up* the interval $[c, \infty)$.

Example 1. Consider

$$1 - u_c(t) = \begin{cases} 1, & 0 \le t < c, \\ 0, & c \le t. \end{cases}$$

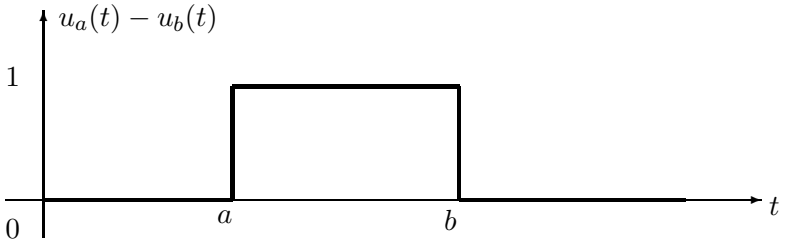
A plot of this is given below



Similarly, the function $(1-u_c(t))f(t)$ equals to $f(t)$ on the interval $[0, c)$, and 0 everywhere else.

We say that the function $(1 - u_c(t))$ picks up the interval $[0, c)$.

Example 2. Rectangular pulse. Let $0 < a < b < \infty$ and consider the function $u_a(t)-u_b(t)$. The plot of the function looks like



The function $(u_a(t)-u_b(t))f(t)$ equals to $f(t)$ on the interval $[a, b)$, and 0 everywhere else.
We say that the function $u_a(t) - u_b(t)$ picks up the interval $[a, b)$.

We can now express discontinuous functions in terms of step functions.

Example 3. Consider the function

$$g(t) = \begin{cases} t^2, & 0 \le t < a \\ 0, & a \le t \end{cases}$$

We can rewrite it in terms of the unit step function as

$$g(t) = t^2 \cdot \Big(1 - u_a(t)\Big).$$

If we have another function

$$h(t) = \begin{cases} t^2, & 0 \le t < a \\ \sin t & a \le t < b \\ 0, & b \le t \end{cases}$$

We can rewrite it in terms of the unit step function as

$$h(t) = t^2 \cdot \Big(1 - u_a(t)\Big) + \sin t \cdot (u_a(t) - u_b(t)).$$

If we add another function in it:

$$k(t) = \begin{cases} t^2, & 0 \le t < a \\ \sin t & a \le t < b \\ e^t, & b \le t \end{cases}$$

We can rewrite it in terms of the unit step function as

$$k(t) = t^2 \cdot \Big(1 - u_a(t)\Big) + \sin t \cdot (u_a(t) - u_b(t)) + e^t u_b(t).$$

It shall be apparent to us now, that the functions $t^2, \sin t, e^t$ are "dummies". We could replace them with any function, so

$$g(t) = \begin{cases} f_1(t), & 0 \le t < a \\ f_2(t) & a \le t < b \\ f_3(t), & b \le t \end{cases}$$

can be written as

$$g(t) = f_1(t) \cdot \Big(1 - u_a(t)\Big) + f_2(t) \cdot (u_a(t) - u_b(t)) + f_3(t) u_b(t).$$

Note that in the final form, each function $f_i(t)$ is multiplied by the step function that "picks up" the corresponding interval. Then we add them all up.

Example 4. One also needs to understand how to go the back way, i.e., if a function is given using step functions, we must understand its meaning. Now consider

$$f(t) = u_3(t)t^2 - u(5)(t)e^t.$$

Can you evaluate this function at various values of t, say $t = 2, 4, 6$?

Recalling the definition of $u_c(t)$, this shall be simple:

$$f(2) = u_3(2)2^2 - u(5)(2)e^2 = 0, \qquad (\text{because} \quad u_3(2) = 0, u_5(2) = 0).$$

Once this is apparent, we can speed up

$$f(4) = 4^2 = 16, \qquad f(6) = 6^2 - e^6.$$

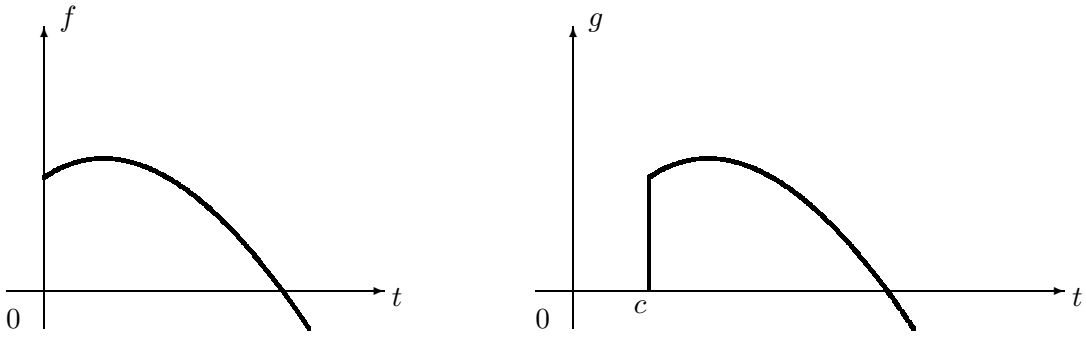
Laplace transform of $u_c(t)$: by definition

$$\mathcal{L}\{u_c(t)\} = \int_0^\infty e^{-st} u_c(t)\, dt = \int_c^\infty e^{-st} \cdot 1\, dt = \frac{e^{-st}}{-s}\bigg|_{t=c}^{\infty} = 0 - \frac{e^{-sc}}{-s} = \frac{e^{-sc}}{s}, \qquad (s > 0).$$

Shift of a function: Given $f(t)$, $t > 0$, then

$$g(t) \doteq u_c(t) \cdot f(t - c) = \begin{cases} f(t - c), & c \le t, \\ 0, & 0 \le t < c, \end{cases}$$

is the shift of f by c units. See figure below.



Let $F(s) = \mathcal{L}\{f(t)\}$ be the Laplace transform of $f(t)$. Then, the Laplace transform of $g(t)$ is

$$\mathcal{L}\{g(t)\} = \mathcal{L}\{u_c(t) \cdot f(t-c)\} = \int_0^\infty e^{-st} u_c(t) f(t-c)\, dt = \int_c^\infty e^{-st} f(t-c)\, dt.$$

Make a variable change, and let $\tau = t - c$, so $t = \tau + c$, and $dt = d\tau$, and we continue

$$\mathcal{L}\{g(t)\} = \int_0^\infty e^{-s(\tau+c)} f(\tau)\, d\tau = e^{-sc} \int_0^\infty e^{-s\tau} f(\tau)\, d\tau = e^{-cs} F(s).$$

So we conclude

$$\mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs} \mathcal{L}\{f(t)\} = e^{-cs} F(s)\,,$$

which is equivalent to

$$\mathcal{L}^{-1}\{e^{-cs} F(s)\} = u_c(t) f(t-c)\,.$$

Note now we are only considering the domain $t \geq 0$. So $u_0(t) = 1$ for all $t \geq 0$.

This is the famous **second shift Theorem**:

$$\mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs} F(s)\,.$$

Remark. Be careful that

$$\mathcal{L}\{u_c(t) f(t-c)\} \neq \mathcal{L}\{u_c(t)\} \cdot \mathcal{L}\{f(t-c)\}\,.$$

In general, the transform of the product is NOT the product of the transform:

$$\mathcal{L}\{f(t) g(t)\} \neq \mathcal{L}\{f(t)\} \cdot \mathcal{L}\{g(t)\}\,.$$

One would need to use *convolution*, which is discussed in Chapter 6.6.

In the remaining part of this Chapter, we will deal with equations with discontinuous (or impulsive) source terms. We shall be convinced by now that the key steps in the computation are taking Laplace transform and inverse Laplace transforms for discontinuous (piecewise continuous) functions, using the second shift Theorem. Once we are fluent with these two steps, we can deal with any equations!

Laplace transform with piecewise continuous functions. In following examples we will compute Laplace transform of piecewise continuous functions with the help of the unit step function and the second shift Theorem.

Example 5. Given

$$f(t) = \begin{cases} \sin t, & 0 \le t < \dfrac{\pi}{4}, \\ \sin t + \cos(t - \dfrac{\pi}{4}), & \dfrac{\pi}{4} \le t. \end{cases}$$

It can be rewritten in terms of the unit step function as

$$f(t) = \sin t + u_{\frac{\pi}{4}}(t) \cdot \cos(t - \frac{\pi}{4}).$$

(Or, if we write out each intervals

$$f(t) = \sin t(1 - u_{\frac{\pi}{4}}(t)) + \left(\sin t + \cos(t - \frac{\pi}{4})\right) u_{\frac{\pi}{4}}(t) = \sin t + u_{\frac{\pi}{4}}(t) \cdot \cos(t - \frac{\pi}{4}).$$

which gives the same answer.)

And the Laplace transform of f is

$$F(s) = \mathcal{L}\{\sin t\} + \mathcal{L}\left\{u_{\frac{\pi}{4}}(t) \cdot \cos(t - \frac{\pi}{4})\right\} = \frac{1}{s^2+1} + e^{-\frac{\pi}{4}s}\frac{s}{s^2+1}\,.$$

Example 6. Given

$$f(t) = \begin{cases} t, & 0 \le t < 1, \\ 1, & 1 \le t. \end{cases}$$

It can be rewritten in terms of the unit step function as

$$f(t) = t(1 - u_1(t)) + 1 \cdot u_1(t) = t - u_1(t) \cdot (t-1)\,.$$

The Laplace transform is

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t\} - \mathcal{L}\{u_1(t) \cdot (t-1)\} = \frac{1}{s^2} - e^{-s}\frac{1}{s^2}\,.$$

Example 7. Given

$$f(t) = \begin{cases} 0, & 0 \le t < 2, \\ t+3, & 2 \le t. \end{cases}$$

We can rewrite it in terms of the unit step function as

$$f(t) = (t+3)u_2(t) = ((t-2)+5)u_2(t) = u_2(t) \cdot (t-2) + 5u_2(t)\,.$$

The Laplace transform is

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{u_2(t) \cdot (t-2)\} + 5\mathcal{L}\{u_2(t)\} = e^{-2s}\frac{1}{s^2} + 5e^{-2s}\frac{1}{s}\,.$$

Example 8. Given

$$g(t) = \begin{cases} 1, & 0 \le t < 2, \\ t^2, & 2 \le t. \end{cases}$$

We can rewrite it in terms of the unit step function as

$$g(t) = 1 \cdot (1 - u_2(t)) + t^2 u_2(t) = 1 + (t^2 - 1)u_2(t)\,.$$

Observe that

$$t^2 - 1 = (t - 2 + 2)^2 - 1 = (t-2)^2 + 4(t-2) + 4 - 1 = (t-2)^2 + 4(t-2) + 3\,,$$

we have

$$g(t) = 1 + \big((t-2)^2 + 4(t-2) + 3\big)u_2(t)\,.$$

The Laplace transform is

$$\mathcal{L}\{g(t)\} = \frac{1}{s} + e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{3}{s}\right).$$

Example 9. Given

$$f(t) = \begin{cases} 0, & 0 \le t < 3, \\ e^t, & 3 \le t < 4, \\ 0, & 4 \le t. \end{cases}$$

We can rewrite it in terms of the unit step function as

$$f(t) = e^t\left(u_3(t) - u_4(t)\right) = u_3(t)e^{t-3}e^3 - u_4(t)e^{t-4}e^4\,.$$

The Laplace transform is

$$\mathcal{L}\{g(t)\} = e^3 e^{-3s}\frac{1}{s-1} - e^4 e^{-4s}\frac{1}{s-1} = \frac{1}{s-1}\left[e^{-3(s-1)} - e^{-4(s-1)}\right].$$

Inverse transform: We use two properties:

$$\mathcal{L}\{u_c(t)\} = e^{-cs}\frac{1}{s}, \qquad \text{and} \qquad \mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\cdot\mathcal{L}\{f(t)\}\,.$$

In the following examples we want to find $f(t) = \mathcal{L}^{-1}\{F(s)\}$.

Example 10.

$$F(s) = \frac{1 - e^{-2s}}{s^3} = \frac{1}{s^3} - e^{-2s}\frac{1}{s^3}.$$

We know that $\mathcal{L}^{-1}\{\frac{1}{s^3}\} = \frac{1}{2}t^2$, so we have

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2}t^2 - u_2(t)\frac{1}{2}(t-2)^2 = \begin{cases} \dfrac{1}{2}t^2, & 0 \le t < 2, \\ \dfrac{1}{2}t^2 - \dfrac{1}{2}(t-2)^2, & 2 \le t. \end{cases}$$

Example 11. Given

$$F(s) = \frac{e^{-3s}}{s^2+7s+12} = e^{-3s}\frac{1}{(s+4)(s+3)} = e^{-3s}\left(\frac{A}{s+4}+\frac{B}{s+3}\right).$$

By partial fraction, we find $A=-1$ and $B=1$. So

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = u_3(t)\left[Ae^{-4(t-3)}+Be^{3(t-3)}\right] = u_3(t)\left[-e^{-4(t-3)}+e^{3(t-3)}\right]$$

which can be written as a p/w continuous function

$$f(t) = \begin{cases} 0, & 0 \le t < 3, \\ -e^{-4(t-3)}+e^{3(t-3)}, & 3 \le t. \end{cases}$$

Example 12. Given

$$F(s) = \frac{se^{-s}}{s^2+4s+5} = e^{-s}\frac{s+2-2}{(s+2)^2+1} = s^{-s}\left[\frac{s+2}{(s+2)^2+1}+\frac{-2}{(s+2)^2+1}\right].$$

So

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = u_1(t)\left[e^{-2(t-1)}\cos(t-1) - 2e^{-2(t-1)}\sin(t-1)\right]$$

which can be written as a p/w continuous function

$$f(t) = \begin{cases} 0, & 0 \le t < 1, \\ e^{-2(t-1)}\left[\cos(t-1) - 2\sin(t-1)\right], & 1 \le t. \end{cases}$$

6.4 Differential equations with discontinuous forcing functions

Topics:

- Solve initial value problems with discontinuous force, examples of various cases,
- Describe behavior of solutions, and make physical sense of them.

Next we study initial value problems with discontinuous force. We will start with an example.

Example 1. (Damped system with force, complex roots) Solve the following initial value problem

$$y''+2y'+2y = g(t), \qquad g(t) = \begin{cases} 0, & 0 \le t < 1, \\ 2, & 1 \le t, \end{cases}, \qquad y(0)=1, \quad y'(0)=0\,.$$

Answer. Let $\mathcal{L}\{y(t)\} = Y(s)$, so $\mathcal{L}\{y'\} = sY-1$ and $\mathcal{L}\{y''\} = s^2Y - s$. Also we have $\mathcal{L}\{g(t)\} = 2\mathcal{L}\{u_1(t)\} = e^{-s}\frac{2}{s}$. Then

$$s^2Y - s + 2sY - 2 + 2Y = e^{-s}\frac{2}{s},$$

which gives

$$Y(s) = \frac{2e^{-s}}{s(s^2+2s+2)} + \frac{s+2}{s^2+2s+2}.$$

Note that the first term is caused by the source (forced response), and the second term is from the solution of the homogeneous equation, without source.

Now we need to find the inverse Laplace transform for $Y(s)$. We rewrite

$$Y(s) = e^{-s}\frac{2}{s((s+1)^2+1)} + \frac{(s+1)+1}{(s+1)^2+1}.$$

We have to do partial fraction first. We have

$$\frac{2}{s((s+1)^2+1)} = \frac{A}{s} + \frac{B(s+1)}{(s+1)^2+1} + \frac{C}{(s+1)^2+1}.$$

Compare the numerators on both sides:

$$2 = A((s+1)^2+1) + (B(s+1)+C)\cdot s$$

Set $s=0$, we get $A=1$.
Set $s=-1$, we get $2 = A - C$, so $C = A - 2 = -1$.
Compare s^2-term: $0 = A + B$, so $B = -A = -1$.
We now have

$$Y(s) = e^{-s}\left(\frac{1}{s} - \frac{(s+1)+1}{(s+1)^2+1}\right) + \frac{(s+1)+1}{(s+1)^2+1}.$$

We now take the inverse Laplace transform. The second term is easy, we have

$$\mathcal{L}^{-1}\left\{\frac{(s+1)+1}{(s+1)^2+1}\right\} = e^{-t}(\cos t + \sin t).$$

For the first term, we need to apply the 2nd shift Theorem because of the e^{-s} term. We get

$$y(t) = u_1(t)\left[1 - e^{-(t-1)}(\cos(t-1) + \sin(t-1))\right] + e^{-t}(\cos t + \sin t).$$

Remark: There are other ways to work out the partial fractions.

Extra question: What happens when $t \to \infty$?

Answer: We see all the terms with the exponential function will go to zero, so $y \to 1$ in the limit. We can view this system as the spring-mass system with damping. Since $g(t)$ becomes constant 1 for large t, and the particular solution (which is also the steady state) with 1 on the right hand side is 1, which provides the limit for y.

Further observation:

- We see that the solution to the homogeneous equation is

  $$e^{-t}\left[c_1\cos t + c_2 \sin t\right],$$

  and these terms do appear in the solution.

- Actually the solution consists of two part: the forced response and the homogeneous solution.

- Furthermore, the g has a discontinuity at $t = 1$, and we see a response in the solution also for $t = 1$, in the term $u_1(t)$.

Example 2. (Undamped system with force, pure imaginary roots) Solve the following initial value problem

$$y'' + 4y = g(t) = \begin{cases} 0, & 0 \le t < \pi, \\ 4, & \pi \le t < 2\pi, \\ 0, & 2\pi \le t, \end{cases} \qquad y(0) = 1, \quad y'(0) = 0\,.$$

Rewrite

$$g(t) = 4(u_\pi(t) - u_{2\pi}(t)), \qquad \mathcal{L}\{g\} = e^{-\pi s}\frac{4}{s} - e^{-2\pi s}\frac{4}{s}\,.$$

So

$$s^2Y - s + 4Y = \frac{4}{s}\left(e^{-\pi} - e^{-2\pi}\right).$$

Solve it for Y:

$$Y(s) = \left(e^{-\pi} - e^{-2\pi}\right)\frac{4}{s(s^2+4)} + \frac{s}{s^2+4} = \frac{4e^{-\pi}}{s(s^2+4)} - \frac{4e^{-2\pi}}{s(s^2+4)} + \frac{s}{s^2+4}\,.$$

Work out partial fraction

$$\frac{4}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}, \qquad A = 1, \quad B = -1, \quad C = 0.$$

So

$$\mathcal{L}^{-1}\{\frac{4}{s(s^2+4)}\} = 1 - \cos 2t\,.$$

Now we take inverse Laplace transform of Y

$$\begin{aligned} y(t) &= u_\pi(t)\,(1 - \cos 2(t-\pi)) - u_{2\pi}(t)\,(1 - \cos 2(t-2\pi)) + \cos 2t \\ &= (u_\pi(t) - u_{2\pi}(t))(1 - \cos 2t) + \cos 2t \\ &= \cos 2t + \begin{cases} 1 - \cos 2t, & \pi \le t < 2\pi, \\ 0, & \text{otherwise}, \end{cases} \\ &= \text{homogeneous solution} + \text{forced response} \end{aligned}$$

Example 3. In Example 2, let

$$g(t) = \begin{cases} 0, & 0 \le t < 4, \\ e^t, & 4 \le 5 < 2\pi, \\ 0, & 5 \le t. \end{cases}$$

Find $Y(s)$.

Answer. Rewrite

$$g(t) = e^t(u_4(t) - u_5(t)) = u_4(t)e^{t-4}e^4 - u_5(t)e^{t-5}e^5\,,$$

so

$$G(s) = \mathcal{L}\{g(t)\} = e^4e^{-4s}\frac{1}{s-1} - e^5e^{-5s}\frac{1}{s-1}\,.$$

Take Laplace transform of the equation, we get

$$(s^2+4)Y(s) = G(s) + s, \qquad Y(s) = \left(e^4e^{-4s} - e^5e^{-5s}\right)\frac{1}{(s-1)(s^2+4)} + \frac{s}{s^2+4}\,.$$

Remark: We see that the first term will give the forced response, and the second term is from the homogeneous equation.

The students may work out the inverse transform as a practice.

Example 4. (Undamped system with force, example 2 from the book p. 334)

$$y'' + 4y = g(t), \quad y(0) = 0, \quad y'(0) = 0, \quad g(t) = \begin{cases} 0, & 0 \le t < 5, \\ (t-5)/5, & 5 \le 5 < 10, \\ 1, & 10 \le t. \end{cases}$$

Let's first work on $g(t)$ and its Laplace transform

$$g(t) = \frac{t-5}{5}(u_5(t) - u_{10}(t)) + u_{10}(t) = \frac{1}{5}u_5(t)(t-5) - \frac{1}{5}u_{10}(t)(t-10),$$

$$G(s) = \mathcal{L}\{g(t)\} = \frac{1}{5}e^{-5s}\frac{1}{s^2} - \frac{1}{5}e^{-10s}\frac{1}{s^2}$$

Let $Y(s) = \mathcal{L}\{y(t)\}$, then

$$(s^2+4)Y(s) = G(s), \qquad Y(s) = \frac{G(s)}{s^2+4} = \frac{1}{5}e^{-5s}\frac{1}{s^2(s^2+4)} - \frac{1}{5}e^{-10s}\frac{1}{s^2(s^2+4)}$$

Work out the partial fraction:

$$H(s) \doteq \frac{1}{s^2(s^2+4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+2D}{s^2+4}$$

one gets $A = 0$, $B = \frac{1}{4}$, $C = 0$, $D = -\frac{1}{8}$. So

$$h(t) \doteq \mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+4)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{4}\cdot\frac{1}{s^2} - \frac{1}{8}\cdot\frac{2}{s^2+2^2}\right\} = \frac{1}{4}t - \frac{1}{8}\sin 2t.$$

Go back to $y(t)$

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y\} = \frac{1}{5}u_5(t)h(t-5) - \frac{1}{5}u_{10}(t)h(t-10) \\ &= \frac{1}{5}u_5(t)\left[\frac{1}{4}(t-5) - \frac{1}{8}\sin 2(t-5)\right] - \frac{1}{5}u_{10}(t)\left[\frac{1}{4}(t-10) - \frac{1}{8}\sin 2(t-10)\right] \\ &= \begin{cases} 0, & 0 \le t < 5, \\ \frac{1}{20}(t-5) - \frac{1}{40}\sin 2(t-5), & 5 \le 5 < 10, \\ \frac{1}{4} - \frac{1}{40}(\sin 2(t-5) - \sin 2(t-10)), & 10 \le t. \end{cases} \end{aligned}$$

Note that for $t \ge 10$, we have $y(t) = \frac{1}{4} + R \cdot \cos(2t + \delta)$ for some amplitude R and phase δ.

The plots of g and y are given in the book. Physical meaning and qualitative nature of the solution:

The source $g(t)$ is known as *ramp loading*. During the interval $0 < t < 5$, $g = 0$ and initial conditions are all 0. So solution remains 0. For large time t, $g = 1$. A particular solution is $Y = \frac{1}{4}$. Adding the homogeneous solution, we should have $y = \frac{1}{4} + c_1 \sin 2t + c_2 \cos 2t$ for t large. We see this is actually the case, the solution is an oscillation around the constant $\frac{1}{4}$ for large t.

6.5 Impulse functions

Definition of the **unit impulse function** $\delta(t)$:

$$\delta(t) = 0, \quad (t \ne 0), \qquad \int_{-\tau}^{\tau} \delta(t)\, dt = 1, \quad (\tau > 0)$$

One can think of this function as the limit of a rectangular wave with area equals to 1:

$$\delta(t) = \lim_{\tau \to 0+} \frac{1}{2\tau} \cdot (u(t+\tau) - u(t-\tau)).$$

Recall $u(t)$ is the unit step function. One can visualize this with graphs.

The impulse function can be shifted:

$$\delta(t-a) = 0, \quad (t \ne a), \qquad \int_{a-\tau}^{a+\tau} \delta(t-a)\, dt = 1, \quad (\tau > 0)$$

The most useful property of $\delta(t-a)$ in integration:

$$\int_{a-\epsilon}^{a+\epsilon} f(t)\delta(t-a)\, dt = f(a), \qquad \epsilon > 0$$

In fact, this can be easily proved, using the limit. We have

$$\begin{aligned} \int_{a-\epsilon}^{a+\epsilon} f(t)\delta(t-a)\, dt &= \lim_{\tau \to 0} \int_{a-\tau}^{a+\tau} f(t)\frac{1}{2\tau}\, dt = \lim_{\tau \to 0} \frac{1}{2\tau} \int_{a-\tau}^{a+\tau} f(t)\, dt \\ &= \lim_{\tau \to 0} \{\text{average of } f(t) \text{ on interval } [a-\tau, a+\tau]\} \\ &= f(a). \end{aligned}$$

This mean: Integrating $f(t)\delta(t-a)$ over any interval that contains $t=a$, one gets the value of f evaluated at $t=a$.

Laplace transform of the unit impulse $\delta(t-a)$ with $a>0$:

$$\mathcal{L}\{\delta(t-a)\} = \int_0^\infty e^{-st}\delta(t-a)\,dt = e^{-as}.$$

Example 1. Solve

$$y'' + 4y' + 5y = \delta(t-\pi), \qquad y(0)=0, \quad y'(0)=0.$$

Physical interpretation of the equation: Think of the spring-mass system, initially at rest. Then at time $t=\pi$, it gets a hit. BANG!

Answer. Take Laplace transform on both sides of the equation, we get

$$(s^2+4s+5)Y(s) = e^{-\pi s}$$

which gives

$$Y(s) = \frac{e^{-\pi s}}{s^2+4s+5} = e^{-\pi s}\frac{1}{(s+2)^2+1}.$$

Taking the inverse transform, we get

$$y(t) = u_\pi(t)\cdot e^{-2(t-\pi)}\sin(t-\pi) = \begin{cases} 0, & 0<t<\pi \\ e^{-2(t-\pi)}\sin(t-\pi) & t\geq\pi. \end{cases}$$

We see clearly the response to the impulsive hit at $t=\pi$!

6.6 Convolution*

We observe that we often need to find

$$\mathcal{L}^{-1}\{F(s)G(s)\}.$$

In general, we don't have $h(t)=f(t)g(t)$. We need the convolution Theorem.

Theorem (Convolution): Let

$$H(s)=F(s)G(s), \qquad f(t)=\mathcal{L}^{-1}\{F(s)\}, \quad g(t)=\mathcal{L}^{-1}\{G(s)\}, \quad h(t)=\mathcal{L}^{-1}\{H(s)\}.$$

Then

$$\mathcal{L}^{-1}\{F(s)G(s)\} = (f*g)(t) \doteq \int_0^t f(t-\tau)g(\tau)\,d\tau, \quad 0\leq t<\infty, \tag{A}$$

provided that the integral exists. The integral in (A) is known as the **convolution** of f and g.

Properties of the convolution integral:

$$\begin{aligned} f*g &= g*f && \text{(commutative law)} \\ f*(g_1+g_2) &= f*g_1+f*g_2, && \text{(distributive law)} \\ (f*g)*h &= f*(g*h) && \text{(associative law)} \\ f*(cg) &= c(f*g) && \text{(linear operator)} \\ f*0 &= 0*f=0. \end{aligned}$$

These properties could be easily proved by the definition of the convolution integral. We skip the proofs.

Watch out! In general, $(f*1)(t)\neq f(t)$. For example, let $f(t)=\cos t$, then

$$(f*1)(t)=\int_0^t \cos(t-\tau)\,d\tau = -\sin(t-\tau)\Big|_{\tau=0}^{\tau=t} = \sin t.$$

This is because

$$(f*1)(t)=\int_0^t f(t-\tau)\,d\tau \neq f(t).$$

Example 1. Calculate $h(t)=(f*g)(t)$ where $f(t)=t$ and $g(t)=e^{-t}$.

Answer. By definition, we have

$$\begin{aligned} h(t) &= (f*g)(t)=\int_0^t f(t-\tau)e^{-\tau}\,d\tau=\int_0^t (t-\tau)e^{-\tau}d\tau=\int_0^t te^{-\tau}d\tau-\int_0^t \tau e^{-\tau}d\tau \\ &= -te^{-\tau}\Big|_{\tau=0}^{t}-(-\tau-1)e^{-\tau}\Big|_{\tau=0}^{t}=t+e^{-t}-1. \end{aligned}$$

We could verify the Convolution Theorem in this example. We have

$$F(s)=\frac{1}{s^2},\quad G(s)=\frac{1}{s+1},\quad H(s)=\frac{1}{s^2}+\frac{1}{s+1}-\frac{1}{s}=\frac{1}{s^2(s+1)}=F(s)G(s).$$

Example 2. Let $g(t)=\delta(t)$ be the impulse function. Then $G(s)=1$. Following the convolution Theorem, we have

$$\mathcal{L}^{-1}\{F(s)\}=\mathcal{L}^{-1}\{F(s)G(s)\}=(f*g)(t)=\int_0^t f(t-\tau)\delta(\tau)\,d\tau=f(t),$$

which verifies the Theorem.

Example 3*. Calculate $f*g$ where

$$f(t)=\begin{cases} t, & 0\le t<1 \\ 0, & 1\le t<\infty. \end{cases} \qquad g(t)=\begin{cases} 0, & 0\le t<2 \\ 1, & 2\le t<3 \\ 0, & 3\le t<\infty. \end{cases}$$

Answer. The piecewise definition of these functions, in particular, the property of local support of these functions, provides a great opportunity to illustrate the graphical aspects of the convolution integral. The value of the function $(f * g)(t)$ depends on the value of t in a piecewise way.

If $0 \le t < 2$, the graphs of $f(t-\tau)$ and $g(\tau)$ do not have common support, therefore the product is 0, and so is $f * g$.

If $2 \le t \le 3$, the functions $f(t-\tau)$ and $g(\tau)$ are both non-zero on the interval $2 \le \tau < t$. Then $f * g$ is the area of the overlapped triangle, i.e., $(f * g)(t) = \frac{1}{2}(t-2)^2$.

If $3 \le t \le 4$, the functions $f(t-\tau)$ and $g(\tau)$ are both non-zero on the interval $t \le \tau < 4$. Then $f * g$ is the area of the overlapped trapezoid, i.e., $(f * g)(t) = \frac{1}{2} - \frac{1}{2}(t-3)^2$.

If $4 \le t < \infty$, the graphs of $f(t-\tau)$ and $g(\tau)$ do not have common support, therefore the product is 0, and so is $f * g$.

The graph of $(f * g)(t)$ is given in plot (f) in Figure 6.1.

We now prove that

$$\mathcal{L}\{f * g\} = F(s)G(s), \qquad \text{i.e.} \qquad (f * g)(t) = \mathcal{L}^{-1}\{F(s)G(s)\}.$$

Proof. We have

$$F(s) = \int_0^\infty e^{-s\xi} f(\xi)\, d\xi, \quad G(s) = \int_0^\infty e^{-s\tau} g(\tau)\, d\tau,$$

Then

$$\begin{aligned} F(s)G(s) &= \int_0^\infty e^{-s\xi} f(\xi)\, d\xi \cdot \int_0^\infty e^{-s\tau} g(\tau)\, d\tau = \int_0^\infty e^{-s\tau} g(\tau) \left[\int_0^\infty e^{-s\xi} f(\xi)\, d\xi \right] d\tau \\ &= \int_0^\infty g(\tau) \left[\int_0^\infty e^{-s\tau} e^{-s\xi} f(\xi)\, d\xi \right] d\tau = \int_0^\infty g(\tau) \left[\int_0^\infty e^{-s(\tau+\xi)} f(\xi)\, d\xi \right] d\tau. \end{aligned}$$

We now make a variable change. For fixed τ, we let $t = \xi + \tau$, so $dt = d\xi$, and $t = \tau$ when $\xi = 0$. We have

$$F(s)G(s) = \int_0^\infty g(\tau) \left[\int_\tau^\infty e^{-st} f(t-\tau)\, dt \right] d\tau = \int_0^\infty \left[\int_\tau^\infty e^{-st} f(t-\tau) g(\tau)\, dt \right] d\tau.$$

We now switch the order of integration, (add a picture), and get

$$\begin{aligned} F(s)G(s) &= \int_0^\infty \left[\int_0^t e^{-st} f(t-\tau) g(\tau)\, d\tau \right] dt = \int_0^\infty e^{-st} \left[\int_0^t f(t-\tau) g(\tau)\, d\tau \right] dt \\ &= \int_0^\infty e^{-st} (f * g)(t)\, dt = \mathcal{L}\{f * g\} \end{aligned}$$

completing the proof.

Immediately, one can extend this result to multiple convolutions. We have

$$\mathcal{L}\{f_1 * f_2 * \cdots * f_n\} = F_1(s)F_2(s)\cdots F_n(s), \qquad \mathcal{L}\{f_i\} = F_i(s).$$

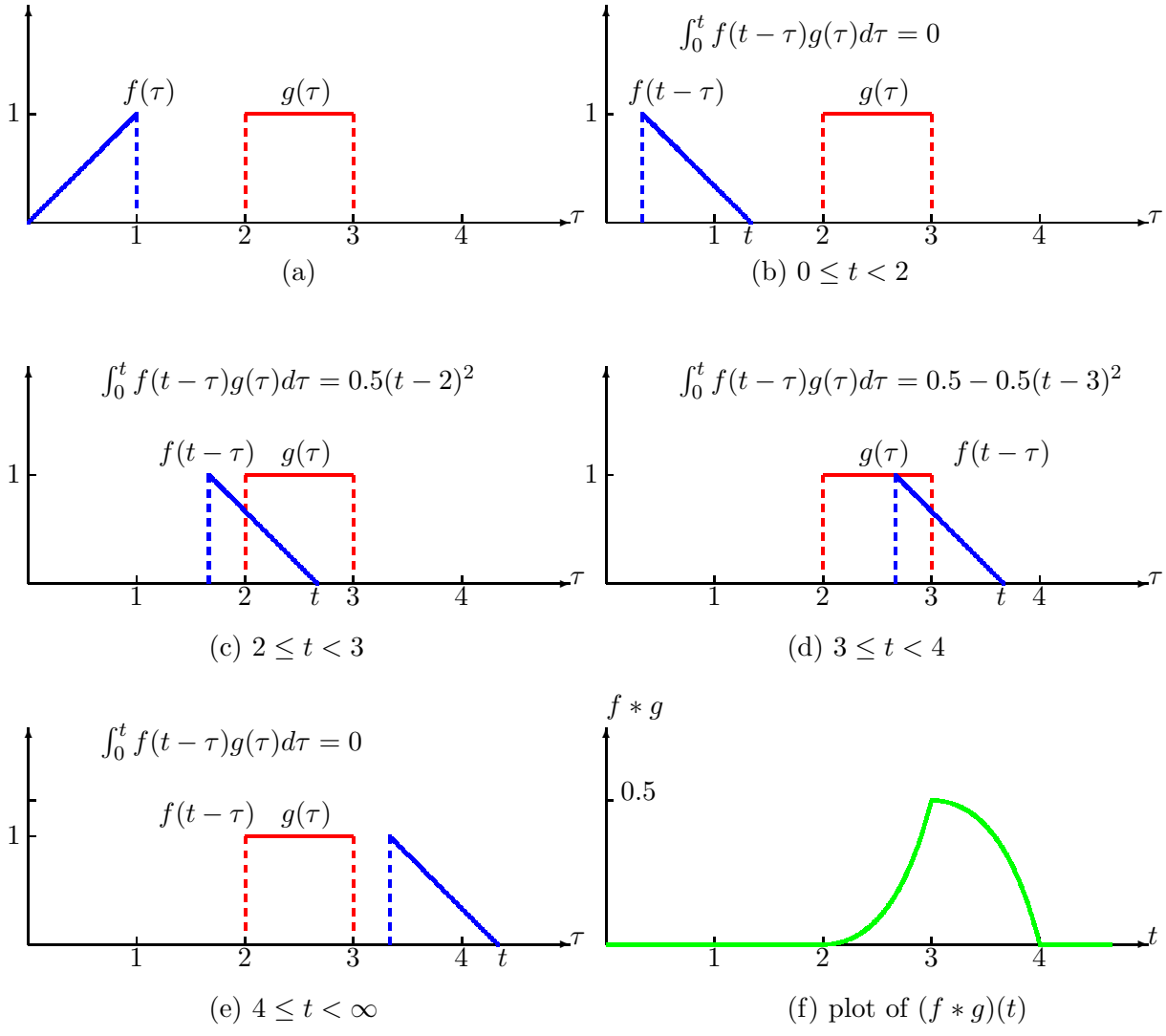


Figure 6.1: Convolution of locally supported functions.

Example Find the Laplace inverse transform of

$$H(s) = \frac{a}{s^2(s^2+a^2)}.$$

Answer. We see that we can rewrite

$$H(s) = F(s)G(s), \qquad F(s) = \frac{1}{s^2}, \quad G(s) = \frac{a^2}{s^2+a^2}$$

and

$$f(t) = t, \qquad g(t) = \sin(at).$$

By the convolution Theorem, we have

$$\mathcal{L}^{-1}\{H\} = (f*g)(t) = \int_0^t (t-\tau)\sin(a\tau)\,d\tau = \frac{at-\sin(at)}{a^2}.$$

Of course we also can do

$$\mathcal{L}^{-1}\{H\} = (g*f)(t) = \int_0^t \tau \sin a(t-\tau)\,d\tau = \frac{at-\sin(at)}{a^2},$$

which gives the same result.

Example 3. Consider the IVP

$$y' = ay + g(t), \qquad y(0) = y_0.$$

We can solve it by the method of integrating factor, where $\mu(t) = e^{-at}$. We have

$$(\mu(t)y(t))' = \mu(t)g(t), \qquad \Big(e^{-at}y(t)\Big)' = e^{-at}g(t).$$

Integrate both sides from 0 to t, we get

$$\int_0^t e^{-a\tau}y(\tau))'\,d\tau = \int_0^t e^{-a\tau}g(\tau)\,d\tau,$$

so

$$e^{-at}y(t) - y(0) = \int_0^t e^{-a\tau}g(\tau)\,d\tau.$$

So the solution is

$$\begin{aligned} y(t) &= e^{at}\Big(\int_0^t e^{-a\tau}g(\tau)\,d\tau + y_0\Big) \\ &= y_0e^{at} + \int_0^t e^{at}e^{-a\tau}g(\tau)\,d\tau \\ &= y_0e^{at} + \int_0^t e^{a(t-\tau)}g(\tau)\,d\tau \\ &= y_0e^{at} + e^{at} * g(t) \end{aligned}$$

The Laplace transform of $y(t)$, computed from this expression, is

$$Y(s) = y_0\mathcal{L}\{e^{at}\} + \mathcal{L}\{e^{at}\}\mathcal{L}\{g(t)\} = \frac{y_0}{s-a} + \frac{G(s)}{s-a}.$$

We can now compare. Taking Laplace transform directly on the differential equation, we get

$$sY(s) - y(0) = aY(s) + G(s), \qquad Y(s) = \frac{y_0 + G(s)}{s-a},$$

which is of course gives us the same result.

Example 4. Consider the second order equation

$$au'' + bu' + cu = \delta(t), \quad u(0) = 0, \quad u'(0) = 0. \tag{A}$$

Here $\delta(t)$ is the impulse function. Let $U(s) = \mathcal{L}\{u(t)\}$. Then,

$$as^2U(s) + bsU(s) + cU(s) = 1, \qquad U(s) = \frac{1}{as^2+bs+c}, \quad u(t) = \mathcal{L}^{-1}\{U(s)\}.$$

Now consider a more general equation

$$ay'' + by' + cy = g(t), \quad y(0) = y_0, \quad y'(0) = y_0'. \tag{B}$$

We have

$$(as^2 + bs + c)Y(s) - (as + b)y_0 - ay_0' = G(s)$$

so

$$Y(s) = \frac{(as+b)y_0 + ay_0'}{as^2+bs+c} + \frac{G(s)}{as^2+bs+c} \doteq \Phi(s) + \Psi(s).$$

Take inverse transform, we get

$$y(t) = \phi(t) + \psi(t), \qquad \phi(t) = \mathcal{L}^{-1}\{\Phi(s)\}, \quad \psi(t) = \mathcal{L}^{-1}\{\Psi(s)\}.$$

We see clearly now that $\phi(t)$ is the solution caused by the non-zero initial condition, and $\psi(t)$ is the system's response to the source term $g(t)$. We observe that we can write

$$\Psi(s) = U(s)G(s).$$

where $U(s)$ solution for the problem in Example 3. This function H is known as the **transfer function**, and we can express the solution using convolution

$$\psi(t) = (u * g)(t) = \int_0^t u(t-\tau)g(\tau)\,d\tau.$$

Chapter 7

Systems of Two Linear Differential Equations

7.1 Introduction to systems of differential equations

Write out the general form of a system of first order ODE, with x_1, x_2 as unknowns.

Given

$$ay'' + by' + cy = g(t), \qquad y(0) = \alpha, \quad y'(0) = \beta$$

we can do a variable change: let

$$x_1 = y, \qquad x_2 = x_1' = y'$$

then

$$\begin{cases} x_1' &= x_2 \\ x_2' &= y'' = \dfrac{1}{a}(g(t) - bx_2 - cx_1) \end{cases} \qquad \begin{cases} x_1(0) &= \alpha \\ x_2(0) &= \beta \end{cases}$$

Observation: For any 2nd order equation, we can rewrite it into a system of 2 first order equations.

Example 1. Given

$$y'' + 5y' - 10y = \sin t, \qquad y(0) = 2, \quad y'(0) = 4$$

Rewrite it into a system of first order equations: let $x_1 = y$ and $x_2 = y' = x_1'$, then

$$\begin{cases} x_1' &= x_2 \\ x_2' &= y'' = -5x_2 + 10x_1 + \sin t \end{cases} \qquad \text{I.C.'s:} \quad \begin{cases} x_1(0) &= 2 \\ x_2(0) &= 4 \end{cases}$$

We can do the same thing to any high order equations. For n-th order differential equation:

$$y^{(n)} = F(t, y, y', \cdots, y^{(n-1)})$$

define the variable change:

$$x_1 = y, \quad x_2 = y', \quad \cdots \quad x_n = y^{(n-1)}$$

we get

$$\begin{cases} x_1' &= y' = x_2 \\ x_2' &= y'' = x_3 \\ \vdots & \\ x_{n-1}' &= y^{(n-1)} = x_n \\ x_n' &= y^{(n)} = F(t, x_1, x_2, \cdots, x_n) \end{cases}$$

with corresponding source terms.

(Optional) Reversely, we can convert a 1st order system into a high order equation.

Example 2. Given

$$\begin{cases} x_1' &= 3x_1 - 2x_2 \\ x_2' &= 2x_1 - 2x_2 \end{cases} \qquad \begin{cases} x_1(0) &= 3 \\ x_2(0) &= \frac{1}{2} \end{cases}$$

Eliminate x_2: the first equation gives

$$2x_2 = 3x_1 - x_1', \qquad x_2 = \frac{3}{2}x_1 - \frac{1}{2}x_1'.$$

Plug this into second equation, we get

$$\left(\frac{3}{2}x_1 - \frac{1}{2}x_1'\right)' = 2x_1 - 2x_2 = -x_1 + x_1'$$

$$\frac{3}{2}x_1' - \frac{1}{2}x_1'' = -x_1 + x_1'$$

$$x_1'' - x_1' - 2x_1 = 0$$

with the initial conditions:

$$x_1(0) = 3, \quad x_1'(0) = 3x_1(0) - 2x_2(0) = 8.$$

This we know how to solve!

Definition of a solution: a set of functions $x_1(t), x_2(t), \cdots, x_n(t)$ that satisfy the differential equations and the initial conditions.

7.2 Review of matrices

A matrix of size $m \times n$:

$$\mathbf{A} = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & & \\ a_{m,1} & \cdots & a_{m,n} \end{pmatrix} = (a_{i,j}), \quad 1 \le i \le m,\ 1 \le j \le n.$$

We consider only square matrices, i.e., $m = n$, in particular for $n = 2$ and 3.

Basic operations: $\mathbf{A}, \mathbf{B}$ are two square matrices of size n.

- Addition: $\mathbf{A}+\mathbf{B}=(a_{ij})+(b_{ij})=(a_{ij}+b_{ij})$
- Scalar multiple: $\alpha\mathbf{A}=(\alpha\cdot a_{ij})$
- Transpose: $\mathbf{A}^T$ switch the $a_{i,j}$ with a_{ji}. $(\mathbf{A}^T)^T=\mathbf{A}$.
- Product: For $\mathbf{A}\cdot\mathbf{B}=\mathbf{C}$, it means $c_{i,j}$ is the inner product of (ith row of $\mathbf{A}$) and (jth column of $\mathbf{B}$). Example:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\cdot\begin{pmatrix} x & y \\ u & v \end{pmatrix}=\begin{pmatrix} ax+bu & ay+bv \\ cx+du & cy+dv \end{pmatrix}$$

We can express system of linear equations using matrix product.

Example 1.

$$\begin{cases} x_1-x_2+3x_3 &= 4 \\ 2x_1+5x_3 &= 0 \\ x_2-x_3 &= 7 \end{cases} \quad \text{can be expressed as:} \quad \begin{pmatrix} 1 & -1 & 3 \\ 2 & 0 & 5 \\ 0 & 1 & -1 \end{pmatrix}\cdot\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 4 \\ 0 \\ 7 \end{pmatrix}$$

Example 2.

$$\begin{cases} x_1' &= a(t)x_1+b(t)x_2+g_1(t) \\ x_2' &= c(t)x_1+d(t)x_2+g_2(t) \end{cases} \quad \Rightarrow \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}'=\begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}\cdot\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}+\begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix}$$

Some properties:

- Identity $\mathbf{I}$: $\mathbf{I}=\text{diag}(1,1,\cdots,1)$, $\mathbf{AI}=\mathbf{IA}=\mathbf{A}$.
- Determinant $\det(\mathbf{A})$:

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}=ad-bc,$$

$$\det\begin{pmatrix} a & b & c \\ u & v & w \\ x & y & z \end{pmatrix}=avz+bwx+cuy-xvc-ywa-zub.$$

- Inverse $\text{inv}(\mathbf{A})=\mathbf{A}^{-1}$: $\mathbf{A}^{-1}\mathbf{A}=\mathbf{A}\mathbf{A}^{-1}=\mathbf{I}$.
- The following statements are all equivalent: (optional)
  - (1) $\mathbf{A}$ is invertible;
  - (2) $\mathbf{A}$ is non-singular;
  - (3) $\det(\mathbf{A})\neq 0$;
  - (4) row vectors in $\mathbf{A}$ are linearly independent;
  - (5) column vectors in $\mathbf{A}$ are linearly independent.
  - (6) All eigenvalues of $\mathbf{A}$ are non-zero.

7.3 Eigenvalues and eigenvectors

Eigenvalues and eigenvectors of A (only for **A** as a 2×2 real matrix)

λ: scalar value, $\vec{v}$: column vector, $\vec{v} \not\equiv 0$.

Definition: If $\mathbf{A}\vec{v} = \lambda\vec{v}$, then $(\lambda, \vec{v})$ is the (eigenvalue, eigenvector) of **A**.

They are also called the eigen-pair of **A**.

Remark: If $\vec{v}$ is an eigenvector, then $\alpha\vec{v}$ for any $\alpha \neq 0$ is also an eigenvector, because

$$\mathbf{A}(\alpha\vec{v}) = \alpha\mathbf{A}\vec{v} = \alpha\lambda\vec{v} = \lambda(\alpha\vec{v}).$$

How to find $(\lambda, \vec{v})$?

$$\mathbf{A}\vec{v} - \lambda\vec{v} = \vec{0}, \qquad (\mathbf{A} - \lambda I)\vec{v} = \vec{0}, \qquad \det(\mathbf{A} - \lambda I) = 0.$$

We see that $\det(\mathbf{A} - \lambda I)$ is a polynomial of degree 2 (if **A** is 2×2) in λ, and it is also called the characteristic polynomial of **A**. We need to find its roots.

Example 1: Find the eigenvalues and the eigenvectors of A where

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}.$$

Answer. Let's first find the eigenvalues.

$$\det(\mathbf{A} - \lambda I) = \det \begin{pmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 4 = 0, \qquad \lambda_1 = -1,\ \lambda_2 = 3.$$

Now, let's find the eigenvector $\vec{v}_1$ for $\lambda_1 = -1$: let $\vec{v}_1 = (a, b)^T$

$$(\mathbf{A} - \lambda_1 I)\vec{v}_1 = \vec{0}, \quad \Rightarrow \quad \begin{pmatrix} 1-(-1) & 1 \\ 4 & 1-(-1) \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$\Rightarrow \quad \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

so

$$2a + b = 0, \qquad \text{choose } a = 1, \text{ then we have } b = -2, \quad \Rightarrow \quad \vec{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Finally, we will compute the eigenvector $\vec{v}_2 = (c, d)^T$ for $\lambda_2 = 3$:

$$(A - \lambda_1 I)\vec{v}_2 = \vec{0}, \quad \Rightarrow \quad \begin{pmatrix} 1-3 & 1 \\ 4 & 1-3 \end{pmatrix} \cdot \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$\Rightarrow \qquad \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} \cdot \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

so

$$2c - d = 0, \qquad \text{choose} c = 1, \text{ then we have } d = 2, \qquad \Rightarrow \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Example 2. Eigenvalues can be complex numbers.

$$\mathbf{A} = \begin{pmatrix} 2 & -9 \\ 4 & 2 \end{pmatrix}.$$

Let's first find the eigenvalues.

$$\det(\mathbf{A} - \lambda I) = \det \begin{pmatrix} 2-\lambda & -9 \\ 4 & 2-\lambda \end{pmatrix} = (2-\lambda)^2 + 36 = 0, \quad \Rightarrow \quad \lambda_{1,2} = 2 \pm 6i$$

We see that $\lambda_2 = \bar{\lambda}_1$, complex conjugate. The same will happen to the eigenvectors, i.e., $\vec{v}_1 = \bar{\vec{v}}_2$. So we need to only find one. Take $\lambda_1 = 2 + 6i$, we compute $\vec{v} = (v^1, v^2)^T$:

$$(\mathbf{A} - \lambda_1 I)\vec{v} = \vec{0}, \qquad \begin{pmatrix} -i6 & -9 \\ 4 & -i6 \end{pmatrix} \cdot \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = \vec{0},$$

$$-6iv^1 - 9v^2 = 0, \qquad \text{choose } v^1 = 1, \text{ so } v^2 = -\frac{2}{3}i,$$

so

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -\frac{2}{3}i \end{pmatrix}, \qquad \vec{v}_2 = \bar{\vec{v}}_1 = \begin{pmatrix} 1 \\ \frac{2}{3}i \end{pmatrix}.$$

Example 3. Eigenvalues can be repeated, i.e., $\lambda_1 = \lambda_2$. For example, let

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}.$$

Then

$$\det(\mathbf{A} - \lambda I) = \det \begin{pmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{pmatrix} = (1-\lambda)(3-\lambda)+1 = \lambda^2 - 4\lambda + 3 + 1 = (\lambda_2)^2 = 0, \quad \Rightarrow \quad \lambda_1 = \lambda_2 = 2.$$

Now we try to find the corresponding eigenvector $\vec{v} = (v^1, v^2)$,

$$(\mathbf{A} - 2I)\vec{v} = \vec{0}, \qquad \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = \vec{0}, \qquad v^1 + v^2 = 0.$$

Therefore only one eigenvector could be found

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Example 4. But, sometimes repeated eigenvalues could have more than 1 corresponding eigenvectors. Consider the identity matrix $\mathbf{A} = \mathbf{I}$, then we obviously have $\lambda_1 = \lambda_2 = 1$. In order to find the eigenvector, we compute

$$(\mathbf{A} - \mathbf{I})\vec{v} = \vec{0}, \qquad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = \vec{0}.$$

This is automatically satisfied, without any constraint on v^1, v^2. We can then choose any two linearly independent vectors, for example

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We say that, in this case, the double eigenvalue has two linearly independent eigenvectors. Note this behavior is essentially different from that in Example 3!

7.4 Basic theory of systems of first order linear equation

General form of a system of first order equations written in matrix-vector form:

$$\vec{x}' = \mathbf{P}(t)\vec{x} + \vec{g}.$$

If $\vec{g} = 0$, it is homogeneous. We only consider this case, so

$$\vec{x}' = \mathbf{P}(t)\vec{x}.$$

Superposition: If $\vec{x}_1(t)$ and $\vec{x}_2(t)$ are two solutions of the homogeneous system, then any linear combination $c_1\vec{x}_1 + c_2\vec{x}_2$ is also a solution.

Wronskian of vector-valued functions are defined as

$$W[\vec{x}_1(t), \vec{x}_2(t), \cdots, \vec{x}_n(t)] = \det\Big(\mathbf{X}(t)\Big)$$

where $\mathbf{X}$ is a matrix whose columns are the vectors $\vec{x}_1(t), \vec{x}_2(t), \cdots, \vec{x}_n(t)$.

If $\det \mathbf{X}(t) \neq 0$, then $\big(\vec{x}_1(t), \vec{x}_2(t), \cdots, \vec{x}_n(t)\big)$ is a set of linearly independent functions.

A set of linearly independent solutions $\big(\vec{x}_1(t), \vec{x}_2(t), \cdots, \vec{x}_n(t)\big)$ is said to be a *fundamental set of solutions.*

The general solution is the linear combination of these solutions, i.e.

$$\vec{x} = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \cdots + c_n\vec{x}_n(t).$$

7.5 Homogeneous systems of two equations with constant coefficients.

We consider the following initial value problem:

$$\begin{cases} x_1' &= ax_1 + bx_2 \\ x_2' &= cx_1 + dx_2 \end{cases} \qquad \text{I.C.'s:} \quad \begin{cases} x_1(0) &= \bar{x}_1 \\ x_2(0) &= \bar{x}_2 \end{cases}$$

In matrix vector form:

$$\vec{x}' = \mathbf{A}\vec{x}, \qquad \text{where} \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \vec{x}(0) = \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Claim: If $(\lambda, \vec{v})$ is an eigen-pair for $\mathbf{A}$, then $\vec{z} = e^{\lambda t}\vec{v}$ is a solution to $\vec{x}' = \mathbf{A}\vec{x}$.

Proof.

$$\vec{z}' = (e^{\lambda t}\vec{v})' = (e^{\lambda t})'\vec{v} = \lambda e^{\lambda t}\vec{v}$$

$$\mathbf{A}\vec{z} = \mathbf{A}(e^{\lambda t}\vec{v}) = e^{\lambda t}(\mathbf{A}\vec{v}) = e^{\lambda t}\lambda\vec{v}$$

Therefore $\vec{z}' = \mathbf{A}\vec{z}$ so $\vec{z}$ is a solution.

Steps to solve the initial value problem:

- Step I: Find eigenvalues of $\mathbf{A}$: λ_1, λ_2.
- Step II: Find the corresponding eigenvectors $\vec{v}_1, \vec{v}_2$.

- Step III: Form two solutions: $\vec{z}_1 = e^{\lambda_1 t}\vec{v}_1$, $\vec{z}_2 = e^{\lambda_2 t}\vec{v}_2$.
- Step IV: Check that $\vec{z}_1, \vec{z}_2$ are linearly independent: the Wronskian

$$W(\vec{z}_1, \vec{z}_2) = \det(\vec{z}_1, \vec{z}_2) \neq 0.$$

  (This step is usually OK in our problems.)
- Step V: Form the general solution: $\vec{x} = c_1\vec{z}_1 + c_2\vec{z}_2$.
- If initial condition $\vec{x}(0)$ is given, then use it to determine c_1, c_2.

We will start with an example.

Example 1. Solve

$$\vec{x}' = \mathbf{A}\vec{x}, \qquad \mathbf{A} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}.$$

First, find out the eigenvalues of $\mathbf{A}$. By an example in 7.3, we have

$$\lambda_1 = -1, \quad \lambda_2 = 3, \quad \vec{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \qquad \vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

So the general solution is

$$\vec{x} = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2 = c_1 e^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Write it out in components:

$$\begin{cases} x_1(t) &= c_1 e^{-t} + c_2 e^{3t} \\ x_2(t) &= -2c_1 e^{-t} + 2c_2 e^{3t} \end{cases}.$$

Qualitative property of the solutions:

- What happens when $t \to \infty$?

  If $c_2 > 0$, then $x_1 \to \infty, x_2 \to \infty$.

  If $c_2 < 0$, then $x_1 \to -\infty, x_2 \to -\infty$.

  Asymptotic relation between x_1, x_2: look at $\frac{x_1}{x_2}$:

$$\frac{x_1}{x_2} = \frac{c_1 e^{-t} + c_2 e^{3t}}{-2c_1 e^{-t} + 2c_2 e^{3t}}.$$

  As $t \to \infty$, we have

$$\frac{x_1}{x_2} = \frac{c_2 e^{3t}}{2c_2 e^{3t}} = \frac{1}{2}.$$

  This means, $x_1 \to \frac{1}{2}x_2$ asymptotically.

- What happens when $t \to -\infty$?

  Looking at $\frac{x_1}{x_2}$, we see as $t \to -\infty$ we have

  $$\frac{x_1}{x_2} = \frac{c_1 e^{-t}}{-2c_1 e^{-t}} = -\frac{1}{2},$$

  which means, $x_1 \to -\frac{1}{2}x_2$ asymptotically as $t \to -\infty$.

Phase portrait. is the trajectories of various solutions in the $x_2 - x_1$ plane.

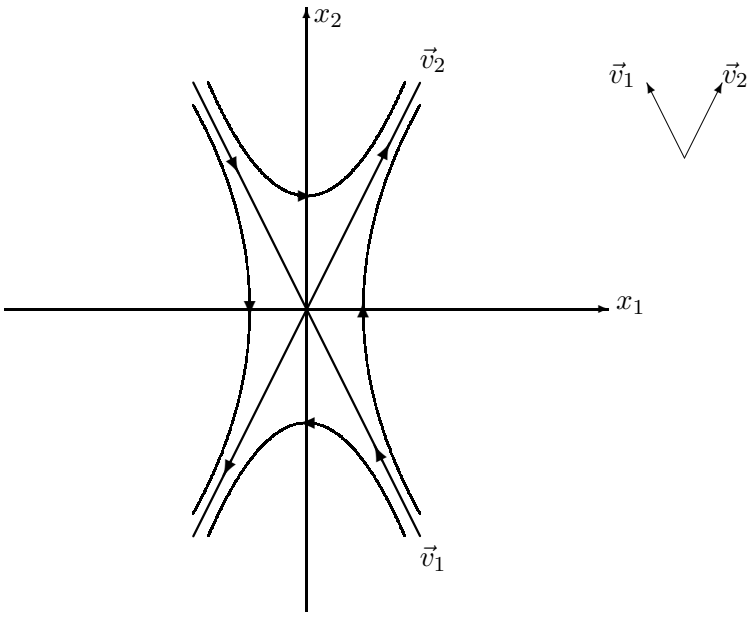
- Since $\mathbf{A}$ is non-singular, then $\vec{x} = \vec{0}$ is the only critical point such that $\vec{x}' = \mathbf{A}\vec{x} = 0$.
- If $c_1 = 0$, then

  $$\frac{x_1}{x_2} = \frac{c_2 e^{3t}}{2c_2 e^{3t}} = \frac{1}{2},$$

  so the trajectory is a straight line $x_1 = \frac{1}{2}x_2$.
  Note that this is exactly the direction of $\vec{v}_2$.
  Since $\lambda_2 = 3 > 0$, the trajectory is going away from 0.
- If $c_2 = 0$, then

  $$\frac{x_1}{x_2} = \frac{c_1 e^{-t}}{-2c_1 e^{-t}} = -\frac{1}{2},$$

  so the trajectory is another straight line $x_1 = -\frac{1}{2}x_2$.
  Note that this is exactly the direction of $\vec{v}_1$.
  Since $\lambda_2 = -1 < 0$, the trajectory is going towards 0.
- For general cases where c_1, c_2 are not 0, the trajectories should start (asymptotically) from line $x_1 = -\frac{1}{2}x_2$, and goes to line $x_1 = \frac{1}{2}x_2$ asymptotically as t grows.



Definition: If A has two real eigenvalues of opposite signs, the origin (critical point) is called a **saddle point**.

Notion of stability: (in layman's term). For solutions nearby a critical point, as time goes,
(1). If the solutions go away: then it is unstable;
(2). If the solutions approach the critical point: it is asymptotically stable;
(3). If the solution stays nearby, but not approaching the critical point: it is stable, but not asymptotically.

A saddle point is unstable.

Tips for drawing phase portrait for saddle point: only need the eigenvalues and eigenvectors!

General case: If two eigenvalues of A are $\lambda_1 < 0$ and $\lambda_2 > 0$, with two corresponding eigenvectors $\vec{v}_1, \vec{v}_2$. To draw the phase portrait, we follow these guidelines:

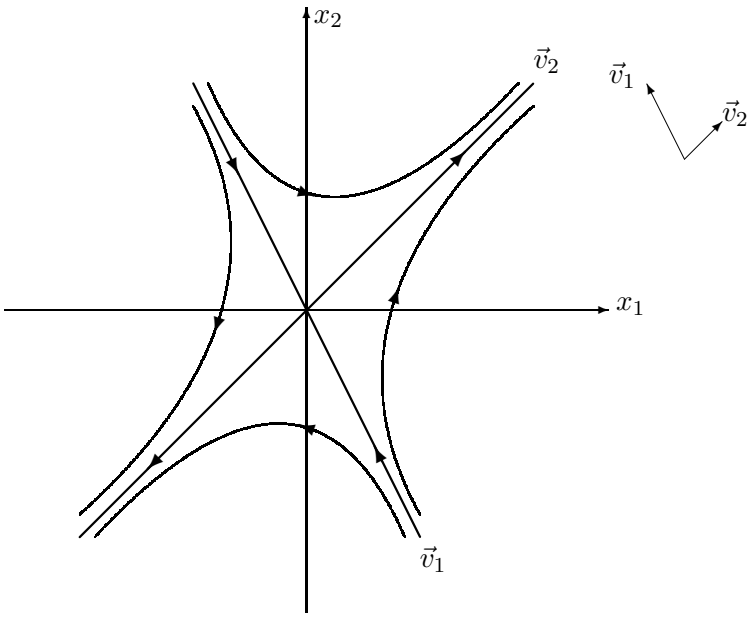
- The general solution is
$$\vec{x} = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2.$$

- If $c_1 = 0$, then the solution is $\vec{x} = c_2 e^{\lambda_2 t}\vec{v}_2$. We see that the solution vector is a scalar multiple of $\vec{v}_2$. This means a line parallel to $\vec{v}_2$ through the origin is a trajectory. Since $\lambda_2 > 0$, solutions $|\vec{x}| \to \infty$ along this line, so the arrows are pointing away from the origin.

- The similar other half: if $c_2 = 0$, then the solution is $\vec{x} = c_1 e^{\lambda_1 t}\vec{v}_1$. We see that the solution vector is a scalar multiple of $\vec{v}_1$. This means a line parallel to $\vec{v}_1$ through the origin is a trajectory. Since $\lambda_1 < 0$, solutions approach 0 along this line, so the arrows are pointing toward the origin.

- Now these two lines cut the plane into 4 regions. We need to draw at least one trajectory in each region. In the region, we have the general case, i.e., $c_1 \neq 0$ and $c_2 \neq 0$. We need to know the asymptotic behavior. We have
$$\begin{aligned} t \to \infty, \quad &=> \quad \vec{x} \to c_2 e^{\lambda_2 t}\vec{v}_2 \\ t \to -\infty, \quad &=> \quad \vec{x} \to c_1 e^{\lambda_1 t}\vec{v}_1 \end{aligned}$$

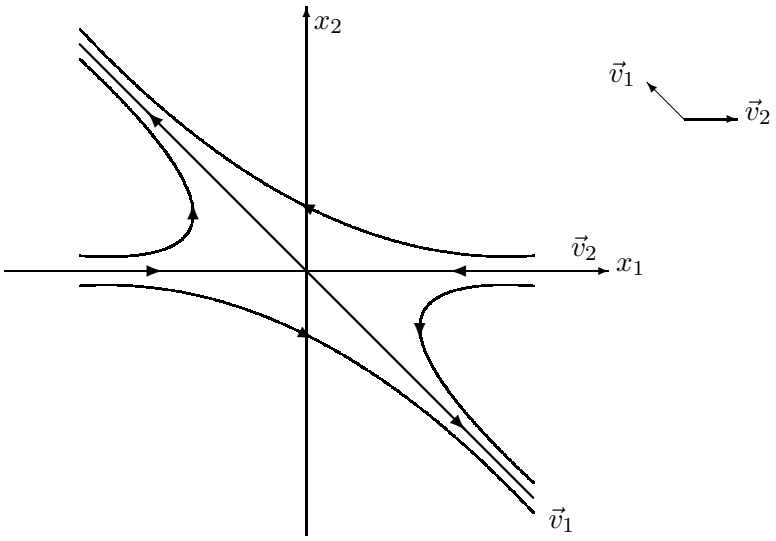
  We see these are exactly the two straight lines we just made. This means, all trajectories come from the direction of $\vec{v}_1$, and will approach $\vec{v}_2$ as t grows. See the plot below.



Example 2. Suppose we know the eigenvalues and eigenvectors of $\mathbf{A}$:

$$\lambda_1 = 3, \qquad \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad \lambda_1 = -3, \qquad \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Then the phase portrait looks like this:



If the two real distinct eigenvalue have the same sign, the situation is quite different.

Example 3. Consider the homogeneous system

$$\vec{x}' = \mathbf{A}\vec{x}, \qquad \mathbf{A} = \begin{pmatrix} -3 & 2 \\ 1 & -2 \end{pmatrix}.$$

Find the general solution and sketch the phase portrait.

Answer.

- Eigenvalues of $\mathbf{A}$:

  $$\det(\mathbf{A}-\lambda\mathbf{I}) = \det\begin{pmatrix} -3-\lambda & 2 \\ 1 & -2-\lambda \end{pmatrix} = (-3-\lambda)(-2-\lambda)-2 = \lambda^2+5\lambda+4 = (\lambda+1)(\lambda+4) = 0,$$

  So $\lambda_1 = -1, \lambda_2 = -4$. (Two eigenvalues are both negative!)

- Find the eigenvector for λ_1. Call it $\vec{v}_1 = (a,b)^T$,

  $$(\mathbf{A}-\lambda_1\mathbf{I})\vec{v}_1 = \begin{pmatrix} -3+1 & 2 \\ 1 & -2+1 \end{pmatrix}\cdot\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}\cdot\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

  This gives $a = b$. Choose it to be 1, we get $\vec{v}_1 = (1,1)^T$.

- Find the eigenvector for λ_2. Call it $\vec{v}_2 = (c,d)^T$,

  $$(\mathbf{A}-\lambda_2\mathbf{I})\vec{v}_1 = \begin{pmatrix} -3+4 & 2 \\ 1 & -2+4 \end{pmatrix}\cdot\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\cdot\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

  This gives $c+2d = 0$. Choose $d = 1$, then $c = -2$. So $\vec{v}_2 = (-2,1)^T$.

- General solution is

  $$\vec{x}(t) = c_1e^{\lambda_1 t}\vec{v}_1 + c_2e^{\lambda_2 t}\vec{v}_2 = c_1e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2e^{-4t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

  Write it out in components:

  $$\begin{cases} x_1(t) &= c_1e^{-t} - 2c_2e^{-4t} \\ x_2(t) &= c_1e^{-t} + c_2e^{-4t} \end{cases}.$$

Phase portrait:

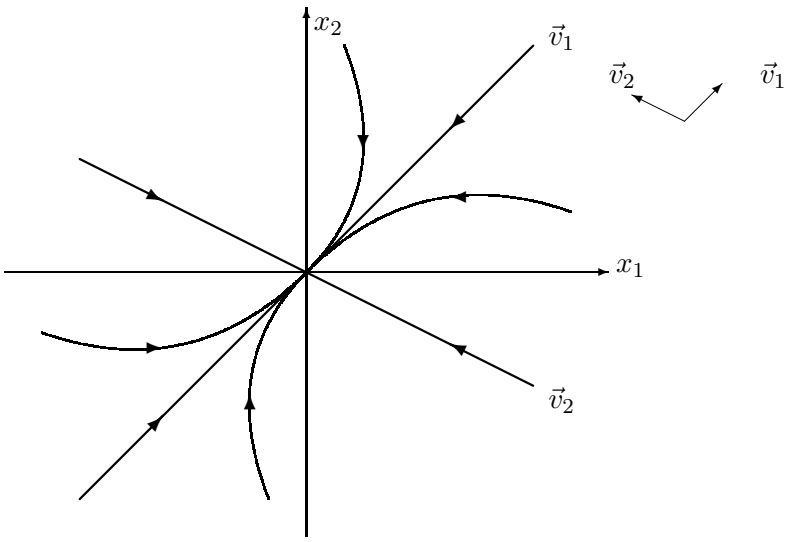
- If $c_1 = 0$, then $\vec{x} = c_2e^{\lambda_2 t}\vec{v}_2$, so the straight line through the origin in the direction of $\vec{v}_2$ is a trajectory. Since $\lambda_2 < 0$, the arrows point toward the origin.

- If $c_2 = 0$, then $\vec{x} = c_1e^{\lambda_1 t}\vec{v}_1$, so the straight line through the origin in the direction of $\vec{v}_1$ is a trajectory. Since $\lambda_1 < 0$, the arrows point toward the origin.

- For the general case, when $c_1 \neq 0$ and $c_2 \neq 0$, we have

  $$\begin{aligned} t \to -\infty, \quad &=> \quad |\vec{x}| \to \infty, \qquad \vec{x} \to c_2e^{\lambda_2 t}\vec{v}_2 \\ t \to \infty, \quad &=> \quad |\vec{x}| \to 0, \qquad \vec{x} \to c_1e^{\lambda_1 t}\vec{v}_1 \end{aligned}$$

  So all trajectories come into the picture in the direction of $\vec{v}_2$, and approach the origin in the direction of $\vec{v}_1$. See the plot below.



In the previous example, if $\lambda_1 > 0$, $\lambda_2 > 0$, say $\lambda_1 = 1$ and $\lambda_2 = 4$, and $\vec{v}_1, \vec{v}_2$ are the same, then the phase portrait will look the same, but with all arrows going away from 0.

Definition: If $\lambda_1 \neq \lambda_2$ are real with the same sign, the critical point $\vec{x} = 0$ is called a *node*.

If $\lambda_1 > 0$, $\lambda_2 > 0$, this node is called a *source*.

If $\lambda_1 < 0$, $\lambda_2 < 0$, this node is called a *sink*.

A sink is stable, and a source is unstable.

Example 4. (Source node) Suppose we know the eigenvalues and eigenvectors of $\mathbf{A}$ are

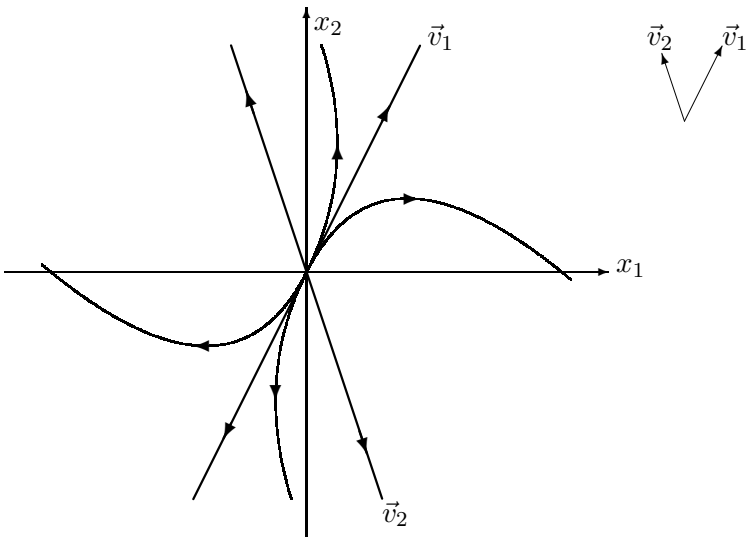
$$\lambda_1 = 3, \quad \lambda_2 = 4, \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

(1) Find the general solution for $\vec{x}' = A\vec{x}$, (2) Sketch the phase portrait.

Answer. (1) The general solution is simple, just use the formula

$$\vec{x} = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2 = c_1 e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{4t}\begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

(2) Phase portrait: Since $\lambda_2 > \lambda_1$, then the solution approach $\vec{v}_2$ as time grows. As $t \to -\infty$, $\vec{x} \to c_1 e^{\lambda_1 t}\vec{v}_1$. See the plot below.



Summary:
(1). If λ_1 and λ_2 are real and with opposite sign: the origin is a saddle point, and it's unstable;
(2). If λ_1 and λ_2 are real and with same sign: the origin is a node.
If $\lambda_1, \lambda_2 > 0$, it's a source node, and it's unstable;
If $\lambda_1, \lambda_2 < 0$, it's a sink node, and it's asymptotically stable;

7.6 Complex eigenvalues

If **A** has two complex eigenvalues, they will be a pair of complex conjugate numbers, say $\lambda_{1,2} = \alpha \pm i\beta$, $\beta \neq 0$.

The two corresponding eigenvectors will also be complex conjugate, i.e,

$$\vec{v}_1 = \bar{\vec{v}}_2.$$

We have two solutions

$$\vec{z}_1 = e^{\lambda_1 t}\vec{v}_1, \qquad \vec{z}_2 = e^{\lambda_2 t}\vec{v}_2.$$

They are complex-valued functions, and they also are complex conjugate. We seek real-valued solutions. By the principle of superposition,

$$\vec{y}_1 = \frac{1}{2}(\vec{z}_1 + \vec{z}_2) = \mathrm{Re}(\vec{z}_1), \qquad \vec{y}_2 = \frac{1}{2i}(\vec{z}_1 - \vec{z}_2) = \mathrm{Im}(\vec{z}_1)$$

are also two solutions, and they are real-valued.

One can show that they are linearly independent, so they form a set of fundamental solutions. The general solution is then $\vec{x} = c_1\vec{y}_1 + c_2\vec{y}_2$.

Now let's derive the formula for the general solution. We have two eigenvalues: λ and $\bar{\lambda}$, two eigenvectors: $\vec{v}$ and $\bar{\vec{v}}$, which we can write

$$\lambda = \alpha + i\beta, \qquad \vec{v} = \vec{v}_r + i\vec{v}_i.$$

One solution can be written

$$\begin{aligned}\vec{z} &= e^{\lambda t}\vec{v} = e^{(\alpha+i\beta)t}(\vec{v}_r + i\vec{v}_i) = e^{\alpha t}(\cos\beta t + i\sin\beta t)\cdot(\vec{v}_r + i\vec{v}_i)\\ &= e^{\alpha t}\left(\cos\beta t\cdot\vec{v}_r - \sin\beta t\cdot\vec{v}_i\right) + ie^{\alpha t}\left(\sin\beta t\cdot\vec{v}_r + \cos\beta t\cdot\vec{v}_i\right).\end{aligned}$$

The general solution is

$$\vec{x} = c_1 e^{\alpha t}\left(\cos\beta t\cdot\vec{v}_r - \sin\beta t\cdot\vec{v}_i\right) + c_2 e^{\alpha t}\left(\sin\beta t\cdot\vec{v}_r + \cos\beta t\cdot\vec{v}_i\right).$$

Notice now if $\alpha = 0$, i.e., we have pure imaginary eigenvalues. Then $\vec{x}$ is a harmonic oscillation, which is a periodic function. This means in the phase portrait all trajectories are closed curves.

Example 1. (pure imaginary eigenvalues.) Find the general solution and sketch the phase portrait of the system:

$$\vec{x}\,' = \mathbf{A}\vec{x}, \qquad \mathbf{A} = \begin{pmatrix} 0 & -4 \\ 1 & 0 \end{pmatrix}.$$

Answer. First find the eigenvalues of $\mathbf{A}$:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 + 4 = 0, \qquad \lambda_{1,2} = \pm 2i.$$

Eigenvectors: need to find one $\vec{v} = (a, b)^T$ for $\lambda = 2i$:

$$(\mathbf{A} - \lambda\mathbf{I})\vec{v} = 0, \qquad \begin{pmatrix} -2i & -4 \\ 1 & -2i \end{pmatrix}\cdot\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$a - 2ib = 0, \qquad \text{choose } b = 1, \text{ then } a = 2i,$$

then

$$\vec{v} = \begin{pmatrix} 2i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + i\begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

The general solution is

$$\vec{x} = c_1\left[\cos 2t\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \sin 2t\cdot\begin{pmatrix} 2 \\ 0 \end{pmatrix}\right] + c_2\left[\sin 2t\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \cos 2t\cdot\begin{pmatrix} 2 \\ 0 \end{pmatrix}\right].$$

Write out the components, we get

$$\begin{aligned} x_1(t) &= -2c_1\sin 2t + 2c_2\cos 2t \\ x_2(t) &= c_1\cos 2t + c_2\sin 2t. \end{aligned}$$

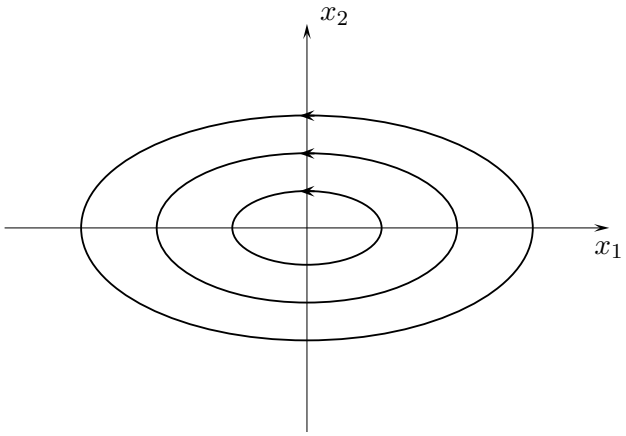
Phase portrait:

- $\vec{x}$ is a periodic function, so all trajectories are closed curves around the origin.
- They do not intersect with each other. This follows from the uniqueness of the solution.
- They are ellipses. Because we have the relation:

$$(x_1/2)^2 + (x_2)^2 = \text{ constant}.$$

- The arrows are pointing either clockwise or counter clockwise, determined by $\mathbf{A}$. In this example, take $\vec{x} = (1,0)^T$, a point on the x_1-axis. By the differential equations, we get $\vec{x}\,' = \mathbf{A}\vec{x} = (0,1)^T$, which is a vector pointing upward. So the arrows are counter-clockwise.

See plot below.



Definition. The origin in this case is called a *center*. A center is stable (b/c solutions don't blow up), but is not asymptotically stable (b/c solutions don't approach the origin as time goes).

If the complex eigenvalues have non-zero real part, the situation is still different.

Example 2. Consider the system

$$\vec{x}\,' = \mathbf{A}\vec{x}, \qquad \mathbf{A} = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}.$$

First, we compute the eigenvalues:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (3-\lambda)(-1-\lambda) + 8 = \lambda^2 - 2\lambda + 5 = 0,$$

$$\lambda_{1,2} = 1 \pm 2i, \quad \Rightarrow \quad \alpha = 1, \quad \beta = 2.$$

Eigenvectors: need to compute only one $\vec{v} = (a,b)^T$. Take $\lambda = 1 + 2i$,

$$(\mathbf{A} - \lambda\mathbf{I})\vec{v} = \begin{pmatrix} 2-2i & -2 \\ 4 & -2-2i \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$(2-2i)a - 2b = 0.$$

Choosing $a = 1$, then $b = 1 - i$, so

$$\vec{v} = \begin{pmatrix} 1 \\ 1-i \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

So the general solution is:

$$\begin{aligned} \vec{x} &= c_1 e^t \left[\cos 2t \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \sin 2t \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right] + c_2 e^t \left[\sin 2t \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \cos 2t \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right] \\ &= c_1 e^t \begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix} + c_2 e^t \begin{pmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{pmatrix}. \end{aligned}$$

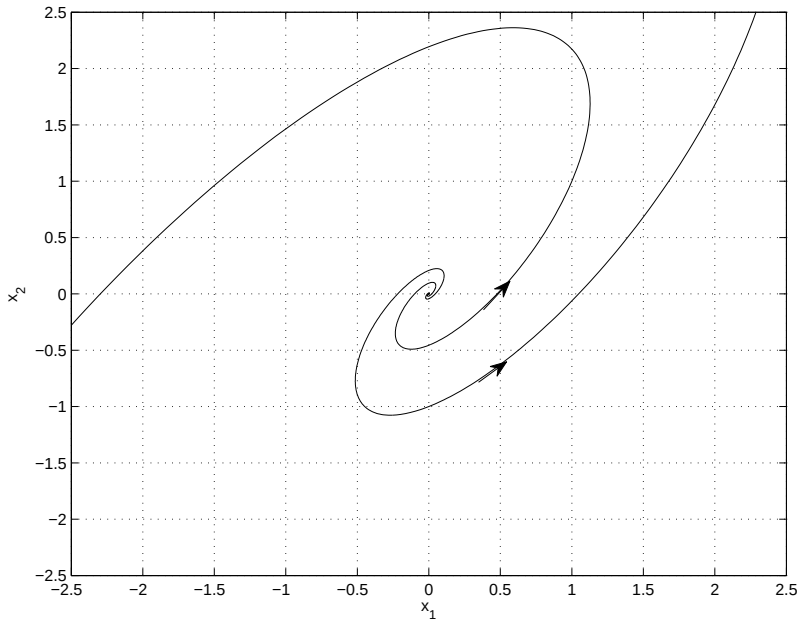
Phase portrait. Solution is growing oscillation due to the e^t. If this term is not present, (i.e., the eigenvalues would be pure imaginary), then the solutions are perfect oscillations, whose trajectory would be closed curves around origin, as the center. But with the e^t term, we will get spiral curves. Since $\alpha = 1 > 0$, all arrows are pointing away from the origin.

To determine the direction of rotation, we need to go back to the original equation and take a look at the directional field.

Consider the point $(x_1 = 1, x_2 = 0)$, then $\vec{x}' = \mathbf{A}\vec{x} = (3, 4)^T$. The arrow should point up with slope 4/3.

At the point $\vec{x} = (0, 1)^T$, we have $\vec{x}' = (-2, -1)^T$.

Therefore, the spirals are rotating counter clockwise. We don't stress on the exact shape of the spirals. See plot below.



In this case, the origin (the critical point) is called the **spiral point**. The origin in this example is an unstable critical point since $\alpha > 0$.

Remark: If $\alpha < 0$, then all arrows will go towards the origin. The origin will be a stable critical point. An example is provided in the text book. We will go through it here.

Example 3. Consider

$$\vec{x}' = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix} \vec{x}.$$

The eigenvalues and eigenvectors are:

$$\lambda_{1,2} = -\frac{1}{2} \pm i, \qquad \vec{v} = \begin{pmatrix} 1 \\ \pm i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \pm i \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Since the formula for the general solution is not so "friendly" to memorize, we use a different approach.

We know that one solution is

$$\vec{z} = e^{\lambda_1 t}\vec{v}_1 = e^{-(\frac{1}{2}+i)t}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \pm i \begin{pmatrix} 0 \\ 01 \end{pmatrix}\right].$$

This is a complex values function. We know the real part and the imaginary part are both solutions, so work them out:

$$\vec{z} = e^{-\frac{1}{2}t}\left[\cos t \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \sin t \begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \sin t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \cos t \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right].$$

The general solution is:

$$\vec{x} = c_1 e^{-\frac{1}{2}t}\left[\cos t \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \sin t \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] + c_2 e^{-\frac{1}{2}t}\left[\sin t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \cos t \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right],$$

and we can write out each component

$$\begin{aligned} x_1(t) &= e^{-\frac{1}{2}t}(c_1 \cos t + c_2 \sin t) \\ x_2(t) &= e^{-\frac{1}{2}t}(-c_1 \sin t + c_2 \cos t) \end{aligned}$$

Phase portrait: If $c_1 = 0$, we have

$$x_1^2 + x_2^2 = (e^{-\frac{1}{2}t})^2 c_2^2(\sin^2 t + \cos^2 t) = (e^{-\frac{1}{2}t})^2 c_2^2.$$

If $c_2 = 0$, we have

$$x_1^2 + x_2^2 = (e^{-\frac{1}{2}t})^2 c_1^2.$$

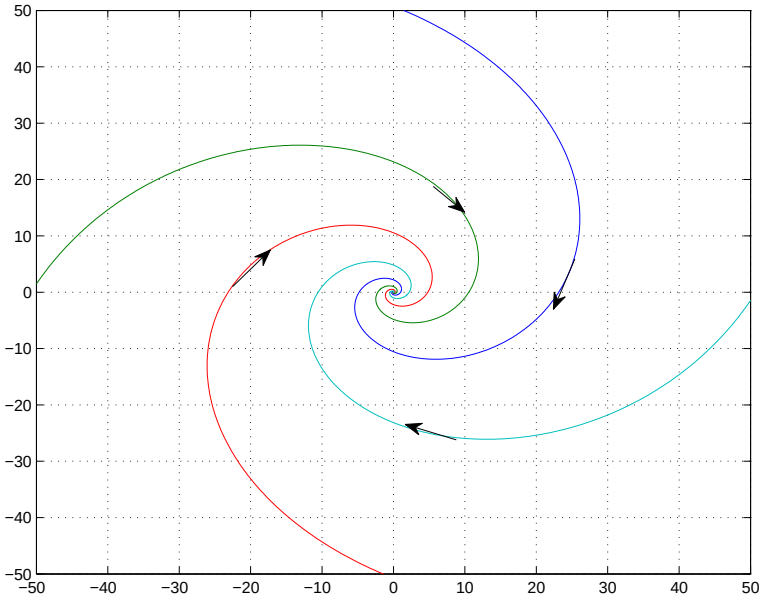
In general, if $c_1 \neq 0$ and $c_2 \neq 0$, we can show:

$$x_1^2 + x_2^2 = (e^{-\frac{1}{2}t})^2 (c_1^2 + c_1^2).$$

The trajectories will be spirals, with arrows pointing toward the origin. To determine with direction they rotate, we check a point on the x_1 axis:

$$\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \vec{x}' = A\vec{x} = \begin{pmatrix} -\frac{1}{2} \\ -1 \end{pmatrix}.$$

So the spirals rotate clockwise. And the origin is a stable equilibrium point. See the picture below.



Summary: For complex roots: $r_{1,2} = \alpha \pm i\beta$.
(i) If $\alpha = 0$: the origin is a **center**. It's stable, but not asymptotically stable.
(ii) If $\alpha > 0$: the origin is a **spiral point**. It's unstable.
(iii) If $\alpha < 0$: the origin is a **spiral point**. It's asymptotically stable.

Connections. At this point, we could make some connections between the 2nd order equations and the 2×2 system of 1st order equations. Consider a 2nd order homogeneous equation with constant coefficients

$$ay'' + by' + cy = 0. \tag{7.1}$$

The characteristic equation is

$$ar^2 + br + c = 0. \tag{7.2}$$

We know that the solutions depend mainly on the roots of the characteristic equation. We had detailed discussions in Chapter 3.

We can perform the standard variable change, and rewrite this into a system. Indeed, let

$$x_1 = y, \qquad x_2 = y'$$

we get

$$\begin{cases} x_1' &= x_2 \\ x_2' &= -\frac{b}{a}x_2 - \frac{c}{a}x_1 \end{cases} \tag{7.3}$$

In matrix-vector notation, this gives

$$\vec{x}' = \mathbf{A}\vec{x}, \qquad \mathbf{A} = \begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix} \tag{7.4}$$

We now compute the eigenvalues of $\mathbf{A}$. We have

$$\det(\mathbf{A}) = -\lambda(-\frac{b}{a} - \lambda) + \frac{c}{a} = 0, \qquad \Rightarrow \quad a\lambda^2 + b\lambda + c = 0.$$

We see that the eigenvalues are the same as the roots for the characteristic equation in (7.2).

7.7 Fundamental Matrices*

The concepts discussed here are a bit abstract. it requires some fluent knowledge on linear algebra. We will make the discussions in connection with linear homogeneous system with constant coefficients, i.e., $\vec{x}' = \mathbf{A}\vec{x}$, where $\mathbf{A}$ is an $n \times n$ matrix. (A more general result is possible, for variable coefficients $\mathbf{A}(t)$, but we will not go into that.)

Let $\vec{x}^{(i)}(t)$ for $i = 1, \cdots, n$ be a set of fundamental solutions, i.e.,

$$\frac{d}{dt}\vec{x}^{(i)}(t) = \mathbf{A}\vec{x}^{(i)}(t).$$

We define a matrix $\mathbf{\Psi}(t)$ whose ith-column is the vector $\vec{x}^{(i)}(t)$, for $i = 1, \cdots, n$. This matrix is called the **fundamental matrix** for the system. Note that the fundamental matrix is not unique.

One interesting **property of the fundamental matrix $\mathbf{\Psi}(t)$** is that it satisfies the same ODE, i.e.,

$$\mathbf{\Psi}'(t) = \mathbf{A}\mathbf{\Psi}(t).$$

This is true because each column of $\mathbf{\Psi}(t)$ satisfies the same equation,

$$\mathbf{\Psi}'(t) = \left[\frac{d}{dt}\vec{x}^{(1)}(t); \cdots; \frac{d}{dt}\vec{x}^{(n)}(t)\right] = \left[\mathbf{A}\vec{x}^{(1)}(t); \cdots; \mathbf{A}\vec{x}^{(n)}(t)\right] = \mathbf{A}\left[\vec{x}^{(1)}(t); \cdots; \vec{x}^{(n)}(t)\right] = \mathbf{A}\mathbf{\Psi}(t).$$

The general solution takes the form

$$\vec{x} = c_1\vec{x}^{(1)}(t) + \cdots + c_n\vec{x}^{(n)}(t) = \sum_{i=1}^{n} c_i\vec{x}^{(i)}(t)$$

which could be written in matrix-vector form, such as

$$\vec{x} = \mathbf{\Psi}(t)\vec{c}, \qquad \vec{c} = (c_1, \cdots, c_n)^T.$$

Now, if we are given an initial condition,

$$\vec{x}(0) = \vec{x}_0,$$

then the constant vector $\vec{c}$ could be solved:

$$\mathbf{\Psi}(0)\vec{c}, \qquad \vec{c} = \vec{x}_0, \qquad \vec{c} = \mathbf{\Psi}^{-1}(0)\vec{x}_0.$$

And the solution can be written

$$\vec{x} = \mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(0)\vec{x}_0 = \mathbf{\Phi}(t)\vec{x}_0,$$

where $\mathbf{\Phi}(t)$ is a *special fundamental matrix*, defined as

$$\mathbf{\Phi}(t) \doteq \mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(0),$$

which clearly has the property

$$\mathbf{\Phi}(0) = \mathbf{I}.$$

The Matrix exponential $\exp(\mathbf{A}t)$: Recall the scalar equation and its solution

$$x' = ax, \qquad x(0) = x_0, \qquad x(t) = x_0 e^{at}.$$

Comparing this to the solution of $\vec{x}' = \mathbf{A}\vec{x}$, i.e.,

$$\vec{x} = \mathbf{\Phi}(t)\vec{x}_0,$$

it suggests that the matrix-valued function $\mathbf{\Phi}(t)$ might have an exponential character. We will explorer this aspect.

Recall the power expansion (Taylor series) for the exponential function

$$\exp(at) = 1 + \sum_{k=1}^{\infty} \frac{a^k t^k}{k!},$$

which converges for all t.

Let's replace the value a with our matrix $\mathbf{A}$, and consider the series

$$\mathbf{I} + \sum_{k=1}^{\infty} \frac{\mathbf{A}^k t^k}{k!} = I + \mathbf{A}t + \frac{\mathbf{A}^2}{2!}t^2 + \cdots + \frac{\mathbf{A}^k t^k}{k!} + \cdots .$$

Each term is an $n \times n$ matrix. Assuming that the series converges for all t, (which could actually be proved!), then the series defines a matrix-valued function. We now define

$$\exp(\mathbf{A}t) \doteq \mathbf{I} + \sum_{k=1}^{\infty} \frac{\mathbf{A}^k t^k}{k!}.$$

Some properties of the matrix exponential function, in correspondence to regular exponential function: (1) First, we have

$$\exp(\mathbf{A}t)\Big|_{t=0} = \mathbf{I}.$$

(2) If $\mathbf{A}$ is a diagonal matrix, i.e., $\mathbf{A} = \mathbf{D} = \operatorname{diag}(d_1, \cdots, d_n)$, then each term in the series is a diagonal matrix, and we have

$$\exp(\mathbf{D}t) = \mathbf{I} + \sum_{k=1}^{\infty} \frac{\operatorname{diag}(d_1^k, \cdot, d_n^k)t^k}{k!} = \operatorname{diag}\left(1 + \sum_{k=1}^{\infty} \frac{d_1^k t^k}{k!}, \cdots, 1 + \sum_{k=1}^{\infty} \frac{d_n^k t^k}{k!}\right) = \operatorname{diag}\left(e^{d_1 t}, \cdots, e^{d_n t}\right).$$

(3) Differentiating the series:

$$\frac{d}{dt}\exp(\mathbf{A}t) = \sum_{k=1}^{\infty} \frac{d}{dt}\left(\frac{\mathbf{A}^k t^k}{k!}\right) = \sum_{k=1}^{\infty} \frac{k\mathbf{A}^k t^{k-1}}{k!} = \sum_{k=1}^{\infty} \frac{\mathbf{A}^k t^{k-1}}{(k-1)!} = \mathbf{A}\left[I + \sum_{k=1}^{\infty} \frac{\mathbf{A}^k t^k}{k!}\right] = \mathbf{A}\exp(\mathbf{A}t)$$

which gives

$$\frac{d}{dt}\exp(\mathbf{A}t) = \mathbf{A}\exp(\mathbf{A}t).$$

Recall that the fundamental matrix $\mathbf{\Phi}(t)$ satisfies

$$\mathbf{\Phi}' = \mathbf{A}\mathbf{\Phi}, \qquad \mathbf{\Phi}(0) = \mathbf{I}.$$

Thus, $\mathbf{\Phi}(t)$ and $\exp(\mathbf{A}t)$ solves the same IVP. By uniqueness, we have

$$\mathbf{\Phi}(t) = \exp(\mathbf{A}t). \tag{A1}$$

The solution can now be written

$$\vec{x} = \exp(\mathbf{A}t)\vec{x}_0.$$

Diagonalizable matrices. If $\mathbf{A}$ is not a diagonal matrix, then the system $\vec{x}' = \mathbf{A}\vec{x}$ is called *coupled*, meaning that different x_i's effect each other, and they must be solved simultaneously. This presents some difficulty. For some matrices $\mathbf{A}$, one could *decouple* it through a variable change, and make it a diagonal system (which is uncoupled, such that each equation could be solved separately.)

Assume now A has n eigenvalues λ_i and n linearly independent eigenvectors $\vec{v}^{(i)}$. We form a matrix $\mathbf{T}$ whose columns are eigenvectors,

$$\mathbf{T} \doteq \left(\vec{v}^{(1)}; \cdots ; \vec{v}^{(n)}\right).$$

Since the eigenvectors are linearly independent, the matrix $\mathbf{T}$ is non-singular and invertible.

We have

$$\mathbf{AT} = \left(\mathbf{A}\vec{v}^{(1)}; \cdots ; \mathbf{A}\vec{v}^{(n)}\right) = \left(\lambda_1\vec{v}^{(1)}; \cdots ; \lambda_n\vec{v}^{(n)}\right) = \mathbf{TD}, \tag{$*$}$$

where $\mathbf{D}$ is a diagonal matrix with the eigenvalues on the diagonal

$$\mathbf{D} = \operatorname{diag}(\lambda_1, \cdots, \lambda_n).$$

Multiply both side of (*) by $\mathbf{T}^{-1}$ on the left, we get

$$\mathbf{T}^{-1}\mathbf{AT} = \mathbf{D},$$

so we transformed $\mathbf{A}$ into a diagonal matrix. This process is called *diagonalization*, and such a matrix $\mathbf{A}$ is called *diagonalizable*.

We now consider the system of ODEs

$$\vec{x}' = \mathbf{A}\vec{x}, \qquad \vec{x}(0) = \vec{x}_0. \tag{Ix}$$

where $\mathbf{A}$ is diagonalizable such that

$$\mathbf{T}^{-1}\mathbf{AT} = \mathbf{D}, \qquad \Rightarrow \qquad \mathbf{A} = \mathbf{TDT}^{-1}.$$

The system can be written as

$$\vec{x}' = \mathbf{TDT}^{-1}\vec{x}, \qquad \Rightarrow \qquad \mathbf{T}^{-1}\vec{x}' = \mathbf{DT}^{-1}\vec{x}.$$

We now make a variable change, and let

$$\vec{y} = \mathbf{T}^{-1}\vec{x}, \qquad \Rightarrow \qquad \vec{x} = \mathbf{T}\vec{y}.$$

Then $\vec{y}$ solves the diagonal (decoupled) system

$$\vec{y}' = \mathbf{D}\vec{y}, \qquad \vec{y}(0) = \vec{y}_0 = \mathbf{T}^{-1}\vec{x}_0, \tag{Iy}$$

which could be easily solved

$$\vec{y}(t) = \exp(\mathbf{D}t)\vec{y}_0 = \operatorname{diag}\left(e^{\lambda_1 t}, \cdots, e^{\lambda_n t}\right)\vec{y}_0 = \mathbf{Q}(t)\vec{y}_0, \qquad \mathbf{Q}(t) = \exp(\mathbf{D}t),$$

where $\mathbf{Q}(t)$ is the fundamental matrix for system (Iy). The solution of $\vec{x}$ is then recovered as

$$\vec{x}(t) = \mathbf{T}\vec{y}(t) = \mathbf{T}\exp(\mathbf{D}t)\mathbf{T}^{-1}\vec{x}_0 = \mathbf{\Phi}(t)\vec{x}_0, \qquad \mathbf{\Phi}(t) = \mathbf{T}\exp(\mathbf{D}t)\mathbf{T}^{-1} \tag{$A2$}$$

where $\mathbf{\Phi}(t)$ is the corresponding fundamental matrix for system (Ix).

Remark. Note that, combining the equations (A1) and (A2), we accidentally proved that, if $\mathbf{A}$ is diagonalizable then it holds

$$\exp(\mathbf{A}t) = \mathbf{T}\exp(\mathbf{D}t)\mathbf{T}^{-1}, \qquad \text{where} \qquad \mathbf{A} = \mathbf{T}\mathbf{D}\mathbf{T}^{-1}.$$

7.8 Repeated eigenvalues

Here we study the case where the two eigenvalues are the same, say $\lambda_1 = \lambda_2 = \lambda$. This can happen, as we will see through our first example. We have two cases, depending on whether we could find 2 linearly independent eigenvectors. We will demonstrate this through examples.

Example 1. (Repeated eigenvalues, with 2 linearly independent eigenvectors) Let

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We see that the two eigenvalues and eigenvectors are

$$\lambda_1 = \lambda_2 = 1, \qquad \vec{v}^{(1)} = (1,0)^T, \qquad \vec{v}^{(2)} = (0,1)^T.$$

The general solution is

$$\vec{x}(t) = c_1\vec{z}_1 + c_2\vec{z}_2, \qquad \vec{z}_1 = e^t\vec{v}^{(1)}, \quad \vec{z}_2 = e^t\vec{v}^{(2)}.$$

Writing out, we have

$$x_1(t) = c_1 e^t, \qquad x_2(t) = c_2 e^t.$$

Note that the system is uncoupled!

We see that,

$$\frac{x_2(t)}{x_1(t)} = \frac{c_2 e^t}{c_1 e^t} = \frac{c_2}{c_1} = c$$

where c is arbitrary. Therefore, the trajectories of the solutions are straight lines through the origin in the phase plane, forming a star shape. Such a critical point is called a **proper node** or a **star node**.

Example 2. (Repeated eigenvalues, with one eigenvector) Let

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}.$$

Then

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \det \begin{pmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{pmatrix} = (1-\lambda)(3-\lambda) + 1 = \lambda^2 - 4\lambda + 3 + 1 = (\lambda - 2)^2 = 0,$$

so $\lambda_1 = \lambda_2 = 2$. And we can find only one eigenvector $\vec{v} = (a, b)^T$

$$(\mathbf{A} - \lambda\mathbf{I})\vec{v} = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \vec{0}, \quad a + b = 0.$$

Choosing $a = 1$, then $b = -1$, and we find $\vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Then, one solution is:

$$\vec{z}_1 = e^{\lambda t}\vec{v} = e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

We need to find a second solution. Let's try $\vec{z}_2 = te^{\lambda t}\vec{v}$. We have

$$\vec{z}_2' = e^{\lambda t}\vec{v} + \lambda t e^{\lambda t}\vec{v} = (1 + \lambda t)e^{\lambda t}\vec{v}$$
$$\mathbf{A}\vec{z}_2 = \mathbf{A}te^{\lambda t}\vec{v} = te^{\lambda t}(A\vec{v}) = te^{\lambda t}\lambda\vec{v} = \lambda te^{\lambda t}\vec{v}$$

If $\vec{z}_2$ is a solution, we must have

$$\vec{z}_2' = \mathbf{A}\vec{z}_2 \quad \rightarrow \quad 1 + \lambda t = \lambda t$$

which doesn't work.

Try something else: $\vec{z}_2 = te^{\lambda t}\vec{v} + \vec{\eta}e^{\lambda t}$. (here $\vec{\eta}$ is a constant vector to be determined later). Then

$$\vec{z}_2' = (1 + \lambda t)e^{\lambda t}\vec{v} + \lambda\vec{\eta}e^{\lambda t} = \lambda te^{\lambda t}\vec{v} + e^{\lambda t}(\vec{v} + \lambda\vec{\eta})$$
$$\mathbf{A}\vec{z}_2 = \lambda te^{\lambda t}\vec{v} + \mathbf{A}\vec{\eta}e^{\lambda t}.$$

Since $\vec{z}_2$ is a solution, we must have $\vec{z}_2' = \mathbf{A}\vec{z}_2$. Comparing terms, we see we must have

$$\vec{v} + \lambda\vec{\eta} = \mathbf{A}\vec{\eta}, \qquad (\mathbf{A} - \lambda\mathbf{I})\vec{\eta} = \vec{v}.$$

This is what one uses to solve for $\vec{\eta}$. Such an $\vec{\eta}$ is called a *generalized eigenvector* corresponding to the eigenvalue λ.

Back to the original problem, to compute this $\vec{\eta}$, we plug in $\mathbf{A}$ and λ, and get

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \eta_1 + \eta_2 = -1.$$

We can choose $\eta_1 = 0$, then $\eta_2 = -1$, and so $\vec{\eta} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$.

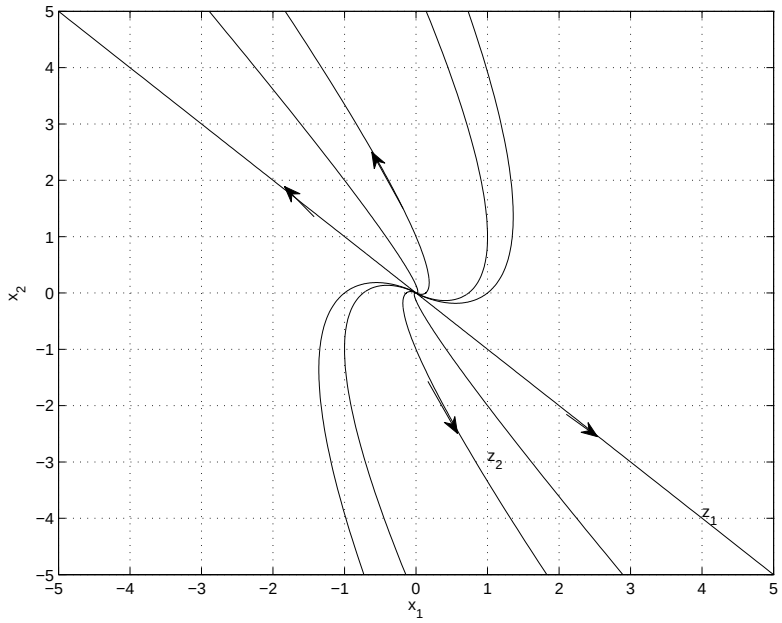
So the general solution is

$$\begin{aligned} \vec{x} &= c_1\vec{z}_1 + c_2\vec{z}_2 = c_1e^{\lambda t}\vec{v} + c_2(te^{\lambda t}\vec{v} + e^{\lambda t}\vec{\eta}) \\ &= c_1e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2\left[te^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{2t}\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right]. \end{aligned}$$

Phase portrait:

- As $t \to \infty$, we have $|\vec{x}| \to \infty$ unbounded.
- As $t \to -\infty$, we have $\vec{x} \to 0$.
- If $c_2 = 0$, then $\vec{x} = c_1 e^{\lambda t}\vec{v}$, so the line through the origin in the direction of $\vec{v}$ is a trajectory. Since $\lambda > 0$, the arrows point away from the origin.
- If $c_1 = 0$, then $\vec{x} = c_2(te^{\lambda t}\vec{v} + e^{\lambda t}\vec{\eta})$. For this solution, as $t \to \infty$, the dominant term in $\vec{x}$ is $te^{\lambda t}\vec{v}$. This means the solution approach the direction of $\vec{v}$. On the other hand, as $t \to -\infty$, the dominant term in $\vec{x}$ is still $te^{\lambda t}\vec{v}$. This means the solution approach the direction of $\vec{v}$. But, due to the change of sign of t, the $\vec{x}$ will change direction and point toward the opposite direction as when $t \to \infty$.

  How does it turn? We need to go back to the system and check the directional field. At $\vec{x} = (1, 0)$, we have $\vec{x}' = (1, 1)^T$, and at $\vec{x} = (0, 1)$, we have $\vec{x}' = (-1, 3)^T$. There it turns kind of counter clockwise. See figure below.
- For the general case, with $c_1 \neq 0$ and $c_2 \neq 0$, a similar thing happens. As $t \to \infty$, the dominant term in $\vec{x}$ is $te^{\lambda t}\vec{v}$. This means the solution approach the direction of $\vec{v}$. As $t \to -\infty$, the dominant term in $\vec{x}$ is still $te^{\lambda t}\vec{v}$. This means the solution approach the direction of $\vec{v}$. But, due to the change of sign of t, the $\vec{x}$ will change direction and point toward the opposite direction as when $t \to \infty$. See plot below.



Remark: If $\lambda < 0$, the phase portrait looks the same except with reversed arrows.

Summary: Let **A** has repeated eigenvalues.
Case (1). If there are 2 linearly independent eigenvectors, the origin is called a *proper node* or a *star point*.

Case (2). If there is only one eigenvector, then the origin is called a *improper node*. The critical point is stable if $\lambda < 0$, and unstable if $\lambda > 0$.

Recipe for solutions of $\vec{x}' = \mathbf{A}\vec{x}$ where $\mathbf{A}$ has repeated eigenvalues.

1. Find the eigenvalue λ, by $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$;
2. Find the eigenvector $\vec{v}$, by $(\mathbf{A} - \lambda\mathbf{I})\vec{v} = \vec{0}$;
3. Find the generalized eigenvector $\vec{\eta}$, by $(\mathbf{A} - \lambda\mathbf{I})\vec{\eta} = \vec{v}$;
4. Form the general solution:
$$\vec{x} = c_1 e^{\lambda t}\vec{v} + c_2(te^{\lambda t}\vec{v} + e^{\lambda t}\vec{\eta}).$$
5. Discuss stability of the critical point: Asymptotically stable if $\lambda < 0$, unstable if $\lambda > 0$.

Example 2. Find the general solution to the system $\vec{x}' = \begin{pmatrix} -2 & 2 \\ -0.5 & -4 \end{pmatrix}\vec{x}$.

We start with finding the eigenvalues:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (-2-\lambda)(-4-\lambda) + 1 = \lambda^2 + 6\lambda + 8 + 1 = (\lambda+3)^2 = 0, \qquad \lambda_1 = \lambda_2 = \lambda = -3$$

We see we have double eigenvalue. The corresponding eigenvector $\vec{v} = (a, b)^T$

$$(\mathbf{A} - \lambda\mathbf{I})\vec{v} = \begin{pmatrix} -2+3 & 2 \\ -0.5 & -4+3 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -0.5 & -1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \vec{0}$$

So we must have $a + 2b = 0$. Choose $a = 2$, then $b = -1$, and we get $\vec{v} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$. To find the generalized eigenvector $\vec{\eta}$, we solve

$$(\mathbf{A} - \lambda I)\vec{\eta} = \vec{v}, \qquad \begin{pmatrix} 1 & 2 \\ -0.5 & -1 \end{pmatrix} \cdot \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

This gives us one relation $\eta_1 + 2\eta_2 = 2$. Choose $\eta_1 = 0$, then we have $\eta_2 = 1$, and so $\vec{\eta} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The general solution is

$$\vec{x} = c_1 e^{\lambda t}\vec{v} + c_2(te^{\lambda t}\vec{v} + e^{\lambda t}\vec{\eta}) = c_1 e^{-3t}\begin{pmatrix} 2 \\ -1 \end{pmatrix} + c_2\left[te^{-3t}\begin{pmatrix} 2 \\ -1 \end{pmatrix} + e^{-3t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right].$$

The origin is an improper node which is asymptotically stable.

Fundamental matrices and Jordan Forms. (skip).

More on eigenvalues. Consider a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Define

$$p = a + d = \text{trace}(A), \qquad q = ad - bc = \det(A).$$

Then, the eigenvalues are the roots of the characteristic polynomial

$$\det(A - \lambda I) = (\lambda - a)(\lambda - b) - bc = \lambda^2 - (a + d)\lambda + ad - bc = \lambda^2 - p\lambda + q$$

Let λ_1, λ_2 be the two eigenvalues. Then we must have

$$\lambda_1 + \lambda_2 = p, \qquad \lambda_1 \lambda_2 = q. \tag{$*$}$$

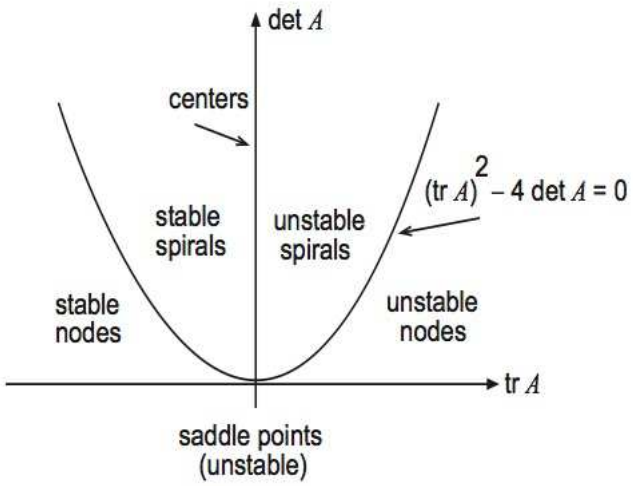
Let $\Delta = p^2 - 4q$ denote the discriminant of the characteristic polynomial. Then, the two roots can be written as

$$\lambda_{1,2} = \frac{p \pm \sqrt{\Delta}}{2}.$$

We now discuss several cases.

- If $\Delta < 0$ and $p = 0$, we have pure imaginary eigenvalues. The equilibrium is a center.
- If $\Delta < 0$ and $p > 0$, we have complex eigenvalues with positive real parts. The equilibrium is a spiral point, which is unstable.
- If $\Delta < 0$ and $p < 0$, we have complex eigenvalues with negative real parts. The equilibrium is a spiral point, which is asymptotically stable.
- If $\Delta > 0$, then the two eigenvalues are real and distinct. Using the relation (*), we see that:
  - If $q < 0$, then λ_1, λ_2 have the opposite sign, and we have a saddle point which is unstable.
  - If $q > 0$, then λ_1, λ_2 have the same sign, and we have a proper node. If $p > 0$, they are positive, so the node is unstable. Otherwise, if $p < 0$, they are negative and we have a stable node.
- Finally, if $\Delta = 0$, the eigenvalues are repeated, and we have either a proper node or an improper node. If $p > 0$, it is unstable, and if $p < 0$, it is asymptotically stable.

See graph below for an illustration.



7.9 Summary of Stabilities and types of critical points for linear systems

For the 2×2 system

$$\vec{x}' = \mathbf{A}\vec{x}$$

we see that $\vec{x} = (0, 0)$ is the only critical point if $\mathbf{A}$ is invertible.

In a more general setting: the system

$$\vec{x}' = \mathbf{A}\vec{x} - \vec{b}$$

would have a critical point at $\vec{x} = \mathbf{A}^{-1}b$.

The type and stability of the critical point is solely determined by the eigenvalues of $\mathbf{A}$. Below is a summary of what we learned in Chapter 7:

Summary of types and stabilities of critical points:

$\lambda_{1,2}$	eigenvalues	type of C.P.	stability
real	$\lambda_1 \cdot \lambda_2 < 0$	saddle point	unstable
real	$\lambda_1 > 0, \lambda_2 > 0, \lambda_1 \neq \lambda_2$	node (source)	unstable
real	$\lambda_1 < 0, \lambda_2 < 0, \lambda_1 \neq \lambda_2$	node (sink)	A.S.(asymptotically stable)
real	$\lambda_1 = \lambda_2$, 2 eigenvectors	proper node/star point	A.S. if $\lambda_1 < 0$, unstable if $\lambda_1 > 0$
real	$\lambda_1 = \lambda_2$, one eigenvector	improper node	A.S. if $\lambda_1 < 0$, unstable if $\lambda_1 > 0$
imaginary	$\lambda_{1,2} = \pm i\beta$	center	stable but not asymptotically
complex	$\lambda_{1,2} = \alpha \pm i\beta$	spiral point	A.S. if $\alpha < 0$, unstable if $\alpha > 0$

As long stability is concerned, the sole factor is the sign of the real part of the eigenvalues. If any of eigenvalue shall have a positive real part, the it is unstable.

9.2: Autonomous systems and their critical points

Let $x(t), y(t)$ be the unknowns, we consider the system

$$\begin{cases} x'(t) &= F(x,y) \\ y'(t) &= G(x,y) \end{cases} \qquad \begin{cases} x(t_0) &= x_0, \\ y(t_0) &= y_0. \end{cases}$$

for some functions $F(x,y), G(x,y)$ that do not depend on t. Such a system is called autonomous. Typical examples are in population dynamics, which we will see in our examples.

Using matrix-vector form, one could also write an autonomous system as

$$\vec{x}'(t) = \vec{F}(\vec{x}), \qquad \vec{x}(t_0) = \vec{x}_0.$$

A critical point is a point such that the righthand-side is 0, i.e.,

$$F(x,y) = 0, \quad G(x,y) = 0$$

or in the vector notation

$$\vec{F}(\vec{x}) = 0.$$

Note that, since now the functions are non-linear, there could be multiple critical points. Finding zeros for a nonlinear vector-valued function could be a non-trivial task.

We first go through some examples on how to find the critical points.

Example 1. Find all critical points for

$$\begin{cases} x'(t) &= -(x-y)(1-x-y) \\ y'(t) &= x(2+y) \end{cases}$$

Answer. We see that the righthand-sides are already in factorized form, which makes our task easier. We must now require

$$x = y, \qquad \text{or} \qquad x - y = 1$$

and

$$x = 0 \qquad \text{or} \qquad y = -2.$$

We see that we have 4 combinations.

$$(1) \quad \begin{cases} x = y \\ x = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} x = 0 \\ y = 0 \end{cases}$$

$$(2) \quad \begin{cases} x = y \\ y = -2 \end{cases} \quad \Rightarrow \quad \begin{cases} x = -2 \\ y = -2 \end{cases}$$

$$(3) \quad \begin{cases} x + y = 1 \\ x = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} x = 0 \\ y = 1 \end{cases}$$

$$(4) \quad \begin{cases} x + y = 1 \\ y = -2 \end{cases} \quad \Rightarrow \quad \begin{cases} x = 3 \\ y = -2 \end{cases}$$

General strategy.
(1) Factorize the righthand as much as you can.
(2) Find the conditions for each equation.
(3) Make all combinations and solve.

Example 2. Find all critical points for

$$\begin{cases} x'(t) &= xy - 6x \\ y'(t) &= xy - 2x + y - 2 \end{cases}$$

Answer. The righthand-sides are not factorized, so we need to do it first. We get

$$\begin{cases} x'(t) &= x(y-6) \\ y'(t) &= (xy-2x)+(y-2) = x(y-2)+(y-2) = (x+1)(y-2) \end{cases}$$

The conditions are:

$$x = 0 \qquad \text{or} \qquad y = 6$$

and

$$x = -1 \qquad \text{or} \qquad y = 2.$$

In principle we could make 4 combinations, but only 2 of them would give us a critical points. We end up with 2 critical points:

$$(x, y) = (0, 2), \quad (x, y) = (-1, 6).$$

Example 3. Find all critical points for

$$\begin{cases} x'(t) &= x^2 - xy \\ y'(t) &= xy - 3x + 2 \end{cases}$$

Answer. We factorize first, and get

$$\begin{cases} x'(t) &= x(x-y) \\ y'(t) &= xy - 3x + 2 \end{cases}$$

The conditions are

$$x = 0 \qquad \text{or} \qquad x = y$$

and

$$xy - 3x + 2 = 0.$$

We have 2 combinations. One is

$$x = 0 \qquad \text{and} \qquad xy - 3x + 2 = 0,$$

which gives no answer. The second combination is

$$x = y \qquad \text{and} \qquad xy - 3x + 2 = 0,$$

which gives

$$x^2 - 3x + 2 = 0, \quad (x-1)(x-2) = 0, \quad x = 1 \text{ or } x = 2.$$

This gives us two critical points

$$(x, y) = (1, 1), \qquad (x, y) = (2, 2).$$

Example 4. (Competing species) Let $x(t), y(t)$ be the population densities of two species living in a common habitat, using the same natural resource. They are not in a prey-predator relation. They simply compete with each other for the resources. The model is

$$\begin{cases} x'(t) &= x(M - ax - by) \\ y'(t) &= y(N - cx - dy) \end{cases}$$

Here a, b, c, d, M, N are positive constants, with physical meanings.

If species x lives alone, i.e., $y = 0$, we get $x' = x(M - ax)$. The growth rate is $M - ax$. At $x = M/a$, the rate will be 0. So one can view M/a as the max population density of x that the habitat could support in absence of y. Adding y, the rate of growth would decrease.

A similar argument holds for the 2nd equation, where $y = N/d$ would be the max density of y in absence of x.

We search for critical points. They must satisfies

$$x = 0 \qquad \text{or} \qquad M - ax - by = 0$$

and

$$y = 0 \qquad \text{or} \qquad N - cx - dy = 0.$$

We now have 4 combinations:

$$\begin{array}{ll} (1) & (x, y) = (0, 0) \\ (2) & x = 0 \text{ and } N - cx - dy = 0, \qquad \to \ (x, y) = (0, N/d) \\ (3) & M - ax - by = 0 \text{ and } y = 0, \qquad \to \ (x, y) = (M/a, 0) \\ (4) & M - ax - by = 0 \text{ and } N - cx - dy = 0. \end{array}$$

In case (4) we need to solve a linear equation for (x, y).

Example 5. Prey-predator model of Lotka-Volterra. Let $x(t)$ be the population density of a prey which lives together with a predator, represented by $y(t)$ as its population density. The predator lives on eating the prey, while the prey lives on natural resource which we assume is abundant. We have the model

$$\begin{cases} x'(t) &= x(a - by) \\ y'(t) &= y(-c + dx) \end{cases}$$

Physical meaning of the model:
(1) If there is no predator, i.e, $y = 0$, then $x' = at$, so the prey grows exponentially with rate a. The existence of the predator has a negative effect on the growth rate of the prey (represented by the term $-by$). (2) If there is no prey, then, $y' = -cy$, the predator decays exponentially

(dies out). The existence of the prey has a positive effect on the growth rate of the predator (represented by the term dx).

Find the critical points: We have the conditions

$$x = 0 \qquad \text{or} \qquad a - by = 0$$

and

$$y = 0 \qquad \text{or} \qquad -c + dx = 0.$$

Only 2 out of the 4 combinations give the critical point. The 2 critical points are

$$(x, y) = (0, 0), \qquad (x, y) = (d/c, b/a).$$

9.3: Stability of Critical points; local linearization

Moral: In a small neighborhood of the critical point, the nonlinear system behaves in a similar way to a linearized system.

Linearization. Consider a scalar function $f(x)$, and let x_0 be a root such that $f(x_0) = 0$. Assuming that f is smooth near $x = x_0$, we see that we can approximate f with a straight line through x_0, with the slope equals to $f'(x_0)$. Draw a graph to see the idea. Then, we have

$$f(x) \approx f'(x_0)(x - x_0)$$

in a small neighborhood of $x = x_0$.

The idea can be extended to vector valued functions. Here, the derivative of the vector-valued function, however, takes a more complicated form. Using our notation, consider the system

$$\left\{ \begin{array}{rcl} x'(t) & = & F(x, y) \\ y'(t) & = & G(x, y) \end{array} \right. \tag{A}$$

Let (x_o, y_o) be a critical point such that $F(x_o, y_o) = 0, G(x_o, y_o) = 0$.

We introduce the concept of the **Jacobian Matrix**, defined as

$$\mathbf{J}(x, y) \doteq \begin{pmatrix} F_x(x, y) & F_y(x, y) \\ G_x(x, y) & G_y(x, y) \end{pmatrix}$$

This matrix serves as the derivative of the vector-valued function on the RHS of the system.

We say that we linearize the system (A) at the point (x_o, y_o) as

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \mathbf{J}(x_o, y_o) \begin{pmatrix} x - x_o \\ y - y_o \end{pmatrix}$$

The type and stability of the critical point (x_o, y_o) is determined by the eigenvalues of the Jacobian matrix $J(x_o, y_o)$, evaluated at the critical point (x_o, y_o).

Example 1. We revisit Example 1 previous chapter,

$$\left\{ \begin{array}{rcl} x'(t) & = & -(x - y)(1 - x - y) \\ y'(t) & = & x(2 + y) \end{array} \right.$$

where the 4 critical points are

$$(0,0), \qquad (-2,-2), \qquad (0,1), \qquad (3,-2).$$

We now determine their type and stability. We first compute the Jacobian matrix. We have

$$F_x = -1+2x, \qquad F_y = 1-2y, \qquad G_x = 2+y, \qquad G_y = x$$

so

$$\mathbf{J}(x,y) = \begin{pmatrix} -1+2x & 1-2y \\ 2+y & x \end{pmatrix}.$$

At $(0,0)$, we have

$$\mathbf{J}(0,0) = \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix}, \qquad \lambda_1 = 1,\ \lambda_2 = -2, \quad \text{saddle point, unstable.}$$

At $(-2,-2)$, we have

$$\mathbf{J}(-2,-2) = \begin{pmatrix} -5 & 5 \\ 0 & -2 \end{pmatrix}, \qquad \lambda_1 = -5,\ \lambda_2 = -2, \quad \text{nodal sink, asymp. stable.}$$

At $(0,1)$, we have

$$\mathbf{J}(0,1) = \begin{pmatrix} -1 & 1 \\ 3 & 0 \end{pmatrix}, \qquad \lambda^2 + \lambda + 3 = 0.$$

The eigenvalues are complex with negative real parts. This is a spiral point. It is asymptotically stable.

At $(3,-2)$, we have

$$\mathbf{J}(3,-2) = \begin{pmatrix} 5 & 5 \\ 0 & 3 \end{pmatrix}, \qquad \lambda_1 = 5,\ \lambda_2 = 3, \quad \text{nodal source, unstable.}$$

Example 2. Consider

$$\begin{cases} x'(t) &= xy - 6x \\ y'(t) &= xy - 2x + y - 2 \end{cases}$$

whose critical points are

$$(0,2), \qquad (-1,6).$$

To check their type and stability, we compute the Jacobian matrix

$$\mathbf{J}(x,y) = \begin{pmatrix} y-6 & x \\ y-2 & x+1 \end{pmatrix}.$$

At $(0,2)$, we have

$$\mathbf{J}(0,2) = \begin{pmatrix} -4 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \lambda_1 = -4,\ \lambda_2 = 1, \quad \text{saddle point, unstable.}$$

At $(0,2)$, we have

$$\mathbf{J}(-1,6) = \begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix}, \qquad \lambda_{1,2} = \pm 2i, \quad \text{center, stable but not asymp..}$$

Example 3. We now consider again the prey-predator model, and set in values for the constants. We consider[1]

$$\begin{cases} x'(t) &= x(10-5y) \\ y'(t) &= y(-6+x) \end{cases}$$

which has 2 critical points $(0,0)$ and $(6,2)$. The Jacobian matrix is

$$\mathbf{J}(x,y) = \begin{pmatrix} 10-5y & -5x \\ y & -6+x \end{pmatrix}.$$

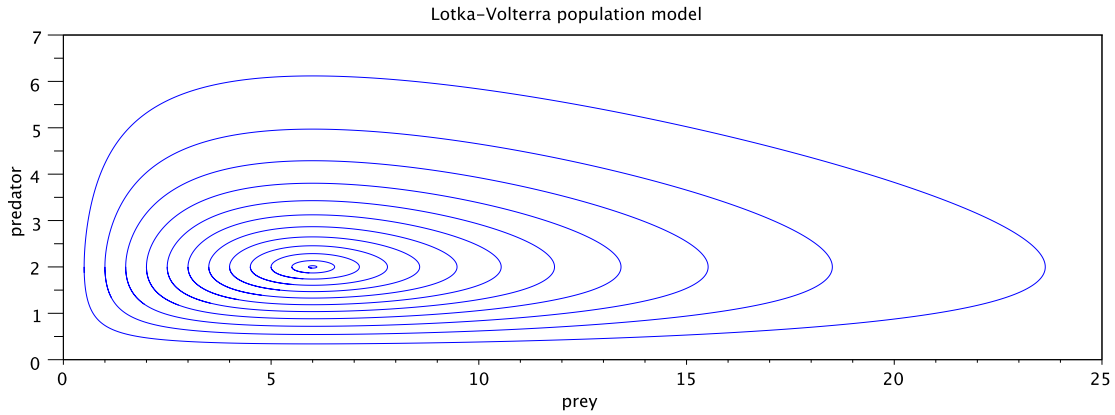
At $(0,0)$ we have

$$\mathbf{J}(0,0) = \begin{pmatrix} 10 & 0 \\ 0 & -6 \end{pmatrix}, \quad \lambda_1 = 10, \ \lambda_2 = -6, \quad \text{saddle point, unstable.}$$

At $(6,2)$ we have

$$\mathbf{J}(6,2) = \begin{pmatrix} 0 & -30 \\ 2 & 0 \end{pmatrix}, \quad \lambda_{1,2} = \pm i\sqrt{60}, \quad \text{center, stable but not asymp..}$$

To see more detailed behavior of the model, we compute the two eigenvector for $\mathbf{J}(0,0)$, and get $\vec{v}_1 = (1,0)$ and $\vec{v}_2 = (0,1)$. We sketch the trajectories of solution in (x_1, x_2)-plane in the next plot, where the trajectories rotate around the center counter clock wise.



One can interpret these as "circles of life".

In particular, the big circles can be interpreted as: When there are very little predators, the prey grows exponentially, very quickly. As the population of the prey becomes very large, there is a lot of food for the prey, and this triggers an sudden growth of the predator. As

[1] We remark that in real models the coefficients are much smaller than the ones we use here. We choose these numbers, so we can have friendly numbers in the computation. One can also view this as a rescaled model where one stretches the time variable.

the predators increase their numbers, the prey population shrinks, until there is very little prey left. Then, the predators starve, and its population decays exponentially (dies out). The circle continuous in a periodic way, forever!

Example 4*. As a final example, we consider the model of two **competing species**. Suppose that in some enclosed environment there are two species that are competing for natural resource, and let $x(t)$ and $y(t)$ be their populations at time t. Assume that, if living alone, each species grow following the logistic equation.

$$\begin{aligned} x'(t) &= x(a_1 - b_1 x) \\ y'(t) &= y(a_2 - b_2 y) \end{aligned}$$

where (a_1, a_2) are the growth rates and $(a_1/b_1, a_2/b_2)$ are the habitat's capacities, for x and y, respectively. When they are competing, then each species will have a negative effect on the other species. Thus, the system is modified into

$$\begin{aligned} x'(t) &= x(a_1 - b_1 x - c_1 y) \\ y'(t) &= y(a_2 - b_2 y - c_2 x) \end{aligned}$$

Here all the coefficients are positive constants.

The Jacobian matrix is

$$\mathbf{J}(x, y) = \begin{pmatrix} a_1 - 2b_1 x - c_1 y & -c_1 x \\ -c_2 y & a_2 - 2b_2 y - c_2 x \end{pmatrix}.$$

We now find all the critical points. We have

$$x = 0, \qquad \text{or} \qquad a_1 - b_1 x - c_1 y = 0$$

and

$$y = 0, \qquad \text{or} \qquad a_2 - b_2 y - c_2 x = 0.$$

We have 4 critical points. The first three are

$$(x_1, y_1) = (0, 0), \quad (x_2, y_2) = \left(0, \frac{a_2}{b_2}\right), \quad (x_3, y_3) = \left(\frac{a_1}{b_1}, 0\right),$$

and the last one is the solution of the system

$$a_1 = b_1 x + c_1 y, \qquad a_2 = b_2 y + c_2 x.$$

Depending on the constants, it might or might not have a solution in the first quadrant where $x > 0, y > 0$. Assuming that $b_1 b_2 - c_1 c_2 \neq 0$, the 4th critical point is (x_4, y_4) where

$$x_4 = \frac{a_2 b_2 - a_1 c_1}{b_1 b_2 - c_1 c_2}, \qquad y_4 = \frac{b_1 a_2 - a_1 c_2}{b_1 b_2 - c_1 c_2}.$$

If $b_1 b_2 - c_1 c_2 = 0$, then there might be no solutions or infinitely many, depending on the values of a_1, a_2.

We see that the critical points depend on the values of the coefficients. We now consider two different sets of constants, which would lead to very different dynamics of the populations.

Case (1). Consider the following model:

$$\begin{aligned} x'(t) &= x(1-x-y) \\ y'(t) &= y(0.75-y-0.5x) \end{aligned}$$

The 4 critical points are

$$(x_1, y_1) = (0,0), \quad (x_2, y_2) = (0, 0.75)\,, \quad (x_3, y_3) = (1,0)\,, \quad (x_4, y_4) = (0.5, 0.5).$$

For CP1, the Jacobian matrix is:

$$\mathbf{J}_1 = \mathbf{J}(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 0.75 \end{pmatrix}.$$

which gives the two eigenpairs

$$\lambda_1 = 1, \quad \vec{v}^{(1)} = (1,0)^T, \qquad \lambda_2 = 0.75, \quad \vec{v}^{(2)} = (0,1)^T.$$

This critical point is unstable.

For CP2, the Jacobian matrix is:

$$\mathbf{J}_2 = \mathbf{J}(0, 0.75) = \begin{pmatrix} 0.25 & 0 \\ -0.375 & -0.75 \end{pmatrix}.$$

which gives the two eigenpairs

$$\lambda_1 = 0.25, \quad \vec{v}^{(1)} = (8,-3)^T, \qquad \lambda_2 = -0.75, \quad \vec{v}^{(2)} = (0,1)^T.$$

It is a saddle point.

For CP3, the Jacobian matrix is:

$$\mathbf{J}_3 = \mathbf{J}(1,0) = \begin{pmatrix} -1 & -1 \\ 0 & 0.25 \end{pmatrix}.$$

which gives the two eigenpairs

$$\lambda_1 = -1, \quad \vec{v}^{(1)} = (1,0)^T, \qquad \lambda_2 = 0.25, \quad \vec{v}^{(2)} = (4,-5)^T.$$

It is also a saddle point.

For CP4, the Jacobian matrix is:

$$\mathbf{J}_4 = \mathbf{J}(0.5, 0.5) = \begin{pmatrix} -0.5 & -0.5 \\ -0.25 & -0.5 \end{pmatrix}.$$

which gives the two eigenpairs

$$\lambda_1 = -0.146, \quad \vec{v}^{(1)} = (\sqrt{2}, -1)^T, \qquad \lambda_2 = -0.854, \quad \vec{v}^{(2)} = (\sqrt{2}, 1)^T.$$

This is asymptotically stable proper node.

One can sketch a phrase portrait based on these information. See Figure 9.4.2 on p. 524 in textbook.

It is apparent now that CP4 is the attractor for the first quadrant. This means, if $x > 0, y > 0$ initially, then after a long time they will approach (x_4, y_4). This is an example of co-existence of two species.

Case (2). Consider the following model:

$$\begin{aligned} x'(t) &= x(1-x-y) \\ y'(t) &= y(0.75-y-0.5x) \end{aligned}$$

The 4 critical points are

$$(x_1, y_1) = (0,0), \quad (x_2, y_2) = (0,2), \quad (x_3, y_3) = (1,0), \quad (x_4, y_4) = (0.5, 0.5).$$

Straight computation gives now their eigenvalues and eigenvectors

$$\begin{aligned} (1)\ (0,0): &\quad \lambda_1 = 1, \quad \vec{v}^{(1)} = (1,0)^T, \qquad \lambda_2 = 0.75, \quad \vec{v}^{(2)} = (0,1)^T \\ (2)\ (0,2): &\quad \lambda_1 = -1, \quad \vec{v}^{(1)} = (1,3)^T, \qquad \lambda_2 = -0.5, \quad \vec{v}^{(2)} = (0,1)^T \\ (3)\ (1,0): &\quad \lambda_1 = -1, \quad \vec{v}^{(1)} = (1,0)^T, \qquad \lambda_2 = -0.25, \quad \vec{v}^{(2)} = (4,-3)^T \\ (4)\ (0.5,0.5): &\quad \lambda_1 = 0.16, \quad \vec{v}^{(1)} = (1,-1.32)^T, \qquad \lambda_2 = -0.78, \quad \vec{v}^{(2)} = (1,0.57)^T \end{aligned}$$

We see both CP1 and CP4 are unstable, but both CP2 and CP3 are stable.

One can use these info to sketch the phase portrait. See Figure 9.4.4 on p. 528 in textbook.

This implies that, for all initial data in the first quadrant, some of them will approach CP2, while some others will approach CP3. There is a curve that separates these two. We call this curve a *separatrix*. For initial data on this separatrix, the solution will approach CP4, but however, such an solution is unstable with respect to small perturbations. This predicts that one of the species will die out, while the other could survive and live up to the habitat's capacity. No co-existence is possible. Note that this dynamics is completely different from Case (1).

We observe that the dynamics depends on the property of the two lines

$$(\text{L1}): \ a_1 - b_1 x - c_1 y = 0, \qquad (\text{L2}): \ a_2 - b_2 y - c_2 x = 0.$$

The intersection of these two line gives the 4th critical point, if it lies in the first quadrant.

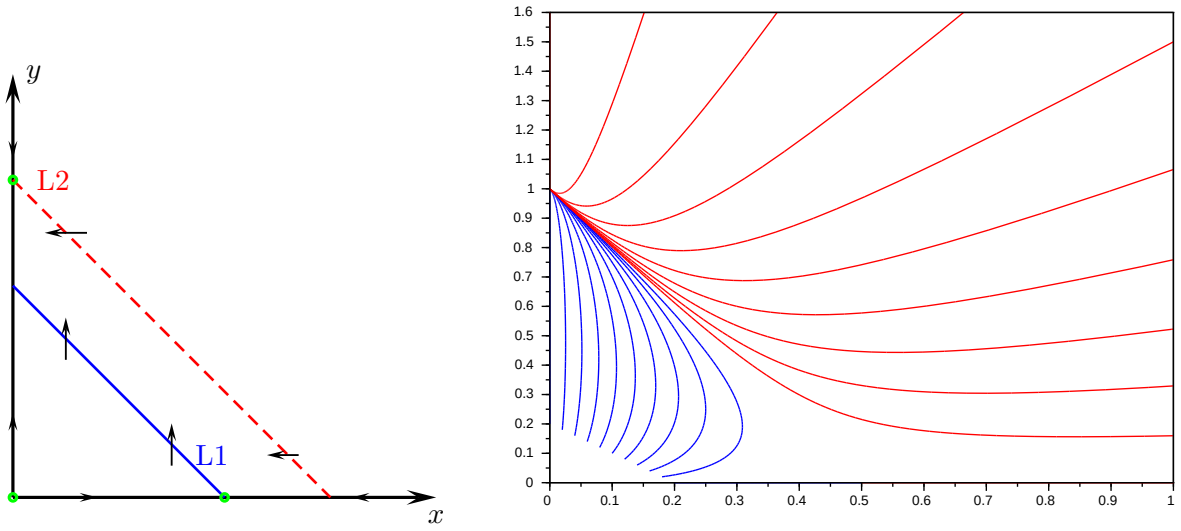
Along L1, we have $x' = 0$. Above L1, we have $x' < 0$. Below L1, we have $x' > 0$.

Along L2, we have $y' = 0$. Above L2, we have $y' < 0$. Below L2, we have $y' > 0$.

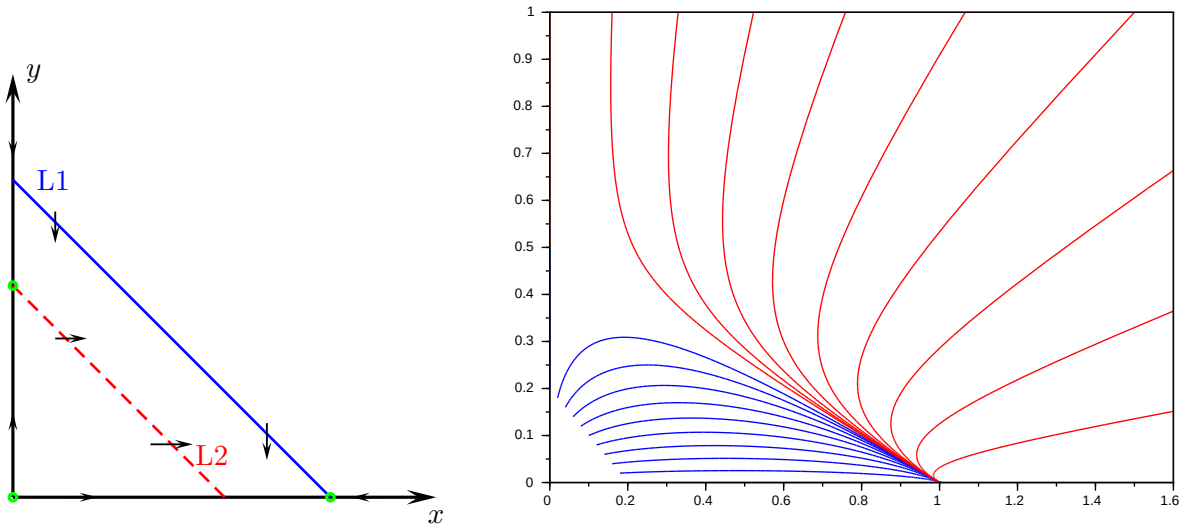
These lines are called the $x-$ and $y-$nullclines.

There are 4 possible configurations. See Figure 9.4.5 on p 529 in textbook. We will take a close look at each, considering only the first quadrant.

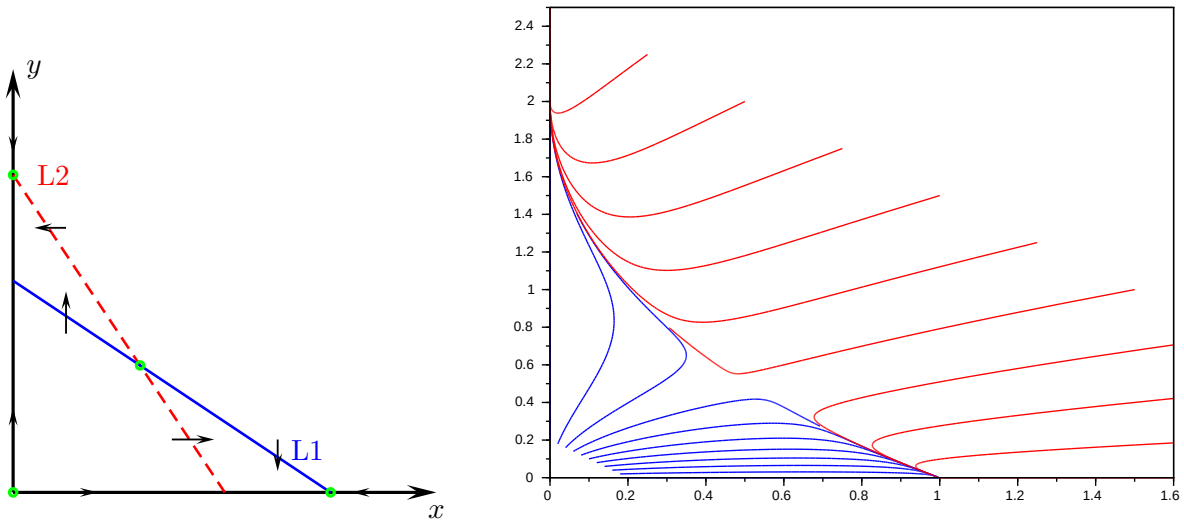
Case (a): L2 lies above L1. There are 3 critical points of interests. We see that along L1, we have $x' = 0, y' > 0$. Along L2, we have $x' < 0, y' = 0$. The directional fields now indicates that critical point on the y-axis with coordinate $(0, a_2/b_2)$ is asymptotically stable. Here is the phase portrait:



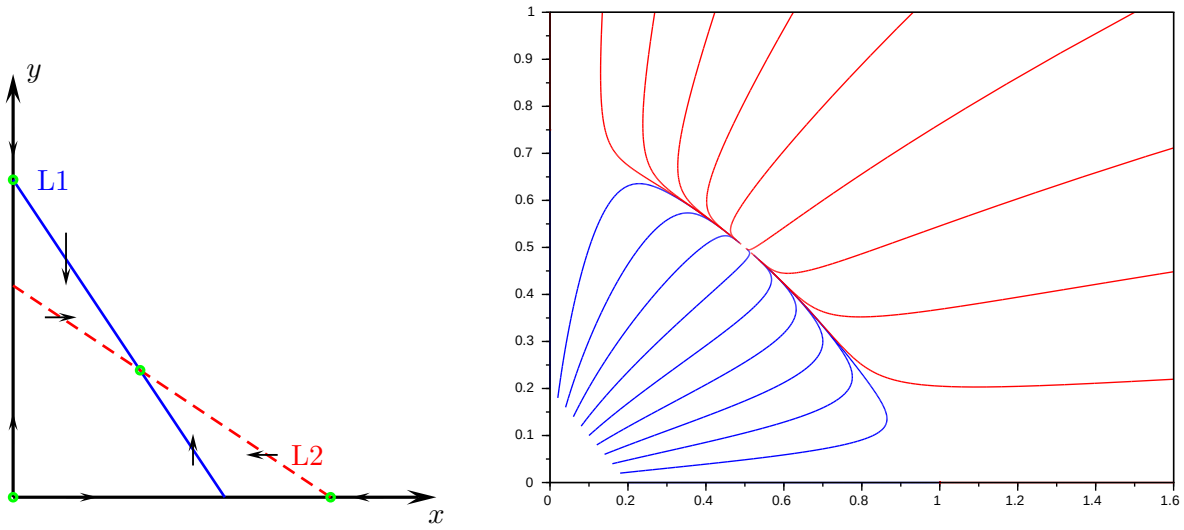
Case (b): L2 lies below L1. There are 3 critical points of interests. We see that along L1, we have $x' = 0, y' < 0$. Along L2, we have $x' > 0, y' = 0$. The directional fields now indicates that critical point on the x-axis with coordinate $(a_1/b_1, 0)$ is asymptotically stable. Here is the phase portrait:



Case (c): L1 crosses L2, with the L2 above L1 for small x values, and L2 below L1 for large x values. There are 4 critical points. Below is the graph of the vector field (on the left) and the phase portrait (on the right).



Case (d): L1 crosses L2, with the L2 below L1 for small x values, and L2 above L1 for large x values. There are 4 critical points. Below is the graph of the vector field (on the left) and the phase portrait (on the right). We see that this is the only case where co-existence is possible.



In the textbook there are detailed discussion analyzing the values of the constants a_1, b_1, c_1 etc. In conclusion, if the competition is weak, i.e., if $b_1 b_2 > c_1 c_2$, then coexist is possible. Otherwise, if the competition is strong, i.e., if $b_1 b_2 < c_1 c_2$ only one species can survive.

Chapter 10

Fourier Series

Fourier series will be useful in series solutions for linear 2nd order partial differential equations.

10.1 Introduction and Basic Fourier Series

Objective: representing periodic functions as a series of sine and cosine functions.

Let $f(x)$ be a periodic function with period P, i.e.,

$$f(x+P) = f(x), \qquad \forall x$$

Note: If P is a period, so are $2P, 3P, 4P, \cdots$,.

The smallest period is called the **fundamental period**.

Observation: If $f(x)$ and $g(x)$ are both periodic with period P, so will any linear combination $af(x) + bg(x)$ for arbitrary constants a, b. Also, the product $f(x)g(x)$ is periodic with the same period.

Known examples of periodic functions: trig functions.

With period 2π:

$$\sin x, \ \ \sin 2x, \ \ \sin 3x, \cdots, \cos x, \ \ \cos 2x, \ \ \cos 3x, \cdots$$

With period $2L$:

$$\sin \frac{\pi x}{L}, \ \ \sin \frac{2\pi x}{L}, \ \ \sin \frac{3\pi x}{L}, \ \ \cdots \ \ \cos \frac{\pi x}{L}, \ \ \cos \frac{2\pi x}{L}, \ \ \cos \frac{3\pi x}{L}, \ \ \cdots$$

We have the trig set:

$$\left\{1, \sin \frac{m\pi x}{L}, \cos \frac{m\pi x}{L}, \right\}, \qquad m = 1, 2, \cdots$$

Plots of some sine functions are given in Figure 10.1.

Definition. Let $f(x)$ be periodic with period $2L$. Fourier series for $f(x)$ is:

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \right). \qquad (*)$$

Here the constants a_0, a_m, b_m are called: Fourier coefficients.

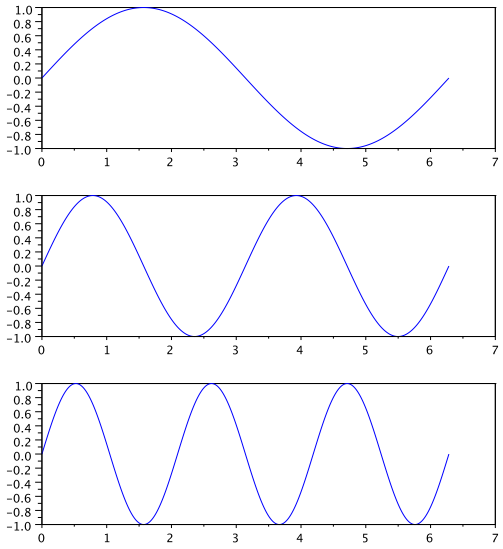


Figure 10.1: Some sine functions.

How to compute the Fourier coefficients? Use the orthogonality of the trig set!

Definition: Given two functions $u(x), v(x)$, define the **inner product** as

$$(u, v) \doteq \int_a^b u(x)v(x)\,dx.$$

In our case we will choose $a = -L, b = L$, i.e.,

$$(u, v) \doteq \int_{-L}^{L} u(x)v(x)\,dx.$$

Definition: The functions u and v are **orthogonal** if $(u, v) = 0$.

Claim: The trig set is mutually orthogonal, i.e, any two distinct functions in the set are orthogonal to each other. This means

$$\begin{aligned} (1, \sin\frac{m\pi x}{L}) = 0, \qquad (1, \cos\frac{m\pi x}{L}) = 0, \qquad & \forall m \\ (\sin\frac{m\pi x}{L}, \sin\frac{n\pi x}{L}) = 0, \qquad (\cos\frac{m\pi x}{L}, \cos\frac{n\pi x}{L}) = 0, \qquad & \forall m \neq n \\ (\cos\frac{m\pi x}{L}, \sin\frac{n\pi x}{L}) = 0, \qquad & \forall m, n \end{aligned}$$

Proof. By direct integration. For example:

$$(1, \sin\frac{m\pi x}{L}) = \int_{-L}^{L} \sin\frac{m\pi x}{L}\,dx = 0.$$

This integration is 0 because it integrates a sine function over several complete periods.

For $m \neq n$, we compute

$$(\sin\frac{m\pi x}{L}, \sin\frac{n\pi x}{L}) = \int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = \frac{1}{2}\int_{-L}^{L}\left[\cos\frac{(m-n)\pi x}{L} - \cos\frac{(m+n)\pi x}{L}\right]dx = 0.$$

Both integrals are 0 because it integrates over several periods of cosine function, when $m \neq n$.

All other identities are proven in a similar way. We skip the details.

Useful identities.

$$(\sin\frac{m\pi x}{L}, \sin\frac{m\pi x}{L}) = (\cos\frac{m\pi x}{L}, \cos\frac{m\pi x}{L}) = L.$$

Proof: Direct computation gives:

$$(\sin\frac{m\pi x}{L}, \sin\frac{m\pi x}{L}) = \frac{1}{2}\int_{-L}^{L}\left[1 - \cos\frac{2m\pi x}{L}\right]dx = \frac{1}{2}(2L) = L.$$

Similarly

$$(\cos\frac{m\pi x}{L}, \cos\frac{m\pi x}{L}) = L.$$

One may also observe that, by the periodic property, we have

$$(\sin\frac{m\pi x}{L}, \sin\frac{m\pi x}{L}) = (\cos\frac{m\pi x}{L}, \cos\frac{m\pi x}{L}).$$

Then, use the trig identity $\sin^2 x + \cos^2 x = 1$, we have

$$(\sin\frac{m\pi x}{L}, \sin\frac{m\pi x}{L}) + (\cos\frac{m\pi x}{L}, \cos\frac{m\pi x}{L}) = \int_{-L}^{L}\left[\sin^2\frac{m\pi x}{L} + \cos^2\frac{m\pi x}{L}\right]dx = 2L$$

which gives the same answer.

Back to Fourier series. We now multiply equation (*) by $\cos\frac{n\pi x}{L}$ and integrate over $[-L, L]$,

$$\begin{aligned}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx &= \int_{-L}^{L}\frac{a_0}{2}\cos\frac{n\pi x}{L}\,dx + \sum_{m=1}^{\infty} a_m\int_{-L}^{L}\cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx \\ &\quad + \sum_{m=1}^{\infty} b_m\int_{-L}^{L}\sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx\end{aligned}$$

All the terms are 0 except one:

$$\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = a_n\int_{-L}^{L}\cos\frac{n\pi x}{L}\cos\frac{n\pi x}{L}\,dx = a_n L$$

This gives us the formula to compute a_n:

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx, \qquad n = 1, 2, 3, \cdots$$

Deriving in a completely similar way, we get

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx, \qquad n = 1, 2, 3, \cdots$$

and

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx.$$

Note that $a_0/2$ (the constant term in Fourier series) is the average of $f(x)$ over a period. The formula for a_0 fit the one for a_n with $n = 0$.

These formulas for computing the Fourier coefficients are called **Euler-Fourier formula**.

If the period is 2π, i.e., $L = \pi$ in the formulas, we get simpler looking formulas

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \qquad n = 0, 1, 2, \cdots \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx, \qquad n = 1, 2, 3, \cdots . \end{aligned}$$

We now take some examples in computing Fourier series.

Example 1. Find the Fourier series for a periodic function $f(x)$ with period 2π

$$f(x) = \begin{cases} -1, & \text{if } -\pi < x < 0 \\ 1, & \text{if } 0 < x < \pi \end{cases}, \qquad f(x + 2\pi) = f(x)$$

Answer. We use the Euler-Fourier formulas with $L = \pi$:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0.$$

We note that $f(x)$ is an odd function, i.e., $f(-x) = -f(x)$. Therefore integrating over a period, one get 0.

For $n \geq 1$, we have

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\ &= \frac{1}{\pi}\int_{-\pi}^{0} -\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}\cos nx\,dx \\ &= \frac{1}{\pi}(-\frac{1}{n})\sin nx|_{x=-\pi}^{0} + \frac{1}{\pi}\frac{1}{n}\sin nx|_{x=0}^{\pi} = 0 \end{aligned}$$

Actually, we could get this integral quickly by observing the following: $f(x)$ is an odd function, and $\cos nx$ is an even function. Then, the product $f(x)\cos nx$ is an odd function. Therefore, the integral over an entire period is 0.

Finally, we compute b_n as

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \;=\; \frac{1}{\pi}\int_{-\pi}^{0} -\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}\sin nx\,dx \\ &= -\frac{1}{\pi}\frac{1}{n}(-\cos nx)|_{x=-\pi}^{0} + \frac{1}{\pi}\frac{1}{n}(-\cos nx)|_{x=0}^{\pi} \\ &= \frac{1}{n\pi}(\cos 0 - \cos(-n\pi)) - \frac{1}{n\pi}(\cos n\pi - \cos 0) \\ &= \frac{1}{n\pi}(1 - \cos n\pi) + \frac{1}{n\pi}(1 - \cos n\pi) \;=\; \frac{2}{n\pi}(1 - \cos n\pi). \end{aligned}$$

The actual computation could be shortened by observing the following: $\sin nx$ is an odd function, so $f(x)\sin nx$ is an even function. The integrals on $[-\pi, 0]$ and $[0, \pi]$ are the same. So one needs to do only one integral, and multiply the result by 2.

We observe

$$\cos\pi = -1,\ \cos 2\pi = 1,\ \cos 3\pi = -1, \cdots, \quad \Rightarrow \quad \cos n\pi = (-1)^n$$

Then

$$b_n = \frac{2}{n\pi}\left(1 - (-1)^n\right) = \begin{cases} \frac{4}{n\pi}, & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases}$$

We can now write out the Fourier series. Since all a_n's are 0, we will only have sine functions. Also, b_n is non-zero only for odd n, and we can write out the first few

$$b_1 = \frac{4}{1\pi}, \quad b_3 = \frac{4}{3\pi}, \quad b_5 = \frac{4}{5\pi}, \quad , b_7 = \frac{4}{7\pi}, \cdots$$

Note that $4/\pi$ is a common factor, which we can take out. This gives:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi = \frac{4}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x + \cdots\right].$$

Partial sum of a series: the sum of the first few terms.

We can write $y_n(x)$ to be the sum of the first n term in the Fourier series. For our example, we have

$$\begin{aligned} y_1(x) &= \frac{4}{\pi}\sin x \\ y_2(x) &= \frac{4}{\pi}\left[\sin x + \frac{1}{3}\sin 3x\right] \\ y_3(x) &= \frac{4}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x\right] \\ &\cdots \end{aligned}$$

Then, the limit $\lim_{n\to+\infty} y_n(x)$ (if it converges) gives the whole Fourier series.

The partial sums of Fourier series and the original function $f(x)$ for this example are plotted together in Fig 10.2.

Example 2. Find the Fourier series of the function

$$f(x) = \begin{cases} 0, & -2 < x < -1 \\ K, & -1 < x < 1 \\ 0, & 1 < x < 2 \end{cases}, \qquad \text{period} = 4.$$

Answer. Since the period is 4, we have $2L = 4$ so $L = 2$. We compute the Fourier coefficients by Euler-Fourier formulas. We have

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx = \frac{1}{2}2K = K,$$

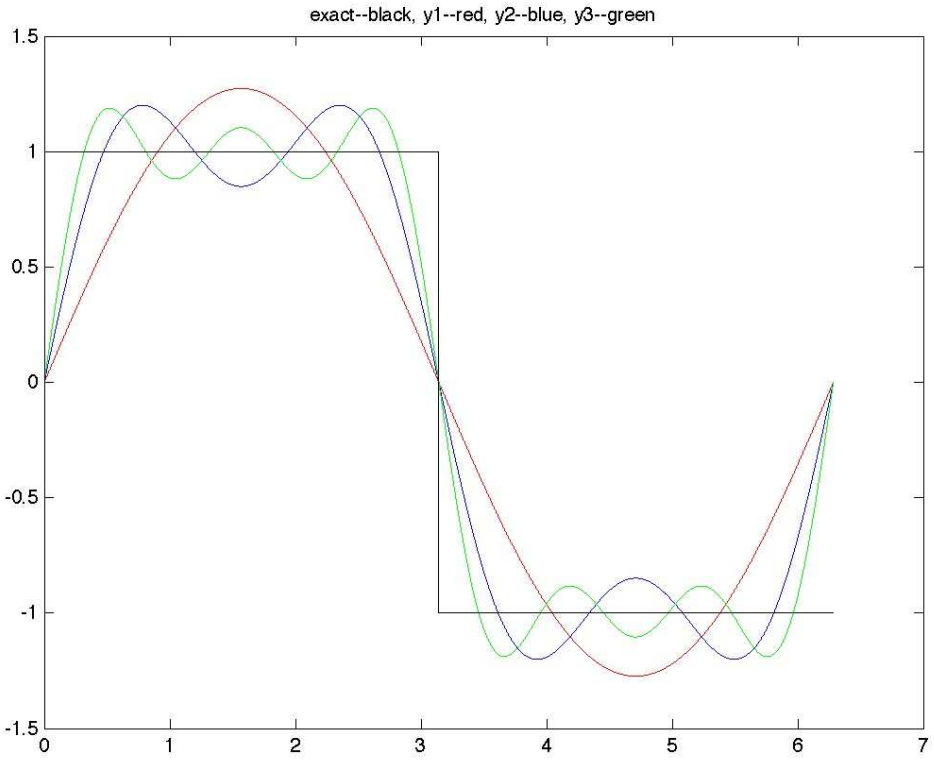


Figure 10.2: Fourier series, the first few terms, Example 1.

and

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}dx = \frac{K}{2}\int -1^1 \cos\frac{n\pi x}{2}dx = \frac{K}{2}\frac{2}{n\pi}\sin\frac{n\pi x}{2}\Big|_{x=-1}^{1} = \frac{2K}{n\pi}\sin\frac{n\pi}{2}.$$

The function $\sin\frac{n\pi}{2}$ takes only values $0, 1, 0, -1$ in a periodic ways, depending on n. We have

$$a_n = \begin{cases} 0, & n \text{ even} \\ \frac{2K}{n\pi}, & n = 1, 5, 9, 13, 17, \cdots \\ -\frac{2K}{n\pi}, & n = 3, 7, 11, 15, 19, \cdots \end{cases}$$

For the b_n, note that $f(x)$ is an even function, and $\sin\frac{n\pi x}{2}$ is an odd function, so the product is an odd function. Integrating over a whole period gives 0, i.e,

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}dx = 0.$$

Note that in this example, there will be no sine functions in the Fourier series!

We can now write out the Fourier series:

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2} = \frac{K}{2} + \frac{2K}{\pi}\left[\cos\frac{\pi x}{2} - \frac{1}{3}\cos\frac{3\pi x}{2} + \frac{1}{5}\cos\frac{5\pi x}{2} - \frac{1}{7}\cos\frac{7\pi x}{2} + \cdots\right]$$

The partial sums of Fourier series and the original function $f(x)$ are plotted together in Fig 10.3, for $K = 1$.

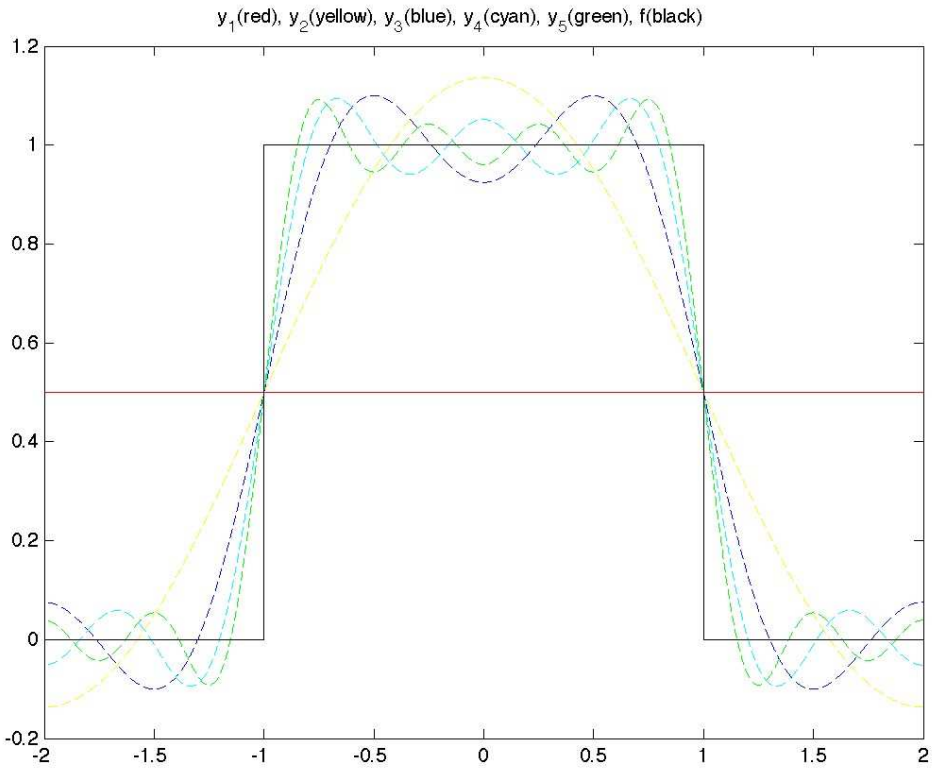


Figure 10.3: Fourier series, the first few terms, Example 2.

Observation.
If $f(x)$ is an odd function, then there are no cosine functions in the Fourier series.
If $f(x)$ is an even function, then there are no sine functions in the Fourier series.
We will see later that this is a general rule!

On tabular method for integration by parts. We see that if $f(x)$ is some kind of polynomial, we will end up with integrations of the form

$$\int f(x) \sin \frac{n\pi x}{L}\, dx, \qquad \int f(x) \sin \frac{n\pi x}{L}\, dx.$$

In this case, we need to perform integration by parts multiple times, which could be quite time consuming. The tabular method makes this computation a bit easier.

In general, this method works best when one of the two functions in the product is a polynomial, and the other is a function that you can easily find antiderivatives (such as $e^x, \sin ax, \cos ax$). Then after differentiating the polynomial several times one obtains zero. The method may also be extended to work for functions that will repeat themselves, such as $e^{ax} \sin bx$ and $e^{ax} \cos bx$.

To fix the idea, let's take an example. Consider the integration

$$\int x^3 \cos nx \, dx.$$

Let $u = x^3$ and $v = \cos nx$. Begin with these functions and we set up a table. In column A we put derivatives of u until it becomes zero. In column B we put in integrals of v. We get the following result

Derivatives of u (column A)	Integrals of v (column B)
x^3	$\cos nx$
$3x^2$	$\frac{1}{n}\sin nx$
$6x$	$-\frac{1}{n^2}\cos nx$
6	$-\frac{1}{n^3}\sin nx$
0	$\frac{1}{n^4}\cos nx$

Now take the product of the 1st entry of column A with the 2nd entry of column B, the 2nd entry of column A with the 3rd entry of column B, in this diagonal way, until you hit the zero term. These products will be multiplied with alternative signs, starting with the positive sign, and then added up. We get

$$\begin{aligned}\int x^3 \cos nx \, dx &= x^3\frac{1}{n}\sin nx - 3x^2(-\frac{1}{n^2}\cos nx) + 6x(-\frac{1}{n^3}\sin nx) - 6\frac{1}{n^4}\cos nx \\ &= \frac{x^3}{n}\sin nx + \frac{3x^2}{n^2}\cos nx - \frac{6x}{n^3}\sin nx - \frac{6}{n^4}\cos nx.\end{aligned}$$

Example 3. Find the Fourier series of

$$r(t) = \begin{cases} t + \pi/2, & -\pi < t < 0 \\ -t + \pi/2, & 0 < t < \pi \end{cases}, \qquad \text{period} = 2\pi.$$

Answer. We seek Fourier series for $r(t)$, i.e.,

$$r(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt.$$

We first note that $r(t)$ is an even function, we can immediately conclude that $b_n = 0$ for all n.

Furthermore, $r(t)\cos nt$ will be an even function. To integrate over a period, we only need to integrate over half period, and multiply the answer by 2. We now compute a_n.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} r(t)\, dt \;=\; 0, \qquad \text{(Look at the graph of } r \text{ over a period!)}$$

and

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} r(t)\cos nt\,dt \;=\; \frac{2}{\pi}\int_0^{\pi}(-t+\frac{\pi}{2})\cos nt\,dt$$

Let's try the tabular method for integration by parts:

$-t+\pi/2$	$\cos nt$
-1	$(1/n)\sin nt$
0	$-(1/n^2)\cos nt$

We have now

$$\begin{aligned} a_n &= \frac{2}{\pi}\left[(-t+\frac{\pi}{2})\frac{1}{n}\sin nt - \frac{1}{n^2}\cos nt\right]_{t=0}^{\pi} \\ &= \frac{2}{\pi}\left[-\frac{1}{n^2}\cos n\pi + \frac{1}{n^2}\right] \;=\; \frac{2}{\pi n^2}(1-(-1)^n) \;=\; \begin{cases} 0, & n \text{ even}, \\ \dfrac{4}{n^2\pi}, & n \text{ odd}. \end{cases} \end{aligned}$$

We can now write out the Fourier series

$$r(t) = \sum_{n=1}^{\infty} a_n \cos nt = \sum_{n \text{ odd}} \frac{4}{n^2\pi}\cos nt = \frac{4}{\pi}\left[\cos t + \frac{1}{9}\cos 3t + \frac{1}{25}\cos 5t + \cdots\right].$$

The plots of several partial sums and their error are included in Figure 10.4.

We make some observations:
(1). In general, the error decreases as we take more terms.
(2). For a fixed partial sum, the error is larger at the point where the function $r(t)$ has a kink, and smaller in the region where $r(t)$ is smooth.
(3). After taking 3 terms, the partial sum is already a very good approximation to $r(t)$.
(4). It seems like $y_n(t)$ converges to $r(t)$ at every point t.

Next example is a dummy one, but might be useful.

Example 4. Find the Fourier coefficients for $f(x)$, periodic with $p = 2\pi$, given as

$$f(x) = 2 + 4\sin x - 0.5\cos 4x - 99\sin 100x.$$

Answer. Since the function f here is already given in terms of sine and cosine function, there is no need to compute the Fourier coefficients. We just need to figure out where each term would fit, by comparing it with a Fourier series. We have

$$a_0 = 4, \quad a_4 = -0.5, \quad a_n = 0, \quad \forall n \neq 0, 4$$

and

$$b_1 = 4, \quad b_{100} = -99, \qquad b_n = 0 \quad \forall n \neq 1, 100.$$

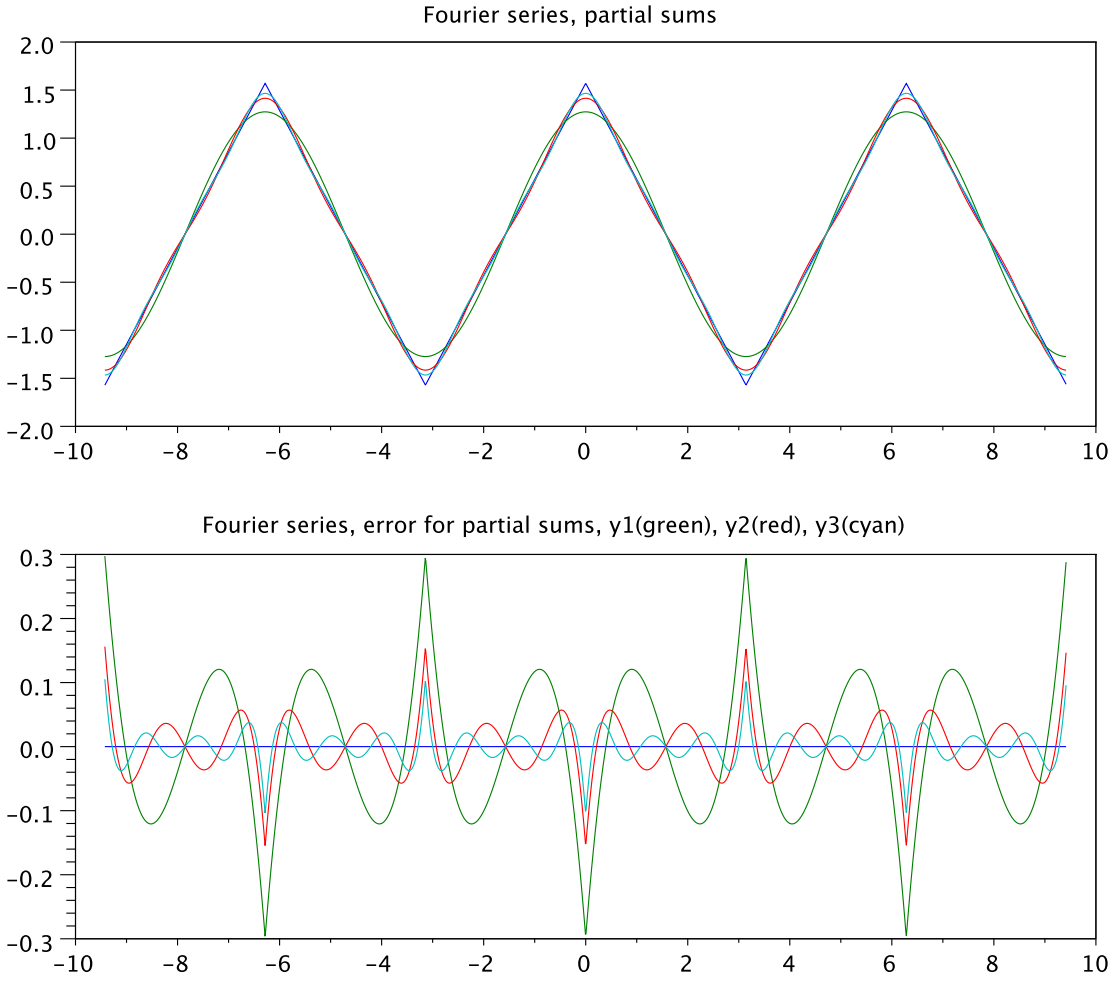


Figure 10.4: Fourier series, the first few terms and the errors, Example 3.

10.2 Even and Odd Functions; Fourier sine and Fourier cosine series.

Through examples we have already observed that, for even and odd functions, the Fourier series takes simpler forms. We will summarize it here.

A function $f(x)$ is **even** if

$$f(-x) = f(x).$$

The graph of the function is symmetric about the y-axis. Examples include $f(x) = 1$ and $f(x) = \cos nx$ for any integer n.

A function $f(x)$ is **odd** if

$$f(-x) = -f(x).$$

The graph of the function is symmetric about the origin. Examples include $f(x) = \sin nx$ for any integer n.

Properties:

- Product of two even functions is even;
- Product of two odd functions is even;
- Product of an even and an odd function is odd.
- Integration of an odd function over $[-L, L]$ is 0.
- Integration of an even function over $[-L, L]$ is twice the integration over $[0, L]$.

We have already observed that, if $f(x)$ is an even function, then its Fourier series will NOT have sine functions. If $f(x)$ is an odd function, then its Fourier series will NOT have cosine functions.

This fits the instinct: One can not represent an odd function with the sum of some even functions, and visa versa.

The formulas for the Fourier coefficients could be simplified, as we have already observed.

- If $f(x)$ is an even, periodic function with $p = 2L$, it has a **Fourier cosine series**

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

where

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx, \qquad n = 0, 1, 2, \cdots$$

- Correspondingly, if $f(x)$ is an odd, periodic function with $p = 2L$, it has a **Fourier sine series**

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

where

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx, \qquad n = 1, 2, 3, \cdots$$

Note now we only need to integrate over half period, i.e., over $[0, L]$, because the product is an even function.

Half-range expansion. If a function $f(x)$ is only defined on an interval $[0, L]$, we can extend/expand the domain into the whole real line by periodic expansion. There are two ways of doing this:

- Extend $f(x)$ onto the interval $[-L, L]$ such that f is an even function, i.e., $f(-x) = f(x)$, then extend it into a periodic function with $p = 2L$;
- Extend $f(x)$ onto the interval $[-L, L]$ such that f is an odd function, i.e., $f(-x) = -f(x)$, then extend it into a periodic function with $p = 2L$.

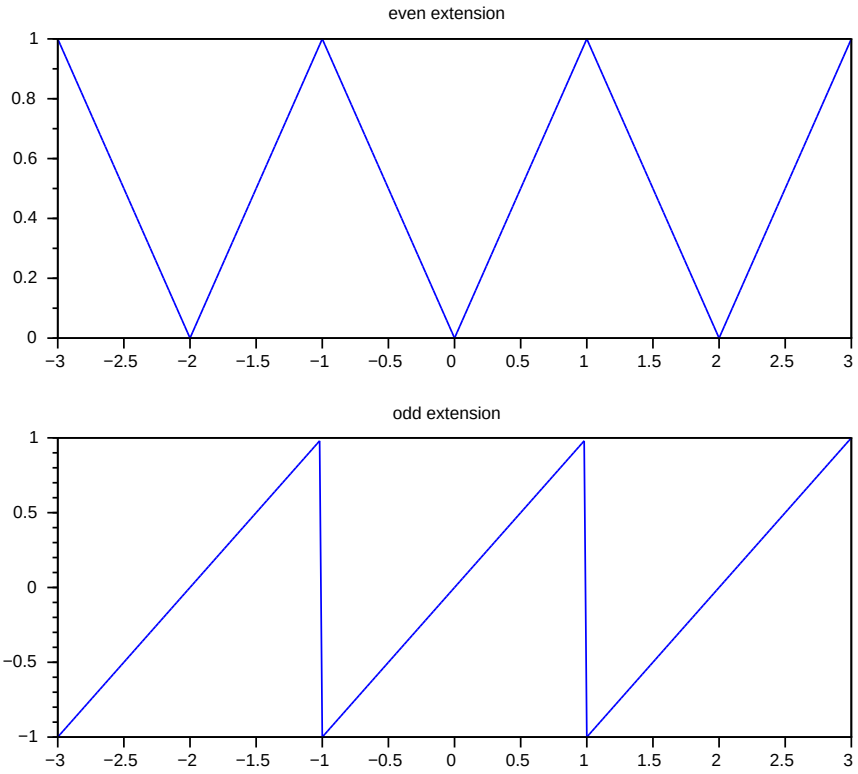


Figure 10.5: Even and odd extension, 3 periods plotted.

These are called **even/odd periodic extensions** of f, or **half-range expansions**.

Example 1. Let $f(x) = x$ be defined on the interval $x \in [0, L]$. Sketch 3 periods of the even and odd extension of f, and then compute the corresponding Fourier sine or cosine series.

Answer. The graph of even and odd extensions are given in Fig 10.5.

The odd periodic extension turns out to be the "sawtooth wave". We have Fourier sine series, with the following coefficients

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L x \sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\left(\frac{L}{n\pi}\right)^2 \left\{\sin\frac{n\pi x}{L} - \frac{n\pi x}{L}\cos\frac{n\pi x}{L}\right\}\Big|_{x=0}^{L} \\ &= \frac{2L}{n\pi}(-1)^{n+1}, \qquad\qquad n = 1, 2, 3, \cdots \end{aligned}$$

therefore

$$f_{odd}(x) = \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{L} = \frac{2L}{\pi}\left[\sin\frac{\pi x}{L} - \frac{1}{2}\sin\frac{2\pi x}{L} + \frac{1}{3}\sin\frac{3\pi x}{L} - \frac{1}{4}\sin\frac{4\pi x}{L} + \cdots\right].$$

The even extension gives triangle waves (similar to Example 3 in ch 1.1). It will have a

Fourier cosine series, with coefficients

$$\begin{aligned} a_0 &= \frac{2}{L}\int_0^L f(x)\,dx = \frac{2}{L}\int_0^L x\,dx = \frac{2}{L}(\frac{L^2}{2}) = L, \\ a_n &= \frac{2}{L}\int_0^L x\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\left\{\left(\frac{L}{n\pi}\right)^2\cos\frac{n\pi x}{L} + \frac{xL}{n\pi}\sin\frac{n\pi x}{L}\right\}\Big|_{x=0}^{L} \\ &= \frac{2L}{n^2\pi^2}(\cos n\pi - 1) = \frac{2L}{n^2\pi^2}((-1)^n - 1), \qquad\qquad n = 1, 2, 3, \cdots \end{aligned}$$

Therefore $a_n = 0$ for n even, and $a_n = -\frac{4L}{n^2\pi^2}$ for odd n. We have the Fourier cosine series

$$f_{even}(x) = \frac{L}{2} - \sum_{n \text{ odd}} \frac{4L}{n^2\pi^2}\cos\frac{n\pi x}{L} = \frac{L}{2} - \frac{4L}{\pi^2}\left[\cos\frac{\pi x}{L} + \frac{1}{9}\cos\frac{3\pi x}{L} + \frac{1}{5^2}\cos\frac{5\pi x}{L} + \frac{1}{7^2}\cos\frac{7\pi x}{L} + \cdots\right].$$

Include the plots of partial sums, and the errors, for even and odd expansions are included in Fig 10.6 and Fig 10.7.

10.3 Properties of Fourier Series

Linearity. Let $f(x)$ and $g(x)$ be 2 periodic functions withe the same period, and each has a Fourier series with coefficients (a_n, b_n) for $f(x)$ and $(\bar{a}_n, \bar{b}_n)$ for $g(x)$. Then, the followings hold.
(1). The function $f(x)+g(x)$ will have Fourier coefficients $(a_n+\bar{a}_n, b_n+\bar{b}_n)$. (2). The function $\alpha f(x)$ for some constant α will have Fourier coefficients $(\alpha a_n, \alpha b_n)$.

Convergence Theorem. Let $f(x)$ be a periodic function and let it have a Fourier series $F(x)$. Assume $f(x)$ is piecewise continuous. Then,
(1). Fourier series converges to $f(x)$ at all points x where f is continuous;
(2). At a point x where f is discontinuous, Fourier series converges to the mid value of the left and right limit, i.e., $\frac{1}{2}[f(x-) + f(x+)]$.

This is confirmed by our example, see Example 1 and 2 in previous section, and Figure 10.2 and Figure 10.3.

Example 1. Find the Fourier series for:

$$h(x) = \begin{cases} 0, & -\pi < x \le 0 \\ K, & 0 < x \le \pi \end{cases}, \qquad h(x + 2\pi) = h(x).$$

Indicate the function that the Fourier series of $h(x)$ converges to.

Answer. Instead of working out the Fourier coefficients by direct computation, we will use the linear property. Let now

$$g(x) = K/2$$

The Fourier coefficients for $g(x)$ are simply

$$\bar{a}_0 = K/2, \qquad \bar{a}_n = \bar{b}_n = 0, \qquad \forall n \ge 1.$$

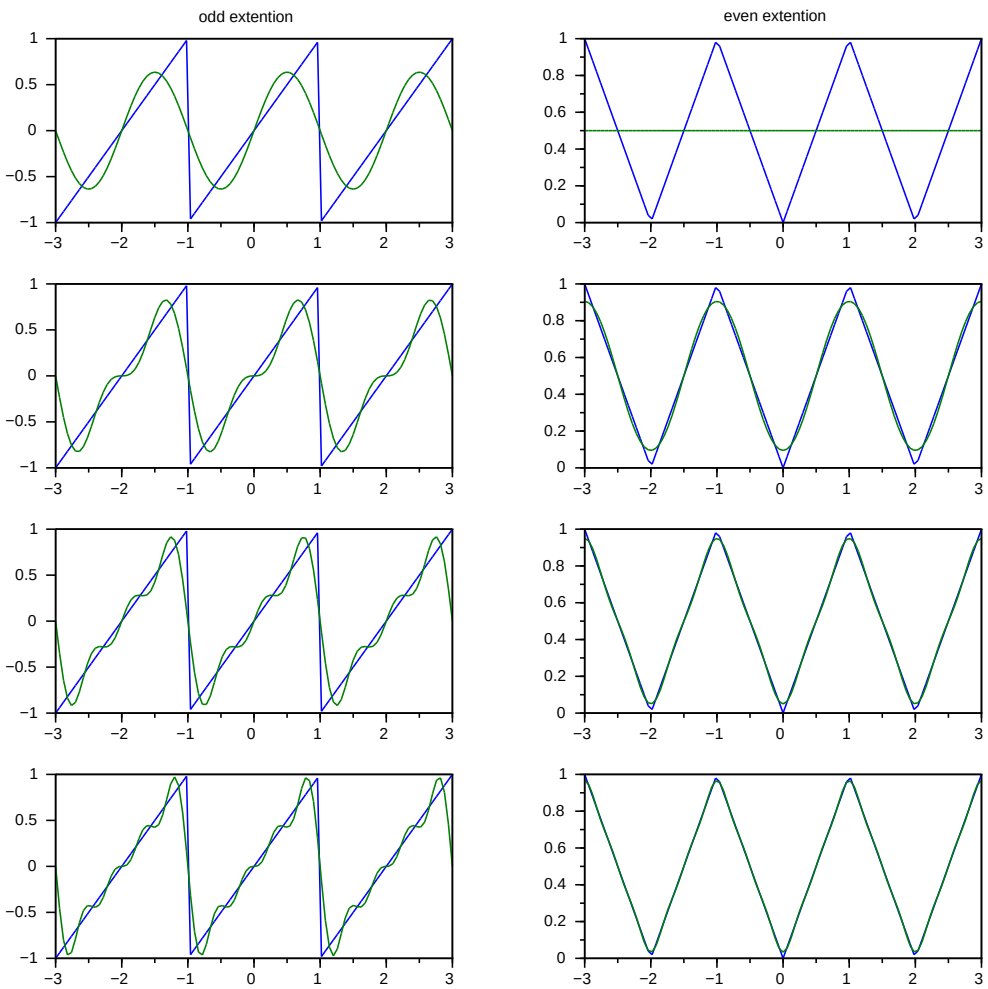


Figure 10.6: Even and odd extensions, first 4 partial sums, over 3 periods.

Then,

$$h(x) - g(x) = \begin{cases} -K/2, & -\pi < x \le 0 \\ K/2, & 0 < x \le \pi \end{cases} = \frac{K}{2} \begin{cases} -1, & -\pi < x \le 0 \\ 1, & 0 < x \le \pi \end{cases} = \frac{K}{2} f(x),$$

where $f(x)$ is the same function as in Example 1 in previous chapter, for which we have already computed the Fourier coefficients, i.e,

$$a_0 = 0, \qquad a_n = 0, \qquad b_n = \frac{2}{n\pi}(1 - (-1)^n).$$

By linearity, for $h(x) = (K/2)f(x) + g(x)$, will have Fourier coefficients

$$\tilde{a}_n = (K/2)a_n + \bar{a}_n = 0, \qquad \tilde{b}_n = (K/2)b_n + \bar{b}_n = \frac{K}{n\pi}(1 - (-1)^n),$$

which gives

$$h(x) = \frac{K}{2} + \frac{2K}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x + \cdots\right].$$

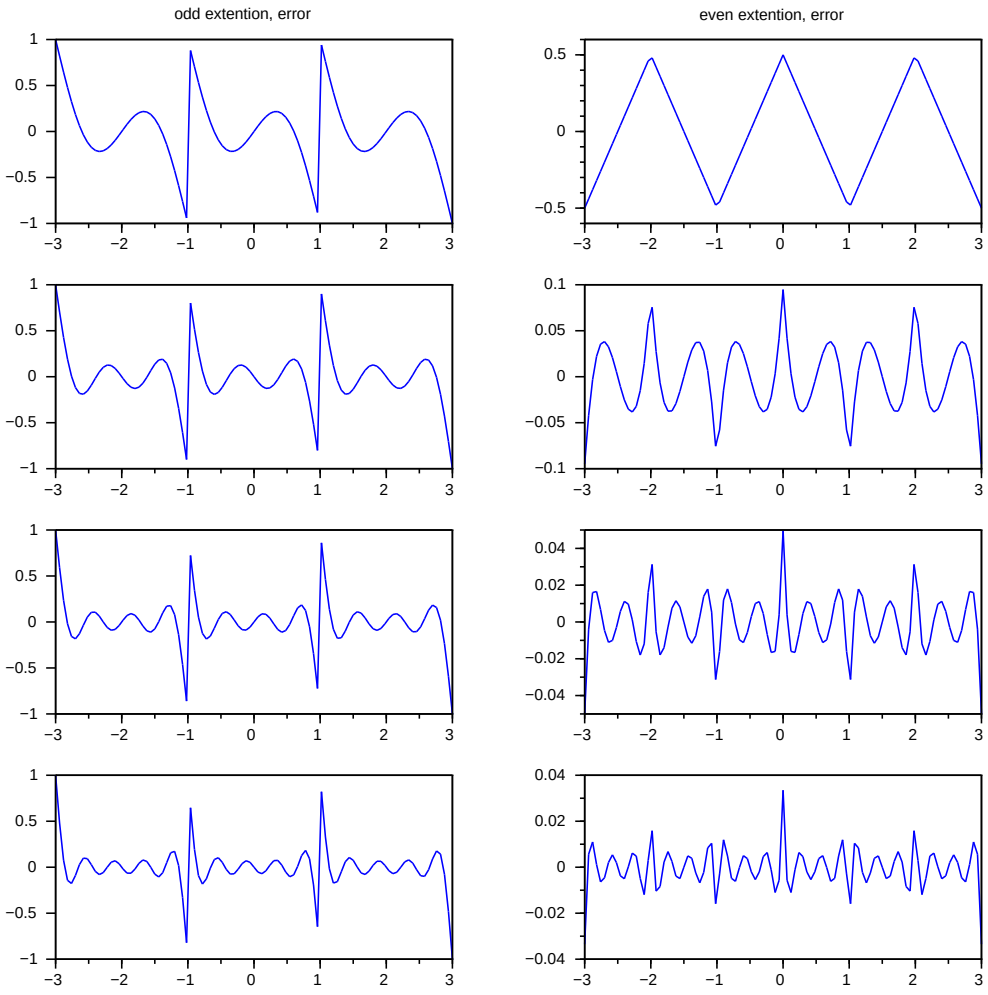


Figure 10.7: Even and odd extensions, errors for the first 4 partial sums, over 3 periods.

The Fourier series of $h(x)$ converges to $h(x)$ where-ever the function is continuous, and to the mid value at discontinuities, i.e.,

$$\bar{h}(x) = \begin{cases} K/2, & x = -\pi, \\ 0, & -\pi < x < 0, \\ K/2, & x = 0, \\ K, & 0 < x < \pi, \end{cases} \qquad \bar{h}(x + 2\pi) = \bar{h}(x).$$

One can sketch a graph to see it more clearly.

Choice of the half range expansion with concerns on convergence. We note that the Fourier cosine series, i.e, the even expansion seems to have smaller error for the same number of terms in the partial sum. This is because the even extension is a continuous function, while the odd extension is a piecewise continuous function with discontinuity points at $x = \pm 1, \pm 3, \pm 5, \cdots$. All sine and cosine functions are smooth. Using smooth functions to represent discontinuous function would give larger error.

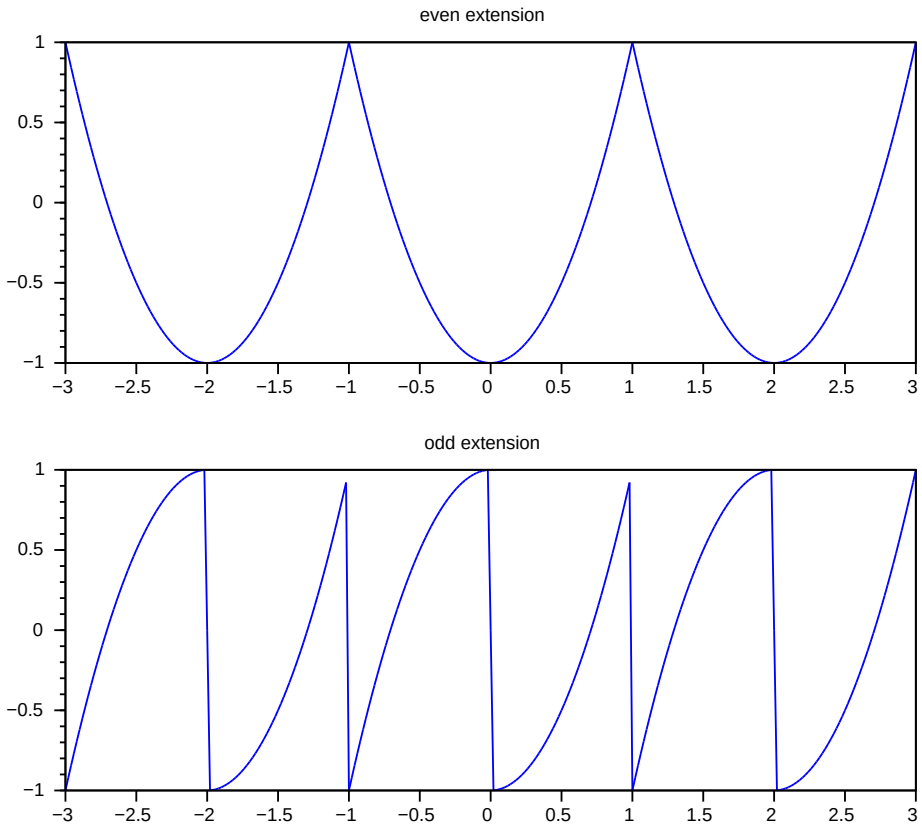
From the convergence Theorem, we know that, at a discontinuous point, the Fourier series converges to the mid value of the left and right limits. This implies an error that is equal to half of the size of the jump at this point. This error will not become smaller by taking more terms in the partial sum.

In practice, when one has the choice, it would always be recommended to choose the expansion that does NOT has discontinuities, if possible. So even expansions should be preferred for accuracy.

We give another example, on convergence of Fourier series, in connection with even and odd periodic extensions.

Example 2. Let $f(x) = 2x^2 - 1$ be defined on the interval $x \in [0, 1]$. Sketch 3 periods of it even and odd periodic extension. Where do their Fourier cosine and sin series converge to at the points $x = 0, 0.5, 1, 100, 151.5$?

Answer. Even and odd extensions are plotted below:



We see that the even extension is a continuous function, so Fourier cosine series converges to the function value, and the convergence is faster.

However, the odd extension is discontinuous at $x = 0, \pm 1, \pm 2, \cdots$, and the Fourier sine series will converge to the mid value of the left and right limits.

We put these value in a table.

x	0	0.5	1	100	151.5
even	-1	-0.5	1	-1	-0.5
odd	0	-0.5	0	0	0.5

10.4 Two-Point Boundary Value Problems; Eigenvalue Problems

Let $y(x)$ be the unknown, and x is the space variable. We consider 2nd order linear ODE

$$y'' + p(x)y' + q(x)y = g(x)$$

over the interval $x \in [x_1, x_2]$, with the boundary conditions

$$y(x_1) = y_1, \qquad y(x_2) = y_2.$$

Since now the conditions are given at the two boundary points, this is called a *two-point boundary value problem*.

If $y_1 = y_2 = 0$, we called this homogeneous boundary conditions.

NB! Boundary conditions could of other forms, such as $y'(x_1) = 0$ etc.

Solution strategy: Find the general solution (as in Chapter 3), then, use the boundary conditions to determine the constants c_1, c_2.

Example 1. Solve the boundary value problem

$$y'' + y = 0, \qquad y(0) = 1, \quad y(\pi/2) = 0.$$

Answer. By characteristic equation $r^2 + 1 = 0$, $r_{1,2} = \pm i$, we get the general solution

$$y(x) = c_1 \cos x + c_2 \sin x.$$

We now put in the boundary conditions, and get

$$c_1 = 1, \quad c_2 = 0$$

so the solution is $y(x) = \cos x$.

Example 2. Boundary conditions could change the solution. In the previous example, we now assume the boundary conditions

$$y(0) = 0, \qquad y(\pi) = 2.$$

From $y(0) = 0$ we get $c_1 = 0$. From $y(\pi) = 2$, we get $c_1 = 2$, which is contradictory to the first condition. Therefore, there is no solution.

Now we assume different boundary conditions

$$y(0) = 0, \qquad y(\pi) = 0.$$

Then, $c_1 = 0$, and c_2 can be arbitrary, so $y(x) = c_2 \sin x$ is a solution for any c_2.

Example 3. Solve

$$y'' + 4y = \cos x, \qquad y'(0) = 0, \quad y'(\pi) = 0.$$

Answer. Since $r^2 + 4 = 0$, and $r_{1,2} = \pm 2i$, the general solution for the homogeneous equation is

$$y_H(x) = c_1 \cos 2x + c_2 \sin 2x.$$

We now find a particular solution for the non-homogeneous equation. We guess the form $Y = A \cos x$. Then $Y'' = -A \cos x$, so

$$-A \cos x + 4A \cos x = \cos x, \quad \Rightarrow \quad 3A = 1, \quad A = \frac{1}{3}.$$

This gives the general solution

$$y(x) = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{3} \cos x.$$

To check the boundary condition, we first differentiate y

$$y'(x) = -2c_1 \sin 2x + 2c_2 \cos 2x - \frac{1}{3}\sin x.$$

Then,

$$y'(0) = 0 \quad \Rightarrow \quad 2c_2 = 0, \quad \Rightarrow \quad c_2 = 0$$

and

$$y'(\pi) = 0, \quad \Rightarrow \quad 2c_2 = 0, \quad \Rightarrow \quad c_2 = 0.$$

Then, c_1 remain arbitrary. We conclude

$$y(x) = c\cos 2x + \frac{1}{3}\cos x$$

is the solution, for arbitrary c.

Important observation: For two-point boundary value problems, the existence and uniqueness theorem is automatic! This is very different from the initial value problems we studied in Chapter 3. For linear two-point boundary value problems, there are 3 possibilities: (i) There is one uniqueness solution, (ii) There are no solutions at all; or (iii) There are infinitely many solutions.

Comparing this to the solution of two linear equations with two unknowns, do we see the similarity?

Eigenvalue problems. (compare to eigenvalues of a matrix.) Consider the problem:

$$y'' + \lambda y = 0, \qquad y(0) = 0, \quad y(L) = 0. \tag{$*$}$$

We are interested in non-trivial solutions (i.e., $y \equiv 0$ is not considered because it is trivial), and possible values of λ that would give us non-trivial solutions.

Note that it is important to have **homogeneous boundary conditions** for eigenvalue problems!

This type of problems is an important building block in series solutions of partial differential equations. For example, this is part of the solution for a vibrating string, where $u(x,t)$ is the vertical displacement of the string at position x, and the string is horizontally placed. There, it turns out (as we will study later), that the solution takes the form $u(x,t) = y(x)G(t)$, where $y(x)$ satisfies the eigenvalue problem.

The eigenvalue problem is a two-point boundary value problem. Depending on the boundary condition, it might or might not have non-trivial solutions.

If for certain value λ_n we find a nontrivial solution $y_n(x)$, then, λ_n is called an **eigenvalue**, and y_n is the corresponding **eigenfunction**.

Example 1. We now attempt to solve the problem in (*). The general solution depends on the roots, i.e., on the sign of λ. We have 3 situations:

(1). If $\lambda < 0$, we write $\lambda = -k^2$, where $k > 0$. Then, $r^2 = k^2$, so $r_1 = -k, r_2 = k$, and the general solution is

$$y(x) = c_1 e^{kt} + c_2 e^{-kt}$$

By boundary conditions, we must have

$$c_1 + c_2 = 0, \qquad c_1 e^{kL} + c_2 e^{-kL} = 0$$

which gives the solution $c_1 = c_2 = 0$. Then $y(x) = 0$, which is a trivial solution. We discard it.

(2). If $\lambda = 0$, then $y'' = 0$, and $y(x)$ must be a linear function. With the zero boundary conditions, we conclude $y(x) = 0$, therefore trivial.

(3). If $\lambda > 0$, we write $\lambda = k^2$, for $k = \sqrt{\lambda} > 0$. Then, $r^2 = -k^2$, and $r_{1,2} = \pm ik$, and the general solution is

$$y(x) = c_1 \cos kx + c_2 \sin kx$$

We now check the boundary conditions. By $y(0) = 0$, we have

$$y(0) = c_1 = 0$$

which means $y = c_2 \sin kx$. Then, by $y(L) = 0$, we get

$$y(\pi) = c_2 \sin kL.$$

We can either require $c_2 = 0$ or $\sin kL = 0$. If we require $c_2 = 0$, then $y(x) = 0$ which is trivial. So we must require $c_2 \neq 0$ and $\sin kL = 0$. This gives a constraint on the values of k (i.e., λ). Indeed, we must have

$$kL = n\pi, \quad \Rightarrow \quad k = \frac{n\pi}{L}, \quad n = 1, 2, 3, \cdots,$$

We see that we have found a family (infinite size) of eigenvalues and eigenfunctions! Using n as the index, they are

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad y_n(x) = \sin \frac{n\pi x}{L}, \qquad n = 1, 2, 3, \cdots$$

Note that we let the arbitrary constant $c_2 = 1$, since for any arbitrary c_2 will yield an eigenfunction.

Note that these eigenfunctions are precisely part of the trig set used for Fourier series. We can observe these eigenfunction by playing with the slinky.

Example 2. Consider the problem with different boundary conditions

$$y'' + \lambda y = 0, \qquad y'(0) = 0, \quad y'(L) = 0. \tag{$*$}$$

We will find very different eigenvalues and eigenfunctions. We still consider the same 3 cases.

(1). If $\lambda = -k^2 < 0$, then

$$y(x) = c_1 e^{kt} + c_2 e^{-kt}, \qquad y'(x) = kc_1 e^{kt} - kc_2 e^{-kt}.$$

The boundary conditions give

$$kc_1 - kc_2 = 0, \quad kc_1 e^{kL} - kc_2 e^{-kL} = 0, \quad \Rightarrow \quad c_1 = c_2 = 0$$

which gives only the trivial solution.

(2). If $\lambda = 0$, then $y'' = 0$, so $y(x) = Ax + B$, and $y'(x) = A$. By boundary condition, we must have $A = 0$, but B remains arbitrary. So we found an eigenpair:

$$\lambda_0 = 0, \qquad y_0(x) = 1.$$

(3). If $\lambda = k^2 > 0$, then

$$y(x) = c_1 \cos kx + c_2 \sin kx, \qquad y'(x) = -kc_1 \sin kx + kc_2 \cos kx.$$

We now check the boundary conditions:

$$y'(0) = 0, \quad \Rightarrow \quad kc_2 = 0, \quad \Rightarrow \quad c_2 = 0$$

and

$$y'(L) = 0, \quad \Rightarrow \quad -kc_1 \sin kL = 0$$

If $c_1 = 0$, we get trivial solution. So $c_1 \neq 0$. Then, we must have

$$\sin kL = 0, \quad \Rightarrow \quad kL = n\pi, \quad \Rightarrow \quad k = \frac{n\pi}{L}, \quad n = 1, 2, 3, \cdots$$

For each k, we get a pair of eigenvalue and eigenfunction

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad y_n(x) = \cos \frac{n\pi}{L} x, \qquad n = 1, 2, 3, \cdots$$

One could combine the results in (2) and (3), and get

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad y_n(x) = \cos \frac{n\pi}{L} x, \qquad n = 0, 1, 2, \cdots$$

Note that these are also a part of the trig set used in Fourier series!

Observation.

- We notice that, different types of boundary conditions would give very different eigenvalues and eigenfunctions!
- In these two examples, the eigenfunctions are sine and cosine functions, in the same form as the trig set we use in Fourier series. Recall that the trig set is a mutually orthogonal set. So, for each of these two eigenvalue problems, the set of eigenfunctions are mutually orthogonal. In fact, this is a more general property for eigenfunctions. One can define proper inner product such that eigenfunctions for the same eigenvalue problem would always form a mutually orthogonal set. (If you are familiar with eigenvectors of a symmetric matrix, they also form an orthogonal basis!)

Example 3. Find all positive eigenvalues and their corresponding eigenfunctions of the problem

$$y'' + \lambda y = 0, \qquad y(0) = 0, \quad y'(L) = 0.$$

Answer. Since $\lambda > 0$, we write $\lambda = k^2$ where $k > 0$, and the general solution is

$$y(x) = c_1 \cos kx + c_2 \sin kx, \qquad y'(x) = -kc_1 \sin kx + kc_2 \cos kx.$$

We now check the boundary conditions. First,

$$y(0) = 0 \quad \Rightarrow \quad c_1 = 0$$

so the answer is simplified to

$$y(x) = c_2 \sin kx, \qquad y'(x) = kc_2 \cos kx.$$

The 2nd boundary condition gives

$$y'(L) = 0 \quad \Rightarrow \quad kc_2 \cos kL = 0, \quad \Rightarrow \quad \cos kL = 0$$

which implies

$$kL = (n + \frac{1}{2})\pi \quad \Rightarrow \quad k_n = \frac{\pi}{L}(n + \frac{1}{2}), \qquad n = 1, 2, 3, \cdots$$

We get the eigenvalues λ_n and the corresponding eigenfunction y_n as

$$\lambda_n = k_n^2 = \left(\frac{\pi(n + 1/2)}{L}\right)^2, \qquad y_n = \sin \frac{\pi(n + 1/2)x}{L}, \qquad n = 1, 2, 3, \cdots .$$

(Optional) Sometimes, with more complicated boundary conditions, eigenvalues could be obtained through graphs.

Example 4. (optional) Find all <u>positive</u> eigenvalues λ_n and their corresponding eigenfunctions $u_n(x)$ of the problem

$$u'' + \lambda u = 0, \qquad u'(0) = 0, \quad u(\pi) + u'(\pi) = 0.$$

Answer. Since $\lambda > 0$, we write $\lambda = k^2$ where $k > 0$. The general solution is

$$u(x) = c_1 \cos kx + c_2 \sin kx, \qquad u'(x) = -kc_1 \sin kx + kc_2 \cos kx.$$

We now check the boundary conditions. First,

$$u'(0) = 0 \quad \Rightarrow \quad kc_2 = 0, \quad \Rightarrow \quad c_2 = 0$$

so the general solution is simplified

$$u(x) = c_1 \cos kx, \qquad u'(x) = -kc_1 \sin kx.$$

By the 2nd boundary condition

$$u(\pi) + u'(\pi) = 0 \quad \Rightarrow \quad c_1 \cos k\pi - kc_1 \sin k\pi = 0, \quad \Rightarrow \quad c_1 \cos k\pi = kc_1 \sin k\pi.$$

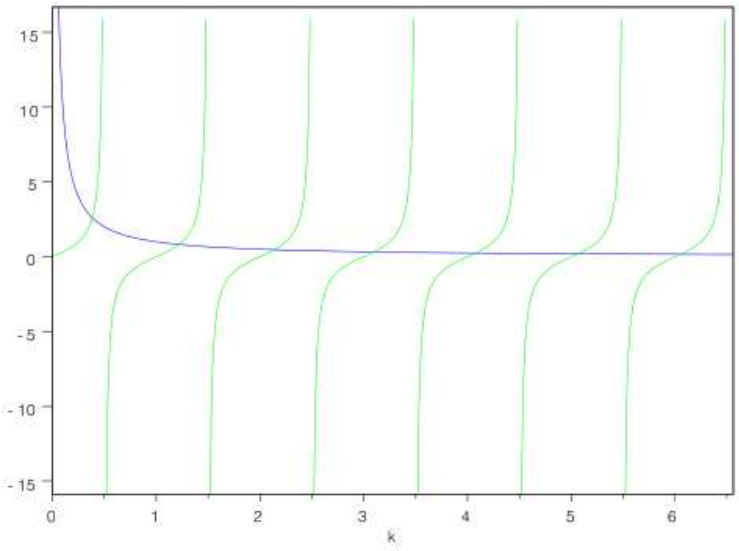
Since $c_1 \neq 0$ (otherwise trivial), we must have

$$\cos k\pi = k \sin k\pi, \quad \Rightarrow \quad \frac{1}{k} = \tan k\pi.$$

To find the values of k that satisfy this relation, we can plot the two functions

$$f_1(k) = \frac{1}{k}, \qquad f_2(k) = \tan k\pi$$

on the same graph, for $k > 0$, and we look for intersection points. See plot below:



In this case, one finds infinitely many intersection points. The k-coordinates of all these points give all the eigenvalues. Marking the p-coordinate of these intersection points as p_n for $n = 1, 2, \cdots$, we get the eigenvalues and the eigenfunctions

$$\lambda_n = p_n^2, \qquad u_n = \cos p_n x, \qquad n = 1, 2, \cdots .$$

We observe that, as n grows bigger, the interception point gets closer to the k-axis, so $\lambda_n \approx n-1$ for large n.

Summary: It would be useful to memorize the solutions to the following simple eigenvalue problems:

- Homogeneous Dirichlet boundary condition:

$$y'' + \lambda y = 0, \qquad y(0) = 0, \quad y(L) = 0.$$

The solutions are

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad y_n(x) = \sin \frac{n\pi x}{L}, \qquad n = 1, 2, 3, \cdots .$$

- Homogeneous Neumann boundary condition:

$$y'' + \lambda y = 0, \qquad y'(0) = 0, \quad y'(L) = 0.$$

The solutions are

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad y_n(x) = \cos\frac{n\pi x}{L}, \quad n = 0, 1, 2, \cdots .$$

Chapter 11

Partial Differential Equations

11.1 Basic Concepts

Definition of PDE: an equation with partial derivatives of the unknown u that depends on several variables, e.g., $u(x,t)$ or $u(x,y)$ etc.

Order of PDE = the highest order of derivatives

Linear PDE: the terms with u and its derivatives are in a linear form

Non-liear PDE: otherwise

Homogeneous: each term contains u or its derivatives

Non-homogeneous: otherwise

Notations:

$$u_t = \frac{\partial u}{\partial t}, \quad u_x = \frac{\partial u}{\partial x}, \quad u_{xx} = \frac{\partial^2 u}{\partial x^2}, \quad \text{etc.}$$

Examples of 2nd order linear PDEs:

- $u_{tt} = c^2 u_{xx}$, 1D wave equation
- $u_t = c^2 u_{xx}$, 1D heat equation
- $u_{xx} + u_{yy} = 0$, 2D Laplace equation
- $u_{xx} + u_{yy} = f(x,y)$, 2D Poisson equation (non-homogeneous)
- $u_t = u_{xx} + u_{yy}$, 2D heat equation

Concept of solution: u is a solution if it satisfies the equation, and any boundary or initial conditions if given.

The students should be able to verify if a given function is a solution of a certain equation.

Fundamental principle of superposition for linear PDEs:

- Homogeneous:

$$\mathcal{L}(u) = 0 \tag{11.1}$$

Principle of superposition: If u_1 and u_2 are two solutions, i.e., $\mathcal{L}(u_1) = 0$ and $\mathcal{L}(u_2) = 0$, so is $u = c_1 u_1 + c_2 u_2$ with arbitrary constants c_1, c_2.

- Non-homogeneous:
$$\mathcal{L}(u) = f, \tag{11.2}$$
  - If u_H solves (11.1) and u_p solves (11.2), then $u = u_H + u_p$ solves (11.2).
  - If u_1 solves $\mathcal{L}(u_1) = f_1$ and $\mathcal{L}(u_2) = f_2$, then $u = c_1u_1 + c_2u_2$ solves $\mathcal{L}(u) = c_1f_1 + c_2f_2$.

How to separate variables?

Let $u(x,t)$ be the solution of some PDE, with suitable boundary and initial conditions. We seek solutions of the form
$$u(x,t) = F(x)G(t)$$
where $F(x)$ is only a function of x, and $G(t)$ is only a function of t. Then, the partial derivatives are
$$u_x = F'(x)G(t), \quad u_{xx} = F''(x)G(t), \quad u_t = F(x)G'(t), \quad u_{tt} = F(x)G''(t), \quad u_{xt} = F'(x)G'(t).$$
We will then put these into the equation, and try to separate x and t on different sides of the equation.

11.2 Heat Equation in 1D; Solution by Separation of Variable and Fourier series

Consider the heat equation in 1D
$$u_t = c^2u_{xx}, \quad 0 < x < L, \quad t > 0. \tag{11.3}$$
Here $u(x,t)$ measures the temperature of a rod with length L. We first assign the boundary conditions
$$u(0,t) = 0, \quad u(L,t) = 0, \qquad t > 0$$
This means, we fix the temperature of the two end-points of the rod to be 0. This type of boundary condition is called Dirichlet condition.

We also have the initial condition
$$u(x,0) = f(x), \quad 0 < x < L$$
gives the initial temperature distribution.

The main technique to find an explicit form of the solution is **Separation of variables**.

Step 1. Separating variables. Seek solution of the form
$$u(x,t) = F(x) \cdot G(t)$$
Then
$$u_x = F'(x)G(t), \quad u_{xx} = F''(x)G(t), \quad u_t = F(x)G'(t),$$

Plug these into the equation (11.3),

$$F(x)G'(t) = c^2F''(x)G(t), \quad \rightarrow \quad \frac{F''(x)}{F(x)} = \frac{G'(t)}{c^2G(t)} = \text{constant} = -k$$

We end up with 2 ODEs,

$$F''(x) + kF(x) = 0, \qquad G'(t) + c^2kG(t) = 0.$$

Step 2. Solve for $F(x)$. Fit in the boundary conditions

$$u(0,t) = F(0)G(t) = 0, \quad u(L,t) = F(L)G(t) = 0, \quad t > 0$$

which implies

$$F(0) = 0, \quad F(L) = 0$$

We now have the following *eigenvalue problem* for $F(x)$

$$F'' + pF = 0, \qquad F(0) = 0, \quad F(L) = 0.$$

This is an example we had earlier. We have

$$p_n = w_n^2, \quad w_n = \frac{n\pi}{L}, \qquad F_n(x) = \sin w_n x, \qquad n = 1, 2, 3, \cdots$$

Step 3. Solution for $G(t)$. For any given n, we get a solution $G_n(t)$, which solves

$$G'(t) + c^2w_n^2G(t) = 0$$

Call now $\lambda_n = cw_n = \frac{n\pi c}{L}$, then

$$G'(t) + \lambda_n^2G(t) = 0, \qquad G_n(t) = C_ne^{-\lambda_n^2 t}$$

where C_n is arbitrary.

Step 4. We now get the *eigenvalues* and their *eigenfunctions*

$$\lambda_n = \frac{n\pi c}{L}, \qquad u_n(x,t) = C_ne^{-\lambda_n^2 t}\sin w_n x, \qquad n = 1, 2, 3, \cdots$$

The sum of them is also a solution. This gives the *formal solution*

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} C_ne^{-\lambda_n^2 t}\sin w_n x, \qquad w_n = \frac{n\pi}{L}, \qquad \lambda_n = \frac{n\pi c}{L}. \tag{11.4}$$

Step 5. Find C_n by initial condition. Fit in the initial condition

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin w_n x = f(x)$$

we conclude that C_n must the Fourier sine coefficients for the odd periodic half-range extension of $f(x)$, i.e.,

$$C_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx, \qquad n = 1, 2, 3, \cdots. \tag{11.5}$$

Summary: The formal solution for initial and boundary value problem for the heat equation

$$\begin{aligned} &u_t = c^2 u_{xx}, \quad 0 < x < L, \quad t > 0. \\ &u(0,t) = 0, \quad u(L,t) = 0, \qquad t > 0 \\ &u(x,0) = f(x), \quad 0 < x < L \end{aligned}$$

is

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} C_n e^{-(\frac{n\pi c}{L})^2 t} \sin \frac{n\pi x}{L}$$

where

$$C_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx, \qquad n = 1, 2, 3, \cdots.$$

Discussions on solutions:

- Harmonic oscillation in x, exponential decay in t:
- Speed of decay depending on $\lambda_n = n\pi c/L$. Faster decay for larger n, meaning the high frequency components are 'killed' quickly. After a while, what remain in the solution are the terms with small n.
- As $t \to \infty$, we have $u(x,t) = 0$ for all x. This is called **asymptotic solution** or **steady state** of the heat equation.

Example 1. Let $c = 1$ and $L = 1$. If $f(x) = 10 \sin \pi x$, then we have $C_1 = 10$ and all other $C_n = 0$, so the solution is

$$u(x,t) = 10 e^{-\pi^2 t} \sin \pi x.$$

At $t = 1$, the amplitude of the solution is

$$\max_x |u(x,1)| = 10 e^{-\pi^2} \approx 5.17 \times 10^{-4}.$$

If now we let $f(x) = 10 \sin 3\pi x$, then $C_3 = 10$ and all other $C_n = 0$, and the solution is

$$u(x,t) = 10 e^{-9\pi^2 t} \sin 3\pi x.$$

At $t = 1$, the amplitude is

$$\max_x |u(x,1)| = 10 e^{-9\pi^2} \approx 2.65 \times 10^{-38}.$$

Note that this amplitude is much smaller.

If the initial temperature is

$$f(x) = 10 \sin \pi x + 10 \sin 3\pi x,$$

the solution would be

$$u(x,t) = 10 e^{-\pi^2 t} \sin \pi x + 10 e^{-9\pi^2 t} \sin 3\pi x$$

and the amplitude at $t = 1$ is

$$\max_x |u(x,1)| = 5.17 \times 10^{-4} + 2.65 \times 10^{-38}.$$

Clearly, the first term dominates.

Neumann boundary condition. We now consider a new BC: (insulated)

$$u_x(0,t) = 0, \quad u_x(L,t) = 0$$

This means that the 2 ends are insulated, and no heat flows through.

Following the same setting, we get the eigenvalue problem for $F(x)$ as

$$F''(x) + pF(x) = 0, \qquad F'(0) = 0, \quad F'(L) = 0$$

From Example 2 in the last Chapter, we have only non-negative p. Let $p = w^2$ with $w \geq 0$. We have the eigenvalues w_n and the eigenfunctions $F_n(x)$:

$$w_n = \frac{n\pi}{L}, \quad F_n(x) = \cos w_n x, \quad n = 0, 1, 2, \cdots$$

Note that $n = 0$ is permitted in this solution. The solution for $G(t)$ remains the same

$$G_n(t) = C_n e^{-\lambda_n^2 t}, \quad \lambda_n = \frac{cn\pi}{L}, \quad n = 0, 1, 2, \cdots$$

This gives the eigenfunctions

$$u_n(x,t) = C_n e^{-\lambda_n^2 t} \cos w_n x, \quad n = 0, 1, 2, \cdots$$

which leads to the formal solution

$$u(x,t) = C_0 + \sum_{n=1}^{\infty} C_n e^{-\lambda_n^2 t} \cos w_n x, \qquad w_n = \frac{n\pi}{L}, \qquad \lambda_n = \frac{cn\pi}{L}. \tag{11.6}$$

Finally, we fit in the initial condition

$$u(x,0) = C_0 + \sum_{n=1}^{\infty} C_n \cos w_n x = f(x).$$

So, C_n must be the Fourier cosine coefficient for the even half-range expansion of $f(x)$, i.e.,

$$C_0 = \frac{1}{L}\int_0^L f(x)\,dx, \quad C_n = \frac{2}{L}\int_0^L f(x) \cos\frac{n\pi x}{L}\,dx, \qquad n = 1, 2, 3, \cdots. \tag{11.7}$$

Conclusion: The formal solution is given in (11.6) + (11.7).

Discussion on the formal solution:

- harmonic oscillation in x,
- exponential decay in t, except the term C_0. Decay faster for larger n.

- As $t \to \infty$, we get $u \to C_0$, which is the average of $f(x)$ (initial temperature). This is reasonable b/c the bar is insulated.

Steady State: As $t \to \infty$, solution does not change in time anymore, as it reaches a steady state. Call it $U(x)$. Then $U_t = 0$, so $U_{xx} = 0$. So $U(x)$ must be a linear function, i.e., $U(x) = Ax + B$, where A, B are determined by boundary conditions.

Example 2: (add graphs)

(1) If $u(0,t) = a, u(L,t) = b$, then $U(0) = a, U(L) = b$, we get

$$U(x) = a + \frac{b-a}{L}x.$$

(2) If $u(0,t) = a, u_x(L,t) = 0$, then $U(0) = a, U'(L) = 0$, we get

$$U(x) = a.$$

(3) If $u(0,t) = a, u_x(L,t) = b$, then $U(0) = a, U'(L) = b$, we get

$$U(x) = a + bx.$$

Non-homogeneous boundary condition. Take for example

$$u_t = c^2 u_{xx}, \qquad u(0,t) = a, \quad u(L,t) = b.$$

We know that the steady state is $U(x) = a + \frac{b-a}{L}x$. Now define a new variable

$$w(x,t) = u(x,t) - U(x).$$

Then, we have

$$w_t = u_t, \quad w_x = u_x - U'(x), \quad w_{xx} = u_{xx} - U''(x) = u_{xx}$$

so w solve the heat equation:

$$w_t = c^2 w_{xx}$$

Now, we check the BCs for w:

$$w(0,t) = u(0,t) - U(0) = a - a = 0, \quad w(L,t) = u(L,t) - U(L) = b - b = 0$$

which are homogeneous. Then, one can find the solution for w by the standard separation of variables and Fourier series. Once this is done, one can go back to u by

$$u(x,t) = w(x,t) + U(x).$$

We now take an example with non-homogeneous BCs.

Example 3. Consider the heat equation $u_t = u_{xx}$ with the following BCs

$$u(0,t) = 2, \quad u(1,t) = 4,$$

and IC

$$u(x,0) = 2 + 2x - \sin \pi x - 3 \sin 3\pi x.$$

Find the solution $u(x,t)$.

Answer. : Step 1. We first find the steady state. Call it $w(x)$, it satisfies the following two-point boundary value problem

$$w'' = 0, \quad w(0) = 2, \quad w(1) = 4,$$

which gives the solution

$$w(x) = 2 + 2x.$$

Step 2: Let now U be the solution of the heat equation with homogeneous boundary condition

$$U_t = U_{xx}, \qquad U(0,t) = 0, \quad U(1,t) = 0$$

and the initial condition

$$U(x,0) = u(x,0) - w(x) = -\sin \pi x - 3 \sin 3\pi x.$$

The formal solution for U is

$$U(x,t) = \sum_{n=1}^{+\infty} C_n e^{-\lambda_n^2 t} \sin w_n x,$$

where we have

$$L = 1, \qquad w_n = \frac{n\pi}{L} = n\pi, \qquad \lambda_n = \frac{nc\pi}{L} = n\pi$$

so

$$U(x,t) = \sum_{n=1}^{+\infty} C_n e^{-n^2\pi^2 t} \sin n\pi x.$$

Here C_n are Fourier coefficients of the initial data $U(x,0)$. We find only two coefficients that are not 0, namely

$$C_1 = -1, \quad C_3 = -3.$$

This gives us

$$U(x,t) = -e^{-\pi^2 t} \sin \pi x - 3e^{-9\pi^2 t} \sin 3\pi x.$$

Step 3. Putting them together, we get the solution

$$u(x,t) = w(x) + U(x,t) = 2 + 2x - e^{-\pi^2 t} \sin \pi x - 3e^{-9\pi^2 t} \sin 3\pi x.$$

More on separation of variables. This technique could be applied to a more general class of PDEs. It is not difficult to check whether an equation is separable. After separating the variables, one needs to fit in the boundary condition.

Example 1. Consider

$$x^2 u_{tt} - (x+1)t^2 u_{xx} = 0$$

with boundary conditions

$$u(0,t) = 0, \qquad u(L,t) = 5,$$

and initial condition

$$u(x,0) = \cos(x).$$

Letting $u = F(x)G(t)$, we have

$$x^2 F(x)G''(t) - (x+1)t^2 F''(x)G(t) = 0, \quad \Rightarrow \quad x^2 F(x)G''(t) = (x+1)t^2 F''(x)G(t),$$

which is separable

$$\frac{G''(t)}{t^2 G(t)} = \frac{(x+1)F''(x)}{x^2 F(x)} = -\lambda$$

which gives two ODEs:

$$(x+1)F''(x) + \lambda x^2 F(x) = 0, \qquad F(0) = 0, \quad F(L) = 5$$

and

$$G''(t) + \lambda t^2 G(t) = 0.$$

Example 2. Consider

$$u_{xx} - 6u_{tx} + 9u_{tt} = 0$$

Letting $u = F(x)G(t)$, we have

$$F''(x)G(t) - 6F'(x)G'(t) + 9F(x)G''(t) = 0$$

It is not possible to separate the variables.

Example 3. Consider

$$u_{tt} = 4u_{xx} - 5u_x + u$$

Then,

$$FG'' = 4F''G - 5F'G + FG, \quad \Rightarrow \quad FG'' = G(4F'' - 5F' + F)$$

which is separable

$$\frac{G''}{G} = \frac{4F'' - 5F' + F}{F} = -\lambda$$

We get two ODEs:

$$4F'' - 5F' + F + \lambda F = 0$$

and

$$G'' + \lambda G = 0.$$

How to solve these eigenvalue problem, that is a different question.

11.3 Solutions of Wave Equation by Fourier Series

Consider the 1D wave equation

$$u_{tt} = c^2 u_{xx}, \qquad t > 0, \quad 0 < x < L. \tag{11.8}$$

Here $u(x,t)$ is the unknown variable. If this is a model of vibrating string, then the constant c^2 has physical meaning, namely, $c^2 = T/\rho$ where T is the tension and ρ is the density such that $\rho\Delta x$ is the mass of the string segment.

For the first example, we assign the following boundary and initial conditions

$$\text{(B.C.'s)} \qquad u(0,t) = 0, \quad u(L,t) = 0, \qquad t > 0. \tag{11.9}$$

$$\text{(I.C.'s)} \qquad u(x,0) = f(x), \quad u_t(x,0) = g(x), \qquad 0 < x < L. \tag{11.10}$$

We now solve this equation by separation of variables and Fourier series.

Step 1. Let $u(x,t) = F(x) \cdot G(t)$, then

$$u_{tt} = F(x)G''(t), \qquad u_{xx} = F''(x)G(t)$$

and we get

$$F(x)G''(t) = c^2 F''(x)G(t), \quad \rightarrow \quad \frac{F''(x)}{F(x)} = \frac{G''(t)}{c^2 G(t)} = -p \text{ (constant)}.$$

This gives us 2 ODEs

$$F''(x) + pF(x) = 0, \qquad G''(t) + c^2 pG(t) = 0.$$

Step 2. We now solve for $F(x)$. The BCs in (11.9) gives

$$F(0) = 0, \qquad F(L) = 0.$$

We are now familiar with this eigenvalue problem, and the solutions are

$$p = w_n^2, \quad w_n = \frac{n\pi}{L}, \quad F_n(x) = \sin w_n x, \qquad n = 1, 2, 3, \cdots .$$

Step 3. Now, for a given n, the ODE for $G(t)$ takes the form,

$$G''(t) + \lambda_n^2 G(t) = 0, \qquad \lambda_n = n w_n.$$

The general solution is

$$G_n(t) = C_n \cos \lambda_n t + D_n \sin \lambda_n t, \qquad n = 1, 2, 3, \cdots .$$

We let

$$u_n(x,t) = F_n(x)G_n(t) = (C_n \cos \lambda_n t + D_n \sin \lambda_n t) \sin w_n x, \qquad n = 1, 2, 3, \cdots$$

Here λ_n are *eigenvalues*, and $u_n(x,t)$ are *eigenfunction*. The set of eigenvalues $\{\lambda_1, \lambda_2, \cdots\}$ are called *the spectrum*.

Discussion on eigenfunctions:

- Harmonic oscillation in x.
- $G(t)$ gives change of amplitude in t, harmonic oscillation. Draw a figure (for ex. with $n = 2$) and explain.
- Different n gives different motion. These are called *modes*. Draw figures of modes with $n = 1, 2, 3, 4$. With $n = 1$, we have the fundamental mode. $n = 2$ gives an octave, $n = 3$ gives an octave and a fifth, $n = 4$ gives 2 octaves.

Step 4. We now construct solution of the wave equation (11.8). The formal solution is

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} (C_n \cos \lambda_n t + D_n \sin \lambda_n t) \sin w_n x. \tag{11.11}$$

The coefficients C_n, D_n are chosen such that the ICs (11.10) are satisfied.

We first check the IC $u(x,0) = f(x)$. This gives

$$\sum_{n=1}^{\infty} C_n \sin w_n x = f(x),$$

therefore, C_n must be Fourier sine coefficient of $f(x)$, namely

$$C_n = \frac{2}{L} \int_0^L f(x) \sin w_n x \, dx, \qquad n = 1, 2, 3, \cdots . \tag{11.12}$$

For the 2nd IC $u_t(x,0) = g(x)$, we first differentiate the solution

$$u_t(x,t) = \sum_{n=1}^{\infty} (-\lambda_n C_n \sin \lambda_n t + \lambda_n D_n \cos \lambda_n t) \sin w_n x.$$

Then, we have

$$u_t(x,0) = \sum_{n=1}^{\infty} \lambda_n D_n \sin w_n x = g(x),$$

which is a Fourier sine series for $g(x)$, and we have

$$D_n = \frac{1}{\lambda_n} \cdot \frac{2}{L} \int_0^L g(x) \sin w_n x \, dx, \qquad n = 1, 2, 3, \cdots . \tag{11.13}$$

In summary, the formal solution for the wave equation (11.8) with BC (11.9) and IC (11.10) is given in (11.11) with (11.12)+(11.13).

NB! Note that here the function $f(x)$ and $g(x)$ are extended to the whole real line by the odd half-range expension.

Discussion on the formal solution:

- If $g(x) = 0$, then $D_n = 0$ for all n.
- If $f(x) = 0$, then $C_n = 0$ for all n.

Remark: This solution takes the form of a Fourier series. It is very hard to see what's going on in the solution. In particular, one can not get any intuition on wave phenomenon in the solution.

On the other hand, we can manipulate the solution by some trig identity. Consider the simpler case where $g(x) = 0$ so $D_n = 0$, the solution takes the form

$$u(x,t) = \sum_{n=1}^{\infty} C_n \cos \lambda_n t \sin w_n x, \qquad \lambda_n = cw_n, \quad w_n = \frac{n\pi}{L}.$$

Recall the trig identity

$$\sin A \cos B = \frac{1}{2}\sin(A-B) + \frac{1}{2}\sin(A+B).$$

We have

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} C_n \frac{1}{2}\left[\sin w_n(x+ct) + \sin w_n(x-ct)\right] \\ &= \frac{1}{2}\sum_{n=1}^{\infty} C_n \sin w_n(x+ct) + \frac{1}{2}\sum_{n=1}^{\infty} C_n \sin w_n(x-ct) \\ &= \frac{1}{2}f^*(x+ct) + \frac{1}{2}f^*(x-ct), \end{aligned} \tag{11.14}$$

where f^* is the odd half-range periodic expansion of f.

Note that $f(0) = 0 = f(l)$, then f^* is continuous.

Wave interpretation: (make some graphs.)

- $f^*(x-ct)$: f^* travels with speed c as time t goes. (wave travels to the right.)
- $f^*(x+ct)$: f^* travels with speed $-c$ as time t goes. (wave travels to the left.)

New meaning of solution of wave equation with $g(x) = 0$: The initial deflection f^* is split into two equal parts, one travels to the left, one to the right, with the speed c, and superposition of them gives the solution.

Remark: Such a phenomenon can also be observed when $g(x)$ is not 0. The computation would be somewhat more involving. Try it on your own for practice.

Example : Do the triangle wave.

First write out solution in Fourier series.

Then, use (11.14), and sketch the graphs of solutions at various time t, as follows.

- First sketch $f(x)$ on $0 < x < L$, then extend it to odd, and periodic.
- Sketch for $t = 0, \frac{L}{4c}, \frac{2L}{4c}, \frac{3L}{4c}, \frac{4L}{4c}$, the graph of $f^*(x+ct)$ (in white) and $f^*(x-ct)$ (in color) on the same graph, then the sum of them on a separate graph.
- summarize the behavior.

11.4 Laplace Equation in 2D (probably skip)

Consider the Laplace equation in 2D

$$u_{xx} + u_{yy} = 0$$

Some applications of this equation: steady state of 2D heat equation, electrostatic potential, minimum surface problem, etc.

To begin with, we consider a rectangular domain R: $0 < x < a, 0 < y < b$.

There are different types of BCs one can assign:

- Dirichlet BC: u is prescribed on the boundary.
- Neumann BC: $u_n = \frac{\partial u}{\partial n}$ normal derivative is given on the boundary.
- Robin BC: mixed BC, combine the Dirichlet and Neumann BCs.

We now start with Dirichlet BC. Let

$$u(0, y) = 0, \quad u(a, y) = 0, \qquad u(x, 0) = 0, \quad u(x, b) = f(x). \tag{11.15}$$

Note that on 3 sides we have homogeneous conditions, and only on one side the condition is non-homogeneous.

This will take several steps.

Step 1. Separating variables. Let

$$u(x, y) = F(x) \cdot G(y).$$

Then

$$u_{xx} = F''(x)G(y), \qquad u_{yy} = F(x)G''(y),$$

so

$$\frac{F''(x)}{F(x)} = -\frac{G''(y)}{G(y)} = -p \quad \text{(constant)}$$

This gives us 2 ODEs

$$F''(x) + pF(x) = 0, \qquad G''(y) - pG(y) = 0.$$

Step 2. Solve for F. From the BCs, we have

$$F(0) = 0, \quad F(a) = 0.$$

We solve this eigenvalue problem for F. This we have done several times. We have

$$p = w_n^2, \quad w_n = \frac{n\pi}{a}, \qquad F_n(x) = \sin w_n x, \qquad n = 1, 2, 3, \cdots .$$

Step 3. Solve for $G(y)$. For each n, we have

$$G''(y) - w_n^2 G(y) = 0$$

which gives the general solution

$$G_n(y) = A_n e^{w_n y} + B_n e^{-w_n y}.$$

The BC at $y = 0$ gives the condition for G

$$G(0) = 0$$

Then, we have

$$A_n + B_n = 0, \quad B_n = -A_n$$

so

$$G_n(y) = A_n(e^{w_n y} - e^{-w_n y}) = 2A_n \sinh w_n y.$$

Recall that

$$2\sinh x = e^x - e^{-x}, \quad 2\cosh x = e^x + e^{-x}.$$

Since A_n is arbitrary so far, so is $2A_n$, and we will simply call it A_n. Then

$$G_n(y) = A_n \sinh w_n y.$$

Step 4. Put together, we get the eigenvalues and eigenfunctions

$$\lambda_n = w_n^2, \quad w_n = \frac{n\pi}{a}, \qquad u_n = F_n(x)G_n(y) = A_n \sinh w_n y \sin w_n x, \qquad n = 1, 2, 3, \cdots .$$

The formal solution is

$$u(x, y) = \sum_{n=1}^{\infty} u_n(x, y) = \sum_{n=1}^{\infty} A_n \sinh w_n y \sin w_n x. \tag{11.16}$$

Now we fit in the last BC, i.e., $u(x, b) = f(x)$:

$$\sum_{n=1}^{\infty} A_n \sinh w_n b \sin w_n x = f(x)$$

This is a Fourier sine series for $f(x)$, and we must have

$$A_n \sinh w_n b = \frac{2}{a} \int_0^1 f(x) \sin w_n x \, dx$$

which gives the coefficients A_n

$$A_n = \frac{2}{a \sinh(w_n b)} \int_0^1 f(x) \sin w_n x \, dx \tag{11.17}$$

In summary, the formal solution is given in (11.16) + (11.17).

How does the solution look like? Consider the minimum surface problem. There is the maximum principle: The max or min value only occur on the boundary.

Example : with a different boundary condition: (graph..)

$$u(0,y)=0, \quad u(a,y)=0, \qquad u(x,0)=g(x), \quad u(x,b)=0. \tag{11.18}$$

One can carry out the same steps, and reach

$$w_n=\frac{n\pi}{a}, \quad F_n(x)=\sin w_n x, \quad G_n(y)=A_n e^{w_n y}+B_n e^{-w_n y}, \qquad n=1,2,3,\cdots .$$

Now, fit in the boundary condition $u(x,b)=0$, which gives $G_n(b)=0$, we have

$$A_n e^{w_n b}+B_n e^{-w_n b}=0, \qquad \to \quad A_n e^{w_n b}=-B_n e^{-w_n b}.$$

We may write

$$A_n e^{w_n b}=C_n, \qquad B_n e^{-w_n b}=-C_n$$

so

$$A_n=C_n e^{-w_n b}, \qquad B_n=-C_n e^{w_n b}.$$

which gives

$$G_n(y)=C_n e^{-w_n b}e^{w_n y}-C_n e^{w_n b}e^{-w_n y}=C_n\left[e^{w_n(y-b)}-e^{-w_n(y-b)}\right]=2C_n\sinh w_n(y-b).$$

Since $2C_n$ is arbitrary, we call it C_n, so

$$G_n(y)=C_n\sinh w_n(y-b), \qquad n=1,2,3,\cdots .$$

This gives the formal solution

$$u(x,y)=\sum_{n=1}^{\infty}u_n(x,y)=\sum_{n=1}^{\infty}C_n\sinh w_n(y-b)\sin w_n x. \tag{11.19}$$

We now fit in the last BC, i.e., $u(x,0)=g(x)$, and we get

$$\sum_{n=1}^{\infty}C_n\sinh(-w_n b)\sin w_n x=g(x).$$

This is a Fourier sine series for $g(x)$, requiring

$$C_n\sinh(-w_n b)=\frac{2}{a}\int_0^a g(x)\sin w_n x\,dx, \qquad n=1,2,3,\cdots .$$

This gives the formula for the coefficient C_n:

$$C_n=-\frac{2}{a\sinh(w_n b)}\int_0^a g(x)\sin w_n x\,dx. \qquad n=1,2,3,\cdots \tag{11.20}$$

In summary, the formal solution is given in (11.19)+(11.20).

Example : If the BCs are now (graphs)

$$u(0,y)=0, \quad u(a,y)=h(y), \qquad u(x,0)=0, \quad u(x,b)=0. \tag{11.21}$$

or

$$u(0,y) = k(y), \quad u(a,y) = 0, \qquad u(x,0) = 0, \quad u(x,b) = 0. \tag{11.22}$$

We can simply switch the roles of x and y, and carry out the whole procedure.

Non-homogeneous BCs: If we have boundary values assigned non-zeros everywhere, what do we do?

$$u(0,y) = k(y), \quad u(a,y) = h(y), \quad u(x,0) = g(x), \quad u(x,b) = f(x). \tag{11.23}$$

Let u_1, u_2, u_3, u_4 be the solutions with BCs (11.15), (11.18), (11.21) and (11.22), respectively. (Make a graph.) Then, set

$$u = u_1 + u_2 + u_3 + u_4$$

By superposition, u solves the Laplace equation, and satisfies the boundary condition in (11.23).

Possible examples on Neumann BC, or an example of mixed BC. ?

11.5 D'Alembert's Solution of Wave Equation

In this section we derive the D'Alembert's solution of wave equation. Consider the wave equation in 1D

$$u_{tt} = c^2 u_{xx} \tag{11.24}$$

We claim that

$$u_1(x,t) = \phi(x+ct), \qquad u_2(x,t) = \psi(x-ct)$$

are solutions for (11.24) for arbitrary functions of ϕ, ψ. This will then imply that

$$u(x,t) = \phi(x+ct) + \psi(x-ct) \tag{11.25}$$

is a solution. This is called *D'Alembert's solution* of wave equation.

Proof. We need to plug u_1 and u_2 into the wave equation (11.24) and check is the equation holds. By the Chain Rule, we have

$$(u_1)_x = \phi'(x+ct), \quad (u_1)_{xx} = \phi''(x+ct),$$

and

$$(u_1)_t = c\phi'(x+ct), \quad (u_1)_{tt} = c^2\phi''(x+ct),$$

We clearly have $(u_1)_{tt} = c^2(u_1)_{xx}$. The proof for u_2 is completely similar.

We now assign the initial conditions

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x), \tag{11.26}$$

and derive the formula for the solution, i.e., determine the functions ϕ, ψ by these ICs in (11.26).

By the condition $u(x,0) = f(x)$, we get

$$\phi(x) + \psi(x) = f(x). \tag{11.27}$$

We differentiate u in t

$$u_t(x,t) = c\phi'(x+ct) - c\psi'(x-ct).$$

Then, the IC $u_t(x,0) = g(x)$ gives

$$c\phi'(x) - c\psi'(x) = g(x), \quad \rightarrow \quad (\phi(x) - \psi(x))' = \frac{1}{c}g(x). \tag{11.28}$$

Integrate (11.28) from x_0 to x, where x_0 is arbitrary,

$$\int_{x_0}^{x} (\phi(s) - \psi(s))'\, ds = \frac{1}{c}\int_{x_0}^{x} g(s)\, ds.$$

we get

$$\phi(x) - \psi(x) = \phi(x_0) - \psi(x_0) + \frac{1}{c}\int_{x_0}^{x} g(s)\, ds.$$

Call $M = \phi(x_0) - \psi(x_0)$, we have

$$\phi(x) - \psi(x) = M + \frac{1}{c}\int_{x_0}^{x} g(s)\, ds. \tag{11.29}$$

Add (11.29) to (11.27) and divide by 2, we get

$$\phi(x) = \frac{M}{2} + \frac{1}{2}f(x) + \frac{1}{2c}\int_{x_0}^{x} g(s)\, ds. \tag{11.30}$$

Then, we can recover ψ

$$\psi(x) = -\frac{M}{2} + \frac{1}{2}f(x) - \frac{1}{2c}\int_{x_0}^{x} g(s)\, ds. \tag{11.31}$$

Plug (11.30)-(11.31) back into (11.25), we get

$$\begin{aligned}
u(x,t) &= \phi(x+ct) + \psi(x-ct) \\
&= \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_{x_0}^{x+ct} g(s)\, ds + \frac{1}{2}f(x-ct) - \frac{1}{2c}\int_{x_0}^{x-ct} g(s)\, ds \\
&= \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x_0}^{x+ct} g(s)\, ds + \frac{1}{2c}\int_{x-ct}^{x_0} g(s)\, ds \\
&= \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\, ds.
\end{aligned}$$

In summary, the solution is

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\, ds. \tag{11.32}$$

The solution consists of two parts, where the first term is caused by the initial deflection $f(x)$, and the second term is from the initial velocity $g(x)$.

Note that, if this is a vibrating string problem, then f, g here in (11.32) are the odd half-range expansions onto the whole real line.

Remark: The solution (11.32) is more general than Fourier Series solution. In fact, there is no requirement for $f(x), g(x)$ to be periodic. For any f, g defined on the whole real line, the formula (11.32) gives the solution of the wave equation.

Example . Solve the wave equation by D'Alembert's formula.

$$u_{tt} = u_{xx}, \qquad u(x,0) = \sin 5x, \quad u_t(x,0) = \frac{1}{5}\cos x$$

Here we have $f(x) = \sin 5x$ and $g(x) = \frac{1}{5}\cos x$. This is just a practice of using the formula (11.32) with $c = 1$. We first work out the integral

$$\int_{x-t}^{x+t} g(s)\,ds = \frac{1}{5}\int_{x-t}^{x+t} \cos s\,ds = \frac{1}{5}(\sin(x+t) - \sin(x-t)).$$

Then, we get the solution

$$\begin{aligned} u(x,t) &= \frac{1}{2}\sin 5(x-t) + \frac{1}{2}\sin(x+t) + \frac{1}{10}(\sin(x+t) - \sin(x-t)) \\ &= \left[\frac{1}{2}\sin 5(x+t) + \frac{1}{10}\sin(x+t)\right] + \left[\frac{1}{2}\sin 5(x-t) - \frac{1}{10}\sin(x-t)\right]. \end{aligned}$$

Note the first term is a function of $x+t$, i.e., $\phi(x+t)$, where

$$\phi(u) = \frac{1}{2}\sin 5u + \frac{1}{10}\sin u,$$

and the second term is a function of $x-t$, i.e., $\psi(x-t)$ where

$$\psi(u) = \frac{1}{2}\sin 5u - \frac{1}{10}\sin u.$$

Characteristics. As we see, solutions of the wave equation can be written as

$$u(x,t) = \phi(x+ct) + \psi(x-ct).$$

This implies:

–$\phi(x+ct)$ is constant along lines of $x+ct = K$ where K is constant;

–$\psi(x-ct)$ is constant along lines of $x-ct = K$ where K is constant;

In the $t-x$ plan,

– $x+ct$ =constant: are straight lines with slope $-1/c$,

– $x-ct$ =constant: are straight lines with slope $1/c$

Draw a graph.

These lines are paths where information is being carried along. They are called *characteristics* of this problem.

Remark:

- This is a more general property for many PDEs, including non-linear PDEs.
- Characteristics lines might not be parallel to each other, or straight lines. The situation could be very complicated...

11.6 Method of Characteristics; Classification of 2nd order linear PDEs.

General form of 2nd order quasi-linear PDEs: Let $u(x, y)$ be the unknown (NB! It could be $u(x, t)$ also...)

$$Au_{xx} + BU_{xy} + Cu_{yy} = F(x, y, u, u_x, u_y). \tag{11.33}$$

Define the *discriminant*

$$\Delta = B^2 - 4AC. \tag{11.34}$$

Type of PDE is determined only by Δ

- If $\Delta > 0$, the PDE is *hyperbolic*. Example: wave equation $u_{tt} - c^2 u_{xx} = 0$;
- If $\Delta = 0$, the PDE is *parabolic*. Example: heat equation $u_t = c^2 u_{xx}$;
- If $\Delta < 0$, the PDE is *elliptic*. Example: Laplace equation $u_{xx} + u_{yy} = 0$.

If $A(x, y), B(x, y), C(x, y)$ are functions, then the equation can changed type. This is called *mixed type*. In general, mixed type equations are difficult.

We now consider the homogeneous case

$$Au_{xx} + BU_{xy} + Cu_{yy} = 0. \tag{11.35}$$

Assume that $y + rx =$constant are the characteristics, i.e., we assume that

$$u(x, y) = \phi(y + rx)$$

is a solution for (11.35) for an arbitrary function ϕ. We now try to find values of r.

By the Chain Rule, we have

$$u_x = -r\phi'(y + rx), \quad u_{xx} = r^2\phi''(y + rx),$$

and

$$u_y = \phi'(y + rx), \quad u_{yy} = \phi''(y + rx), \quad u_{xy} = -r\phi''(y + rx).$$

Plug these back into (11.35), we get

$$Ar^2\phi''(y + rx) - Br\phi''(y + rx) + C\phi''(y + rx) = 0.$$

Since ϕ is arbitrary, then ϕ'' is not zero in general. We can drop the common factor $\phi''(y+rx)$, and get

$$Ar^2 - Br + C = 0. \tag{11.36}$$

This is called the *characteristic equation*. Solve it for r. The types of the roots depend on the discriminant Δ:

- If $\Delta > 0$ (hyperbolic), then r_1, r_2 are real and distinct. (meaningful)
- If $\Delta = 0$ (parabolic), then $r_1 = r_2$ are real and repeated. (kind of meaningful)
- If $\Delta < 0$ (elliptic), then r_1, r_2 are complex conjugate pair. (not meaningful)

From now on, we will focus on the hyperbolic case, with constant coefficients. Let r_1, r_2 be the roots. Then, the solution is

$$u(x,y) = \phi_1(y - r_1 x) + \phi_2(y - r_2 x)$$

where ϕ_1, ϕ_2 are arbitrary functions, to be determined by BCs (or ICs if it is a time depending problem).

Example . Find the solution by method of characteristics for the following equation

$$u_{xx} + 3u_{xy} + 2u_{yy} = 0$$

with BCs

$$u(0,y) = f(y), \quad u_x(0,y) = 0.$$

To solve this, we set up the characteristic equation

$$r^2 - 3r + 2 = 0, \quad r_1 = 1, \quad r_2 = 2,$$

and the solution is

$$u(x,y) = \phi_1(y - r_1 x) + \phi_2(y - r_2 x) = \phi_1(y - x) + \phi_2(y - 2x).$$

We now determine ϕ_1, ϕ_2 by BCs. The first BC gives

$$\phi_1(y) + \phi_2(y) = f(y).$$

For the 2nd BC, we first differentiate the solution

$$u_x(x,y) = -\phi_1'(y - x) - 2\phi_2'(y - 2x).$$

Then, we have

$$-\phi_1'(y) - 2\phi_2'(y) = 0, \quad (\phi_1(y) + 2\phi_2(y))' = 0.$$

Integrating this, we get

$$\phi_1(y) + 2\phi_2(y) = M$$

where M is any integration constant. We can now solve

$$\phi_2(y) = M - f(y), \qquad \phi_1(y) = f(y) - \phi_1(y) = 2f(y) - M.$$

Putting these back into the solution, we get

$$u(x,y) = (2f(y - x) - M) + (M - f(y - 2x)) = 2f(x - y) - f(y - 2x).$$

Example . One more example on Method of Characteristics. Consider

$$u_{tt} - u_{tx} = 0,$$

where $u(t,x)$ is the unknown, and the ICs are

$$u(0,x) = f(x), \quad u_t(0,x) = g(x).$$

To solve, we set up the char equation

$$r^2 + r = 0, \quad r_1 = 0, \quad r_2 = 1$$

Then

$$u(t,x) = \phi(x - r_1 t) + \psi(x - r_2 t) = \phi(x) + \psi(x - t).$$

By the first IC, we have

$$\phi(x) + \psi(x) = f(x). \tag{11.37}$$

For the 2nd IC, we differentiate

$$u_t(t,x) = 0 - \psi'(x - t)$$

then

$$u_t(0,x) = -\psi'(x) = g(x).$$

Integrating this equation from x_0 (arbitrary) to x, we get

$$\psi(x) = \psi(x_0) - \int_{x_0}^{x} g(s)\, ds.$$

Then, we get

$$\phi(x) = f(x) - \psi(x) = f(x) - \psi(x_0) + \int_{x_0}^{x} g(s)\, ds.$$

Putting then into solution

$$\begin{aligned} u(t,x) &= \phi(x) + \psi(x - t) \\ &= \left[f(x) - \psi(x_0) + \int_{x_0}^{x} g(s)\, ds \right] + \left[\psi(x_0) - \int_{x_0}^{x-t} g(s)\, ds \right] \\ &= f(x) + \int_{x_0}^{x} g(s)\, ds + \int_{x-t}^{x_0} g(s)\, ds \\ &= f(x) + \int_{x-t}^{x} g(s)\, ds. \end{aligned}$$

Chapter 12

Homeworks

12.1 Homework 1

Problem 1: In each problem, determine the order of the equation and state whether the equation is linear or nonlinear.

(a) $t^2y'' + ty' + 2y = \sin t$;

(b) $y^{(4)} + y^{(3)} + y'' + y' + y = 1$;

(c) $y'' + \sin(t + y) = \sin t$.

Problem 2: In each problem, verify that each given function is a solution of the differential equation.

(a) $ty' - y = t^2$; $\quad y(t) = 3t + t^2$;

(b) $y'''' + 4y''' + 3y = t$; $\quad y_1(t) = t/3$, $\quad y_2(t) = e^{-t} + t/3$;

(c) $2t^2y'' + 3ty' - y = 0$, $\quad t > 0$; $\qquad y_1(t) = \sqrt{t}$, $\quad y_2(t) = t^{-1}$.

Problem 3: In each problem,draw a direction field for the given differential equation. Based on the direction field, determine the behavior of y as $t \to \infty$. Describe how this behavior depends on the initial value of y at $t = 0$.

(a) $y' = 1 + 2y$;

(b) $y' = y + 2$;

(c) $y' = y^2$;

(d) $y' = y(y - 2)^2$;

(e) $y' = -2 + t - y$;

(f) $y' = t + 2y$;

(g) $y' = -t/y$. (You don't need to discuss asymptotical behavior for this problem.)

Problem 4: Exam I, Spring 2013, Problem 2.

Problem 5: In each problem, solve the differential equation.

(a) $y' = x^2/y$;

(b) $y' = (\cos^2 x)(\cos^2 y)$;

(c) $xy' = (1 - y^2)^{1/2}$;

Problem 6: In each problem, solve the differential equation, and determine approximately the interval in which the solution is defined. You might use graphing tools if needed.

(a) $y' = (1 - 2x)/y, \qquad y(1) = -2$;

(b) $x\,dx + ye^{-x}\,dy = 0, \qquad y(0) = 1$;

(c) $y' = (e^{-x} - e^x)/(3 + 4y), \quad y(0) = 1$;

(d) $y^2(1 - x^2)^{1/2}\,dy = \arcsin x\,dx, \quad y(0) = 1$;

(e) $y' = 3x^2/(3y^2 - 4), \qquad y(1) = 0$.

Problem 7: Exam I, Fall 2013, Problem 3.

A collection of previous exams could be found at the coordinator's web:
`http://www.math.psu.edu/tseng/class/M251samples.html`

12.2 Homework 2

Problem 1. Find the general solution of the given differential equation. If an initial condition is given, find the particular solution which satisfies this initial condition.

a. $y' + 3y = e^t, \quad y(0) = -2;$

b. $y' - 2y = e^{2t}, \quad y(0) = 4;$

c. $ty' + y = e^t, \quad y(1) = 0;$

d. $y' = -\dfrac{y}{t} + \cos(t^2);$

e. $y' - 3y = 25\cos(4t);$

f. $z' = 2t(z - t^2);$

g. $y' + y\cos t = \cos t, \quad y(0) = 0;$

h. $y' - \dfrac{2}{t}y = \dfrac{t+1}{t}, \quad y(1) = -3;$

i. $y' + ay = e^{bt}$, where a and b are constant and $b + a \neq 0;$

j. $t^2y' + 2ty = 1, \quad y(2) = a$, where a is a constant.

Problem 2. Previous Exam Problems:

a. Fall 2013, Exam I, problem 2.

b. Fall 2012, Exam I, problem 2.

Problem 3. Without solving the problems, determine an interval in which the solution of the given initial value problem is certain to exist.

a. $(t-3)y' + (\ln t)y = 2t, \qquad y(1) = 2$

b. $t(t-4)y' + y = 0, \qquad y(2) = 1$

c. $y' + (\tan t)y = \sin t, \qquad y(\pi) = 0$

d. $(4 - t^2)y' + 2ty = 3t^2, \qquad y(-3) = 1$

e. $(4 - t^2)y' + 2ty = 3t^2, \qquad y(1) = -3$

f. $(\ln t)y' + y = \cot t, \qquad y(2) = 3$

Problem 4. Previous Exam Problems:

a. Fall 2013, Exam I, problem 9.

b. Fall 2012, Exam I, problem 4.

Problem 5. A tank contains 10gal of brine in which 2 lb of salt is dissolved. Brine containing 1 lb of salt per gallon flows into the tank at the rate of 3 gal/min, and the stirred mixture is drained off the tank at the rate of 4 gal/min. Find the amount $y(t)$ of salt in the tank at any time t.

Problem 6. A 30-L container initially contains 10L of pure water. A brine solution containing 20 g of salt per liter flows into the container at a rate of 4 L/min. The well-stirred mixture is pumped out of the container at a rate of 2 L/min.

(a) How long does it take for the container to overflow?

(b) How much salt is in the tank at the moment the tank begins to overflow?

Problem 7. Previous Exam Problems:

a. Fall 2012, Exam I, problem 5

b. Fall 2013, Exam I, problem 5

A collection of previous exams could be found at the coordinator's web:
`http://www.math.psu.edu/tseng/class/M251samples.html`

12.3 Homework 3

Problem 1. Spring 2013, Exam I, problem 10.

Problem 2. Find the critical points and equilibrium solutions of the given ODEs. Sketch the graphs of the solutions with the given initial conditions. Determine on the stabilities of these critical points.

a. $y' = 300y - 2y^2$; $y(0) = 50$; $y(0) = 100$; $y(0) = 200$

b. $y' = 15y - y^2/2$; $y(0) = 10$; $y(0) = 20$; $y(0) = 40$

c. $y' = 8y - 2y^2 - 6$; $y(0) = 0.5$; $y(0) = 1.5$; $y(0) = 3.5$

d. $y' = 2y^2 - 80y$; $y(0) = 10$; $y(0) = 30$; $y(0) = 50$

e. $y' = y^2(y - 1)$; $y(0) = -1$; $y(0) = 0.5$; $y(0) = 1.5$

Problem 3. Consider the equation $dy/dt = f(y)$ and suppose that y_1 is a critical point, i.e., $f(y_1) = 0$, and f is a continuous function. Show that $y(t) = y_1$ is asymptotically stable if $f'(y_1) < 0$ and unstable if $f'(y_1) > 0$.

Problem 4. Previous Exam Problems.

a. Fall 2013, Exam I, Problem 8;

b. Spring 2013, Exam I, Problem 11;

c. Fall 2012, Exam I, Problem 11.

Problem 5. Verify that the following problems are exact. Then solve the IVP (Initial Value Problems).

a. $4xy + \left[2x^2 - 6y\right] y' = 0$, $y(1) = 1$;

b. $3x^2y^{-2} + x^{-2}y^2 - 2x^{-3} + \left[6y^{-4} - 2x^3y^{-3} - 2x^{-1}y\right] y' = 0$, $y(1) = -1$.

c. $2e^{2x} - 2\sin(2x)\sin y + \left[2y^{-3} + \cos(2x)\cos y\right] y' = 0$, $y(0) = \pi/2$.

Problem 6. Show that any separable equation written as $M(x) + N(y)y' = 0$ is exact.

Problem 7. Previous Exam Problems.

a. Fall 2013, Exam I, Problem 7 and Problem 11;

b. Spring 2013, Exam I, Problem 5;

c. Fall 2012, Exam I, Problem 3.

A collection of previous exams could be found at the coordinator's web:
`http://www.math.psu.edu/tseng/class/M251samples.html`

12.4 Homework 4

Problem 1. Find the general solution of the given differential equations.

a. $y'' + 2y' - 3y = 0$

b. $6y'' - y' - y = 0$

c. $y'' + 5y' = 0$

d. $y'' - 9y' + 9y = 0$

Problem 2. Find the solution $y(t)$ of the given IVP (Initial Value Problems).

a. $y'' + y' - 6y = 0, \quad y(0) = 2, \quad y'(0) = 9$

b. $2y'' + y' - y = 0, \quad y(0) = 4, \quad y'(0) = -5/2$

c. $y'' + 4y' = 0, \quad y(0) = -2, \quad y'(0) = -4$

d. $6y'' - y' - y = 0, \quad y(0) = 2, \quad y'(0) = -3/2$

Problem 3. Find the differential equation whose general solution is given.

a. $y(t) = c_1 + c_2 e^t$

b. $y(t) = c_1 e^{2t} + c_2 e^{3t}$

Problem 4. Solve the IVP

$$y'' - y' - 2y = 0, y(0) = a, y'(0) = 2.$$

Then find a so that the solution approach 0 as $t \to \infty$.

Problem 5. Find the Wronskian of the given pair of functions

a. $e^{2t}, \quad e^{-3t/2}$

b. $\cos t, \quad \sin t$

c. $e^{-2t}, \quad te^{-2t}$

d. $e^t \sin t, \quad e^t \cos t$

Problem 6. Without solving the equation, find the longest interval in which the given IVP is certain to have a unique twice differential solution.

a. $ty'' + 3y = t, \quad y(1) = 1, \quad y'(1) = 2$

b. $(t-1)y'' - 3ty' + 4y = \sin t, \quad y(-2) = 2, \quad y'(-2) = 1$

c. $y'' + (\cos t)y' + 3(\ln|t|)y = 0, \quad y(2) = 3, \quad y'(2) = 1$

d. $(x-3)y'' + xy' + (\ln|x|)y = 0, \quad y(1) = 0, \quad y'(1) = 1$

Problem 7.

a. If $W(f,g) = 3e^{4t}$ and if $f(t) = e^{2t}$, find $g(t)$.

b. If $W(f,g) = t^2e^t$ and if $f(t) = t$, find $g(t)$.

Problem 8. Assume that y_1 and y_2 are a fundamental set of solutions of $y'' + p(t)y' + q(t)y = 0$ and let $y_3 = ay_1 + by_2$ and $y_4 = cy_1 + dy_2$, where a, b, c, d are any constants. Show that

$$W(y_3, y_4) = (ad - bc) \cdot W(y_1, y_2).$$

Are y_3, y_4 also a fundamental set of solutions? Why or why not?

Problem 9. Previous Exam Problems.

a. Fall 2013, Exam I, Problem 4, 9, 13.

b. Spring 2013, Exam I, Problem 6, 7;

c. Fall 2012, Exam I, Problem 7, 8.

A collection of previous exams could be found at the coordinator's web:
`http://www.math.psu.edu/tseng/class/M251samples.html`

12.5 Homework 5

Problem 1. Find the general solution of the given equations.

a. $9y'' + 6y' + y = 0$

b. $4y'' - 4y' - 3y = 0$

c. $4y'' + 12y' + 9y = 0$

Problem 2. Solve the following IVP (Initial Value Problems). Sketch the graph of the solution and describe the asymptotic behavior as $t \to \infty$.

a. $9y'' - 12y' + 4y = 0, \qquad y(0) = 2, \quad y'(0) = -1$

b. $y'' - 6y' + 9y = 0, \qquad y(0) = 0, \quad y'(0) = 2$

c. $y'' + 4y' + 4y = 0, \qquad y(-1) = 2, \quad y'(-1) = 1.$

Problem 3. Consider the following IVP

$$y'' - y' + 0.25y = 0, \qquad y(0) = 2, \quad y'(0) = b.$$

Find the solution (which will contain the constant b). Then determine the critical value of b that separates solutions that grow positively from those that eventually grow negatively.

Problem 4. Use the method of reduction of order to find a second solution of the given differential equations.

a. $t^2y'' + 2ty' - 2y = 0, \qquad t > 0; \qquad y_1(t) = t$

b. $t^2y'' + 3ty' + y = 0, \quad t > 0; \qquad y_1(t) = t^{-1}$

c. $t^2y'' - t(t+2)y' + (t+2)y = 0, \quad t > 0; \qquad y_1(t) = t$

d. $(x-1)y'' - xy' + y = 0, \quad x > 1; \qquad y_1(x) = e^x$

Problem 5. Find the general solution of the given equations.

a. $y'' - 2y' + 6y = 0$

b. $y'' + 2y' + 2y = 0$

c. $y'' + 4y' + 6.25y = 0$

Problem 6. Solve the following IVP (Initial Value Problems). Sketch the graph of the solution and describe the asymptotic behavior as $t \to \infty$.

a. $y'' + 4y' + 5y = 0, \qquad y(0) = 1, \quad y'(0) = 0$

b. $y'' - 2y' + 5y = 0, \qquad y(\pi/2) = 0, \quad y'(\pi/2) = 2$

c. $y'' + y' + 1.25y = 0, \qquad y(0) = 3, \quad y'(0) = 1$

Problem 7. Consider the IVP

$$y'' + 2y' + 6y = 0, \qquad y(0) = 2, \quad y'(0) = a \geq 0.$$

a. Find the solution $y(t)$ of this problem.

b. Find a so that $y = 0$ when $t = 1$. (You may use a calculator)

c. Find, as a function of a, the smallest positive value of t for which $y = 0$.

d. Determine the limit of the expression found in part (c) as $a \to \infty$.

Problem 8. Previous Exam Problems.

a. Fall 2012, Exam I, Problem 12.

b. Spring 2012, Exam I, Problem 7, 12.

c. Fall 2011, Exam I, Problem 8, 13.

A collection of previous exams could be found at the coordinator's web:
`http://www.math.psu.edu/tseng/class/M251samples.html`

12.6 Homework 6

Problem 1. Find the general solution of the given equations.

a. $y'' - 2y' - 3y = 3e^{2t}$

b. $y'' + 2y' + 5y = 3\sin 2t$

c. $y'' + 9y = t^2e^{3t} + 6$

Problem 2. Solve the following IVP (Initial Value Problems). Be patient! These problems take a lot of time!

a. $y'' + y' - 2y = 2t, \qquad y(0) = 0, \quad y'(0) = 1$

b. $y'' + 4y = t^2 + 3e^t, \qquad y(0) = 0, \quad y'(0) = 2$

c. $y'' + 4y = 3\sin 2t \qquad y(0) = 2, \quad y'(0) = -1.$

Problem 3. Determine a suitable form for $Y(t)$ for the following non-homogeneous equations, if the method of undetermined coefficients is to be used. (Do not solve for the coefficients.)

a. $y'' + 3y' = 2t^4 + t^2e^{-3t} + \sin 3t$

b. $y'' - 5y' + 6y = e^t\cos 2t + e^{2t}(3t + 4)\sin t$

c. $y'' - 4y' + 4y = 2t^2 + 4te^{2t} + t\sin 2t$

d. $y'' + 3y' + 2y = e^t(t^2 + 1)\sin 2t + 3e^{-t}\cos t + 4e^t$

Problem 4. Determine the general solution of

$$y'' + w^2y = \sum_{m=1}^{N} a_m \sin m\pi t$$

where $w > 0$, $N > 0$ is a positive integer, and $w \neq m\pi$ for any $m = 1, 2, \cdots, N$

Problem 5. Write the following functions into the form $u = R\cos(\omega_0 t - \delta)$.

a. $u = 3\cos 2t + 4\sin 2t$

b. $u = -\cos t + \sqrt{3}\sin t$

c. $u = 4\cos 3t - 2\sin 3t$

d. $u = -2\cos \pi t - 3\sin \pi t$

Problem 6. A mass weighing 3 lb stretches a spring 3 in. If the mass is pushed upward, contracting the spring a distance of 1 in., and then set in motion with a downward velocity of 2 ft/s, and if there is no damping, find the position u of the mass at any time $t > 0$. Determine the frequency, period, amplitude, and phase of the motion.

Problem 7. A mass weighing 8 lb stretches a spring 1.5 in. The mass is also attached to a damper with coefficient γ. Determine the value of γ for which the system is critically damped.

Problem 8. The position of a certain spring-mass system satisfies the initial value problem

$$\frac{3}{2}u'' + ku = 0, \qquad u(0) = 2, \quad u'(0) = \nu$$

If the period and amplitude of the resulting motion are observed to be π and 3, respectively, find the values of k and ν.

Problem 9. Previous Exam Problems.

a. Fall 2013, Exam I, Problem 10, 12;

b. Spring 2013, Exam I, Problem 12;

c. Fall 2012, Exam I, Problem 13.

A collection of previous exams could be found at the coordinator's web:
`http://www.math.psu.edu/tseng/class/M251samples.html`

12.7 Homework 7

Problem 1. Write the given expression as a product of two trigonometric functions of different frequencies.

a. $\cos 9t - \cos 7t$

b. $\sin 7t - \sin 6t$

c. $\cos \pi t + \cos 2\pi t$

Problem 2. Determine intervals in which solutions are sure to exist.

a. $y^{(4)} + 4y''' + 3y = t$,

b. $ty''' + (\sin t)y'' + 3y = \cos t$

c. $t(t-1)y^{(t)} + e^t y'' + 4t^2 y = 0$

Problem 3. Find the general solution of the given differential equation.

a. $y''' - y'' - y' + y = 0$

b. $y''' - 3y'' + 3y' - y = 0$

c. $y^{(6)} - y'' = 0$

d. $y^{(4)} + 2y'' + y = 0$

Problem 4. Previous Exam Problems.

a. Fall 2013, **Exam II**, Problem 1; (Note these are not Exam I).

b. Spring 2013, Exam I, Problem 8, 13;

c. Fall 2012, Exam I, Problem 10.

d. Fall 2012, **Exam II**, Problem 1.

e. Spring 2012, **Exam II**, Problem 1, 2, 5, 9a.

A collection of previous exams could be found at the coordinator's web:
`http://www.math.psu.edu/tseng/class/M251samples.html`

12.8 Homework 8

Problem 1. Find the Laplace transform of the given functions by definition (without using the Table 6.2.1). This means you should find a way to work out the integrals.

a. $f(t) = \cosh(bt)$, recall $\cosh x = \frac{1}{2}(e^x + e^{-x})$;

b. $f(t) = \sinh(bt)$, recall $\sinh x = \frac{1}{2}(e^x - e^{-x})$;

c. $f(t) = e^{at}\cosh(bt)$;

d. $f(t) = e^{at}\sinh(bt)$.

Problem 2. Use Euler's formula $e^{ix} = \cos x + i\sin x$ to find the Laplace transform of $f(t) = e^{at}\sin(bt)$ and $g(t) = e^{at}\cos(bt)$.

Problem 3. Use Integration by parts to find the Laplace transform of the following functions.

a. $f(t) = te^{at}$

b. $f(t) = t\sin(at), \qquad g(t) = t\cos(at)$

c. $f(t) = t\sinh(at), \qquad g(t) = t\cosh(at)$

Problem 4. Recall the following property of the Laplace transform:

$$\text{If} \quad F(s) = \mathcal{L}\{f(t)\}, \qquad \text{then} \quad F'(s) = \mathcal{L}\{-tf(t)\}.$$

a. Show that $F''(s) = \mathcal{L}\{(-t)^2 f(t)\}$, and furthermore $F^{(n)}(s) = \mathcal{L}\{(-t)^n f(t)\}$ for all integer $n \geq 1$.

b. Use these observations to find the Laplace transforms of the functions

$$f(t) = t^n e^{at}, \qquad g(t) = te^{at}\sin(bt), \qquad h(t) = te^{at}\cos(bt).$$

Problem 5. Find the inverse Laplace transform of the given functions.

a. $F(s) = \dfrac{3}{s^2+4}$;

b. $F(s) = \dfrac{4}{(s-1)^3}$;

c. $F(s) = \dfrac{2}{s^2+3s-4}$;

d. $F(s) = \dfrac{3s}{s^2-s-6}$:

e. $F(s) = \dfrac{2s+2}{s^2+2s+5}$;

f. $F(s) = \dfrac{8s^2-4s+12}{s(s^2+4)}$;

g. $F(s) = \dfrac{2s-3}{s^2+2s+10}$.

Problem 6. Use Laplace transform to solve the following IVPs. Be patient! These problems take a lot of work!

a. $y'' - y' - 6y = 0; \qquad y(0) = 1, \quad y'(0) = -1;$

b. $y'' - 2y' + 2y = 0; \qquad y(0) = 0, \quad y'(0) = 1;$

c. $y'' - 4y' + 4y = 0; \qquad y(0) = 1, \quad y'(0) = 1;$

d. $y^{(4)} - 4y''' + 6y'' - 4y' + y = 0; \qquad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 0, \quad y'''(0) = 1;$

e. $y^{(4)} - y = 0; \qquad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 1, \quad y'''(0) = 0;$

f. $y'' + w^2 y = \cos(2t), \quad w^2 \neq 4; \qquad y(0) = 1, \quad y'(0) = 0;$

Problem 7. Previous Exam Problems. (Note all the problems are now from Exam II.)

a. Fall 2013, Exam II, Problem 2, 5;

b. Spring 2013, Exam II, Problem 2;

c. Fall 2012, Exam II, Problem 2, 10a.

d. Spring 2012, Exam II, Problem 8a.

A collection of previous exams could be found at the coordinator's web:
`http://www.math.psu.edu/tseng/class/M251samples.html`

12.9 Homework 9

Problem 1. Sketch the graph of the given functions and express $f(t)$ in terms of the unit step function $u_c(t)$.

(a). $f(t) = \begin{cases} 0, & 0 \le t < 3 \\ -2, & 3 \le t < 5 \\ 2, & 5 \le t < 7 \\ 1, & 7 \le t. \end{cases}$

(b). $f(t) = \begin{cases} 1, & 0 \le t < 2 \\ e^{-(t-2)}, & t \ge 2 \end{cases}$

Problem 2. Find the Laplace transform of the given functions.

a. $f(t) = \begin{cases} 0, & 0 \le t < 2 \\ (t-2)^2, & t \ge 2 \end{cases}$

b. $f(t) = \begin{cases} 0, & 0 \le t < 1 \\ t^2 - 2t + 2, & t \ge 1 \end{cases}$

c. $f(t) = \begin{cases} 0, & 0 \le t < \pi \\ t - \pi, & \pi \le t \le 2\pi \\ 0, & 2\pi < t \end{cases}$

d. $f(t) = u_1(t) + 2u_3(t) - 6u_4(t)$

e. $f(t) = (t-3)u_2(t) - (t-2)u_3(t)$.

f. $f(t) = t - u_1(t)(t-1)$, $t \ge 0$.

Problem 3. Find the inverse Laplace transform of the given functions.

a. $F(s) = \dfrac{e^{-2s}}{s^2 + s - 2}$

b. $F(s) = \dfrac{2(s-1)e^{-2s}}{s^2 - 2s + 2}$

c. $F(s) = \dfrac{2e^{-2s}}{s^2 - 4}$

d. $F(s) = \dfrac{(s-2)e^{-s}}{s^2 - 4s + 3}$

e. $F(s) = \dfrac{e^{-s} + e^{-2s} - e^{-3s} - e^{-4s}}{s}$

Problem 4. Suppose that $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > d \geq 0$.

a. Show that if c is a positive constant, then $\mathcal{L}\{f(ct)\} = \dfrac{1}{c}F\left(\dfrac{s}{c}\right), \quad s > cd.$

b. Show that if k is a positive constant, then $\mathcal{L}^{-1}\{F(ks)\} = \dfrac{1}{k}f\left(\dfrac{t}{k}\right).$

c. Show that if a and b are constants with $a > 0$. Then

$$\mathcal{L}^{-1}\{F(as+b)\} = \frac{1}{a}e^{-bt/a}f\left(\frac{t}{a}\right).$$

d. Utilize these results to find the inverse Laplace transform of the following functions.

$$F_1(s) = \frac{2^{n+1}n!}{s^{n+1}}, \qquad F_2(s) = \frac{2s+1}{4s^2+4s+5}, \qquad F_3(s) = \frac{e^2e^{-4s}}{2s-1}.$$

Problem 5. Find the solution of the given IVP. Be patient! These problems take long time!

a. $y'' + y = f(t); \quad y(0) = 0, \quad y'(0) = 1; \qquad f(t) = \begin{cases} 1, & 0 \leq t < 3\pi \\ 0, & 3\pi \leq t \end{cases}$

b. $y'' + 4y = \sin t + u_\pi(t)\sin(t-\pi); \qquad y(0) = 0, \quad y'(0) = 0$

c. $y'' + 3y' + 2y = u_2(t); \qquad y(0) = 0, \quad y'(0) = 1$

d. $y'' + y = g(t); \qquad y(0) = 0, \quad y'(0) = 1; \qquad g(t) = \begin{cases} t/2, & 0 \leq t < 6 \\ 3, & 6 \leq t \end{cases}$

Problem 6. Find the solution of the given IVP. (Expect a lot of work!)

a. $y'' + 2y' + 2y = \delta(t-\pi); \qquad y(0) = 1, \quad y'(0) = 0;$

b. $y'' + 4y = \delta(t-\pi) - \delta(t-2\pi); \qquad y(0) = 0, \quad y'(0) = 0;$

c. $y'' + 3y' + 2y = \delta(t-5) + u_{10}(t); \qquad y(0) = 0, \quad y'(0) = 1/2;$

d. $y'' + y = \delta(t-2\pi)\cos t; \qquad y(0) = 0, \quad y'(0) = 1;$

Problem 7. Previous Exam Problems. (Note all the problems are now from Exam II.)

a. Fall 2013, Exam II, Problem 3, 4, 11, 13;

b. Spring 2013, Exam II, Problem 1ab, 4, 8, 9;

A collection of previous exams could be found at the coordinator's web:
`http://www.math.psu.edu/tseng/class/M251samples.html`

12.10 Homework 10

Problem 1. In each of the problem, transform the given equation into a system of first order equations. If ICs are given, specify these as well.

a. $u'' + 0.5u' + 2u = 0$;

b. $t^2u'' + tu' + (t^2 - 0.25)u = 0$;

c. $u^{(4)} - u = 0$;

d. $u'' + 0.25u' + 4u = 2\cos 3t, \quad u(0) = 1, \quad u'(0) = -2$;

e. $u'' + p(t)u' + q(t)u = g(t), \quad u(0) = u_0, \quad u'(0) = u_0'$.

Problem 2. Find the eigenvalues and their corresponding eigenvectors of the following matrices.

$$(a). \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}, \qquad (b). \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}, \qquad (c). \begin{pmatrix} -3 & 3/4 \\ -5 & 1 \end{pmatrix} \qquad (d). \begin{pmatrix} -3 & 4 \\ 0 & 1 \end{pmatrix}$$

Problem 3. For each problem, find the general solution and sketch the phase portrait. You may use some of the results in Problem 2.

$$(a). \ \vec{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \vec{x}, \qquad (b). \ \vec{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \vec{x}, \qquad (c). \ \vec{x}' = \begin{pmatrix} -3 & 4 \\ 0 & 1 \end{pmatrix} \vec{x},$$

Problem 4. For each problem, find the general solution and sketch the phase portrait. Note that some of the eigenvalues are 0.

$$(a). \ \vec{x}' = \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix} \vec{x}, \qquad\qquad (b). \ \vec{x}' = \begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix} \vec{x},$$

Problem 5. Consider the IVP $\vec{x}' = A\vec{x}$, $x_1(0) = 2, x_2(0) = 3$. Given the eigenvalues and eigenvectors of A, (i) write out the general solution; (ii) solve the IVP; (iii) sketch the phase portrait; and (iv) sketch the trajectory passing through the initial point $(2, 3)$.

$$(a). \ \lambda_1 = -1, \vec{v}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \quad \lambda_2 = -2, \vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$(b). \ \lambda_1 = 1, \ \vec{v}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \quad \lambda_2 = -2, \vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Problem 6. Previous Exam Problems.

a. Fall 2013, Exam II, Problem 6, 14;

b. Spring 2013, Exam II, Problem 3, 5, 6;

c. Fall 2012, Exam II, Problem 5;

d. Spring 2012, Exam II, Problem 6, 10.

A collection of previous exams could be found at the coordinator's web:
`http://www.math.psu.edu/tseng/class/M251samples.html`

12.11 Homework 11

Problem 1. For each of the problem, find the eigenvalues and eigenvectors of the coefficient matrix, find the general solution, state the type of the equilibrium and the stability, and sketch the phase portrait.

$$(a).\ \vec{x}' = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}\vec{x}, \qquad (b).\ \vec{x}' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}\vec{x}, \qquad (c).\ \vec{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}\vec{x}.$$

Problem 2. Find the solution for the IVPs.

$$(a).\ \vec{x}' = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix}\vec{x}, \quad \vec{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad (b).\ \vec{x}' = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix}\vec{x}, \quad \vec{x}(0) = \begin{pmatrix} 1 \\ -2 \end{pmatrix},$$

Problem 3. For each of the problem, find the eigenvalues and eigenvectors of the coefficient matrix, find the general solution, state the type of the equilibrium and the stability, and sketch the phase portrait.

$$(a).\ \vec{x}' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}\vec{x}, \qquad (b).\ \vec{x}' = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix}\vec{x}.$$

Problem 4. Find the solution for the IVPs.

$$(a).\ \vec{x}' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix}\vec{x}, \quad \vec{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \qquad (b).\ \vec{x}' = \begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix}\vec{x}, \quad \vec{x}(0) = \begin{pmatrix} 2 \\ 4 \end{pmatrix},$$

Problem 5. Find all the critical points.

a. $x' = x - xy; \quad y' = y + 2xy;$

b. $x' = 1 + 2y, \quad y' = 1 - 3x^2;$

c. $x' = 2x - x^2 - xy, \quad y' = 3y - 2y^2 - 3xy;$

d. $x' = (2 + x)(y - x), \quad y' = y(2 + x - x^2).$

Problem 6. For each problem: (i) Find all the critical points; (ii) Find the corresponding linear system near each critical point; (iii) Find the eigenvalues for each system; (iv) Classify the type of each critical point and its stability. (NB! These problems take long time!)

a. $x' = (2 + x)(y - x); \quad y' = (4 - x)(y + x);$

b. $x' = 1 - y, \quad y' = x^2 - y^2;$

c. $x' = (2 + y)(y - 0.5x), \quad y' = (2 - x)(y + 0.5x);$

Problem 7. Previous Exam Problems.

a. Fall 2013, Exam II, Problem 7, 8, 9, 10, 12;

b. Spring 2013, Exam II, Problem 1cd, 6, 7, 10, 11;

c. Fall 2012, Exam II, Problem 6, 7, 8, 12;

A collection of previous exams could be found at the coordinator's web:
`http://www.math.psu.edu/tseng/class/M251samples.html`

12.12 Homework 12

Problem 1. For each of the problem, (i) sketch the graph of the given function for three periods, (ii) find the Fourier series, (iii) write out the first 5 non-zero terms in the series, and (iv) describe how the Fourier series seems to be converging.

(a). $f(x) = -x$, $-L \le x < L$; $\quad f(x+2L) = f(x)$;

(b). $f(x) = \begin{cases} 1, & -L \le x < 0, \\ 0, & 0 \le x < L; \end{cases} \quad f(x+2L) = f(x)$;

(c). $f(x) = \begin{cases} x, & -\pi \le x < 0, \\ 0, & 0 \le x < \pi; \end{cases} \quad f(x+2\pi) = f(x)$;

(d). $f(x) = \begin{cases} 0, & -2 \le x \le -1, \\ x, & -1 \le x < 1, \\ 0, & 1 \le x < 2; \end{cases} \quad f(x+4) = f(x)$;

(e). $f(x) = \begin{cases} -1, & -2 \le x < 0, \\ 1, & 0 \le x < 2; \end{cases} \quad f(x+4) = f(x)$;

Problem 2*. (optional) Suppose f is an integrable and differentiable periodic function with period T.

(a). Show that f' is periodic with the same period T.

(b). Show that for any a, b, we have

$$\int_a^{a+T} f(t)\,dt = \int_b^{b+T} f(t)\,dt.$$

(c). Determine whether $F(t) = \int_0^t f(s)\,ds$ is always periodic.

Problem 3. Assume that f has a Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right),$$

show formally that

$$\frac{1}{L} \int_{-L}^{L} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2 \right).$$

This relation is known as the Parseval's identity, and is an important result. (Hint: Recall how we derived the formula for computing a_n, b_n.)

Problem 4. In each of the problem, a function f is given on and interval of length L. Sketch the graphs of the even and odd extension of f of period $2L$.

(a). $f(x) = \begin{cases} x, & 0 \le x < 2, \\ 1, & 2 \le x < 3; \end{cases}$

(b). $f(x) = \begin{cases} 0, & 0 \le x < 1, \\ x-1, & 1 \le x < 2; \end{cases}$

(c). $f(x) = 2 - x$, $0 < x < 2$.

(d). $f(x) = x - 3$, $\quad 0 < x < 4$.

Problem 5. In each of the following problem, either solve the given boundary value problem or else show that it has no solution.

(a). $y'' + y = 0, \qquad y(0) = 0, \quad y'(\pi) = 1;$
(b). $y'' + 2y = 0, \qquad y'(0) = 1, \quad y'(\pi) = 0;$
(c). $y'' + y = 0, \qquad y(0) = 0, \quad y'(L) = 0;$
(d). $y'' + y = x, \qquad y(0) = 0, \quad y(\pi) = 0;$
(e). $y'' + 4y = \cos x, \qquad y'(0) = 0, \quad y'(\pi) = 0.$

Problem 6. Solve the eigenvalue problems. Assume that all eigenvalues are real.
(a). $y'' + \lambda y = 0, \qquad y(0) = 0, \quad y'(\pi) = 0;$
(b). $y'' + \lambda y = 0, \qquad y'(0) = 0, \quad y(\pi) = 0;$
(c). $y'' + \lambda y = 0, \qquad y'(0) = 0, \quad y'(\pi) = 0;$
(d). $y'' + \lambda y = 0, \qquad y'(0) = 0, \quad y(L) = 0;$

Problem 7. Previous Exam Problems. Note that these are Final Exams!

a. Fall 2013, Final Exam, Problem 1, 11, 13, 14;

b. Spring 2013, Final Exam, Problem 2, 10, 12, 13;

c. Fall 2012, Final Exam, Problem 2, 14, 15;

A collection of previous exams could be found at the coordinator's web:
`http://www.math.psu.edu/tseng/class/M251samples.html`

12.13 Homework 13

Problem 1. For each of the problem, determine whether the method of separation of variables can be used to replace the given PDE by a pair of ODEs. If so, find the ODEs.

(a). $xu_{xx} + u_t = 0$
(b). $tu_{xx} + xu_t = 0$
(c). $u_{xx} + u_{xt} + u_t = 0$
(d). $[p(x)u_x]_x - r(x)u_{tt} = 0$
(e). $u_{xx} + (x + y)u_{yy} = 0$
(f). $u_{xx} + u_{yy} + xu = 0$

Problem 2. Find the solution of the heat conduct problem

$$\begin{aligned} &100u_{xx} = u_t, \qquad 0 < x < 1, \quad t > 0 \\ &u(0, t) = 0, \qquad u(1, t) = 0, \quad t > 0 \\ &u(x, 0) = \sin 2\pi x - \sin 5\pi x, \qquad 0 \le x \le 1. \end{aligned}$$

Problem 3. Find the **formal** solution of the heat conduct problem

$$\begin{aligned} &u_{xx} = u_t, \qquad 0 < x < 40, \quad t > 0 \\ &u(0, t) = 0, \qquad u(40, t) = 0, \quad t > 0 \\ &u(x, 0) = 50, \qquad 0 \le x \le 40. \end{aligned}$$

Problem 4*. (Optional) The heat conduct equation in two space dimension is

$$\alpha^2(u_{xx} + u_{yy}) = u_t.$$

Assuming that $u(x, y, t) = X(x)\,Y(y)\,T(t)$, find ODEs that are satisfied by the functions $X(x), Y(y)$, and $T(t)$.

Problem 5*. (Optional) The heat conduct equation in two space dimensions may be expressed in terms of polar coordinates as

$$\alpha^2 \left[u_{rr} + (1/r)u_r + (1/r^2)u_{\theta\theta}\right] = u_t.$$

Assuming that $u(r, \theta, t) = R(r)\,\Theta(\theta)\,T(t)$, find ODEs that are satisfied by $R(r), \Theta(\theta)$, and $T(t)$.

Problem 6. In each of problem, find the steady state solution of the heat conduct equation $c^2u_{xx} = u_t$ that satisfies the given set of boundary conditions.

(a). $u(0, t) = 10, \quad u(50, t) = 40$
(b). $u_x(0, t) = 0, \quad u(L, t) = T$
(c). $u_x(0, t) - u(0, t) = 0, \quad u(L, t) = T$
(d). $u(0, t) = T, \quad u_x(L, t) + u(L, t) = 0$

Problem 7. Let an aluminum rod of length 20cm be initially at the uniform temperature of 25^oC. Suppose that at time $t = 0$, the end $x = 0$ is cooled to 0^OC while the end $x = 20$ is heated to 60^oC, and both are thereafter maintained at those temperatures. Find the temperature distribution in the rod at any time t.

Problem 8. Previous Exam Problems. Note that these are Final Exams!

a. Fall 2013, Final Exam, Problem 9, 10, 15;

b. Spring 2013, Final Exam, Problem 8, 9, 14, 16;

A collection of previous exams could be found at the coordinator's web:
`http://www.math.psu.edu/tseng/class/M251samples.html`

12.14 Homework 14

Problem 1. (This problem takes a lot of work!) Consider the wave equation for an vibrating string with length $L = 10$, as

$$u_{tt} = u_{xx}, \qquad 0 < x < 10, \quad t > 0$$

The ends of the string are fixed. The initial displacement of the string is given as $u(x, 0) = f(x)$, where $f(x) = x$ on $0 < x < 5$, and $f(x) = 10 - x$ on $5 < x < 10$. The string is initially at rest.
(a). Plot the initial displacement;
(b). Find the formal solution $u(x, t)$ in term of Fourier series;
(c*) (optional). Write out the D'Alembert solution of this wave equation;
(d*) (optional). Plot $u(x, t)$ as a function of x, for several t values, such as $t = 0, t = 2.5, t = 5, t = 7.5, t = 10$.

Problem 2. Consider the same problem as in Problem 1, but with a different function $f(x)$. Let $0 < a < 10$ be given. Then, $f(x) = x/a$ on $0 < x < a$, and $f(x) = (10-x)/(10-a)$. Answer all the questions in Problem 1.

Problem 3. Previous Exam Problems. Note that these are Final Exams!

a. Fall 2013, Final Exam, Problem 16;

b. Spring 2013, Final Exam, Problem 15;

c. Fall 2012, Final Exam, Problem 11, 17;

d. Spring 2012, Final Exam, Problem 14.

A collection of previous exams could be found at the coordinator's web:
`http://www.math.psu.edu/tseng/class/M251samples.html`

Chapter 13

Answers to Homework Problems

13.1 Answer/keys for homework 1

Problem 1: (a) 2nd order linear, (b) 4th order linear, (c) 2nd order nonlinear

Problem 5:

(a). $3y^2 - 2x^3 = c$; $\quad y \neq 0$;

(b). $2\tan 2y - 2x - \sin 2x = c$ if $\cos 2y \neq 0$, i.e., $y \neq \pm(2n+1)\pi/4$ for any integer n; everywhere

(c). $y = \sin[\ln|x| + c]$ if $x \neq 0$ and $|y| < 1$; also $y = \pm 1$.

Problem 6:

(a) $y = -\sqrt{2x - 2x^2 + 4}$, $\quad -1 < x < 2$;

(b) $y = [2(1-x)e^x - 1]^{1/2}$, $\quad -1.68 < x < 0.77$ approximately

(c) $y = -\dfrac{3}{4} + \dfrac{1}{4}\sqrt{65 - 8e^x - 8e^{-x}}$, $\quad |x| < 2.08$ approximately

(d) $y = [\frac{3}{2}(\arcsin x)^2 + 1]^{1/3}$, $\quad -1 < x < 1$;

(e) $y^3 - 4y - x^3 = -1$, $\quad |x^3 - 1| < 16/3\sqrt{3}$ or $-1.28 < x < 1.60$;

13.2 Answer/keys to homework 2

Problem 1:

a. $y = \frac{1}{4}e^t - \frac{9}{4}e^{-3t}$

b. $y = te^{2t} + 4e^{2t}$

c. $y = \frac{e^t}{t} - \frac{e}{t}$

d. $y = \frac{\sin(t^2)}{2t} + \frac{c}{t}$

e. $y = 4\sin 4t - 3\cos 4t + ce^{3t}$

f. $z = t^2 + 1 + ce^{t^2}$

g. $y = 1 - e^{-\sin t}$

h. $y = -t - \frac{1}{2} - \frac{3}{2}t^{-2}$

i. $y = \frac{1}{a+b}e^{bt} + ce^{-at}$

j. $y = \frac{1}{t} + (4a-2)t^{-2}$.

Problem 3.

a. $0 < t < 3$

b. $0 < t < 4$

c. $\pi/2 < t < 3\pi/2$

d. $-\infty < t < -2$

e. $-2 < t < 2$

f. $1 < t < \pi$

Problem 5.

$y = (10-t) - \frac{8}{10000}(10-t)^4$, for $0 \le t \le 10$. After 10 min, there is no salf in the tank.

Problem 6.

(a) 10 min; (b) ≈ 533.33g.

13.3 Answer/keys to homework 3

Problem 2.

a. $y = 0$ (unstable), $y = 150$ (asymptotically stable),

b. $y = 0$ (unstable), $y = 30$ (asymptotically stable)

c. $y = 1$ (unstable), $y = 3$ (asymptotically stable)

d. $y = 0$ (asymptotically stable), $y = 40$ (unstable)

e. $y = 0$ (semi-stable), $y = 1$ (unstable)

Problem 3. If $f'(y_1) < 0$ and $f(y_1) = 0$, then in a small neighborhood around $y = y_1$, we have $y' = f(y) < 0$ for $y > y_1$ and $y' = f(y) > 0$ for $y < y_1$. Therefore nearby solutions approach $y = y_1$, therefore it is asymptotically stable. On the other hand, if $f'(y_1) >$, then $y' = f(y) > 0$ for $y > y_1$ and $y' = f(y) < 0$ for $y < y_1$, and nearby solutions go away from $y = y_1$, therefore it is unstable.

Problem 5.

a. $2x^2y - 3y^2 + 1 = 0$

b. $x^{-2} - 2y^{-3} + x^3y^{-2} - x^{-1}y^2 = 3$

c. $\cos(2x)\sin y + e^{2x} - y^{-2} = 2 - 4/\pi^2$

Problem 6. Since $M_y(x) = 0$ and $N_x(y) = 0$, the equation is exact.

13.4 Answer/keys to homework 4

Problem 1.

a. $y = c_1 e^{t} + c_2 e^{-3t}$

b. $y = c_1 e^{t/2} + c_2 e^{-t/3}$

c. $y = c_1 + c_2 e^{-5t}$

d. $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ where $r_{1,2} = \dfrac{9 \pm 3\sqrt{5}}{2}$

Problem 2.

a. $y(t) = 3e^{2t} - e^{-3t}$

b. $y(t) = 3e^{-t} + e^{t/2}$

c. $y(t) = -3 + 4e^{-4t}$

d. $y(t) = 3e^{-t/3} - e^{t/2}$

Problem 3.

a. $y'' - y' = 0$

b. $y'' - 5y' + 6y = 0$

Problem 4. $a = -2$.
Problem 5. (a). $-\dfrac{7}{2}e^{t/2}$, (b). 1, (c). e^{-4t}, (d). $-e^{2t}$
Problem 6. (a) $0 < t < \infty$, (b) $-\infty < t < 1$, (c) $0 < t < \infty$, (d) $0 < x < 3$
Problem 7. (a) $3te^{2t} + ce^{2t}$, (b) $te^{t} + ct$ (One can choose $c = 0$ here.)
Problem 8. They are a fundamental set of solutions if and only if $ad - bc \neq 0$.

13.5 Answer/keys to homework 5

Problem 1.

a. $y = c_1 e^{-t/3} + c_2 t e^{-t/3}$

b. $y = c_1 e^{-t/2} + c_2 e^{3t/2}$

c. $y = c_1 e^{-3t/2} + c_2 t e^{-3t/2}$

Problem 2.

a. $y = 2e^{2t/3} - \dfrac{7}{3} t e^{2t/3}, \quad y \to -\infty$ as $t \to \infty$

b. $y = 2te^{3t}, \quad y \to \infty$ as $t \to \infty$

c. $y = 7e^{-2(t+1)} + 5te^{-2(t+1)}, \quad y \to 0$ as $t \to \infty$

Problem 3. $y = 2e^{t/2} + (b-1)te^{t/2}; \quad b = 1$

Problem 4.

a. $y_2(t) = t^{-2}$

b. $y_2(t) = t^{-1} \ln t$

c. $y_2(t) = te^t$

d. $y_2(x) = x$

Problem 5.

a. $y = c_1 e^t \cos \sqrt{5}t + c_2 e^t \sin \sqrt{5}t$

b. $y = c_1 e^{-t} \cos t + c_2 e^{-t} \sin t$

c. $y = c_1 e^{-2t} \cos(3t/2) + c_2 e^{-2t} \sin(3t/2)$

Problem 6.

a. $y = e^{-2t} \cos t + 2e^{-2t} \sin t;$ decaying oscillation

b. $y = -e^{t-\pi/2} \sin(2t);$ growing oscillation

c. $y = 3e^{-t/2} \cos t + \dfrac{5}{2} e^{-t/2} \sin t;$ decaying oscillation

Problem 7.

a. $y = 2e^{-t} \cos \sqrt{5}t + \dfrac{a+2}{\sqrt{5}} e^{-t} \sin \sqrt{5}t$

b. $a \approx 1.50878$

c. $t = \dfrac{1}{\sqrt{5}} \left[\pi - \arctan \dfrac{2\sqrt{5}}{2+a} \right]$

d. $\pi/\sqrt{5}$

13.6 Answer/keys to homework 6

Problem 1.

a. $y = c_1 e^{3t} + c_2 e^{-t} - e^{2t}$

b. $y = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t + \dfrac{3}{17}\sin 2t - \dfrac{12}{17}\cos 2t$

c. $y = c_1 \cos 3t + c_2 \sin 3t + \dfrac{1}{162}(9t^2 - 6t + 1)e^{3t} + \dfrac{2}{3}$

Problem 2.

a. $y = e^t - \dfrac{1}{2}e^{-2t} - t - \dfrac{1}{2}$

b. $y = \dfrac{7}{10}\sin 2t - \dfrac{19}{40}\cos 2t + \dfrac{1}{4}t^2 - \dfrac{1}{8} + \dfrac{3}{5}e^t$

c. $y = 2\cos 2t - \dfrac{1}{8}\sin 2t - \dfrac{3}{4}t\cos 2t$

Problem 4.

a. $Y = t(A_4 t^4 + A_3 t^3 + a_2 t^2 + A_1 t + a_0) + t(B_2 t^2 + B_1 t + B_0)e^{-3t} + D\sin 3t + E\cos 3t$

b. $Y = e^t(A\cos 2t + B\sin 2t) + (D_1 t + D_0)e^{2t}\sin t + (E_1 t + E_0)e^{2t}\cos t.$

c. $Y = (A_2 t^2 + A_1 t + A_0) + t^2(B_1 t + B_0)e^{2t} + (D_1 t + D_0)\sin 2t + (E_1 t + E_0)\cos 2t$

d. $Y = (A_2 t^2 + A_1 t + A_0)e^t \sin 2t + (B_2 t^2 + B_1 t + B_0)e^t \cos 2t + e^{-t}(D\cos t + E\sin t) + Fe^t$

Problem 5.

a. $u = 5\cos(2t - \delta)$, $\delta = \arctan(4/3) \approx 0.9273$

b. $u = 2\cos(t - 2\pi/3)$

c. $u = 2\sqrt{5}\cos(3t - \delta)$, $\delta = -\arctan(1/2) \approx -0.4636$

d. $u = \sqrt{13}\cos(\pi t - \delta)$, $\delta = \pi + \arctan(3/2) \approx 4.1244$

Problem 6. $u = \dfrac{1}{4\sqrt{2}}\sin(8\sqrt{2}t) - \dfrac{1}{12}\cos(8\sqrt{2}t)$ ft, $\omega = 8\sqrt{2}$ rad/s, $T = \frac{\pi}{4\sqrt{2}}$ s, $R = \sqrt{11/288} \approx 0.1954$ ft, $\delta = \pi - \arctan\frac{3}{\sqrt{2}} \approx 2.0113$

Problem 7. $\gamma = 8$ lb $\cdot$ s/ft

Problem 8. $k = 6$, $\nu = \pm 2\sqrt{5}$.

13.7 Answer/keys to homework 7

Problem 1. a. $-2\sin(8t)\sin t$, b. $2\sin(t/2)\cos(13t/2)$, c. $2\cos(3\pi t/2)\cos(\pi t/2)$

Problem 2. a. $-\infty < t < \infty$, b. $t > 0$ or $t < 0$, c. $t > 1$, or $0 < t < 1$, or $t < 0$.

Problem 3.

a. $y = c_1 e^t + c_2 t e^t + c_3 e^{-t}$

b. $y = c_1 e^t + c_2 t e^t + c_3 t^2 e^t$

c. $y = c_1 + c_2 t + c_3 e^{2t} + c_4 e^{-t} + c_5 \cos t + c_6 \sin t$

d. $y = (c_1 + c_2 t)\cos t + (c_3 + c_4 t)\sin t$

13.8 Answer/keys to homework 8

Problem 1.

a. $F(s) = \dfrac{s}{s^2 - b^2}, \quad s > |b|;$ b. $F(s) = \dfrac{b}{s^2 - b^2}, \quad s > |b|;$

c. $F(s) = \dfrac{s-a}{(s-a)^2 - b^2}, \quad s > a + |b|;$ d. $F(s) = \dfrac{b}{(s-a)^2 - b^2}, \quad s > a + |b|;$

Problem 2.

$F(s) = \dfrac{b}{(s-a)^2 + b^2} \quad (s > a) \quad \text{and} \quad G(s) = \dfrac{s-a}{(s-a)^2 + b^2} \quad (s > a).$

Problem 3.

a. $F(s) = \dfrac{1}{(s-a)^2}, \ \ s > a;$ b. $F(s) = \dfrac{2as}{(s^2 + a^2)^2}, \ \ s > 0;$

c. $F(s) = \dfrac{s^2 + a^2}{(s-a)^2(s+a)^2}, \ \ s > |a|.$

Problem 4.

(b). $F(s) = \dfrac{n!}{(s-a)^{n+1}}, \quad G(s) = \dfrac{2b(s-a)}{[(s-a)^2 + b^2]^2}, \quad H(s) = \dfrac{(s-a)^2 - b^2}{[(s-a)^2 + b^2]^2}.$

Problem 5.

a. $f(t) = \dfrac{3}{2}\sin 2t;$ b. $f(t) = 2t^2 e^t;$

c. $f(t) = \dfrac{2}{3}e^t - \dfrac{2}{5}e^{-4t};$ d. $f(t) = \dfrac{9}{5}e^{3t} + \dfrac{6}{5}e^{-2t};$

e. $f(t) = 2e^{-t}\cos 2t;$ f. $f(t) = 3 - 2\sin 2t + 5\cos 2t;$

g. $f(t) = 2e^{-t}\cos 3t - \dfrac{5}{3}e^{-t}\sin 3t.$

Problem 6.

a. $y(t) = \dfrac{1}{5}(e^{3t} + 4e^{-2t});$ b. $y(t) = e^t \sin t;$

c. $y(t) = e^{2t} - te^{2t};$ d. $y(t) = te^t - t^2 e^t + \dfrac{2}{3}t^3 e^t;$

e. $y(t) = \cosh t = \dfrac{1}{2}(e^t + e^{-t});$ f. $y(t) = \dfrac{1}{w^2 - 4}[(w^2 - 5)\cos wt + \cos 2t].$

13.9 Answer/keys to homework 9

Problem 1.

a. $f(t) = -2u_3(t) + 4u_5(t) - u_7(t)$

b. $f(t) = 1 + u_2(t)[e^{-(t-2)} - 1]$

Problem 2.

a. $F(s) = 2e^{-s}/s^3$

b. $F(s) = e^{-s}(s^2+2)/s^3$

c. $F(s) = \dfrac{e^{-\pi s}}{s^2} - \dfrac{e^{-2\pi s}}{s^2}(1+\pi s)$

d. $F(s) = \dfrac{1}{s}(e^{-s} + 2e^{-3s} - 6e^{-4s})$

e. $F(s) = s^{-2}[(1-s)e^{-2s} - (1+s)e^{-3s}]$

f. $F(s) = (1-e^{-s})/s^2$

Problem 3.

a. $f(t) = \dfrac{1}{3}u_2(t)[e^{t-2} - e^{-2(t-2)}]$

b. $f(t) = 2u_2(t)e^{t-2}\cos(t-2)$

c. $f(t) = u_2(t)\sinh 2(t-2)$

d. $f(t) = u_1(t)e^{2(t-1)}\cosh(t-1)$

e. $f(t) = u_1(t) + u_2(t) - u_3(t) - u_4(t)$

Problem 4. (d). $f_1(t) = 2(2t)^n, \quad f_2(t) = \dfrac{1}{2}e^{-t/2}\cos t, \quad$ and $f_3(t) = \dfrac{1}{2}e^{t/2}u_2(t/2).$

Problem 5.

a. $y(t) = 1 - \cos t + \sin t - u_{3\pi}(t)(1+\cos t)$

b. $y(t) = \dfrac{1}{6}(2\sin t - \sin 2t) - \dfrac{1}{6}u_\pi(t)(2\sin t + \sin 2t)$

c. $y(t) = e^{-t} - e^{-2t} + u_2(t)[\dfrac{1}{2} - e^{-(t-2)} + \dfrac{1}{2}e^{-2(t-2)}]$

d. $y(t) = \dfrac{1}{2}\sin t + \dfrac{1}{2}t - \dfrac{1}{2}u_6(t)[t - 6 - \sin(t-6)]$

Problem 6.

a. $y(t) = e^{-t}\cos t + e^{-t}\sin t + u_\pi(t)e^{-(t-\pi)}\sin(t-\pi)$

b. $y(t) = \dfrac{1}{2}u_\pi(t)\sin 2(t-\pi) - \dfrac{1}{2}u_{2\pi}(t)\sin 2(t-2\pi)$

c. $y(t) = -\dfrac{1}{2}e^{-2t} + \dfrac{1}{2}e^{-t} + u_5(t)[-e^{-2(t-5)} + e^{-(t-5)}] + u_{10}(t)[\dfrac{1}{2} + \dfrac{1}{2}e^{-2(t-10)} - e^{-(t-10)}]$

d. $y(t) = \sin t + u_{2\pi}(t)\sin(t-2\pi)$

13.10 Answer/keys to homework 10

Problem 1.

a. $x_1 = u,\ x_2 = u',\quad \to \quad x_1' = x_2,\quad x_2' = -2x_1 - 0.5x_2$

b. $x_1 = u,\ x_2 = u',\quad \to \quad x_1' = x_2,\quad x_2' = -(1 - 0.25t^{-2})x_1 - t^{-1}x_2$

c. $x_1 = u,\ x_2 = u',\ x_3 = u'',\ x_4 = u''' \quad \to \quad x_1' = x_2,\quad x_2' = x_3,\quad x_3' = x_4,\quad x_4' = x_1$

d. $x_1 = u,\ x_2 = u',\quad \to \quad x_1' = x_2,\quad x_2' = -4x_1 - 0.25x_2 + 2\cos 3t,\quad x_1(0) = 1,\quad x_2(0) = -2$

e. $x_1 = u,\ x_2 = u',\quad \to \quad x_1' = x_2,\quad x_2' = -q(t)x_1 - p(t)x_2 + g(t),\quad x_1(0) = u_0,\quad x_2(0) = u_0'$

Problem 2.

$(a).\ \lambda_1 = 2, \vec{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix},\quad \lambda_2 = 4, \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$(b).\ \lambda_1 = -3, \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix},\quad \lambda_2 = -1, \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$(c).\ \lambda_1 = -1/2, \vec{v}_1 = \begin{pmatrix} 3 \\ 10 \end{pmatrix},\quad \lambda_2 = -3/2, \vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$(d).\ \lambda_1 = -3, \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\quad \lambda_2 = 1, \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Problem 3. Use the eigenpairs found in Problem 2, and the formula: $(a).\ \vec{x} = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$, one can simply write out the general solutions.

Problem 4.

$(a).\ \vec{x} = c_1 \begin{pmatrix} 3 \\ 4 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$(b).\ \vec{x} = c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2 e^{t} \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

Problem 5.

$(a).(i).\ \vec{x} = c_1 e^{-t} \begin{pmatrix} -1 \\ 2 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix},\qquad (ii).\ c_1 = -1/4, c_2 = 7/4$

$(b).(i).\ \vec{x} = c_1 e^{t} \begin{pmatrix} -1 \\ 2 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix},\qquad (ii).\ c_1 = -1/4, c_2 = 7/4$

13.11 Answer/keys to homework 11

Problem 1.

a. $\vec{x} = c_1 e^t \begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix} + c_2 e^t \begin{pmatrix} \sin 2t \\ -\cos 2t + \sin 2t \end{pmatrix}$

b. $\vec{x} = c_1 e^{-t} \begin{pmatrix} 2\cos 2t \\ \sin 2t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -2\sin 2t \\ \cos 2t \end{pmatrix}$

c. $\vec{x} = c_1 \begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} + c_2 \begin{pmatrix} 5\sin t \\ -\cos t + 2\sin t \end{pmatrix}$

Problem 2.

(a). $\vec{x} = e^{-t} \begin{pmatrix} \cos t - 3\sin t \\ \cos t - \sin t \end{pmatrix}$

(b). $\vec{x} = e^{-2t} \begin{pmatrix} \cos t - 5\sin t \\ -2\cos t - 3\sin t \end{pmatrix}$

Problem 3.

(a). $\vec{x} = c_1 e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \left[t e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]$

(b). $\vec{x} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \left[t \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 0.5 \end{pmatrix} \right]$

Problem 4.

(a). $\vec{x} = e^{-3t} \begin{pmatrix} 3 + 4t \\ 2 + 4t \end{pmatrix}$

(b). $\vec{x} = 2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 14t \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

Problem 5.

(a). $(-0.5, 1)$, saddle point, unstable;
$(0, 0)$, proper node, unstable

(b). $(0, 0)$, node, unstable;
$(2, 0)$, node, asymptotically stable;
$(0, 1.5)$, saddle point, unstable;
$(-1, 3)$, node, asymptotically stable

(c). $(0, 0)$, spiral point, asymptotically stable;
$(1 - \sqrt{2}, 1 + \sqrt{2})$, saddle point, unstable;
$(1 + \sqrt{2}, 1 - \sqrt{2})$, saddle point, unstable

13.12 Answer/keys to homework 12

Problem 1.

a. $f(x) = \dfrac{2L}{\pi} \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin \frac{n\pi x}{L}$

b. $f(x) = \dfrac{1}{2} - \dfrac{2}{\pi} \displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1} \sin[(2n-1)\pi x/L]$

c. $f(x) = -\dfrac{\pi}{4} + \displaystyle\sum_{n=1}^{\infty} \left[\frac{2}{\pi(2n-1)^2} \cos(2n-1)x + \frac{(-1)^{n+1}}{n} \sin nx \right]$

d. $f(x) = \displaystyle\sum_{n=1}^{\infty} \left[-\frac{2}{n\pi} \cos \frac{n\pi}{2} + \frac{4}{(n\pi)^2} \sin \frac{n\pi}{2} \right] \sin \frac{n\pi x}{2}$

e. $f(x) = \dfrac{4}{\pi} \displaystyle\sum_{n=1}^{\infty} \frac{\sin[(2n-1)\pi x/2]}{2n-1}$

Problem 2. (c). $\displaystyle\int_0^x f(t)\,dt$ may not be periodic; for example, consider $f(t) = 1 + \sin t$.

Problem 5.

a. $y = -\sin x$

b. $y = (\cot \sqrt{2}\pi \cos \sqrt{2}x + \sin \sqrt{2}x)/\sqrt{2}$;

c. $y = c_2 \sin x$ if $\cos L = 0$, and $y = 0$ for all other L;

d. No solution

e. $y = c_1 \cos 2x + \dfrac{1}{3} \cos x$.

13.13 Answer/keys to homework 13

Problem 1.

a. Let $u(x,t)=F(x)G(t)$, we have $xF''(x)-\lambda F(x)=0, \quad G'(t)+\lambda G(t)=0$

b. Let $u(x,t)=F(x)G(t)$, we have $F''(x)-\lambda xF(x)=0, \quad G'(t)+\lambda tG(t)=0$

c. Let $u(x,t)=F(x)G(t)$, we have $F''(x)-\lambda(F'(x)+F(x))=0, \quad G'(t)+\lambda G(t)=0$

d. Let $u(x,t)=F(x)G(t)$, we have $[p(x)F'(x)]'+\lambda r(x)F(x)=0, \quad G''(t)+\lambda G(t)=0$

e. Not separable.

f. Let $u(x,t)=F(x)G(y)$, we have $F''(x)+(x+\lambda)F(x)=0, \quad G''(y)-\lambda G(y)=0$

Problem 2.

$$u(x,t)=e^{-400\pi^2 t}\sin 2\pi x-e^{-2500\pi^2 t}\sin 5\pi x$$

Problem 3.

$$u(x,t)=\frac{100}{\pi}\sum_{n=1}^{\infty}\frac{1-\cos(n\pi)}{n}e^{-n^2\pi^2 t/1600}\sin\frac{n\pi x}{40}$$

Problem 4*.

$$X''+\mu^2 X=0, \quad Y''+(\lambda^2-\mu^2)Y=0, \quad T'+\alpha^2\lambda^2 T=0$$

Problem 5*.

$$r^2R''+rR'+(\lambda^2r^2-\mu^2)R=0, \quad \Theta''+\mu^2\Theta=0, \quad T'+\alpha^2\lambda^2T=0.$$

Problem 6.

(a). $U(x)=10+\dfrac{3}{5}x$ (b). $U(x)=T$

(c). $U(x)=\dfrac{T(1+x)}{1+L}$ (d). $U(x)=\dfrac{T(1+L-x)}{1+L}$

Problem 7. The IVBP for the heat equation is:

$$u_t=c^2u_{xx}, \quad u(0,t)=0, \quad u(20,t)=60, \quad u(x,0)=25.$$

Note that this has non-homogeneous B.C.s! Carrying out the computation, we get

$$u(x,t)=3x+\sum_{n=1}^{\infty}C_n e^{-(\frac{cn\pi}{20})^2 t}\sin\frac{n\pi x}{20}, \qquad C_n=\frac{1}{10}\int_0^{20}(25-3x)\sin\frac{n\pi x}{20}\,dx=\frac{1}{n\pi}(70(-1)^n+50)$$