CHAPTER TWO

2. CONDITIONAL PROBABILITY AND INDEPENDENCE

2.1. Conditional Probability

2.2 Multiplication theorem, Bayes' Theorem and Total probability theorem

2.3 Independent Events

2.1. Conditional Probability

It is to be recalled that up to this moment we were talking about events that are independent. In the real world variables are known to be interrelated or dependent to one another, and so do events. Accordingly, in most cases it seems rare to find probabilities of events that do not involve conditional probabilities.

The conditional probability will helps us to calculate the probability of events when it known that another event has occurred. In particular, it is known that some event has already occurred, you will find the probability of another event. Conditional probability refers to those probabilities that arise in the case of dependent events. To introduce this important concept let us consider the following example.

Example: A box of 80 candles consists of 30 defective and 50 non-defective candles. Suppose that we choose two candles from this box one after the other.

a) with replacement b) with out replacement

Define two events A and B as follows

A=the first candle is defective B=the second candle is defective

If we are choosing with replacement, P(A) =P(B)= 30/80 =3/8 because each time the selected candle will be replaced back and hence the composition remains the same.

However, if we are choosing without replacement, the results are not quite immediate.

P(A)=3/8 but what about P(B)=? It is clear that P(B) depends on whether A did occur or didn't occur. If A did not occur then P(B)=30/79. On the other hand, if A did occur then on the second draw there are only 79 candles left of which 29 are defective. In this case P(B)= P(B/A)=29/79. Such types of probability are called conditional probabilities.

Definition: Conditional Probability: Let A and B be two events in the sample space S with $P(B) > 0$. Then the probability that event A will given that event B had already occurred is called the conditional probability of A given that B which we denote it by $P(B/A)$ is defined as

$$P(A/B) = \frac{P\ (A \cap B)}{P(B)}; where\ P\ (B)\ > 0$$

Example: Consider the above example when it is without replacement

It is easily observed that $P(A) = 3/8$

$$P(A \cap B) = P(\text{Both are defectives}) = \frac{\binom{30}{2}\binom{50}{0}}{\binom{80}{2}} = \frac{435}{3160} = \frac{87}{632}$$

$$\Longrightarrow P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{87/632}{3/8} = \frac{29}{79}$$

Remark:

1. $P(A/B)$ for fixed B satisfies all of the various axioms of probability.
   I. $0 \leq P(A/B) \leq 1$
   II. $P(S/B) = 1$; where S denotes the Sample Space.
   III. $P((A_1 \cup A_2)/B) = P(A_1/B) + P(A_2/B)$; where $A_1\ and\ A_2$ are mutually exclusive events
   IV. If $A_1, A_2 \ldots, A_n,$ are a sequence mutually exclusive events in S, then

$$P(\bigcup_{i=1}^{n} A_i/B) = \sum_{i=1}^{n} P(A_i/B)$$

2. If the possible outcomes are finitely many and equally likely, we have

$$P(A/B) = \frac{number\ of\ elements\ of\ A \cap B}{\text{Number of elements of B}} = \frac{n(A \cap B)}{n(B)}$$

(I.e. Conditional probability can also be viewed as a probability law on a new universe B, because all of the conditional probability is concentrated on B)

3. For every events $A_1\ and\ A_2$ in S, then

$$P((A_1 \cup A_2)/B) = P(A_1/B) + P(A_2/B) - P((A_1 \cap A_2)/B)$$

4. If $A_1\ and\ A_2$ are events in S and $A_1 \subseteq A_2$ then $P(A_1/B) \leq P(A_2/B)$
5. If A is an event in S, then $P(A'/B) = 1 - P(A/B)$

Example 1: Toss a fair coin three times successively. Let

$$A = \{More\ heads\ than\ tails\ come\ up\}$$

$$B = \{first\ toss\ is\ a\ head\}$$

Then compute $P(A/B)$.

Example 2: For a student enrolling at freshman at certain university the probability that he/she will get scholarship is 0.25 and the probability that he/she that he/she will graduate is 0.75. If the probability that he/she will get scholarship and will also graduate is 0.2. What is the probability that a student who get a scholarship will graduate?

Example 3: Let probability that a policy will be correctly formulated is 0.60 and the probability that a policy will be correctly formulated and correctly executed is 0.54. Then fined the probability that:

a. the policy will be correctly executed given that it is correctly formulated
b. The policy will not be correctly executed given that it is correctly formulated.

Example 4: A lot consists of 20 defective and 80 non-defective items from which two items are chosen without replacement. Events A & B are defined as

A = {the first item chosen is defective}, B = {the second item chosen is defective}

a. What is the probability that both items are defective?
b. What is the probability that the second item is defective?

2.2. Multiplication theorem, Bayes' Theorem and Total probability theorem

2.2.1. Multiplication theorem of probability

Definition: If A and B are two events in S and $P(A)$ is not zero, then the probability that A and B will both occur is the product of the probability of A and the conditional probability of B given A. i.e. $P(A \cap B) = P(A)P(B/A)$. Similarly, $P(A \cap B) = P(B)P(A/B)$.

Multiplication rule of probability is used to find the probability of simultaneous occurrence of events.

The above multiplication rule can be generalized to more than two events in the following way.

Let $A_1, A_2, \ldots, A_n$ be a sequence of events and assume $\bigcap_{i=1}^{n} A_i \neq \phi$. Then

$$P(\bigcap_{i=1}^{n} A_i) = P(A_1)\, P(A_2/A_1)\, P(A_3/A_1, A_2) \ldots .. P(A_n/A_1, A_2, \ldots, A_{n-1})$$

Example 1: suppose there is a box containing 20 fuses of which 5 are defective. If 3 fuses are selected at random and removed successively, what is the probability that all are defective?

Example 2: Three balls are drawn at random one after the other without replacement from a box containing 3 white and 4 red balls. Find the probability that the balls alternate in color.

Solution

Let W: {white ball is drawn}

R: {red ball is drawn}

B: {the balls alternate in color}

$B= (W \cap R \cap W) \cup (R \cap W \cap R)$

$P(B)= P(W \cap R \cap W) + P(R \cap W \cap R)$

$= P(W)P(R/W)\ P(W/(W\cap R)) + P(R)\ P(W/R)\ P(R/(R \cap W))$

$= (3/7)(4/6)(1/5) + (4/7)(3/6)(3/5)$

$= 2/7$

Example 3: Suppose that our sample space is the population of adults in a small town who have completed the requirements for a college degree. They are categorized according to sex and employment status as follows.

	Employed(E)	Unemployed	Total
Male(M)	30	40	70
Female	20	70	90
Total	50	110	160

A person is selected at random. What is the probability that the selected person is

a) a man given that he is employed b)unemployed given that she is female

c) a male or a female d)unemployed male e)employed

Solution: Let E: The person is employed $\Rightarrow$ $E^{/}$: The person is unemployed

M: The person is male $\Rightarrow$ $M^{/}$: The person is female

a) $P(M/E) = \dfrac{P(M \cap E)}{P(E)} = \dfrac{30/160}{50/160} = \dfrac{30}{50} = \dfrac{3}{5}$

b) $P(E'/M') = \dfrac{P(M' \cap E')}{P(M')} = \dfrac{70/160}{90/160} = \dfrac{70}{90} = \dfrac{7}{9}$

c) $P(M \cup M') = P(S) = 1$

d) $P(E' \cap M) = \dfrac{n(E' \cap M)}{n(S)} = \dfrac{40}{160} = \dfrac{1}{4}$

e) $P(E) = \dfrac{n(E)}{n(S)} = \dfrac{50}{160} = \dfrac{5}{16}$

2.2.2. Bayes' Theorem

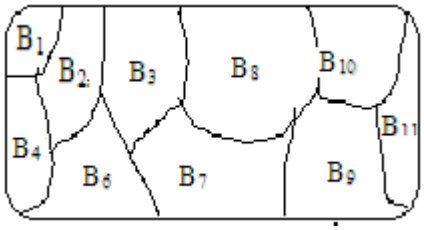
Definition (Partition of Sample Space): We say that the events $B_1, B_2, \ldots B_k$ represents partition of the sample space S if

a) $B_i \cap B_j = \phi$ i.e. if the events are mutually exclusive

b) $\bigcup_{i=1}^{k} B_i = S$

c) $P(B_i) > 0 \quad \forall i$

Thus the events $B_1, B_2, \ldots B_k$ are mutually exclusive and exhaustive. Graphically:



Example: Let $S = \{1,2,3,4,5,6\}$. $B_1 = \{1,2\}$, $B_2 = \{3,4,5\}$, $B_{31} = \{6\}$

Let A be some event associated with S and let $B_1, B_2, \ldots B_k$ be partitions of S. Hence we can write

$A = (A \cap B_1) \cup (A \cap B_2) \cup \ldots \cup (A \cap B_k)$

Of course some of $A \cap B_i$ may be empty. Note that since Bis are partitions and have no intersection (mutually exclusive events) all sets $(A \cap B_1)$, $(A \cap B_2)$, …,$(A \cap B_k)$ are pair wise mutually exclusive.

Hence $P(A) = P(A \cap B_1) + P(A \cap B_2) + \ldots + P(A \cap B_k)$

But $P(A \cap B_i)$, each for each and every i, may be expressed as $P(A/B_i)\, P(B_i)$ and we obtain what is called the theorem of total probability.

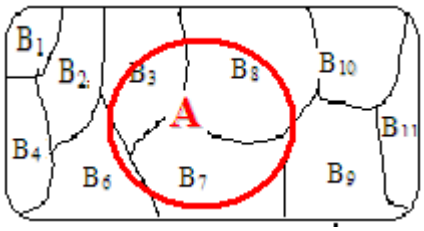
Total Probability Theorem

If we have $B_1, B_2, \ldots B_k$ to be partitions of S then for any event A

$P(A) = P(A/B_1)\, P(B_1) + P(A/B_2)\, P(B_2) + \ldots + P(A/B_k)\, P(B_k)$

$$= \sum_{i=1}^{k} P(A/B_i)\, P(B_i)$$

Graphically:



Example: A coin, is weighted so that $P(H) = \frac{2}{3}$ and $P(T) = \frac{1}{3}$, is tossed. If heads appears, then a number is selected at random from the numbers 1 through 9; if tails appears, then a number is selected at random from the numbers 1 through 5. Find the probability that an even number is selected.

Example: Three machines A, B and C produce 50%, 30% and 20% of the total output respectively. The percentages of defective output of these machines are 3%, 4% and 5% respectively. If an item is selected at random, what is the probability that the item is defective?

Solution:

Let A: The selected item is defective

B_1: Outputs produced by A $\quad P(B_1) = 0.5 \quad P(A/B_1) = 0.03$

B_2: Outputs produced by B $\quad P(B_2) = 0.3 \quad P(A/B_2) = 0.04$

B_3: Outputs produced by C $\quad P(B_3) = 0.2 \quad P(A/B_3) = 0.05$

$P(A) = P(A/B_1)\,P(B_1) + P(A/B_2)\,P(B_2) + P(A/B_3)\,P(B_3)$

$= (0.5)(0.03)+(0.3)(0.04)+(0.2)(0.05)$

$= \mathbf{0.037}$

The above example may motivate us to ask one question. *Suppose one item is chosen and is found to be defective. What is the probability that it is from machine A?* The following well-known theorem, called Bayes' Theorem, provides the answer!

Bayes' Theorem: Let $B_1, B_2, \ldots, B_k$ be a partition of the sample space S and let A be an event associated with S. Then

$$P(B_i / A) = \frac{P(A/B_i)P(B_i)}{\sum_{i=1}^{k} P(A/B_i)P(B_i)}$$

Proof

$$P(B_i / A) = \frac{P(B_i \cap A)}{P(A)} = \frac{P(A/B_i)P(B_i)}{P(A)}$$

But then from total probability theorem we have $P(A) = \sum_{i=1}^{k} P(A/B_i)P(B_i)$, *thus*

$$P(B_i / A) = \frac{P(A/B_i)P(B_i)}{\sum_{i=1}^{k} P(A/B_i)P(B_i)}$$

Example: Consider the previous example. An item was selected and is found to be defective. What is the probability that it is from B_1?

Solution:

$P(B_1) = 0.5$ $P(A/B_1) = 0.03$

$P(B_2) = 0.3$ $P(A/B_2) = 0.04$

$P(B_3) = 0.2$ $P(A/B_3) = 0.05$

$$P(B_1 / A) = \frac{P(A/B_1)P(B_1)}{\sum_{i=1}^{3} P(A/B_i)P(B_i)} = \frac{P(A/B_1)P(B_1)}{P(A/B_1)P(B_1)+P(A/B_2)P(B_2)+P(A/B_3)P(B_3)}$$

$$= \frac{(0.03)(0.5)}{0.037} = \frac{15}{37}$$

In other words, we divide the probability of the required path, those paths which lead to a defective item, by the probability of the reduced sample space.

2.3. Independent events: Two events A and B are said to be independent iff

$$P(A \cap B) = P(A)\,P(B)$$

Verbally speaking, A and B are independent provided knowledge of the occurrence of A by no means influences the probability of occurrence of B.

Remarks

1) If A and B are mutually exclusive when do they become independent?

   $P(A \cap B) = P(\phi) \Rightarrow P(A)\,P(B) = 0$

   $\therefore$ A and B are independent and mutually exclusive iff $A \cap B = \phi$ and either $P(A) = 0$ or $P(B) = 0$

2) If $A \subseteq B$ then $A \cap B = A$

   $\Rightarrow P(A \cap B) = P(A)$

   $\Rightarrow$ A and B cannot be independent unless $P(B) = 1$

   $\Rightarrow B = S$

3) ϕ and S are independent of any event

4) Three events A, B, and C are jointly independent iff

   i. $P(A \cap B) = P(A)P(B)$
   ii. $P(A \cap C) = P(A)P(C)$
   iii. $P(B \cap C) = P(B)P(C)$
   iv. $P(A \cap B \cap C) = P(A)P(B)P(C)$

   It is worth noting that three events can be pair wise independent in the sense of definition of independency of two events but not jointly independent. This is to mean that the pairwise independency is not a sufficient condition for three events to be mutually independent. In general, for n events to be jointly independent $2^n - n - 1$ independence conditions have to be satisfied.

Theorem: Let A and B be independent. Then

I. A and B' are independent
II. A' and B' are independent
III. A' and B are independent