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CHAPTER 1:TRIGONOMETRY

Objectives: By the end of this chapter student should be able

- Know radian measure and be able to convert to degrees and vice versa.
- Sketch and use the graphs of sine, cosine, tangent graphs up to $4\pi^c$ or 720^0
- Solve simple trigonometric equations in the range $0^0 \leq x \leq 360^0$.
- Use the exact values of the sine, cosine and tangent of 30^0,45^0 and 60^0 (eg. sin 150 = $-\sqrt{3}/2$ for angles > 90^0)
- Sketch & use the graphs of sec, cosec and cotan.
- Use the identities $\frac{sinx}{cosx} \equiv \tan x$, $\sin^2 x + \cos^2 x \equiv 1$, $\sec^2 x - \tan^2 x \equiv 1$ in proofs and equations.
- Use the expansions of sin (A±B), cos (A ± B) tan (A±B), sin 2A, cos2A, tan2A , asinx±bsinx ≡ Rsin(x±α) or Rcos(x±α) in proofs and equations
- Use the principal values of inverse trig. functions, $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$
- Write. the trigonometry into polar form

The word Trigonometry is derived from two Greek words - 'trigon' (meaning a triangle) and `metron' (meaning a measure) and hence literally Trigonometry means the measurement of a triangle. Thus it is that branch of mathematics which deals with the measurement of the sides and angles of a triangle and the investigation of various relations which exist among them

1.1 Angles in Degrees and Radians

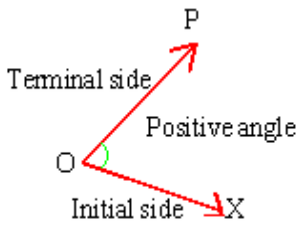


Figure 1.1

The plane angle XOP is formed by intersecting half lines(i.e. rays) OX and OP. The point O is called the vertex and half lines are called the sides of the angle. More often a plane angle is to be thought of as generated by revolving (in plane) a half line (i.e. ray) from the initial position (OX) to terminate positon ($\overrightarrow{OP}$).Then O is again the vertex, $\overrightarrow{OX}$ is the initial side (arm), and OP is the terminal side (or arm) of the angle.

Note that the definition says "an angle". This is because in the given configuration the initial side may be rotated clock-wise or counter clockwise, and it may be rotated completely around once or several times before resting at the terminal side.

Following figure shows a few angles determined by the same configuration.

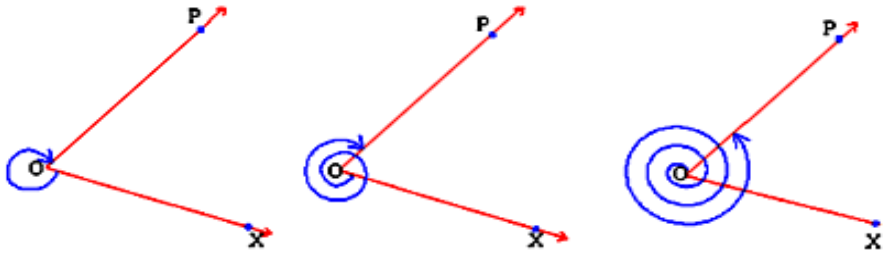


Figure 1.1.2

An angle so generated is called positive (+ve) if the direction of rotation (indicated by curved arrows) is counter clockwise and negative (-ve) if the direction of rotation is clockwise.

- **Zero angle and straight angle:** If the ray OX does not rotate at all i.e. ray OP coincides with OX without rotation then $\angle XOP$ is a zero angle. When rotation takes place, the angle so formed can be measured in terms of the amount of rotation. One full rotation in counter clockwise sense describes an angle of 360^0 of Sexagesimal system (or the English System). Hence, one half of complete rotation is the angle of $\frac{1}{2}\left(360^0\right)$.This angle is called straight angle since ray OX and OP are in straight line but in the opposite direction. Similarly quarter rotation i.e. $\frac{1}{4}\left(360^0\right)$ describes an angle of 90^0 (right angle) and so on.

ANGLES IN STANDARD POSITION

An angle in the coordinate plane is said to be in standard position, if

1. its vertex is at the origin, and
2. its initial side lies on the positive *x*-axis.

Example 2 The following angles are all in standard position:

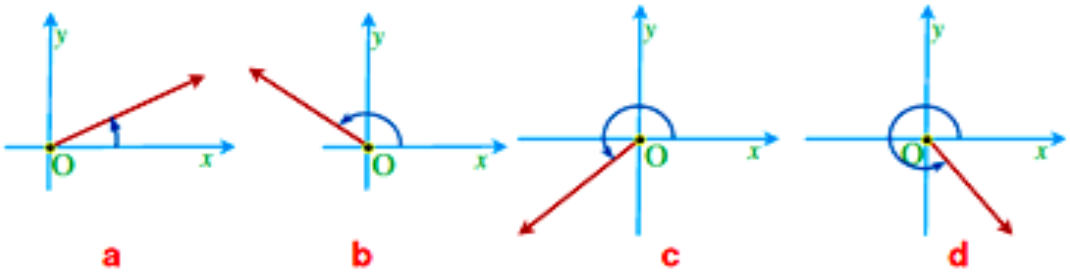


Figure 1.1.3

First, second, third and fourth quadrant angles

• If the terminal side of an angle in standard position lies in the first quadrant, then it is called a first quadrant angle.

• If the terminal side of an angle in standard position lies in the second quadrant, then it is called a second quadrant angle.

• If the terminal side of an angle in standard position lies in the third quadrant, then it is called a third quadrant angle.

• If the terminal side of an angle in standard position lies in the fourth quadrant, then it is called a fourth quadrant angle.

Example 3 The following are angles in different quadrants:

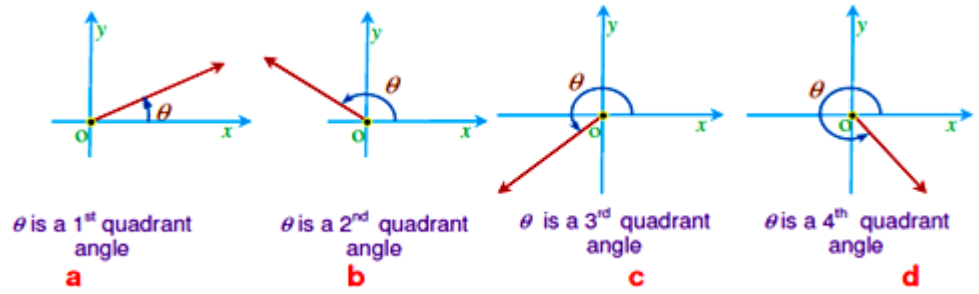


Figure 1.1.4

Quadrantal angles

If the terminal side of an angle in standard position lies along the *x*-axis or the *y*-axis, then the angle is called a quadrantal angle.

Example 4 The following are all quadrantal angles.

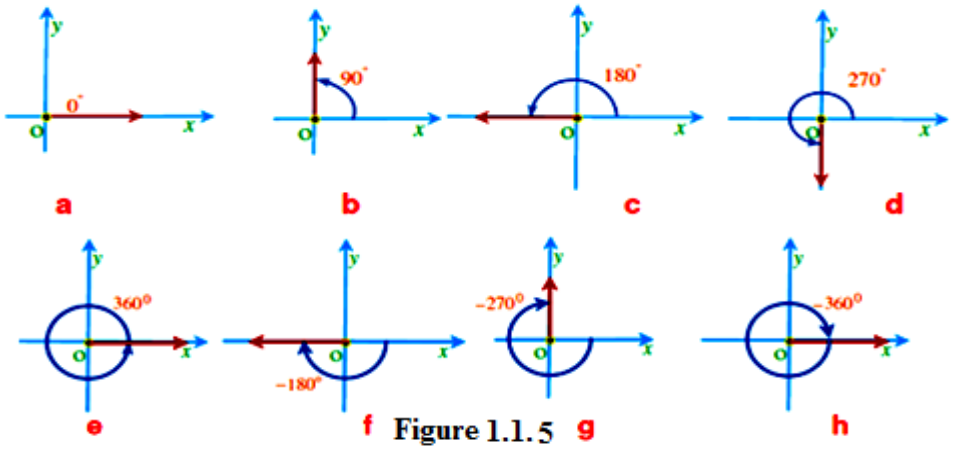


Figure 1.1.5

Angles with measures of -360^0, -270^0, -180^0, -90^0, 0^0, 90^0, 180^0, 270^0, and 360^0 are examples of quadrantal angles because their terminal sides lie along the *x*-axis or the *y*-axis.

Example 5

The following are measures of different angles. Put the angles in standard position and indicate to which quadrant they belong:

a. 200^0 **b.** 1125^0 **c.** -900^0

Solution:

a. $200^0 = 180^0 + 20^0$

$\therefore$ an angle with measure of 200o is a third quadrant angle.

b. $1125^0 = 3(360)^0 + 45^0$

1125o is a measure of a first quadrant angle.

c. $-900^0 = 2(-360)^0 + (-180^0)$

-900^0 is a measure of a quadrantal angle.

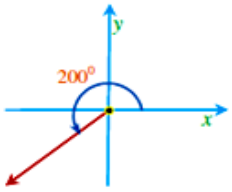


Figure 1.1.6

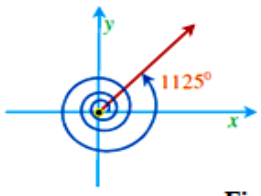


Figure 1.1.7

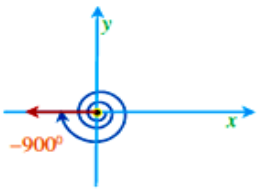


Figure 1.1.8

Exercise 1.1.1

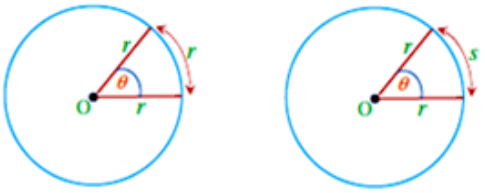
The following are measures of different angles. Put the angles in standard position and indicate to which quadrant they belong:

a.	240^0	**c.**	620^0	**e.**	-350^0	**g.**	550^0
b.	350^0	**d.**	666^0	**f.**	-480^0	**h.**	-1080^0

Radian measure of angles and Degree measures

So far we have measured angles in degrees. However, angles can also be measured in radians. Scientists, engineers, and mathematicians usually work with angles in radians.

The angle θ subtended at the centre of a circle by an arc equal in length to the radius is 1 radian. That is $\theta = \frac{r}{r} = 1$ radian (see **Figure 1.1.9 a**)



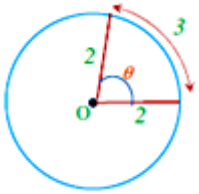
a Figure 1.1.9 b

In general, if the length of the arc is *s* units and the radius is *r* units, then $\theta = \frac{s}{r}$ radians. (*See* Figure 1.1.9 b.) This indicates that the size of the angle is the ratio of the arc length to the length of the radius.

Example 6 if *s* = 3 cm and *r* = *2cm*, calculate θ in radians

Solution

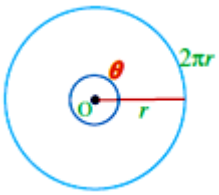
$\theta = \frac{s}{r} \Rightarrow \frac{3}{2} = 1.5$ radians



Example 7 Convert 360^0 to radians

Solution

A circle with radius r units as a circumference of $2\pi r$



In this case $\theta = \frac{s}{r}$ becomes $\theta = \frac{2\pi r}{r} \Rightarrow \theta = 2\pi$, i.e. $360^0 = 2\pi$ radians

Example 8 Can you convert 180^0 to *radian* measures?

Solution

Since $360^0 = 2\pi$ radians, $180^0 = \pi$ radians ....because $180^0 = \frac{360^0}{2}$

It follows that $1 \text{ radians} = \frac{180^0}{\pi} \cong 57.3^0$

Rule 1

To convert degrees to radians, multiply by $\frac{\pi}{180^o}$

i.e., *radians* = *degrees* $\times \frac{\pi}{180^o}$.

Example 9

a. Convert 30^0 to radians. **b.** Convert 240^0 to radians.

Solution

a. $30^0 = 30^0 \times \frac{\pi}{180^0} = \frac{\pi}{6}$ radians

b. $240^0 = 240^0 \times \frac{\pi}{180^0} = \frac{4\pi}{3}$ radians

Rule 2

To convert radians to degrees, multiply by $\frac{180^o}{\pi}$.

i.e., *degrees* = *radians* $\times \frac{180^o}{\pi}$.

Example 10

a $\frac{\pi}{2}\text{rad} = \frac{\pi}{2} \times \frac{180^o}{\pi} = 90^o$ **b** $-4\pi \text{ rad} = -4\pi \times \frac{180^o}{\pi} = -720^o$

Exercise

1. Convert each of the following degrees to radians

a. 60^0 **c.** 135^0 **e.** 114^0

b. -120^0 **d.** 1080^0 **f.** 300^0

g. -45^0 **h.** 270^0 **i.** 90^0

2. Convert each of the following *radians* to *degrees*:

a. $\frac{\pi}{12}$ **c.** $\frac{2\pi}{3}$ **e.** $-\frac{10\pi}{3}$

b. $-\frac{\pi}{6}$ **d.** $\frac{5\pi}{6}$ **f.** 3

Arc Length & Area of the Sector

Arc length

O is the centre of the circle of radius '*r*' ($r > 0$). A central angle of θ^c radians intercepts an arc AB of length's'. We know that angle subtended at the centre (is proportional to the length of the arc.

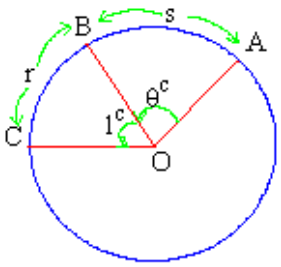


Figure 1.2.1

$\therefore \theta^c = K.\, s$ .....(1). Where K is a constant. Let `C' be any point on this circle such that

$\angle$ BOC = 1 radian so that length of arc BC = r

$\therefore 1^c = K.\, r$..... (2). Comparing the two relations (1) and (2) we have $\frac{Ks}{Kr} = \frac{\theta}{1} \Rightarrow s = r\theta$

$\therefore$ $\boxed{s = r\theta}$ as required.

Note: In this formula θ is always in radians with ***r*** and ***s*** measured in the same units of length. But when θ is measured in degrees, $s = \frac{\theta}{360} \times 2\pi r$

For military purpose this written as W = R m where m is the central angle in mils. R is the radius (range) expressed in thousands of meters and W is the chord (width) expressed in meters.

Example 1

The sum of two angles is 100^0 and their difference is $\frac{2\pi^c}{9}$. Find the angles in radian.

Solution

Let the required angles be *x* rad. and *y* rad. respectively.

$$\text{Also } 100^0 = 100 \times \frac{\pi}{180} rad = \frac{5\pi}{9} rad$$

$$\therefore x + y = \frac{5\pi}{9} rad \text{ and } x - y = \frac{2\pi}{9}$$

Adding we get,

$$2x = \frac{7\pi}{9} \Rightarrow x = \frac{7\pi}{18}$$

And on subtracting we get,

$$2y = \frac{3\pi}{9} \Rightarrow y = \frac{\pi}{6}$$

$\therefore$ the two angles are $\frac{7\pi}{18}$ and $\frac{\pi}{6}$ respectively.

Example 2

Find the approximate diameter of the moon if an observer on the earth measures an angle of 0.5^0 from the top to the bottom of the disk when the moon is 240000 km away.

Solution

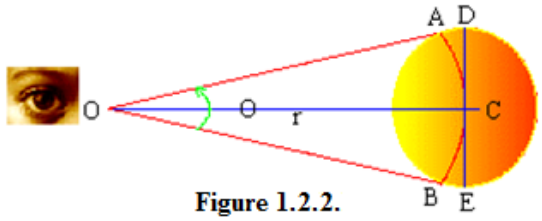


Figure 1.2.2.

Let O be the eye of the observer. C is the centre of the moon, *r* is the distance from O to C, and DE is the diameter of the moon perpendicular to OC. Finally, Let A and B be the points where the circle of radius r centered at O intersects the moon. Then r is the approximate distance at O that intersects the moon. Then r is the approximate distance from the earth to the moon, and DE is approximately S, where S = *l* (ACB).

From s = r

From $s = r\theta$, since $\theta = 0.5^0 = 0.5^0\left(\frac{\pi}{180}\right)$rad

$$\therefore s = r\theta \Rightarrow 240,000(0.5)\left(\frac{\pi}{180}\right)km$$

Estimating $s = 240,000(0.5)\left(\frac{\pi}{180}\right) = 2000km$

Since *r* = 240,000 is probably accurate to only two significant figures, thus the most we can say is that the diameter ≥2100 km

Example 3

Find the interior angles of a regular polygon of 10 sides in radians.

Solution

$$\text{Exterior angle of a polygon} = \frac{360^0}{\text{No. of sides}} = \frac{360^0}{10} = 36^0$$

$$\text{Also each interior angle} = 180^0 - 36^0 = 144^0$$

$$= 144 \times \frac{\pi}{180} \text{ rad} = \frac{4}{5}\pi \text{ rad}$$

Example 4

The length of an arc of a circle is 7 cm. Find the angle (in degrees and radians) it makes at the centre, if the radius is 4 cm. give the answer to three significant figures.

Solution

Given that s = 7cm, r = 4 cm with usual notations.

Now we have $s = r\theta^c \therefore \theta = \theta = \frac{s}{r}$ = 7 cm/4cm = 1.75 rad or $1.75 \times 180/\pi = 100.27^{0.}$

Example 5

From a circle of radius `a', a sector containing a central angle of 60^0 has been cut off, and the remainder is folded over to form a right circular cone, find the radius of the base of the cone

Solution

As shown in the adjoining fig. A sector of central angle 60^0 is cut off from the circle, then the remaining larger sector has a central angle

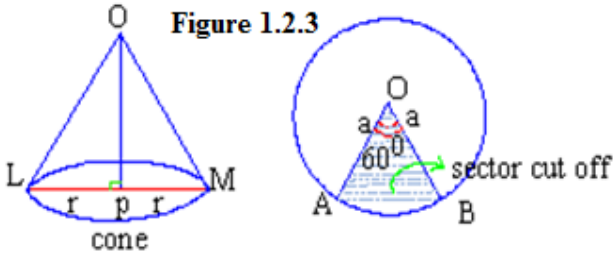


Figure 1.2.3

$$= 360^0 - 60^0 = 300^0$$

$$= 300 \times \frac{\pi}{180} \text{ rad} = \frac{5\pi}{3} \text{ rad}$$

Now OA = OB = a

$\therefore$ If arc ACB = s then $s = a\theta^c = a \times \frac{5\pi}{3}$

Now this sector with central angle $\frac{5\pi}{3}$, folded into a right circular cone. Note that the circumference of the base of the cone = length of arc ACB (s).

Let 'r' be the radius of the base of the cone then circumference = $2\pi r$

$$\therefore \quad s = 2\pi r$$
$$\therefore \quad a \cdot \frac{5\pi}{3} = 2\pi r$$
$$\therefore \quad r = \frac{5}{6} a \text{ units}$$

Example 6

Find the diameter of a pulley which is driven at 360 revolutions per minute by a belt moving at 44cm/sec.

Solution

The given pulley is driven at 360 revolutions per minute.

$\therefore$ It is driven at 6 revolutions per sec.

Let `r' be the radius of the pulley, then its circumference

$= 2\pi r$

The belt is around the pulley, therefore in one revolution the length of the belt unwound is $2\pi r$.

$\therefore$ Length of the belt unwound in 6 revolutions $= 6 \times 2\pi r = 12\pi r$

$$\therefore \quad 12\pi r = 44 \text{ cm}$$
$$\therefore \quad 12 \times \frac{22}{7} r = 44 \text{ cm} \quad (\text{since } \pi = 22/7)$$
$$\therefore \quad r = \frac{7 \text{ cm}}{6} = 1.1667 \text{ cm (approx.)}$$
$$\therefore \quad \text{Diameter of the pulley} = 2r = 2.333 \text{ (approx.)}$$

Exercise 1.2.1

1. Two cities 270 miles apart lie on the same meridian. Find their difference in latitude.(Hints r = 3960 miles (of the earth).

Ans: $\frac{3}{44}$ rad or $3.54^0.4'$.

2. A rail road curve is to be laid out on a circle. What radius should be used if the track is to change direction by 250 in a distance of 120 m. ?

Ans: 275 m. (approx)

3. A field is in the shape of a circular sector containing an angle $\pi / 3$ rad. Its straight sides are 100 feet each. What is its price per sq.foot if it is sold at the rate of $ 100 per foot of the frontage (arc length)?

Ans: 2 $

4.The angles of a triangle are in the ratio 3 : 7 : 8. Find them in degrees as well as in radians?

Ans: - 30^0, 70^0 80^0 $\pi/6$, $7\pi/18$, $4\pi/6$ rad.

5.Find the length of arc of a circle of radius 9cms. The arc subtending an angle of 40^0 at the centre.

Ans: - 2π cms.

6.In a circle of radius 9 cms, an arc of length 2 cms subtends angle 'θ^c ' at the centre. Find 'θ' in degrees?

Ans :- $400/\pi$.

7.A circle has a radius of 30 in. How many radians are there in an angle at the centre subtended by an arc of (a) of 30 in (b) of 20 in (c) of 50 in.

Ans :- (a) 1 rad (b) 2/3 rad (c) 5/3 rad

8. Find the radius of the circle for which an arc 15 in long subtends an angle

(a) of 1 rad, (b) of 2/3 rad, (c) of 3 rad (d) of 20^0 (e) of 50^0

Ans :- (a) 15 in (b) 22.5 in (c) 5 in (d) 43.0 in (e) 17.2 in

9. The end of a 40 in. pendulum describes an arc of 5 in. Through what angle does the pendulum swing?

Ans: - 1/8 rad or 7^0 9' 43".

Area of a sector

Consider a circle with centre O and radius r. Let AB be an arc of this circle which sub tends an angle θ rad at the centre O of the circle. The areas of sectors are directly proportional to the measures of their corresponding central angles

$$\frac{\text{Area of sector OAB}}{\text{Area of circle}} = \frac{\text{TheSize of the central angle of thee sector OAB}}{\text{Size of the angel of circe}}$$

$$\frac{A}{\pi r^2} = \frac{\theta rad}{2\pi}$$

$$A = \frac{\theta}{2\pi} \times \pi r^2$$

$$A = \frac{1}{2} r^2 \theta^c$$

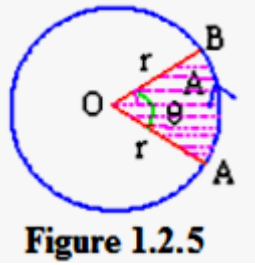
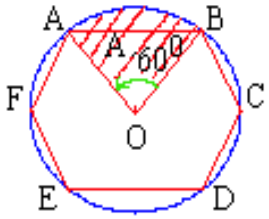


Figure 1.2.5

Example 7

ABCDEF is a regular hexagon inscribed in a circle of radius 9 cm. Find the length of the minor arc AB and the area of the corresponding sector OAB.

Solution



radius $r = 9$ cm. m $\angle AOB = 360^0$

($\because$ no of sides of a regular hexagon$(n) = 6$)

sides of a regular hexagon$(n) = 61$

$\therefore$ m $\angle AOB = 60^0 = \frac{\pi}{3}$ rad

Figure 1.2.6

(i) length of the minor arc AB (s) = $r\,\theta^c$

$9 \times \frac{\pi}{3}$ cm

$= 3\pi$cm $\Rightarrow 3(3.14)$cm $\Rightarrow 9.42$cm(*Appro.*)

(ii) Area of the sector O-AB (A)

$= \frac{1}{2} r^2 \theta^c$

$= \frac{1}{2} (9)^2 \pi/3$

$= \frac{27}{2} \pi$ sq. cm

$= 13.5\ (3.14)$ sq.cm

$= 42.39$ sq.cm (approx)

Example 8

What is the angle measure, in radians of the acute angle formed by the minute hands of a clock at 7.15 with hour hand?

Solution

The hour hand moves $\frac{1}{12}$th of complete rotation in each hour.

Therefore in 15 minutes i.e $\frac{1}{4}$hour, it moves

$\frac{1}{12} \times \frac{1}{4} = \frac{1}{48}$th of a complete revolution.

Therefore, at 7.15, the hour hand and minute hand

are $\frac{4}{12} + \frac{1}{48} = \frac{16+1}{48} = \frac{17}{48}$th of a revolution apart.

Now $\frac{17}{48}$of one revolution $= \frac{17}{48} \times 2\pi = \frac{17 \times \pi}{24}rad=127.5^0$



Figure 1.2.7

Example 9

Two concentric circles with centre at O and radii 5cms and 10 cms are drawn. A straight line OP drawn through the centre cuts them in A and A' respectively. This line is rotated through 600 and in its final position it cuts the circles in B and B' respectively. Find the perimeter and the area of the portion between the two circles and the two lines

Solution

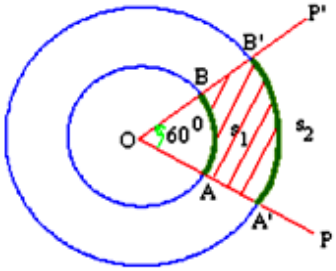


Figure 1.2.8

(i) We have OA = 5 cms, OA' = 10 cms, $\theta = 60^0 = \pi/3$ rad.

$\therefore$ AA' = 5 cms Similarly BB' = 5 cms

Also $\theta = 60^0 = \pi/3$ rad

Let arc AB = s_1 and arc A'B' = s_2

then considering the inner circle and using $s = r\,\theta^c$ we get

$$s_1 = 5 \times \frac{\pi}{3} = \frac{5\pi}{3} cm$$

Similarly for arc A'B' = $s_2 = 10 \times \frac{\pi}{3} = \frac{10\pi}{3} cm$

$\therefore$ the perimeter of the required portion (shaded portion)

= AB + BB' + A'B' + AA'

= s_1 + BB' + s_2 + AA'

$$\frac{5\pi}{3} + 5 + \frac{10\pi}{3} + 5$$

$= 10 + 5\pi = 5\,(\pi + 2) \approx 5\,(3.14 + 2) \approx 5\,(5.14)$

= 25.70 cms.

(ii) Area of the portion ABB'A'A (shaded)

= Area of sector O - A'B' - Area of sector O - AB

$$= \frac{1}{2} OA^2 \theta^c \text{ ....(by using } \frac{1}{2} r^2 \theta^c)$$

$$= \frac{1}{2}\frac{\pi}{3}\left(10^2 - 5^2\right) \text{sq. cms}$$

$\frac{\pi}{6}(100-25)$sq. cms $\Rightarrow \frac{\pi}{6}\times 75$sq. cms

$=12.5\pi$sq. cms

$=(12.5)(3.14)$sq. cms

$A(ABB'A'A) = 39.25$sq. cms (*Approx*)

Example 10

Find the area of the shaded portion of the sector of the circle shown in the given figure.

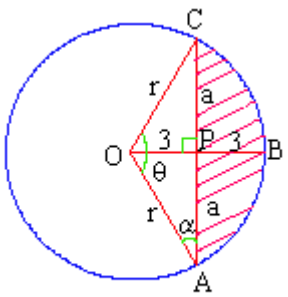


Figure 1.13

Solution

OA = OB = OC = 6..... (Radii of the same circle)

$\therefore$ r = 6

In rt. angled Δ OPA, by Pythagoras theorem,

$r^2 = 3^2 + a^2$

$\therefore$ 62 - $3^2 = a^2$

$\therefore$ $27 = a^2$ $\therefore$ $a = 3\sqrt{3}$

Also $\sin\alpha = \frac{OP}{OA} = \frac{3}{6} = \frac{1}{2}$

$\therefore \alpha = \sin^{-1}(0.5) = 30^0$ $\therefore \alpha = \pi/6$ rad

Since in right angled Δ OPA, $\alpha = 30^0$, $\angle P = 90^0$

we have $\angle POA = 60^0$

Now Area of the sector O-ABC $= \frac{1}{2} r^2\theta^c$

$= \frac{1}{2}(6)^2 \frac{2\pi}{3} = 12\pi$ sq.units

Area of Δ AOC $= \frac{1}{2}$ base x height $= \frac{b\,h}{2} = \frac{1}{2}$ AC x OP

$= \frac{1}{2} 6\sqrt{3} \times 3 = 9\sqrt{3}$ sq. units

Then it follows that $\theta = 120^0 = \frac{2\pi}{3}$

Area of the shaded portion = Area of the sector O-ABC -Area of Δ AOC

$= 12\pi - 9\sqrt{3} = 3(4\pi - 3\sqrt{3})$

≈ 22.11 sq. units.

Exercise 1.2.2

1.In a circle of radius 6 cm. an arc PQ subtends an angle of 30^0 at the centre. Find the area between arc PQ and chord PQ.

Ans : 0.42 sq. cm

2. The area of a circle is 64π sq. cms. Find the length of its arc subtending an angle of 60^0 at the centre. Also find the area of the corresponding sector?

Ans: - $8\pi/3$ cm , $32\pi/3$ sq. cms.

3. A wire of length 10 cms is bent so as to form an arc of a circle of radius 4 cms. What is the angle subtended at the center in degrees?

Ans :- $450^0/\pi$

4. How many degrees are there in a central angle subtended by an arc of length equal to the diameter of the circle ? Also find the area of the corresponding sector?

Ans :- $\left(\frac{360}{\pi}\right) r^2$sq. units.

5. Two concentric circles with centre 'O'. Two radii 6 cms and 12 cms are drawn. A straight line OX is drawn through their centres cuts the circles at A and A'. This line is rotated through 30^0 and cuts the circles at B and B' on the final position. Find the perimeter and area of the portion between the circles and the line?

Ans: - $12.5(\pi - 2)$ sq. cms.

TRIGONOMETRIC FUNCTIONS OF ACUTE ANGLES

Trigonometric Ratios of Acute Angles

The building blocks of trigonometry is the characteristics of similar triangles, originally formulated by '**Euclid**'. If two corresponding angles are congruent then the two triangles are similar. Also corresponding sides of two similar triangles preserve the same ratio. This idea leads to the trigonometric ratios. For trigonometric ratios of acute angles, we need right angled triangles. The measure of an acute angle in a right angled triangle, lower case Greek letters α and θ are usually used.

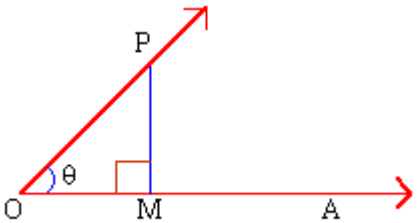


Figure 1.3.1

Let a revolving line OP starts from OA and revolve into a position OP, thus tracing out the angle AOP. Draw segment PM, perpendicular to the initial line OA. Let $\angle$ POM = θ. In the right angled triangle POM, OP is the hypotenuse, PM is the opposite side of θ and OM is the

adjacent side of θ. Now trigonometric ratios or functions are relation between any two sides of a right angled triangle with respect to a given angle. Thus a trigonometric ratio is a function of an angle.

The trigonometric ratios of $\angle$ POM = θ are defined as follows :

1. $\frac{MP}{OP}$ i.e. $\frac{\text{opposite side}}{\text{hypotenuse}}$ is called "sine" of $\angle AOP$ or $\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}}$
2. $\frac{OM}{OP}$ i.e. $\frac{\text{adjacent side}}{\text{hypotenuse}}$ is called "cosine" of $\angle AOP$ or $\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}}$
3. $\frac{MP}{OM}$ i.e. $\frac{\text{opposite side}}{\text{adjacent side}}$ is called "tangent" of $\angle AOP$ or $\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}}$
4. $\frac{OM}{MP}$ i.e. $\frac{\text{adjacent side}}{\text{opposite side}}$ is called "cotangent' of $\angle AOP$ or $\cot\theta = \frac{\text{adjacent side}}{\text{opposite side}}$
5. $\frac{OP}{MP}$ i.e. $\frac{\text{hypotenuse side}}{\text{opposite side}}$ is called "cosecant" of $\angle AOP$ or $\csc\theta = \frac{\text{hypotenuse side}}{\text{opposite side}}$
6. $\frac{MP}{OP}$ i.e. $\frac{\text{hypotenuse side}}{\text{adjacent side}}$ is called "secant" of $\angle AOP$ or $\sec\theta = \frac{\text{hypotenuse side}}{\text{adjacent side}}$

Fundamental Relations between the trigonometric ratios of an angle (from form three books)

Fundamental Identities

1) $\csc\theta = \frac{1}{\sin\theta}$, $\sec\theta = \frac{1}{\cos\theta}$, $\cot\theta = \frac{1}{\tan\theta}$ ....Reciprocal relation

2) $\tan\theta = \frac{\sin\theta}{\cos\theta}$, $\cot\theta = \frac{\cos\theta}{\sin\theta}$ ...Quotient relation

3) $\sin^2\theta + \cos^2\theta = 1$, $sec^2\theta = 1 + \tan^2\theta$.....Square relation or Pythagorean relation

$csc^2\theta = 1 + \cot^2\theta$

4) $\sin\theta = \cos(90^0 - \theta)$, $\cot\theta = \tan(90^0 - \theta)$

$\cos\theta = \sin(90^0 - \theta)$, $\sec\theta = \csc(90^0 - \theta)$ ....Cofunctions relations

$\tan\theta = \cot(90^0 - \theta)$, $\csc\theta = \sec(90^0 - \theta)$

5) $\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos\theta$ $\tan(-\theta) = -\tan\theta$

$\csc(-\theta) = -\csc\theta$, $\sec(-\theta) = \sec\theta$, $\cot(-\theta) = -\cot\theta$ .... identities for negative angle

Values Of The Six Trigonometric Functions For Angles Of Measures 30^0 , 45^0 and 60^0

1) Thetrigonometric functions of $30^0\left(\frac{\pi}{6}\right)$ rad and $60^0\left(\frac{\pi}{3}\right)$rad

in ΔABC, $\angle C=30^0$, $\angle A=60^0$and $\angle B=90^0$

let AC = 2x then AB = $\frac{1}{2}$AC

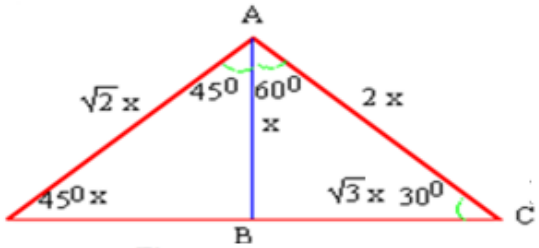


Figure 1.3.2

AB = x .....(opposite side to 30^0)

BC= $\frac{\sqrt{3}}{2}$AC $\therefore$ BC = $\sqrt{3}\cdot$ x .....(opposite side to 60^0)

$\therefore \sin 30^0=\frac{1}{2}$, $\cos 30^0 = \frac{\sqrt{3}}{2}=0.866025$

$\tan 30^0=\frac{1}{\sqrt{3}}=0.577350$, $\cot 30^0=\sqrt{3}=1.73205$

$\csc 30^0= 2$, $\sec 30^0=\frac{2}{\sqrt{3}}=1.15470$

Also $\sin 60^0=\frac{\sqrt{3}}{2}=0.866025$, $\cos 60^0=\frac{1}{2}=0.5$

$\tan 60^0=\sqrt{3}=1.73205$, $\cot 60^0=\frac{1}{\sqrt{3}}=0.577350$

$\csc 60^0=\frac{2}{\sqrt{3}}=1.15470$ and $\sec 60^0=2$

2) t - functions of 45^0 $(\frac{\pi}{4}$ rad $)$

In Δ ABD, $\angle D = \angle A = 45^0$, $\angle B = 90^0$

Let AB = BD = x then

AD = $\sqrt{2}$. x .....(hypotenuse ofan isosceles right Δ is $\sqrt{2}$ timesits congruent side)

$\therefore \sin 45^0 = \frac{1}{\sqrt{2}} = 0.707107$,

$\cos 45^0 = \frac{1}{\sqrt{2}} = 0.707107$

$\therefore \tan 45^0 = 1$, $\cot 45^0 = 1$,

$\csc 45^0 = \sqrt{2} = 1.41421$ and

$\sec 45^0 = \sqrt{2} = 1.41421$

θ	sin θ	cos θ	tan θ	cot θ	csc θ	sec θ
30^0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	2	$\frac{2}{\sqrt{3}}$
45^0	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60^0	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$	$\frac{2}{\sqrt{3}}$	2

Example 1

In Δ ABC, ∠ B = 90^0, BC = 1, AC = 2. Find ∠ A, ∠C and AB.

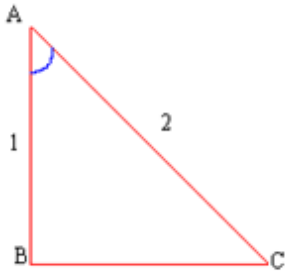


Figure 1.3.3

Solution

$$\cos C = \frac{BC}{AC} = \frac{1}{2}$$

cos C = cos 60^0 ..... [From the table]

$\therefore \angle C = 60^0$

$\therefore \angle A = 60^0$

and by the Pythagoras theorem, $AB^2 = AC^2 - BC^2$

$\therefore$ AB = $\sqrt{(2)^2 - (1)^2}$ $\therefore$ AB = $\sqrt{3}$ units.

Example 2

Triangle ABC is isosceles with ∠ B = ∠ C = π / 6 rad. AD is the median meeting BC in point D. Find AD and AB. Given that BC = 6 cms.

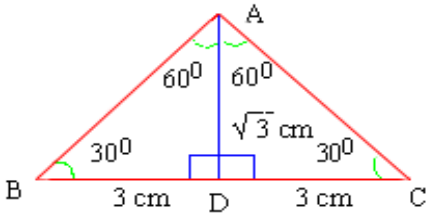


Figure 1.3. 4

Solution

Since∠ B =∠ C= π / 6 rad = 30^0. Since Δ ABC is an isosceles triangle, median AD is perpendicular to side BC Also BD = DC = 3 cm and ∠ BAD = ∠ CAD = 60^0

Now $\tan 30^0 = \frac{AD}{BD} = \frac{AD}{3} \Rightarrow AD = \frac{1}{\sqrt{3}} \times 3 = \sqrt{3}$

Also by the theorem of Pythagoras, $AB^2 = AD^2 + BD^2$

$\therefore \quad AB^2 = (3)^2 + (\sqrt{3})^2 = 12 \ \therefore \ AB = 2\sqrt{3}$

Example 3

If A is acute angle and tan A = x. Determine the values of the remaining trig-ratios.

Solution

Construct the right Δ ABC having, a = x and b = 1 Then $C = \sqrt{x^2+1}$

and $\sin A = \frac{x}{\sqrt{x^2+1}}$, $\cos A = \frac{1}{\sqrt{x^2+1}}$, $\tan A = x$

$\cot A = \frac{1}{x}$, $\sec A = \sqrt{x^2+1}$ and $\csc A = \frac{\sqrt{x^2+1}}{x}$

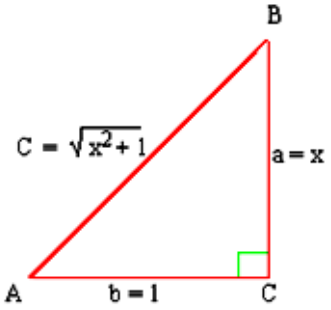


Figure 1.3.5

Example 4

A 6 foot electric cord is stretched taut from the bottom of a clock on a wall to a socket on the same wall. The socket is 1-foot above the floor. If you draw a line from the clock straight down to the floor. The floor would make an angle with the cord whose cosine is 0.9. How high above the floor is the bottom of the clock?

Solution

Given that: AC = 6 ft. BD = 1 ft. and cos C = 0.9

Now $\cos C^2 = 1 - \sin^2 C$

or $\sin^2 C = 1 - \cos^2 C$

$\therefore \quad \sin C = \sqrt{1-(0.9)^2} = \sqrt{1-0.81} = \sqrt{0.19}$

But $\sin C = \frac{AB}{AC}$

$\therefore \quad \sqrt{0.19} = \frac{AB}{6}$ $AB = 6\sqrt{0.19} = 2.62$ ft.

$\therefore \quad AD = AB + BD \ \therefore \ AD = 2.62 + 1 = 3.62$ ft. (approx)

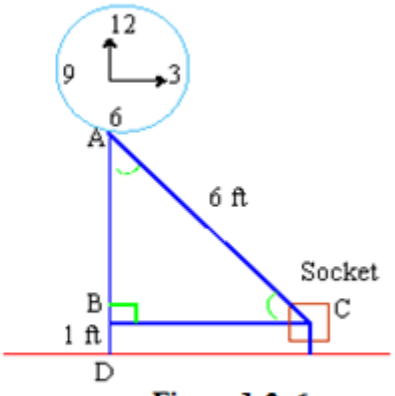


Figure 1.3.6

Trigonometric values of negative angles

Remember that an angle is positive, if measured anticlockwise and negative, if clockwise..

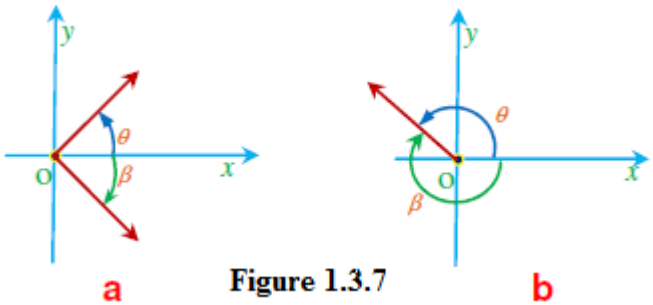


Figure 1.3.7

θ is a positive angle whereas β is a negative angle.

Example 5 Using the unit circle, find the values of the sine, cosine and tangent functions of θ when $\theta = -180^0$.

The terminal side of -180^0 intersects the unit circle at $(-1, 0)$. So $(x, y) = (-1, 0)$.

Hence, $\sin(-180^o) = y = 0$,

$\cos(-180^o) = x = -1$

and $\tan(-180^o) = \frac{y}{x} = \frac{0}{-1} = 0.$

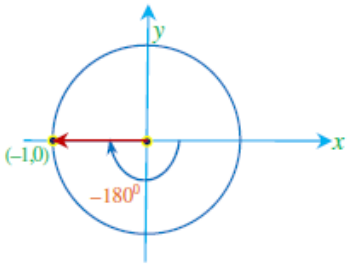


Figure 1.3.8

Example 6 Using the unit circle, find the values of the sine, cosine and tangent functions of θ when $\theta = -45^0$

Solution: Place the –45o angle in standard position. Its terminal side intersects the unit circle at Q (x, y). To determine the coordinates of Q, draw $\overline{QL}$ parallel to the y-axis. ΔOQL is an isosceles right triangle.

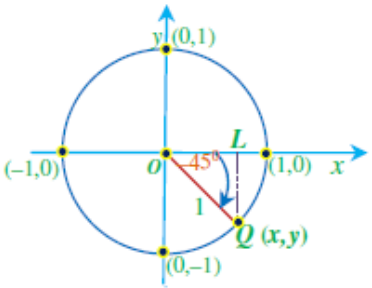


Figure 1.3.9

By Pythagoras Theorem, $(OL)^2 + (QL)^2 = (OQ)^2$

Since $OL = QL$, $(QL)^2 + (QL)^2 = (OQ)^2$.

That is $y^2 + y^2 = 1^2 \Rightarrow 2y^2 = 1 \Rightarrow y^2 = \frac{1}{2} \Rightarrow y = \pm\sqrt{\frac{1}{2}}$

$\therefore y = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$. … ***Remember that*** *y is negative in the fourth quadrant*

Since the triangle is isosceles $OL = QL = \frac{\sqrt{2}}{2}$.

Therefore, the x coordinate of Q is $\frac{\sqrt{2}}{2}$ … *Note that x is positive in the fourth quadrant*

So, the terminal side of the -45^o angle intersects the unit circle at $P(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$

i.e., $(x, y) = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

Hence, $\sin(-45^o) = y = \frac{\sqrt{2}}{2}$; $\cos(-45^o) = x = \frac{\sqrt{2}}{2}$ and $\tan(-45^o) = \frac{y}{x} = \frac{\left(-\frac{\sqrt{2}}{2}\right)}{\left(\frac{\sqrt{2}}{2}\right)} = -1.$

Observe that from the trigonometric values of 45^o and -45^o,

$\sin(-45^o) = -\sin 45^o$, $\cos(-45^o) = \cos 45^o$ and $\tan(-45^o) = -\tan 45^o$.

Negative angles:

If θ is any angle, then sin (–θ) = –sinθ, cos (–θ) = cosθ and tan (–θ) = –tanθ.

Let us refer to Figure 1.3.10 to justify the above

$$\sin\theta = \frac{y}{r},\ \sin(-\theta) = \frac{-y}{r} = -\left(\frac{y}{r}\right) \therefore \sin(-\theta) = -\sin\theta$$

$$\cos\theta = \frac{x}{r},\ \cos(-\theta) = \frac{x}{r} \qquad \therefore \cos(-\theta) = \cos\theta$$

$$\tan\theta = \frac{y}{x},\ \tan(-\theta) = \frac{-y}{x} = -\left(\frac{y}{x}\right) \therefore \tan(-\theta) = -\tan\theta$$

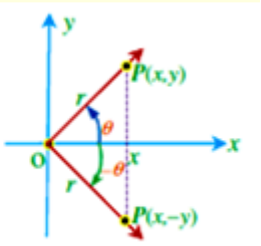


Figure 1.3.10

1.3.1.1 Exercise

1. Find the values of the sine, cosine and tangent functions of θ and complete the following two tables: (Use a dash "–" if it is undefined).

θ	0°	30°	45°	60°	90°	180°	270°	360°
sin θ	0				1		–1	
cos θ						–1		
tan θ					–			

θ	–30°	–45°	–60°	–90°	–180°	–270°	–360°
sin θ	$-\frac{1}{2}$		$-\frac{\sqrt{3}}{2}$				
cos θ	$\frac{\sqrt{3}}{2}$		$\frac{1}{2}$	0			
tan θ	$-\frac{\sqrt{3}}{3}$		$-\sqrt{3}$	–			

2. Which of the following pairs of values are equal?

a. $\sin(-30^0)$ and $\sin(30^0)$
b. $\cos(-30^0)$ and $\cos(30^0)$
c. $\tan(-30^0)$ and $\tan(30^0)$
d. $\sin(-45^0)$ and $\sin(45^0)$
e. $\cos(-45^0)$ and $\cos(45^0)$
f. $\tan(-45^0)$ and $\tan(45^0)$
g. $\sin(-60^0)$ and $\sin(60^0)$
h. $\cos(-60^0)$ and $\cos(60^0)$
i. $\tan(-60^0)$ and $\tan(60^0)$

3. How do you compare the values of:

a. sin (–θ) and sinθ ?
b. cos (–θ) and cosθ ?
c. tan (–θ) and – tanθ ?

Values of Trigonometric Functions for Related Angles

The signs of sine, cosine and tangent functions

In this sub-section you will consider whether the sign for each of the trigonometric functions of an angle is positive or negative. The sign (whether sinθ, cosθ and tanθ are positive or negative) depends on the quadrant to which θ belongs.

Example 1 Consider an angle θ in the first and second quadrants.

If θ is a first quadrant angle, then the sign of

$$\sin\theta = \frac{opp}{hyp} = \frac{y}{r} \text{ is positive}$$

$$\cos\theta = \frac{adj}{hyp} = \frac{x}{r} \text{ is positive}$$

$$\tan\theta = \frac{opp}{adj} = \frac{y}{x} \text{ is positive}$$

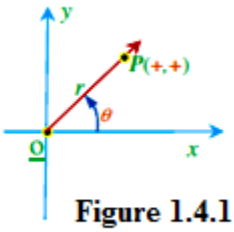


Figure 1.4.1

If θ is a second quadrant angle then, the sign of

$$\sin\theta = \frac{opp}{hyp} = \frac{y}{r} \text{ is positive}$$

$$\cos\theta = \frac{adj}{hyp} = \frac{x}{r} \text{ is negative since } x \text{ is negative}$$

$$\tan\theta = \frac{opp}{adj} = \frac{y}{x} \text{ is negative}$$

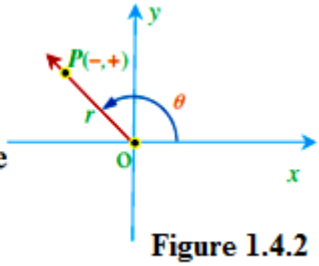


Figure 1.4.2

1. Determine whether the signs of sin θ, cos θ and tan θ are positive or negative:

a. if θ is a third quadrant angle

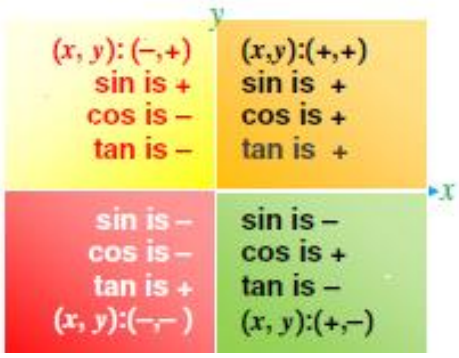
b. if θ is a fourth quadrant angle

2. Decide whether the three trigonometric functions are positive or negative an complete the following table:

	θ has terminal side in quadrant			
	I	II	III	IV
sin θ	+			−
cos θ		−		
tan θ			+	

In general, the signs of the sine, cosine and tangent functions in all of the four quadrants can be summarized as below:





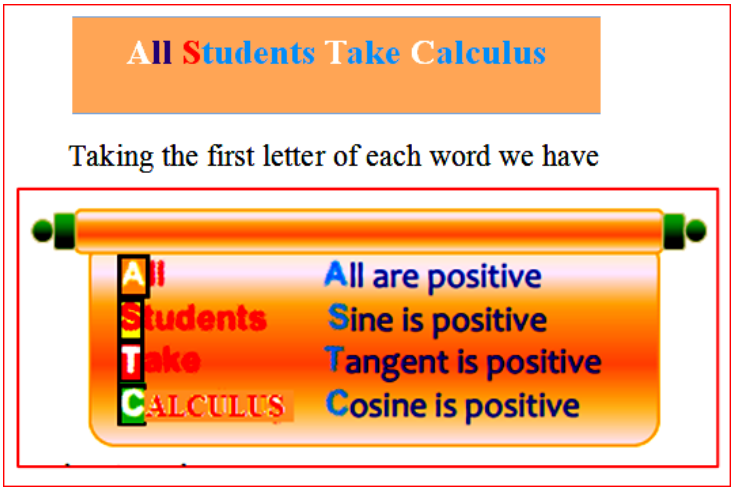
• In the first quadrant all the three trigonometric functions are positive.

• In the second quadrant only sine is positive.

• In the third quadrant only tangent is positive.

• In the fourth quadrant only cosine is positive.

Do you want an easy way to remember this? Keep in mind the following statement:



Example 2 Determine the sign of:

a $\sin 195^\circ$ **b** $\tan 336^\circ$ **c** $\cos 895^\circ$

Solution:

a Observe that $180^\circ < 195^\circ < 270^\circ$. So angle 195° is a third quadrant angle. In the third quadrant the sine function is negative.

$\therefore \sin 195^\circ$ is negative

b Since $270^\circ < 336^\circ < 360^\circ$, the angle whose measure is 336° is a fourth quadrant angle. In the fourth quadrant the tangent function is negative.

Hence $\tan 336^\circ$ is negative.

c Since $2(360)^\circ < 895^\circ < 2(360)^\circ + 180^\circ$, the angle whose measure is 895° is a second quadrant angle. In the second quadrant the cosine function is negative.

Hence, $\cos 895^\circ$ is negative.

Discussion

1. Discuss and answer each of the following:

a. If $\tan\theta > 0$ and $\cos\theta < 0$, then θ is in quadrant_____.

b. If $\sin\theta > 0$ and $\cos\theta < 0$, then θ is in quadrant_____.

c. If $\cos\theta > 0$ and $\tan\theta < 0$, then θ is in quadrant_____.

d. If $\sin\theta < 0$ and $\tan\theta > 0$, then θ is in quadrant_____.

2. Determine the sign of:

a. $\cos 267^0$

b. $\tan(-280^0)$

c. $\sin(-815^0)$

3. Determine the signs of $\sin\theta$, $\cos\theta$ and $\tan\theta$, if $\theta$is an angle in standard position and P (2, - 5) is a point on its terminal side.

Reference angle (θ_R)

If θ is an angle in standard position whose terminal side does not lie on either coordinate axis, then a reference angle θ_R for q is the acute angle formed by the terminal side of q and the *x*-axis as shown in the following figures:

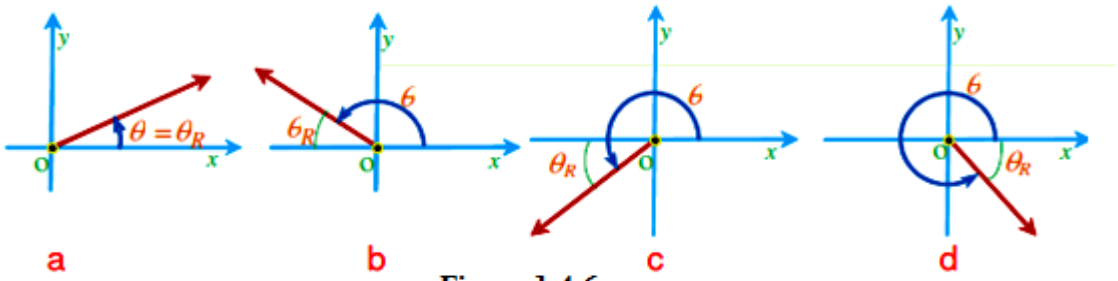


Figure 1.4.6

Example 4 Find the reference angle θ_R for θ if:

a. $\theta = 110°$ **b.** $\theta = 212°$ **c.** $\theta = 280°$

Solution:

a. Since $\theta = 110°$ is a second quadrant angle, $\theta_R = 180 - 110° = 70°$

b. Since $\theta = 212°$ is a third quadrant angle, $\theta_R = 212° - 180° = 32°$

c. Since $\theta = 280°$ is a fourth quadrant angle, $\theta_R = 360° - 280° = 80°$

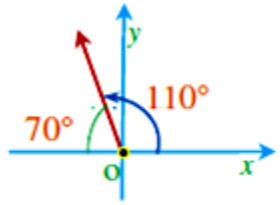


Figure 1.4.7

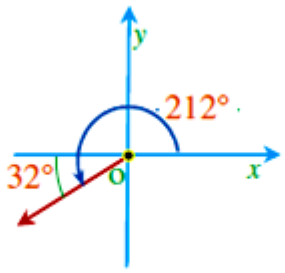


Figure 1.4.8

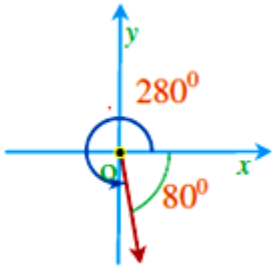


Figure 1.4.9

Exercise 1.4.1.2

Find the reference angle θ_R for θ if:

a. $\theta = 150^0$ **c.** $\theta = 240^0$ **e.** $\theta = 99^0$ **g.** $\theta = 315^0$

b. $\theta = 170^0$ **d.** $\theta = 320^0$ **f.** $\theta = 225^0$ **h.** $\theta = 840^0$

Values of the trigonometric functions of θ and its reference angle θ_R

Let us consider a second quadrant angle q. Put q in standard position as shown in the Figure 1.4.10, and let P $(-x, y)$ be a point on its terminal side. Using the *y*-axis as an axis of symmetry, reflect *P* through the *y*-axis. This will give you another point P'(x, y) which is the image of *P* on the terminal side of θ_R.

This implies that $OP = OP'$, that is $OP = OP' = \sqrt{x^2 + y^2} = r$

Hence, $\sin\theta = \frac{y}{r}$, $\sin\theta_R = \frac{y}{r} \Rightarrow \sin\theta = \sin\theta_R$

$\cos\theta = \frac{-x}{r}$, $\cos\theta_R = \frac{x}{r} \Rightarrow \cos\theta = -\cos\theta_R$

$\tan\theta = \frac{y}{-x} = -\frac{y}{x}$, $\tan\theta_R = \frac{y}{x} \Rightarrow \tan\theta = -\tan\theta_R$

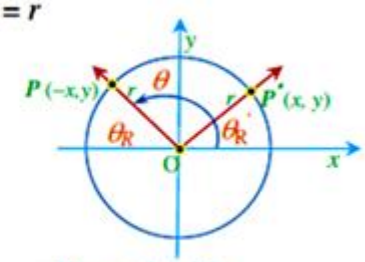


Figure 1.4.10

The values of the trigonometric function of a given angle θ and the values of the corresponding trigonometric functions of the reference angle θ_R are the same in absolute value but they may differ in sign.

Example 5 Express the sine, cosine and tangent functions of 160° in terms of its reference angle.

Solution

Remember that an angle with measure 160^0 is a second quadrant angle .In quadrant II, only sine is positive.

The reference angle $\theta_R = 180^0 - 160^0 = 20^0$

Therefore, $\sin 160^0 = \sin 20^0$, $\cos 160^0 = -\cos 20^0$ and $\tan 160^0 = -\tan 20^0$.

Supplementary angles

Two angles are said to be supplementary, if the sum of their measures is equal to 180o.

Example 6 Pairs of angles with measures of 30^0 and 150^0, 120^0 and 60^0, 45^0 and 135^0, 75^0 and 105^0, 10^0 and 170^0 are examples of supplementary angles.

Example 7 Find the values of $\sin 150^0$, $\cos 150^0$ and $\tan 150^0$.

Solution: The reference angle $\theta_R = 180^0 - 150^0 = 30^0$

Therefore, $\sin 150^0 = \sin 30^0 = \frac{1}{2}$, $\cos 150^0 = -\cos 30^0 = -\frac{\sqrt{3}}{2}$

and $\tan 150^0 = -\tan 30^0 = -\frac{\sqrt{3}}{3}$

Example 8 Find the values of $\sin 240^0$, $\cos 240^0$ and $\tan 240^0$

Solution: The reference angle $\theta_R = 240^0 - 180^0 = 60^0$

$\sin 240^0 = -\sin 60^0 = -\frac{\sqrt{3}}{2}$, $\cos 240^0 = -\cos 60^0 = -\frac{1}{2}$ and

$\tan 240^0 = \tan 60^0 = \sqrt{3}$

… ***remember that*** *in quadrant III only tangent is positive.*

In general,

If θ is a second quadrant angle, then its reference angle will be $(180^\circ - \theta)$. Hence,

$\sin\theta = \sin(180^\circ - \theta)$ $\cos\theta = -\cos(180^\circ - \theta)$ $\tan\theta = -\tan(180^\circ - \theta)$

If θ is a third quadrant angle, its reference angle will be $\theta - 180^0$.

Hence, $\sin\theta = -\sin(\theta - 180^\circ)$ $\cos\theta = -\cos(\theta - 180^\circ)$ and $\tan\theta = \tan(\theta - 180^\circ)$.

Exercise 1.4.1.3

1. Express the sine, cosine and tangent functions of each of the following angle measures in terms of their reference angle:

a. 105^0 **c.** 220^0 **e.** -300^0

b. 175^0 **d.** -260^0 **f.** 380^0

2. Find the values of:

a. $\sin 135^0$, $\cos 135^0$ and $\tan 135^0$ **d.** $\sin 115^0$, if $\sin 65^0 = 0.9063$

b. $\cos 143^0$, if $\cos 37^0 = 0.7986$ **e.** $\tan 159^0$, if $\tan 21^0 = 0.3839$

c. $\tan 138^0$, if $\tan 42^0 = 0.9004$ **f.** $\cos 24^0$, if $\cos 156^0 = -0.9135$

Co–terminal angles

Co–terminal angles are angles in standard position that have a common terminal side.

Example 9

a. The three angles with measures 30^0, -330^0 and 390^0 are co-terminal angles.

(*See* Figure 1.4.11)

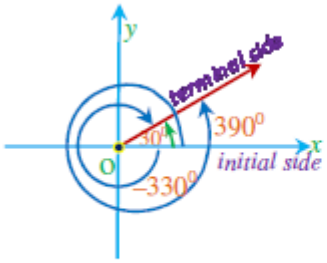


Figure 1.4.11

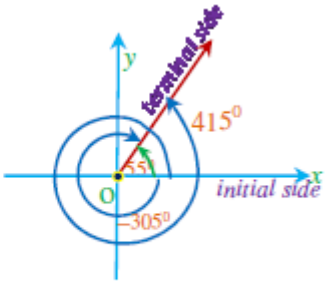


Figure 1.4.12

b. The three angles with measures 55^0, -305^0 and 415^0 are also co-terminal. (See Figure 1.4.12)

Exercise 1.4.1.4

1. With the help of the following table find angles which are coterminal with 60^0.

Angles which are co–terminal with 60°	
$60^\circ + 1(360^\circ) = 420^\circ$	$60^\circ - 1(360^\circ) = -300^\circ$
$60^\circ + 2(360^\circ) = 780^\circ$	$60^\circ - 2(360^\circ) = -660^\circ$
_____	_____
_____	_____
_____	_____
$60^\circ + 6(360^\circ) = 2220^\circ$	$60^\circ - 6(360^\circ) = -2100^\circ$
.	.
.	.
.	.

2. Give a formula to find all angles which are co-terminal with 60^0.

Given an angle θ, all angles which are co–terminal with θ are given by the formula

$\theta \pm n\,(360^\circ)$, where $n = 1, 2, 3, \ldots$

Example 10 Find a positive and a negative angle co-terminal with 75° .

Solution: To find a positive and a negative angle co-terminal with a given angle, you can add or subtract 360°. Hence, $75^\circ - 360^\circ = -285^\circ$; $75^\circ + 360^\circ = 435^\circ$.

Therefore, –285° and 435° are co-terminal with 75°.

There are an infinite number of other angles co-terminal with 75°. They are found by $75^\circ \pm n\,(360^\circ)$, $n = 1, 2, 3, \ldots$

Exercise 1.4.1.5

Find any two co-terminal angles (one of them positive and the other negative) for each of the following angle measures:

a. 70^0	**c.** 220^0	**e.** -90^0	**g.** -60^0
b. 110^0	**d.** 270^0	**f.** -37^0	**h.** -70^0

Trigonometric values of co–terminal angles

Co-terminal angles have the same trigonometric values.

Example 11 Find the trigonometric values of

a -330° and 30° **b** 120° and -240°

Solution:

a Observe that both angles are co-terminal. Their terminal side lies in the first quadrant (*See* Figure 1.4.13).

$-330^\circ = 30^\circ - 1(360^\circ)$. This gives us:

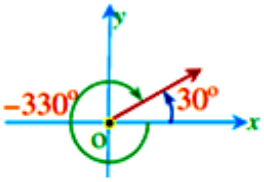


Figure 1.4.13

$$\sin 30^\circ = \sin(-330^\circ) = \frac{1}{2}$$

$$\cos 30^\circ = \cos(-330^\circ) = \frac{\sqrt{3}}{2}$$

$$\tan 30^\circ = \tan(-330^\circ) = \frac{\sqrt{3}}{3}$$

b Both 120° and -240° angles are co-terminal. Their terminal side lies in the second quadrant. (*See* Figure 1.4.14)

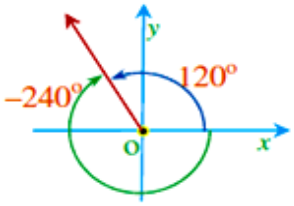


Figure 1.4.14

$-240^\circ = 120^\circ - 360^\circ$. Thus,

$$\sin 120^\circ = \sin(-240^\circ) = \sin 60^\circ = \frac{\sqrt{3}}{2}$$

... a 60^0 angle is the reference angle for a 120^0 angle

$$\cos 120^\circ = \cos(-240^\circ) = -\cos 60^\circ = -\frac{\sqrt{3}}{2}$$

... cosine is negative in quadrant II

$$\tan 120^\circ = \tan(-240^\circ) = -\tan 60^\circ = -\sqrt{3}$$

... tangent is also negative in quadrant II

Angles larger than 360°

Consider the 780° angle

$780^\circ = 360^\circ + 360^\circ + 60^\circ = 2(360^\circ) + 60^\circ$

... a 60° angle is the co-terminal acute angle for a 780° angle

Since an angle and its co–terminal have the same trigonometric value,

$$\sin 780^\circ = \sin 60^\circ = \frac{\sqrt{3}}{2}, \cos 780^\circ = \cos 60^\circ = \frac{\sqrt{3}}{2}$$

and $\tan 780^\circ = \tan 60^\circ = \sqrt{3}$.

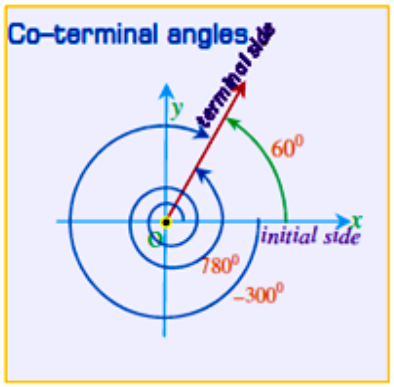


Figure 1.4.15

(***Remember that*** *since 780° is the measure of a first quadrant angle, all three of the functions are positive.*)

Example 12 Find the trigonometric values of 945°.

Solution: $945^\circ = 360^\circ + 360^\circ + 225^\circ = 2(360^\circ) + 225^\circ$

This means 945° and 225° are measures of co-terminal 3rd quadrant angles.

The reference angle for 225° is $\theta_R = 225^\circ - 180^\circ = 45^\circ$. Since an angle and its co-terminal have the same trigonometric value, it follows that

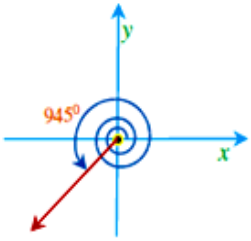


Figure 1.4.16

$$\sin 945^\circ = \sin 225^\circ = -\sin 45^\circ = -\frac{\sqrt{2}}{2}$$ *... sine is negative in quadrant III*

$$\cos 945^\circ = \cos 225^\circ = -\cos 45^\circ = -\frac{\sqrt{2}}{2}$$ *... cosine is negative in quadrant III*

$\tan 945^\circ = \tan 225^\circ = \tan 45^\circ = 1$ *... tangent is positive in quadrant III*

Exercise

1. Find the value of each of the following:

a. $\sin 390^\circ$, $\cos 390^\circ$, $\tan 390^\circ$

b. $\sin(-405^\circ)$, $\cos(-405^\circ)$, $\tan(-405^\circ)$

c. $\sin(-690^\circ)$, $\cos(-690^\circ)$, $\tan(-690^\circ)$

d. $\sin 1395^\circ$, $\cos 1395^\circ$, $\tan 1395^\circ$

2. Express each of the following as a trigonometric function of a positive acute angle:

a. $\sin 130^\circ$

b. $\sin 200^\circ$

c. $\cos 165^\circ$

d. $\cos 310^\circ$

e. $\tan 325^\circ$

f. $\sin(-100^\circ)$

g. $\cos(-305^\circ)$

h. $\tan 415^\circ$

i. $\sin 1340^\circ$

j. $\tan 1125^\circ$

k. $\sin(-330^\circ)$

l. $\cos 14$

1.2. GRAPHS OF TRIGONOMETRIC

Graphs of the Sine, Cosine and Tangent Functions

In this section, you will draw and discuss some properties of the graphs of the three trigonometric functions: sine, cosine and tangent.

Graph of the sine function

ACTIVITY 1.5.A

1 Complete the following table of values for $y = \sin\theta$.

θ in deg	–360	–330	–270	–240	–180	–120	–90
$y = \sin\theta$							

θ in deg	0	90	120	180	240	270	330	360
$y = \sin\theta$								

2 Mark the values of θ on the horizontal axis and the values of y on the vertical axis and plot the points you find.

3 Connect these points using a smooth curve to draw the graph of $y = \sin\theta$.

4 What are the domain and the range of $y = \sin\theta$?

Example 1 Draw the graph of $y = \sin\theta$, where $-360° \le \theta \le 360°$

Solution: To determine the graph of $y = \sin\theta$, we construct a table of values for $y = \sin\theta$, where $-360° \le \theta \le 360°$ (which is the same as $-2\pi \le \theta \le \pi$ in *radians*.)

The tables below show some of the values of $y = \sin\theta$ in the given interval.

θ in *deg*	–360	–330	–300	–270	–240	–210	–180	–150	–120	– 90	– 60	–30
θ in *rad*	-2π	$-\frac{11}{6}\pi$	$-\frac{5}{3}\pi$	$-\frac{3}{2}\pi$	$-\frac{4}{3}\pi$	$-\frac{7}{6}\pi$	$-\pi$	$-\frac{5}{6}\pi$	$-\frac{2}{3}\pi$	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{6}$
$y = \sin\theta$	0	0.5	0.87	1	0.87	0.5	0	– 0.5	– 0.87	– 1	– 0.87	– 0.5

θ in deg	0	30	60	90	120	150	180	210	240	270	300	330	360
θ in rad	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2}{3}\pi$	$\frac{5}{6}\pi$	π	$\frac{7}{6}\pi$	$\frac{4}{3}\pi$	$\frac{3}{2}\pi$	$\frac{5}{3}\pi$	$\frac{11}{6}\pi$	2π
$y = \sin\theta$	0	0.5	0.87	1	0.87	0.5	0	– 0.5	– 0.87	–1	– 0.87	– 0.5	0

To draw the graph we mark the values of θ on the horizontal axis and the values of y on the vertical axis. Then we plot the points and connect them using a smooth curve.

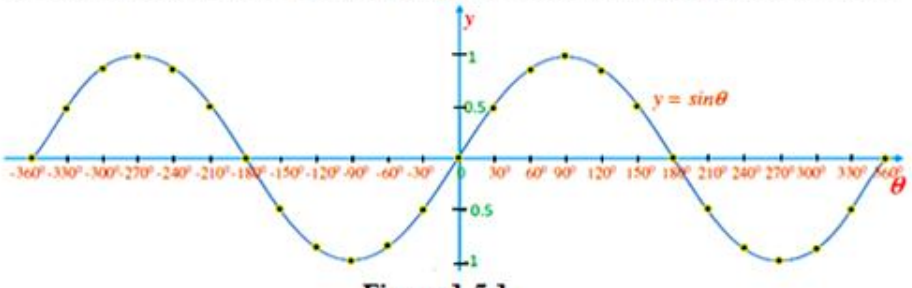


Figure 1.5.1

After a complete revolution (every 360° or 2π radian) the values of the sine function repeat themselves. This means

$\sin 0^\circ = \sin 0^\circ \pm 360^\circ = \sin 0^\circ \pm 2(360^\circ) = \sin 0^\circ \pm 3(360^\circ)$, etc.

$\sin 90^\circ = \sin 90^\circ \pm 360^\circ = \sin 90^\circ \pm 2(360^\circ) = \sin 90^\circ \pm 3(360^\circ)$, etc.

$\sin 180^\circ = \sin 180^\circ \pm 360^\circ = \sin 180^\circ \pm 2(360^\circ) = \sin 180^\circ \pm 3(360^\circ)$, etc.

In general, $\sin \theta = \sin(\theta \pm n(360^\circ))$ where n is an integer.

A function that repeats its values at regular intervals is called a periodic function.
The sine function repeats after every 360° or 2π radians.
Therefore, 360° or 2π is called the period of the sine function.

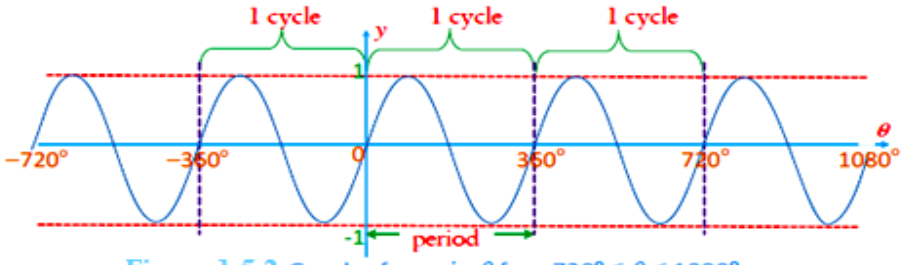


Figure 1.5.2 Graph of $y = \sin\theta$ for $-720^\circ \leq \theta \leq 1080^\circ$

Domain and range

For any angle θ taken on the unit circle, there is some point P (x, y) on its terminal side. Since $\sin\theta = \frac{y}{1} = y$, the function $y = \sin\theta$ is defined for any angle θ taken on the unit circle.

Therefore, the domain of the sine function is the set of all real numbers.

Also, note from the graph that the value of y is never less than -1 or greater than $+1$.

Note: The domain of the sine function is the set of all real numbers

The range of the sine function is $\{y \mid -1 \leq y \leq 1\}$

Graph of the cosine function

ACTIVITY 1.5.B

1 Complete the following tables of values for $y = \cos\theta$.

θ in deg	− 360	− 300	− 270	− 240	− 180	− 120	− 90	− 60
$y = \cos\theta$								

θ in deg	0	60	90	120	180	240	270	300	360
$y = \cos\theta$									

2 Sketch the graph of $y = \cos\theta$.

3 What are the domain and the range of $y = \cos\theta$?

4 What is the period of the cosine function?

From Activity above you can see that $y = \cos\theta$ is never less than –1 or greater than +1.

Just like the sine function, the cosine function is periodic at every 360^0 or 2π radians. Therefore, 360^0 or 2π is called the period of the cosine function.

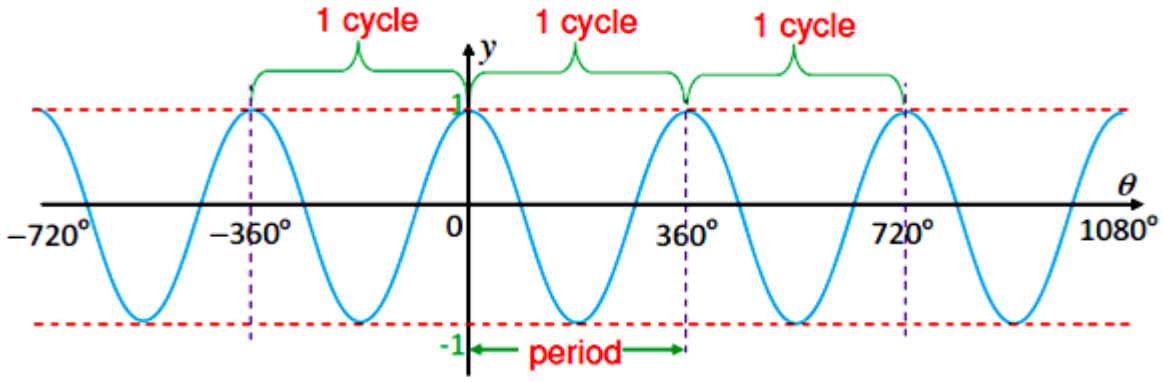


Figure 1.5.3 Graph of $y = \cos\theta$ for $-720° \leq \theta \leq 1080°$

Note: The domain of the cosine function is the set of all real numbers.

The range of the cosine function is $\{y \mid -1 \leq y \leq 1\}$.

Figure 1.5.4 represents the sine and cosine functions drawn on the same co-ordinate system.

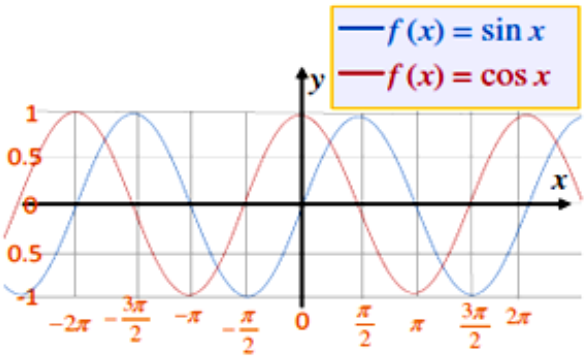


Figure 1.5.4

From this diagram you can see that both sine and cosine curves have the same shape. The curves "follow" each other, always exactly $\frac{\pi}{2}$ radians (90°) apart.

Graph of the tangent function

ACTIVITY 1.5.C

1 Complete the following tables of values for $y = \tan\theta$.

θ in *deg*	–360	–315	–270	–225	–180	–135	–90	–45
$y = \tan\theta$								

θ in *deg*	0	45	90	135	180	225	270	315	360
$y = \tan\theta$									

2 Use the table you constructed above to sketch the graph of $y = \tan\theta$.

3 For which values of θ is $y = \tan\theta$ undefined?

4 What are the domain and the range of $y = \tan\theta$?

5 What is the period of the tangent function?

The Activity above you have done above gives you a hint on what the graph of $y = \tan\theta$ looks like. Next, you will see the graph in detail.

Example 2 Draw the graph of $y = \tan\theta$, where $-360° \leq \theta \leq 360°$.

Solution: The tables below show some of the values of $y = \tan\theta$, where $-2\pi \leq \theta \leq 2\pi$

θ in *deg*	–360	–315	–270	–225	–180	–135	–90	–45	0
θ in *rad*	-2π	$-\frac{7}{4}\pi$	$-\frac{3}{2}\pi$	$-\frac{5}{4}\pi$	$-\pi$	$-\frac{3}{4}\pi$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0
$y = \tan\theta$	0	1	–	–1	0	1	–	–1	0

θ in *deg*	45	90	135	180	225	270	315	360
θ in *rad*	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3}{4}\pi$	π	$\frac{5}{4}\pi$	$\frac{3}{2}\pi$	$\frac{7}{4}\pi$	2π
$y = \tan\theta$	1	–	–1	0	1	–	–1	0

Remember that if θ is in a standard position and P(x, y) is a point where the terminal

side of θ intersects the unit circle, then $\tan\theta = \frac{y}{x}$. However, $\frac{y}{x}$ is not defined if $x = 0$.

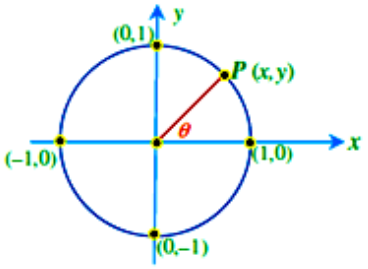


Figure 1.5.5

So $\tan\theta$ is not defined if

$\theta = 90^\circ$, $\theta = 90^\circ \pm 180^\circ$, $\theta = 90^\circ \pm 2(180^\circ)$, $\theta = 90^\circ \pm 3(180^\circ)$, etc.

In general, $\tan\theta$ is undefined if $\theta = 90^\circ \pm n\,(180^\circ)$ or if $\theta = \frac{\pi}{2} + n\pi$, where n is an integer.

The graph of $y = \tan\theta$ does not cross the vertical lines at $\theta = \frac{\pi}{2} + n\pi$, n is integer.

Moreover, if we closely investigate the behaviour of $\tan\theta$ as θ increases from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, we can see that $\tan\theta$ increases from negative infinity to positive infinity (from $-\infty$ to ∞).

A sketch of the graph of $y = \tan\theta$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, is shown in Figure 1.5.6

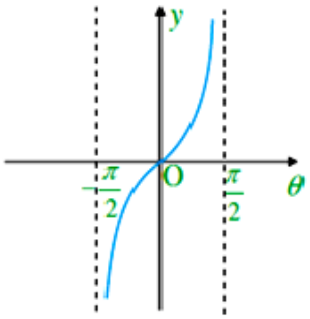


Figure 1.5.6

From the graph we see that the tangent function repeats itself every 180° or π radians.

Therefore, 180° or π is the period for the tangent function.

Since $\tan\theta$ is periodic with period π we can extend the above graph for as many repetitions (cycles) as we want.

For example, the graph of $y = \tan\theta$ for $-2\pi \le \theta \le 2\pi$ is shown below.

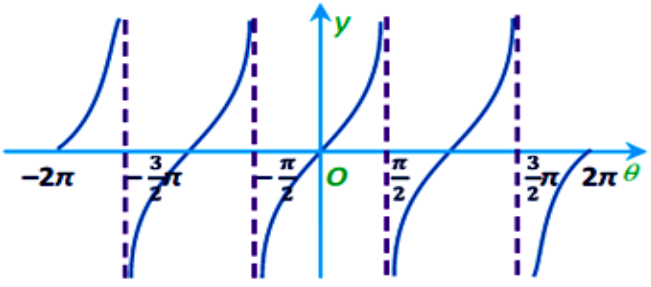


Figure 1.5.7

What are the domain and the range of $y = \tan\theta$?

For which values of θ is $y = \tan\theta$ not defined?

Using a unit circle we can see that $\tan\theta = \frac{y}{x}$ is undefined whenever the x-coordinate on the unit circle is 0.

This happens when $\theta = \pm\frac{\pi}{2}, \pm\frac{3}{2}\pi, \pm\frac{5}{2}\pi, \pm\frac{7}{2}\pi$, etc. Therefore the domain of the tangent function must exclude these odd multiples of $\frac{\pi}{2}$.

Hence, the domain of the tangent function is $\{\theta \mid \theta \neq n\frac{\pi}{2}$, where n is an odd integer$\}$.

The range of $y = \tan\theta$ is the set of real numbers.

GROUPWORK1.5

1. Use the graph of the cosine function to find the values of θ for which $\cos\theta = 0$.
2. From the graph of $y = \sin\theta$, find the values of θ for which $\sin\theta = -1$.
3. Graph the sine curve for the interval $-540^\circ \leq \theta \leq 0^\circ$.

Exercise 1.5

1. Refer to the graph of $y = \sin\theta$ or the table of values for $y = \sin\theta$ to determine how the sine function behaves as θ increases from 0° to 360° and answer the following:
   - a As θ increases from 0° to 90°, $\sin\theta$ increases from __0__ to __1__.
   - b As θ increases from 90° to 180°, $\sin\theta$ decreases from _____ to _____.
   - c As θ increases from 180° to 270°, $\sin\theta$ decreases from _____ to _____.
   - d As θ increases from 270° to 360°, $\sin\theta$ increases from _____ to _____.
2. Refer to the graph of $y = \cos\theta$ or the table of values for $y = \cos\theta$ to determine how the cosine function behaves as θ increases from 0° to 360° and answer the following:
   - a As θ increases from 0° to 90°, $\cos\theta$ decreases from __1__ to __0__.
   - b As θ increases from 90° to 180°, $\cos\theta$ decreases from _____ to _____.
   - c As θ increases from 180° to 270°, $\cos\theta$ increases from _____ to _____.
   - d As θ increases from 270° to 360°, $\cos\theta$ increases from _____ to _____.
3. Determine how the tangent function behaves as θ increases from 0° to 360° and answer the following:
   - a As θ increases from 0° to 90°, $\tan\theta$ increases from 0 to positive infinity $(+\infty)$
   - b As θ increases from 90° to 180° $\tan\theta$ increases from _____ to _____.
   - c As θ increases from 180° to 270° $\tan\theta$ increases from _____ to _____.
   - d As θ increases from 270° to 360° $\tan\theta$ _____ from $-\infty$ to 0.

THE RECIPROCAL FUNCTIONS OF THE BASIC TRIGONOMETRIC FUNCTIONS

Introduction

In this section, you will learn about three more trigonometric functions, which are called the reciprocals of the sine, cosine and tangent functions, named respectively as cosecant, secant and cotangent functions.

Definition 1.2

If θ is an angle in standard position and $\mathbf{P}(x, y)$ is a point on the terminal side of θ, different from the origin $\mathbf{O}(0, 0)$, and r is the distance of point $\mathbf{P}$ from the origin O, then

$$\csc\theta = \frac{HYP}{OPP} = \frac{r}{y}$$

$$\sec\theta = \frac{HYP}{ADJ} = \frac{r}{x}$$

$$\cot\theta = \frac{ADJ}{OPP} = \frac{x}{y}$$

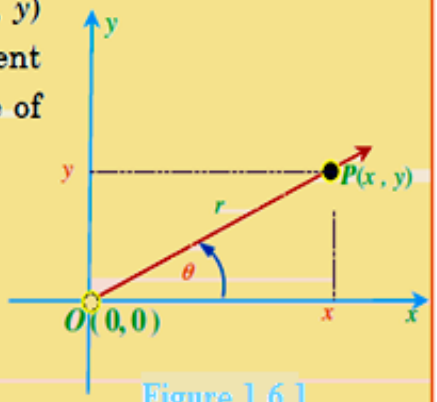


Figure 1.6.1

$\csc\theta$, $\sec\theta$ and $\cot\theta$ are abbreviations for Cosecant θ, Secant θ and Cotangent θ respectively.

Example 1 If θ is an angle in standard position and P(3, 4) is a point on the terminal side of θ, then evaluate the cosecant, secant and cotangent functions.

Solution: The distance $r = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$ units

So, $\csc\theta = \frac{HYP}{OPP} = \frac{5}{4}$,

$\sec\theta = \frac{HYP}{ADJ} = \frac{5}{3}$ and $\cot\theta = \frac{ADJ}{OPP} = \frac{3}{4}$

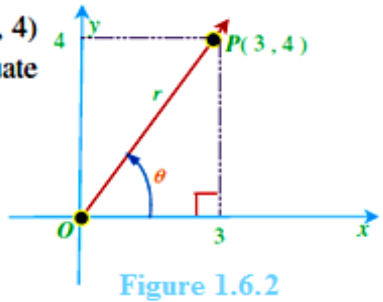


Figure 1.6.2

ACTIVITY 1.6.A

Referring to Figure 1.6.2 find:

1. $\sin\theta$, $\cos\theta$ and $\tan\theta$.
2. Compare $\sin\theta$ with $\csc\theta$; $\cos\theta$ with $\sec\theta$; $\tan\theta$ with $\cot\theta$.
3. How do they relate? Are they equal? Are they opposites? Are they reciprocals?

From the results of Activity above, you can conclude the following:

$\csc\theta = \frac{r}{y}$ whereas $\sin\theta = \frac{y}{r}$

$\sec\theta = \frac{r}{x}$ whereas $\cos\theta = \frac{x}{r}$

$\cot\theta = \frac{x}{y}$ whereas $\tan\theta = \frac{y}{x}$

Have you noticed that one is the reciprocal of the other?

That is,

$$\csc\theta = \frac{r}{y} = \frac{1}{\frac{y}{r}} = \frac{1}{\sin\theta}, \sec\theta = \frac{r}{x} = \frac{1}{\left(\frac{x}{r}\right)} = \frac{1}{\cos\theta} \text{ and}$$

$$\cot\theta = \frac{x}{y} = \frac{1}{\left(\frac{y}{x}\right)} = \frac{1}{\tan\theta}$$

Therefore,

$$\csc\theta = \frac{1}{\sin\theta}, \sec\theta = \frac{1}{\cos\theta} \text{ and } \cot\theta = \frac{1}{\tan\theta}.$$

Hence, $\csc\theta$ and $\sin\theta$ are reciprocals

$\sec\theta$ and $\cos\theta$ are reciprocals

$\tan\theta$ and $\cot\theta$ are reciprocals

Example 2 If $\theta = 30^\circ$, then find $\csc\theta$, $\sec\theta$, $\cot\theta$.

Solution:

$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{\left(\frac{1}{2}\right)} = 2 \quad \text{... remember that } \sin 30^0 = \frac{1}{2} = 0.5$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\left(\frac{\sqrt{3}}{2}\right)} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} \quad \text{... remember that } \cos 30^0 = \frac{\sqrt{3}}{2}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{\left(\frac{\sqrt{3}}{3}\right)} = \frac{3}{\sqrt{3}} = \sqrt{3} \quad \text{... remember that } \tan 30^0 = \frac{\sqrt{3}}{3}$$

Example 3 If $\sin\theta$ is 0.5, then $\csc\theta$ is $\frac{1}{0.5} = 2$

If $\cos\theta$ is -0.1035, then $\sec\theta$ is $\frac{1}{-0.1035} = -9.6618$

If $\tan\theta$ is $-\frac{1}{4}$, then $\cot\theta$ is $\frac{1}{\left(-\frac{1}{4}\right)} = -4$

Example 4 Using a unit circle, find the values of the cosecant, secant and cotangent functions if $\theta = 90^\circ, 180^\circ, 270^\circ$.

Solution: As you can see in the adjacent figure, the terminal side of the 90° angle intersects the unit circle at $(0, 1)$

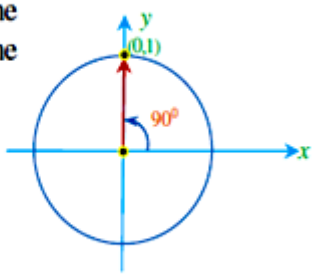


Figure 1.6.3

Hence, $\csc 90^\circ = \frac{r}{y} = \frac{1}{1} = 1$

$\sec 90^\circ = \frac{r}{x} = \frac{1}{0}$ *is undefined*

$\cot 90^\circ = \frac{x}{y} = \frac{0}{1} = 0$

The terminal side of the 180° angle intersects the *unit circle* at $(-1, 0)$.

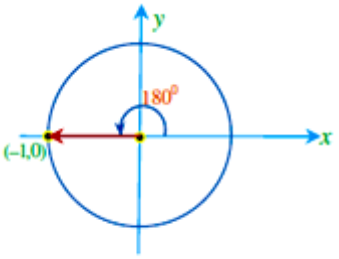


Figure 1.6.4

Hence, $\csc 180^\circ = \frac{r}{y} = \frac{1}{0}$ *is undefined*

$\sec 180^\circ = \frac{r}{x} = \frac{1}{-1} = -1$

$\cot 180^\circ = \frac{x}{y} = \frac{-1}{0}$ *is undefined*

Similarly the terminal side of the 270° angle intersects the *unit circle* at $(0, -1)$.

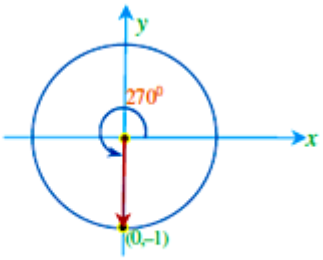


Figure 1.6.5

Hence, $\csc 270^\circ = \frac{r}{y} = \frac{1}{-1} = -1$

$\sec 270^\circ = \frac{r}{x} = \frac{1}{0}$ *is undefined*

$\cot 270^\circ = \frac{x}{y} = \frac{0}{-1} = 0$

Example 5 Using a unit circle, find the values of the cosecant, secant and cotangent functions if $\theta = 360^\circ$.

Solution: The terminal side of angle 360° intersects the unit circle at $(1, 0)$.

Hence, $\csc 360^\circ = \frac{r}{y} = \frac{1}{0}$ *is undefined*

$\sec 360^\circ = \frac{r}{x} = \frac{1}{1} = 1$

$\cot 360^\circ = \frac{x}{y} = \frac{1}{0}$ *is undefined*

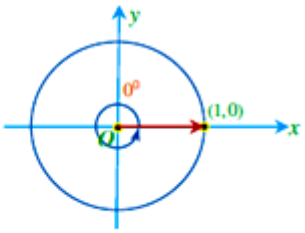


Figure 1.6.6

Remember that these results are also true for 0°, 720°, 1080°, etc.

When do you think the functions $\csc\theta$, $\sec\theta$ and $\cot\theta$ are undefined?

For example, $\csc\theta = \frac{r}{y}$ is undefined when $y = 0$. The value of y on the unit circle will be 0 when $\theta = 0^\circ, \pm 180^\circ, \pm 2(180^\circ), \pm 3(180^\circ), \pm 4(180^\circ)$, etc.

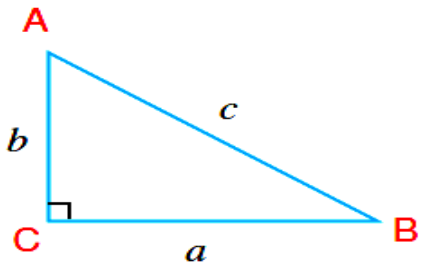
In general, $\csc\theta$ is undefined for $\theta = \pm n\,(180^\circ)$, where n is an integer.

The relationship of the trigonometric functions in a standard right angled triangle is shown below.

$$\csc A = \frac{c}{a} = \frac{1}{\sin A}$$

$$\sec A = \frac{c}{b} = \frac{1}{\cos A}$$

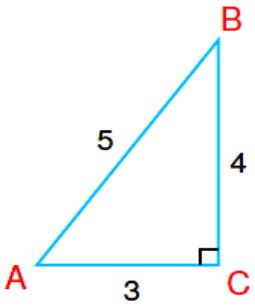
$$\cot A = \frac{b}{a} = \frac{1}{\tan A}$$



Example 6 Given the triangle below, find:

a $\cot A$ **b** $\csc B$

c $\sec A$ **d** $\csc A$



Solution

a $\cot A = \frac{3}{4}$ **b** $\csc B = \frac{5}{3}$

c $\sec A = \frac{5}{3}$ **d** $\csc A = \frac{5}{4}$

Graphs of y = csc x, y = sec x and y = cot x

In the previous section, you studied the graphs of the sine, cosine and tangent functions. In this section you will study the graphs of the remaining three trigonometric functions.

The domain, range and period of the cosecant, secant and cotangent functions are as follows.

1 If $f(x) = \csc x$, then $D_f = \{x \in \mathbb{R}: x \neq k\pi, k \in \mathbb{Z}\}$

Range $= (-\infty, -1] \cup [1, \infty)$

Period, $P = 2\pi$

2 If $f(x) = \sec x$, then $D_f = \left\{x \in \mathbb{R}: x \neq \frac{(2k+1)\pi}{2}; k \in \mathbb{Z}\right\}$

Range $= (-\infty, -1] \cup [1, \infty)$

Period, $P = 2\pi$

3 If $f(x) = \cot x$, then $D_f = \{x \in \mathbb{R}: x \neq k\pi, k \in \mathbb{Z}\}$

Range $= \mathbb{R}$

Period, $P = \pi$

You now want to draw the graph of $f(x) = \csc x$. The domain of cosecant function is restricted, in order to have no division by zero.

By taking the reciprocals of non-zero ordinates on the graph of the sine function as shown in Figure 1.6.7, you obtain the graph of $f(x) = \csc x$. The graph of cosecant function has vertical asymptotes at the point where the graph of the sine function crosses the x-axis.

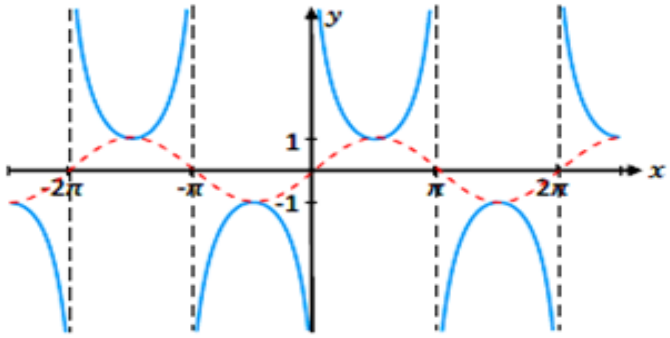


Figure 1.6.7 *Graph of y = csc x*

Applying the same techniques as for the cosecant function, we can draw the graphs of secant and cotangent functions as follows.

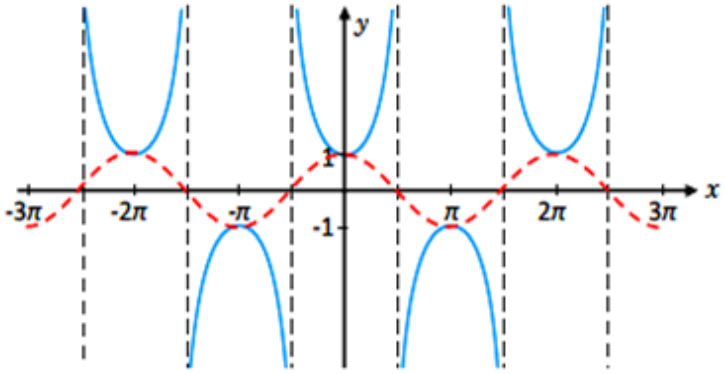


Figure 1.6. 8 *Graph of y = sec x*

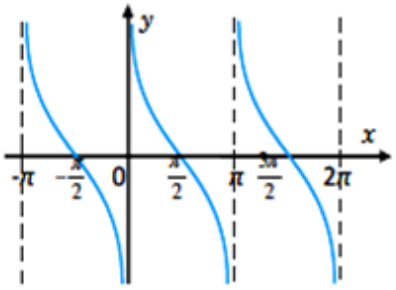


Figure 1.6.9 *Graph of y = cot x*

Exercise 1.6.1

1 Determine each of the following values without the use of tables or calculators.

a $\sec\left(\frac{\pi}{4}\right)$ **b** $\csc\left(-\frac{\pi}{2}\right)$ **c** $\cot\left(\frac{-3\pi}{4}\right)$

d $\sec\left(\frac{\pi}{3}\right)$ **e** $\csc\left(-\frac{\pi}{6}\right)$ **f** $\cot\left(\frac{5\pi}{6}\right)$

g $\sec\left(\frac{2\pi}{3}\right)$ **h** $\csc\left(\frac{7\pi}{3}\right)$ **i** $\cot\left(\frac{7\pi}{6}\right)$

j $\cot(-\pi)$ **k** $\sec\left(\frac{5\pi}{2}\right)$ **l** $\csc(3\pi)$

1.3. Identities of two angles.
The Compound angle Identities

In this part, we will study the trigonometric ratios (functions) of sum and difference of two angles.

$1 \rightarrow \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$2 \rightarrow \sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$3 \rightarrow \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$4 \rightarrow \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

Verification

1) For all real values of α and β $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$.

Consider a unit circle as shown in the Figure 1.8.1 Let R, P, Q, be three points on it such that $\angle ROP = \alpha$, $\angle ROQ = \beta$ so that $\angle QOP = \alpha - \beta$. Set the centre of the circle at the origin of the rectangular Cartesian system. Let x- axis be along OR. Draw Y- axis.

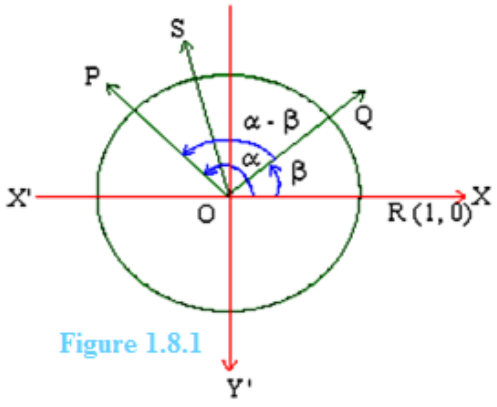


Figure 1.8.1

Here we have OR = OP = OQ = 1

$\therefore$ R $\equiv$ (1, 0), P $\equiv$ (cos α, sin α) and Q $\equiv$ (cos β, sin β) ..... (Using $x = r\cos\theta$, $y = r\sin\theta$ i.e. changing to polar coordinates)

$\therefore$ By distance formula, $PQ^2 = (\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2$

$= (\cos^2\alpha + \sin^2\alpha) + (\cos^2\beta + \sin^2\beta) - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$

$= 1 + 1 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$

$\therefore PQ^2 = 2 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$ ..... (1)

Take point S on the circle so that $\angle ROS = \alpha - \beta$ $\therefore$ S $\equiv$ [cos (α - β), sin (α - β)]

Then by distance formula, $RS^2 = [\cos(\alpha - \beta) - 1]^2 + [\sin(\alpha - \beta)]^2$

$= \cos^2(\alpha - \beta) + \sin^2(\alpha - \beta) - 2\cos(\alpha - \beta) \Rightarrow RS^2 = 2 - 2\cos(\alpha - \beta)$ ..... (2)

Since ∠QOP = ∠ROS, their corresponding arcs PQ and RS are also equal which implies that the corresponding chords PQ and RS are also equal.

∴ From (1) and (2) we get

$2 - 2\cos(\alpha - \beta) = 2 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$

$$\therefore \quad \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

Put $\alpha = x$ and $\beta = -y$ in

$\cos(x - y) = \cos x \cos y + \sin x \sin y,\ x, y \in R$

We get $\cos(\alpha + \beta) = \cos\alpha\cos(-\beta) + \sin\alpha\sin(-\beta)$

but $\cos(-\beta) = \cos\beta$ and $\sin(-\beta) = -\sin\beta$

$$\therefore \quad \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Reduction to Functions of Positive Acute Angles and Negative Acute Angles

(1) If α is real then, putting $x = \pi/2$ and $y = \alpha$ in
$\cos(x - y) = \cos x \cos y + \sin x \sin y$ we get,

$$\therefore \quad \cos\left(\frac{\pi}{2}-\alpha\right) = \cos\frac{\pi}{2}\cos\alpha + \sin\frac{\pi}{2}\sin x$$

$$= 0.\cos\alpha + 1.\sin x$$

$$\therefore \quad \cos\left(\frac{\pi}{2}-\alpha\right) = \sin\alpha$$

(2) $$\therefore \quad \cos\left[\frac{\pi}{2} - \left(\frac{\pi}{2}-\alpha\right)\right] = \sin\left(\frac{\pi}{2}-\alpha\right)$$

$$\therefore \quad \cos\alpha = \sin\left(\frac{\pi}{2}-\alpha\right)$$

$$\therefore \quad \sin\left(\frac{\pi}{2}-\alpha\right) = \cos\alpha$$

(3) We have $\cos(x) = \cos(-x)$

put $x = \pi/2 + \alpha$ then,

$$\cos\left(\frac{\pi}{2}+\alpha\right) = \cos\left[\left(-\frac{\pi}{2}+\alpha\right)\right]$$

$$\therefore \quad \cos\left(\frac{\pi}{2}+\alpha\right) = \cos\left(-\frac{\pi}{2}-\alpha\right) = \cos\left(-\frac{\pi}{2}\right)$$

$$\cos(\alpha) + \sin\left(-\frac{\pi}{2}\right)\sin\alpha$$

$$\therefore \quad \cos\left(\frac{\pi}{2}+\alpha\right) = \cos\frac{\pi}{2}\cos\alpha - \sin\frac{\pi}{2}\sin\alpha$$

$$\therefore \quad \cos\left(\frac{\pi}{2}+\alpha\right) = 0.\cos\alpha - 1.\sin\alpha$$

$$\therefore \quad \cos\left(\frac{\pi}{2}+\alpha\right) = -\sin\alpha$$

(4) We have $\sin\left(\frac{\pi}{2} - x\right) = \cos x$

put $x = -\alpha$ then $\sin\left(\frac{\pi}{2} + \alpha\right) = \cos(-\alpha)$

But $\cos(-\alpha) = \cos\alpha$

$$\therefore \quad \sin\left(\frac{\pi}{2} + \alpha\right) = \cos\alpha$$

(5) $\sin(\pi - \alpha) = \sin\pi\cos\alpha + \cos\pi\sin\alpha$
$= 0\cos\alpha + 1.\sin\alpha$

$$\sin(\pi - \alpha) = +\sin\alpha$$

(6) $\sin(\pi + \alpha) = \sin\pi\cos\alpha + \cos\pi\sin\alpha$
$= 0.\cos\alpha - 1.\sin\alpha$

$$\therefore \quad \sin(\pi + \alpha) = -\sin\alpha$$

(7) $\cos(\pi - \alpha) = \cos\pi\cos\alpha + \sin\pi\sin\alpha$
$= -1\cos\alpha + 0.\sin\alpha$

$$\therefore \quad \cos(\pi - \alpha) = -\cos\alpha$$

(8) $\cos(\pi + \alpha) = \cos\pi\cos\alpha - \sin\pi\sin\alpha$
$= -1\cos\alpha - 0.\sin\alpha$

$$\therefore \quad \cos(\pi + \alpha) = -\cos\alpha$$

Therefore we have a list of **reduction formulas** as,

(I)

$\sin(90^0 - \alpha)$	$= \cos\alpha$,	$\sin(90^0 + \alpha)$	$= \cos\alpha$
$\cos(90^0 - \alpha)$	$= \sin\alpha$,	$\cos(90^0 + \alpha)$	$= -\sin\alpha$
$\tan(90^0 - \alpha)$	$= \cot\alpha$,	$\tan(90^0 + \alpha)$	$= -\cot\alpha$
$\cot(90^0 - \alpha)$	$= \tan\alpha$,	$\cot(90^0 + \alpha)$	$= -\tan\alpha$
$\csc(90^0 - \alpha)$	$= \sec\alpha$,	$\csc(90^0 + \alpha)$	$= \sec\alpha$
$\sec(90^0 - \alpha)$	$= \csc\alpha$,	$\sec(90^0 + \alpha)$	$= -\csc\alpha$

II)

$\sin(180^0 - \alpha)$	$= \sin\alpha$,	$\sin(180^0 + \alpha)$	$= -\sin\alpha$
$\cos(180^0 - \alpha)$	$= -\cos\alpha$,	$\cos(180^0 + \alpha)$	$= -\cos\alpha$
$\tan(180^0 - \alpha)$	$= -\tan\alpha$,	$\tan(180^0 + \alpha)$	$= \cot\alpha$
$\cot(180^0 - \alpha)$	$= -\cot\alpha$,	$\cot(180^0 + \alpha)$	$= \cot\alpha$
$\csc(180^0 - \alpha)$	$= \csc\alpha$,	$\csc(180^0 + \alpha)$	$= -\csc\alpha$
$\sec(180^0 - \alpha)$	$= -\sec\alpha$,	$\csc(180^0 + \alpha)$	$= -\csc\alpha$

Note that

* 1) $\sin(n\,90^0 \pm \alpha) = \pm \cos\alpha$, $n \in N$ and n is odd
* 2) $\cos(n\,90^0 \pm \alpha) = \pm \sin\alpha$, $n \in N$ and n is odd
* 3) $\sin(n\,90^0 \pm \alpha) = \pm \sin\alpha$, $n \in N$ and n is even
* 4) $\cos(n\,90^0 \pm \alpha) = \pm \cos\alpha$, $n \in N$ and n is even

The sign on R.H.S is determined by the Quad. Of the angle in L.H.S using ` A S T C'

Example 1: Evaluate $\sin 80^0 \cos 130^0 + \cos 80^0 \sin 130^0$

Solution

Let $a = 80^0$ and $\beta = 130^0$ then $\sin 80^0 \cos 130^0 + \cos 80^0 \sin 130^0 \Rightarrow \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$= \sin(\alpha + \beta) \Rightarrow \sin(80^0 + 130^0) \Rightarrow \sin(210^0) \Rightarrow \sin(180^0 + 30^0) = -\sin 30^0 = -\frac{1}{2}$

Example 2 Find the value of i) $\cos 480^0$ ii) $\tan(-945^0)$ iii) $\sin 4620^0$

Solution

i. $\cos 480^0 = \cos(5 \times 90^0 + 30^0) = -\sin 30^0 = -1/2$

as angle in Quad II, cosine ratio is negative. .... [Note: $480^0 = 360^0 + 120^0$ coterminal to 120^0. Hence in Quad. II]

ii. $\tan(-945^0) = -\tan(945^0)$ .... $\therefore \tan(-\theta) = -\tan\theta$

$= -\tan(10 \times 90^0 + 45^0)$

$= -\tan 45^0$ .... $\therefore \tan(n90^0 + \alpha)$

= + tan α for n is even and angle is in Quadrant III.

= -1

[**Note:** $945 = 2 \times 360 + 225$ coterminal to 225, Hence in Quad III]

Example 3 Prove that

$$\frac{\cos(180 - \beta)}{\tan(180 + \beta)} \frac{\cot(360 - \beta)}{\csc(360 + \beta)} \frac{\sin(360 - \beta)}{\sec\beta} = \cos^4\beta$$

Solution

$$L.H.S = \frac{(-\cos\beta)}{\tan\beta} \cdot \frac{(-\cot\beta)}{\csc\beta} \cdot \frac{(-\sin\beta)}{\sec\beta}$$

$$= \frac{-\cos\beta}{\frac{\sin\beta}{\cos\beta}} \times \frac{-\cos\beta}{\frac{\sin\beta}{1}} \times \frac{-\sin\beta}{\frac{1}{\cos\beta}} \quad \text{.....[Quotient and reciprocal relations]}$$

$$= \frac{-\cos^2\beta}{\sin\beta} \times \cos\beta \times \sin\beta \times \cos\beta \quad \text{.....(algebraic manipulation)}$$

$$= -\cos^4\beta$$

$$= R.H.S$$

Example 4 Write $\sin 234^0$ in the form $\cos \beta$, $0 < \beta < 90^0$

Solution $\sin 234^0 = \sin (180^0 + 54^0)$

$= - \sin 54^0 \ldots$. [$\sin (18^0 + \theta) = - \sin \theta$]

$= - \sin (90^0 - 54^0) \ldots$. [$\sin \theta = \cos (90^0 - \theta)$]

$= - \cos 36^0$

Example 5 Show that $\cos \beta \cos (\alpha - \beta) - \sin \beta \sin (\alpha - \beta) = \cos \alpha$

Solution

L.H.S $= \cos \beta \cos (\alpha - \beta) - \sin \beta \sin (\alpha - \beta)$

$= \cos [\beta + (\alpha - \beta)]$

$= \cos \alpha$

$=$ R.H.S.

Example 5 prove that $\sin 15^0 = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$

Solution

$$\sin 15^0 = \sin (45^0 - 30^0) = \sin 45^0 \cos 30^0 - \cos 45^0 \sin 30^0$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}} = \text{R .H .S}$$

Example 6

$\dfrac{\sin (A - B)}{\cos A \cos B} + \dfrac{\sin (B - C)}{\cos B \cos C} + \dfrac{\sin (C - A)}{\cos C \cos A} = 0$ Prove this !

Solution

$$LHS = \frac{\sin A \cos B - \cos A \sin B}{\cos A \cos B} + \frac{\sin B \cos C - \cos B \sin C}{\cos B \cos C} + \frac{\sin C \cos A - \cos C \sin A}{\cos C \cos A}$$

$$= \frac{\sin A \cos B}{\cos A \cos B} - \frac{\cos A \sin B}{\cos A \cos B} + \frac{\sin B \cos C}{\cos B \cos C} - \frac{\cos B \sin C}{\cos B \cos C} + \frac{\sin C \cos A}{\cos C \cos A} - \frac{\cos C \sin A}{\cos C \cos A}$$

$$= \tan A - \tan B + \tan B - \tan C + \tan C - \tan A = O$$

$$= RHS$$

Example 6: Find the value of sin 15°.

Solution We have

$$\sin 15^\circ = \sin (45^\circ - 30^\circ)$$

$$= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}.$$

Example 7

Find sin (A + B) if sin A = $-\frac{4}{5}$ and cos B = $\frac{15}{17}$ and A and B are in Quadrant IV. Also state the Quadrant of (A + B).

Solution

$$\sin A = -\frac{4}{5}, A \in \text{Quad IV} \quad \therefore \cos^2 A = 1 - \sin^2 A = 1 - \left(-\frac{4}{5}\right)^2 = 1 - \frac{16}{25} = \frac{9}{25}$$

$$\therefore \cos A = \frac{3}{5} \text{ and } \cos B = \frac{15}{17}, B \in \text{Quad IV}$$

$$\text{using } \sin^2 B = 1 - \cos^2 B = 1 - \left(\frac{15}{17}\right) = 1 - \frac{225}{289} = \frac{64}{269} \therefore \sin B = -\frac{8}{17}$$

$$\sin(A+B) = \sin A \cos B + \cos A \sin B =$$

$$= \left(-\frac{4}{5}\right)\left(\frac{15}{17}\right) + \left(\frac{3}{5}\right)\left(-\frac{8}{17}\right) = \left(-\frac{60}{85}\right) + \left(-\frac{24}{85}\right)$$

$$\therefore \sin(A+B) = -\frac{84}{85} \therefore (A+B) = \sin^{-1}\left(-\frac{84}{85}\right) = 278.80^0$$

$$\therefore \text{the angle}(A+B) \in \text{Quad IV}$$

Example 8: Prove that $\csc(\pi + x) = \csc x$

Solution

$$\csc(\pi + x) = \frac{1}{\sin(\pi + x)} = \frac{1}{\sin\pi\cos x + \cos\pi \sin x}$$

$$= \frac{1}{(0)\cos x + (-1)\sin x} = \frac{1}{-\sin x} = -\csc x$$

Exercise 1.8.1

1. Evaluate the following without using calculator.

a) Sin 75^o

b) Sin 165^o

c) Sin 285^o

d) Cos 25^ocos 5^o – sin 25^o sin 5^o

e) Sin 65^o cos 35^o – cos65^o sin 35^o

2. Verify each identities

a) $\sin\left(270^0 - \theta\right) = -\cos\theta$

b) $\sin\left(\theta + \frac{\pi}{3}\right) - \cos\left(\theta + \frac{\pi}{6}\right) = 0$

c) $\sin\left(x + 30^0\right) + \cos\left(x + 60^0\right) = \cos x$

d) $\sin(x+y)\sin(x-y)=\sin^2 x-\sin^2 y$

e) $\cos(x+y)+\cos(x-y)=2\cos x\cos y$

f) $\dfrac{\cos x+\sin x}{\cos x-\sin x}=\tan\left(\dfrac{\pi}{4}+x\right)$

g) $\cos\left(\dfrac{\pi}{4}-x\right)\cos\left(\dfrac{\pi}{4}-y\right)-\sin\left(\dfrac{\pi}{4}-x\right)\sin\left(\dfrac{\pi}{4}-y\right)=\sin(x+y)$

h) $\cos p+\sin p=\sqrt{2}\sin\left(\dfrac{\pi}{4}+p\right)$

i) $\dfrac{\tan\left(\dfrac{\pi}{4}+x\right)}{\tan\left(\dfrac{\pi}{4}-x\right)}=\left(\dfrac{1+\tan x}{1-\tan x}\right)^2$

3. If $\sin A=\dfrac{5}{13}$ and $\cos B=-\dfrac{4}{5}$. Find cos (A + B). State the Quadrant of (A + B) where A and B are obtuse angles.

4. Given $\sin x=\dfrac{3}{5}$ and that $\sin y=\dfrac{12}{13}$, x and y are acute angles. Find

a. sin (x + y)
b. cos (x – y)
c. tan(x +y)
d. cot(x – y)

Double - Angle

From the sum and difference formulas for sine and cosine, one can get `**double angle**' formulas as :-

1. $\sin 2\theta=\sin(\theta+\theta)$
$=\sin\theta\cos\theta+\cos\theta\sin\theta$
$=2\sin\theta\cos\theta$

$\therefore \sin 2\theta = 2\sin\theta\cos\theta$

2. $\cos 2\theta=\cos(\theta+\theta)$
$=\cos\theta\cos\theta-\sin\theta\sin\theta$
$=\cos^2\theta-\sin^2\theta$ .....(1)

$\therefore \cos 2\theta = \cos^2\theta-\sin^2\theta$

Using Pythagorean identity $\sin^2\theta=1-\cos^2\theta$ in (1)
$\therefore \cos^2\theta=\cos^2\theta-(1-\cos^2\theta)$
$=2\cos^2\theta-1$

$\therefore \cos 2\theta = 2\cos^2\theta-1$

Also using $\cos^2\theta=1-\sin^2\theta$ in (1)
$\cos 2\theta=1-\sin^2\theta-\sin^2\theta$
$=1-2\sin^2\theta$

$\therefore \cos 2\theta = 1-2\sin^2\theta$

Trigonometric Ratios of (3α)

1. $\sin 3\alpha = \sin(2\alpha + \alpha)$
$= \sin 2\alpha \cos\alpha + \cos 2\alpha \sin\alpha$
$= (2 \sin\alpha \cos\alpha)\cos\alpha + (1 - 2\sin^2\alpha)\sin\alpha$
$= 2\sin\alpha\cos^2\alpha + \sin\alpha - 2\sin^3\alpha$
$= 2\sin\alpha(1 - \sin^2\alpha) + \sin\alpha - 2\sin^3\alpha$
$= 2\sin\alpha - 2\sin^3\alpha + \sin\alpha - 2\sin^3\alpha$
$= 3\sin\alpha - 4\sin^3\alpha$

$\therefore \sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$

2. $\cos 3\alpha = \cos(2\alpha + \alpha)$
$= \cos 2\alpha \cos\alpha - \sin 2\alpha \sin\alpha$
$= (2\cos^2\alpha - 1)\cos\alpha - (2\sin\alpha\cos\alpha)\sin\alpha$
$= 2\cos^3\alpha - \cos\alpha - 2\sin^2\alpha\cos\alpha$
$= 2\cos^3\alpha - \cos\alpha - 2(1 - \cos^2\alpha)\cos\alpha$
$= 2\cos^3\alpha - \cos\alpha - 2\cos\alpha + 2\cos^3\alpha$
$= 4\cos^3\alpha - 3\cos\alpha$

$\therefore \cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$

The multiple - angle (Double & Half Angle) Formulas

With the help of the sum and difference (compound angle formulas studied in the previous article, we will express the trigonometric functions of angle θ in terms of $\frac{\theta}{2}$).

(1) $\therefore \sin 2\theta = 2\sin\theta\cos\theta$ then $\therefore \sin\theta = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$

(2) $\therefore \cos 2\theta = \cos^2\theta - \sin^2\theta \quad \therefore \cos\theta = \cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)$

$\therefore \cos^2\theta = 2\cos^2\theta - 1 \quad \therefore \cos\theta = 2\cos^2\left(\frac{\theta}{2}\right) - 1$ and

$\therefore \cos 2\theta = 1 - 2\sin^2\theta \quad \therefore \cos\theta = 1 - 2\sin^2\left(\frac{\theta}{2}\right)$

From the above formulas, we derive the following formulas

1) $\cos\theta = 2\cos^2\left(\frac{\theta}{2}\right) - 1 \Rightarrow 1 + \cos\theta = 2\cos^2\left(\frac{\theta}{2}\right)$

$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos\theta}{2}}$

2) $\cos\theta = 1 - 2\sin^2\left(\frac{\theta}{2}\right) \Rightarrow 1 - \cos\theta = 2\sin^2\left(\frac{\theta}{2}\right)$

$\sin^2\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}}$

3) $1 + \sin\theta = \cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right) + 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$

$\therefore 1 + \sin\theta = \left[\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)\right]^2$

Also $1 - \sin\theta = \left[\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\right]^2$

Example1: Find the value of $\cos 15^0$, using the ratios of 30^0only.

Solution

$$\frac{\sqrt{3}}{2} = 2\cos^2 15^0 - 1$$

$$\therefore \quad \cos 15^0 = \pm \frac{\sqrt{\sqrt{3}+2}}{2} \quad \text{but } 15^0 \text{ is in Quad I}$$

$$\therefore \quad \cos 15^0 = \frac{+\sqrt{\sqrt{3}+2}}{2}$$

Example 2: Find the exact value for $\sin 105^0$using the half- angle identity.

Solution

We have $\sin 105^0 = \sin \frac{210^0}{2}$

Using $\sin(\theta/2) = \pm \sqrt{\frac{1-\cos\theta}{2}}$

$\therefore \sin 105^0 = \sqrt{\frac{1-\cos 210^0}{2}}$ since 105^0 is in Quad. II, sine ratio is +ve.

$\therefore \sin 105^0 = \sqrt{\frac{1-\cos(2\times 90^0 + 30^0)}{2}}$

$\therefore \sin 105^0 = \sqrt{\frac{1-(-\cos 30^0)}{2}}$ $\therefore \sin 105^0 = \sqrt{\frac{1+\sqrt{3}/2}{2}}$

$\therefore \sin 105^0 = \sqrt{\frac{2+\sqrt{3}}{4}}$ $\therefore \sin 105^0 = \sqrt{\frac{2+\sqrt{3}}{2}}$

Example 3

Find the values of sine and cosine of $\left(\frac{1}{2}\theta\right)$, given that $\sin\theta = \frac{5}{13}$, θ is in Quad II

Solution

$$\sin\theta = \frac{5}{13}, \cos^2\theta = 1-\sin^2\theta = 1-\frac{25}{169} = \frac{169-25}{169} = \frac{144}{169}$$

$$\therefore \cos\theta = -\frac{12}{13}, \text{ since } \theta \text{ is in Quad II}$$

$$\text{Now } \sin\left(\frac{1}{2}\theta\right) = \sqrt{\frac{1-\cos\theta}{2}} \text{ since } \frac{1}{2}\theta \text{ is in Quad I}$$

$$\therefore \sin\left(\frac{1}{2}\theta\right) = \sqrt{\frac{1-\left(-\frac{12}{13}\right)}{2}} = \sqrt{\frac{1+\frac{12}{13}}{2}} = \sqrt{\frac{25}{26}} = \frac{5\sqrt{26}}{26}$$

$$\text{And } \cos\left(\frac{1}{2}\theta\right) = \sqrt{\frac{1+\cos\theta}{2}} = \sqrt{\frac{1+\left(-\frac{12}{13}\right)}{2}} = \sqrt{\frac{1-\frac{12}{13}}{2}} = \sqrt{\frac{1}{26}} = \frac{\sqrt{26}}{\sqrt{26}\times\sqrt{26}} = \frac{\sqrt{26}}{26}$$

Example 4: Find the exact value for $\cos 165^0$ using half- angle identity.

Solution:

$\theta/2 = 165^0 = 330^0/2 \quad \therefore \quad \theta = 330^0$ also 165^0 is in Quad II

$\therefore$ cosine is negative.

$$\text{Now } \cos\left(\frac{1}{2}\theta\right) = -\sqrt{\frac{1+\cos\theta}{2}} = -\sqrt{\frac{1+\cos 330^0}{2}}$$

$$= \sqrt{\frac{1+\cos(3\times 90^0 + 60^0)}{2}}$$

$$= -\sqrt{\frac{1+\sin 60^0}{2}} = -\sqrt{\frac{1+\sqrt{3}/2}{2}} = -\sqrt{\frac{2+\sqrt{3}}{4}}$$

$$= -\frac{\sqrt{2}+\sqrt{3}}{2}$$

Example 5: Find $\sin 22.5^0$, $\cos 22.5^0$ and $\tan 22.5^0$ using the half angle formulas.

Solution:

$\frac{1}{2}\theta = 22.5^0 \quad \therefore \theta = 45^0$ also 22.5^0 is in Quad (I)

Now sin

$$\left(\frac{1}{2}\theta\right) = \sqrt{\frac{1-\cos\theta}{2}} \quad \therefore \sin(22.5^0) = \sqrt{\frac{1-\cos 45^0}{2}}$$

$$= \sqrt{\frac{1-1/\sqrt{2}}{2}} = \sqrt{\frac{1-\sqrt{2}/2}{2}} = \sqrt{\frac{2-\sqrt{2}}{4}} = \frac{1}{2}\sqrt{2-\sqrt{2}}$$

$$\text{Also } \cos\left(\frac{1}{2}\theta\right) = \sqrt{\frac{1+\cos\theta}{2}} \quad \therefore \cos 22.5^0 = \sqrt{\frac{1+\cos 45^0}{2}}$$

$$= \sqrt{\frac{2+\sqrt{2}}{4}} = \frac{1}{2}\sqrt{2+\sqrt{2}}$$

$$\text{Now } \tan(22.5^0) = \frac{\sin(22.5^0)}{\cos(22.5^0)} = \frac{\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}}$$

Example6

Prove $\dfrac{\sin x + \sin 2x}{1 + \cos x + \cos 2x} = \tan x$

Solution:

$$\frac{\sin x + \sin 2x}{1 + \cos x + \cos 2x} = \frac{\sin x + 2 \sin x \cos x}{2\cos^2 x + \cos x} = \frac{\sin x(1 + 2\cos x)}{\cos x(2\cos x + 1)}$$

$$= \tan x$$

Example 7: Express $1 - \frac{1}{2}\sin 2x$ in terms sin x and cos x

Solution:

$$1 - \frac{1}{2}\sin 2x = 1 - \frac{1}{2}(2 \sin x \cos x) \quad \ldots\ldots(\sin 2x = 2 \sin x \cos x)$$

$$= 1 - \sin x \cos x$$

$$= \frac{(\sin x + \cos x)(1 - \sin x \cos x)}{\sin x + \cos x} \quad \ldots\ldots\text{algebraic manipulations.}$$

$$= \frac{(\sin x + \cos x)(\sin^2 x + \cos^2 x - \sin x \cos x)}{\sin x + \cos x} \quad \ldots \sin^2 x + \cos^2 x = 1$$

$$= \frac{\sin^3 x + \cos^3 x}{\sin x + \cos x} \quad \ldots (a+b)(a^2 + b^2 - ab) = a^3 + b^3$$

Example 7:

Prove that $\sin^4 \alpha = \frac{3}{8} - \cos 2\alpha + \frac{1}{8}\cos 4\alpha$

$$(\sin^4 \alpha) = (\sin^2 \alpha)^2 = \left(\frac{1 - \cos 2\alpha}{2}\right)^2 \quad \ldots\ldots \cos 2\alpha = 1 - 2\sin^2 \alpha$$

$$= \frac{1 - 2\cos 2\alpha + \cos^2 2\alpha}{4} = \frac{1}{4}\left[1 - 2\cos \alpha + \frac{1 + \cos 4\alpha}{2}\right] \ldots \cos 2\alpha = 2\cos^2 \alpha - 1$$

$$= \frac{1}{4}\left[1 - 2\cos \alpha + \frac{1}{2} + \frac{1}{2}(\cos 4\alpha)\right] = \frac{1}{4}\left[\frac{3}{2} - 2\cos \alpha + \frac{1}{2}\cos 4\alpha\right]$$

$$= \frac{3}{8} - \frac{1}{2}\cos \alpha + \frac{1}{8}\cos 4\alpha$$

Example 8

Prove that $\frac{\sin 2x}{1+\cos 2x} = \tan x$

Solution:

$$\frac{\sin 2x}{1+\cos 2x} = \frac{2\sin x\cos x}{2\cos^2 x} = \tan x$$

Example 9

Prove that $\sqrt{\frac{1-\cos x}{1+\cos x}} = \tan\frac{x}{2}$

Solution

$$\sqrt{\frac{1-\cos x}{1+\cos x}} = \sqrt{\frac{2\sin^2(x/2)}{2\cos^2(x/2)}} = \sqrt{\frac{\tan^2(x/2)}{1}} = \tan\frac{x}{2}$$

Example 10

Prove $\frac{\sin x+\sin 2x}{1+\cos x+\cos 2x} = \tan x$

Solution

$$\frac{\sin x+\sin 2x}{1+\cos x+\cos 2x} = \frac{\sin x+2\sin x\cos x}{2\cos^2 x+\cos x} = \frac{\sin x(1+2\cos x)}{\cos x(2\cos x+1)}$$

$$= \tan x$$

Example 11

1. Suppose x is b/w 90^o and 180^o and $\sin x = \frac{3}{5}$ find sin 2x. Since $\sin^2 x = 2\sin x\cos x$, find cosx first. Use $\cos^2 x + \sin^2 x = 1$

Solution

$\cos^2 x + \sin^2 x = 1 \quad \cos^2 x = 1-\sin^2 x \Rightarrow \cos^2 x = 1-\left(\frac{3}{5}\right)^2 = 1-\frac{9}{25} = \frac{16}{25}$

$\therefore \cos x = \pm\sqrt{\frac{16}{25}} = \pm\frac{4}{5}$. Since x is b/w 90^0 and 180^0, cosx is negative

$\therefore \cos x = -\frac{4}{5}$

$\sin 2x = 2\sin x\cos x = 2\left(\frac{3}{5}\right)\left(-\frac{4}{5}\right)$

$\therefore \sin 2x = -\frac{24}{25}$

Tangent Identities

1) Sum and Difference Identities

i) $\tan(\alpha+\beta)$

$$= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}$$

$$= \frac{\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}-\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$$

$$\therefore \tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha.\tan\beta}$$

Also $\tan(\alpha-\beta) = \tan[\alpha+(-\beta)] = \frac{\tan\alpha+\tan(-\beta)}{1-\tan\alpha\tan(-\beta)}$

ii) $\tan(\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha.\tan\beta}$

Double (Multiple) angle and Half (sub multiple) angle Identities

i) $\tan(2\alpha) = \tan(\alpha+\alpha)$

$$\therefore \tan(2\alpha) = \frac{\tan\alpha+\tan\alpha}{1-\tan\alpha.\tan\alpha} \qquad \therefore \tan(2\alpha) = \frac{2\tan\alpha}{1-\tan^2\alpha}$$

ii) $\tan(\alpha/2) = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$, the sign is determined by the Quad. of $(\alpha/2)$

Also $\frac{\sin\alpha}{1+\cos\alpha} = \frac{2\sin(\alpha/2)\cos(\alpha/2)}{2\cos^2(\alpha/2)}$

$$= \frac{\sin(\alpha/2)}{\cos(\alpha/2)} = \tan(\alpha/2)$$

And $\frac{1-\cos\alpha}{\sin\alpha} = \frac{2\sin^2(\alpha/2)}{\sin\alpha} = \frac{\sin(\alpha/2)}{\cos(\alpha/2)} = \tan(\alpha/2)$

iii) $\tan(3\alpha) = \tan(2\alpha+\alpha) = \frac{\tan 2\alpha+\tan\alpha}{1-\tan(2\alpha)\tan\alpha}$

$$= \frac{\frac{2\tan\alpha}{1-\tan^2\alpha}+\tan\alpha}{1-\left[\frac{2\tan\alpha}{1-\tan^2\alpha}\right].\tan\alpha}$$

$$= \frac{2\tan\alpha+\tan\alpha(1-\tan^2\alpha)}{(1-\tan^2\alpha)-2\tan^2\alpha} = \frac{2\tan\alpha+\tan\alpha-\tan^3\alpha}{1-\tan^2\alpha-2\tan^2\alpha}$$

$$\therefore \tan(3\alpha) = \frac{3\tan\alpha-\tan^3\alpha}{1-3\tan^2\alpha}$$

Example 12: Prove that $\tan 20^0 + \tan 25^0 + \tan 20^0 (\tan 25^0) = 1$

Solution

$$\tan(45^0) = 1$$

$$\therefore \tan(25^0+20^0) = 1$$

$$\therefore \frac{\tan 25^0 + \tan 20^0}{1 - \tan 25^0 \tan 20^0} = 1 \quad = \quad \tan 25^0 + \tan 20^0 = 1 - \tan 25^0 \tan 20^0$$

$$\therefore \tan 25^0 + \tan 20^0 + \tan 20^0 (\tan 25^0) = 1$$

Example 13: Find $\tan 75^0$

Solution

$$\tan 75^0 = \tan(45^0 + 30^0) = \frac{\tan 45^0 + \tan 30^0}{1 - \tan 45^0 \times \tan 30^0}$$

$$= \frac{1 + 1/\sqrt{3}}{1 - 1 \times 1/\sqrt{3}} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = \frac{(\sqrt{3}+1)(\sqrt{3}+1)}{3-1}$$

$$= \frac{3 + 2\sqrt{3} + 1}{2} = \frac{4 + 2\sqrt{3}}{2} = \frac{2(2+\sqrt{3})}{2}$$

$$\therefore \tan 75^0 = 2 + \sqrt{3}$$

Example 14 : Prove that $\tan 50^0 = \tan 40^0 + 2 \tan 10^0$

Solution

$$\tan(50^0) = \tan(40^0 + 10^0) = \frac{\tan 40^0 + \tan 10^0}{1 - \tan 40^0 \tan 10^0}$$

$$\therefore \tan 50^0 (1 - \tan 40^0 \tan 10^0) = \tan 40^0 + \tan 10^0$$

$$\therefore \tan 50^0 - \tan 50^0 \tan 40^0 \tan 10^0 = \tan 40^0 + \tan 10^0$$

$$\therefore \tan 50^0 - \tan 50^0 (\cot 50^0) \tan 10^0 = \tan 40^0 + \tan 10^0$$

$$\ldots\ldots \tan 40^0 = \tan(90^0 - 50^0) = \cot 50^0$$

$$\therefore \tan 50^0 - \tan 50^0 (1/\tan 50^0) \tan 10^0 = \tan 40^0 + \tan 10^0$$

$$\ldots\ldots \cot\theta = \frac{1}{\tan\theta}$$

$$\therefore \tan 50^0 - \tan 10^0 = \tan 40^0 + \tan 10^0$$

Example 15

If A + B = π /4, then prove that (1 + tan A) (1 + tan B) = 2

Solution

$\therefore A + B = \pi/4$ i.e., 45^0

$\therefore \tan(A + B) = \tan(45^0)$

$\therefore A + B = \pi/4$ ie, 45^0

$\therefore \tan(A + B) = \tan(45^0)$

$\therefore \frac{\tan A + \tan B}{1 - \tan A \tan B} = 1 \quad \therefore \tan A + \tan B = 1 - \tan A.\tan B$

$\therefore \tan A + \tan B + \tan A(\tan B) = 1$

Adding 1 on both the sides we get

$\therefore 1(1 + \tan A) + \tan B(1 + \tan A) = 1 + 1$

$\therefore (1 + \tan A)(1 + \tan B) = 2$

Example 16

Evaluate $\frac{\tan 66^0 + \tan 69^0}{1 - \tan 66^0 + \tan 69^0}$

Solution

$$\frac{\tan 66^0 + \tan 69^0}{1 - \tan 66^0 + \tan 69^0} = \tan(66^0 + 69^0) = \tan(135^0)$$

$= \tan(90^0 + 45^0) = -\cot 45^0 = -1$ ..... $\tan(90^0 + \theta) = \cot\theta$

Example 17: Find the value of tan (α + β) where a is in Quad III and b is in Quad .II such that $\cos\alpha = -\frac{3}{5}$ and $\sin\beta = \frac{20}{29}$

Solution

Similarly $\cos^2\beta = 1 - \sin^2\beta = 1 - \frac{400}{841} = \frac{841-400}{841} = \frac{441}{841}$

$\therefore \quad \cos\beta = -21/29$ since cosine is negative in Quad II.

Now $\tan(\alpha+\beta)$

$$= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \cdot \tan\beta} = \frac{\frac{4}{3} + \left(-\frac{20}{21}\right)}{1 - \frac{4}{3}\left(-\frac{20}{21}\right)}$$

$$= \frac{\frac{84-60}{63}}{\frac{63+80}{63}} = \frac{24}{143}$$

$\therefore \tan(\alpha+\beta) = \frac{24}{143}$

$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$

$= \frac{-4/5}{-3/5} = 4/3$ and

$\tan\beta = \frac{\sin\beta}{\cos\beta}$

$= -20/21$

Factorization and Defactorization

Factorization (Sums or Differences into Products)

1. $\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$
2. $\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$
3. $\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$
4. $\cos\alpha + \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$ OR $= 2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\beta-\alpha}{2}\right)$

Defactorization (Products into Sums or Differences)

1. $\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$
2. $\cos\alpha \cdot \cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right]$
3. $\sin\alpha \sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$
4. $\cos\alpha \cdot \cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta) + \cos(\alpha-\beta)\right]$

Note: These identities are valid for both degree and radian measure whenever both sides are defined.

Example 18: Verify that $\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

Solution

We know that $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ .....(i)

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$ .....(ii)

Adding (i) and (ii) we get,

$\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta$ .....(iii)

subtracting (ii) from (i) we get

$\sin(\alpha+\beta) - \sin(\alpha-\beta) = 2\cos\alpha\cos\beta$ .....(iv)

Now let $\alpha+\beta = C$ and $\alpha-\beta = D$

$\therefore \quad C + D = (\alpha+\beta)+(\alpha-\beta) = 2\alpha \quad \therefore \alpha = \left(\frac{C+D}{2}\right)$

and $\quad C - D = (\alpha+\beta)-(\alpha-\beta) = 2\beta \quad \therefore \quad \beta = \left(\frac{C-D}{2}\right)$

$\therefore$ (iii) gives $\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

and (iv) gives $\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

Example 19:

verify that $\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$

Solution

We know that $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha.\sin\beta$ .....(i)

and $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$ .....(ii)

Adding (i) and (ii) we get

$\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha.\cos\beta$

$\therefore \quad \sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$

Example 20: Write cos 7x cos 4x as a sum.

Solution

$$\text{Using } \cos\alpha \cdot \cos\beta = \frac{1}{2}[\cos(\alpha + \beta) + \cos(\alpha - \beta)]$$

and putting $\alpha = 7x$, $\beta = 4x$ we get

$$\therefore \quad \cos 7x \cos 4x = \frac{1}{2}[\cos(7x + 4x) + (7x - 4x)]$$

$$\therefore \quad \cos 7x \cdot \cos 4x = \frac{\cos 11x}{2} + \frac{\cos 3x}{2}$$

Example 21: Write the difference cos 2 α - cos 8α as a product.

Solution

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\beta - \alpha}{2}\right) \text{ we get}$$

$$\cos 2\alpha - \cos 8\alpha = -2\sin\left(\frac{2\alpha + 8\alpha}{2}\right)\sin\left(\frac{8\alpha - 2\alpha}{2}\right)$$

$$= -2\sin 5\alpha \sin 3\alpha$$

Example 22: Find the exact value of $\sin 75^0 + \sin 15^0$

Solution

$$\text{Using } \sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right) \text{ we get}$$

$$\sin 75^0 + \sin 15^0 = 2\sin\left(\frac{75 + 15}{2}\right)\cos\left(\frac{75^0 - 15^0}{2}\right)$$

$$= 2\sin 45^0 \cos 30^0$$

$$= 2 \times \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{\sqrt{3}}{2}$$

$$= \frac{\sqrt{6}}{2}$$

Example 23: Prove that sin 7A + sin 3A = cot 2A

Solution

using $\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$ and

$\cos\alpha - \cos\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\beta-\alpha}{2}\right)$ we get

$$\text{L.H.S} = \frac{2\sin\left(\frac{7A+3A}{2}\right)\cos\left(\frac{7A-3A}{2}\right)}{2\sin\left(\frac{3A+7A}{2}\right)\sin\left(\frac{7A-3A}{2}\right)}$$

$$= \frac{2\sin 5A\cos 2A}{2\sin 5A\sin 2A}$$

$$= \cot 2A = \text{R.H.S}$$

Example 24:

If $\sin\alpha + \sin 3\alpha = \cos\alpha + \cos 3\alpha$ then prove that, either $\tan 2\alpha = 1$ or $\cos\alpha = 0$

Solution

$\sin\alpha + \sin 3\alpha = \cos\alpha + \cos 3\alpha$ ..... (Given)

$\therefore 2\sin 2\alpha\cos\alpha = 2\cos 2\alpha\cos\alpha$

$\therefore \sin 2\alpha\cos\alpha - \cos\alpha.\cos 2\alpha = 0$

$\therefore (\sin 2\alpha - \cos 2\alpha)\cos\alpha = 0$ $\therefore$ either i) $\sin 2\alpha - \cos 2\alpha = 0$ or ii) $\cos\alpha = 0$

$\therefore$ i) $\sin 2\alpha = \cos 2\alpha$ $\therefore \tan 2\alpha = 1$ ii) $\cos\alpha = 0$

Example25:

Find the value of $\cos 52^0 + \cos 68^0 + \cos 172^0$

Solution

$$\cos 52^0 + \cos 68^0 + \cos 172^0 = 2\cos\left(\frac{52+68}{2}\right)\cos\left(\frac{52-68}{2}\right) + \cos 172^0$$

$= 2\cos 60^0 . \cos(-8^0) + \cos 172^0$ .....$\cos 60^0 = 1/2$ and $\cos(-\theta) = \cos\theta$

$= 2\times\frac{1}{2}\cos 8^0 + \cos 172^0$

$= \cos 8^0 + \cos 172^0$

$= 2\cos\left(\frac{8+172}{2}\right).\cos\left(\frac{8-172}{2}\right)$

$= 2\cos(90^0)\cos(-82^0)$

$= 2\times 0\cdot\cos(82^0)$ .....$\cos 90^0 = 0$ and $\cos(-\theta) = \cos\theta$

$= 0$

Example 26:

Express as a sum or difference $\sin 55^0 \sin 40^0$

Solution

Using $\sin\alpha \sin B = 1/2 [\cos(\alpha - \beta) - \cos(\alpha + \beta)]$ we get,

$\therefore \sin 55^0 \sin 40^0 = 1/2 [\cos(56^0 - 40^0) - \cos(55^0 + 40^0)] = 1/2 [\cos 15^0 - \cos 95^0]$

Example 9:

Prove that $\dfrac{\sin A - \sin B}{\sin A + \sin B} = \dfrac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}$

Solution

$$\frac{\sin A - \sin B}{\sin A + \sin B} = \frac{2\cos\frac{1}{2}(A+B)\,2\sin\frac{1}{2}(A-B)}{2\sin\frac{1}{2}(A+B)\;2\cos\frac{1}{2}(A-B)}$$

$$= \cot\frac{1}{2}(A+B)\tan\frac{1}{2}(A-B)$$

$$= \frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}$$

Example 27: $\cos^3 x \sin^2 x = 1/16 (2\cos x - \cos 3x - \cos 5x)$

Solution: $\cos^3 x. \sin^2 x = (\sin x \cos x)^2 \cos x$

$= (1/2 \times 2 \sin x \cos x)^2 \cos x$

$= 1/4 (2 \sin x \cos x)^2 \cos x$

$= 1/4 (\sin 2x)^2 (\cos x)$.... $(\therefore \sin 2x = 2 \sin x \cos x)$

$= 1/4 (\sin 2x)(\sin 2x \cos x)$

$= 1/4 (\sin 2x)[1/2 (\sin 3x + \sin x)]$ ....$\sin\alpha \cos\beta$

$= 1/2[\sin(\alpha + \beta)\cos(\alpha - \beta)]$

$= 1/8 [\sin 3x \sin 2x + \sin 2x \sin x]$

$= 1/8 [1/2 (\cos x - \cos 5x) + 1/2 (\cos x - \cos 3x)]$

$= 1/16 [2\cos x - \cos 3x - \cos 5x]$

Example 28: Simplify cos 3x cos 2x cos x

Solution:

$$
\begin{aligned}
&(\cos 3x \cos 2x)\cos x \\
&= \frac{1}{2}\left[\cos(5x) + \cos(x)\cos x\right] \\
&= \frac{1}{2}\left[\cos 5x \cos x + \cos^2 x\right] \\
&= \frac{1}{2}\left[\frac{1}{2}(\cos 6x \cos 4x) + \frac{1+\cos 2x}{2}\right] \\
&= \frac{1}{4}\cos 6x + \frac{1}{4}\cos 4x + \frac{1}{4} + \frac{1}{4}\cos 2x
\end{aligned}
$$

Review Exercise1.8.4.

For Exercises 1-8, prove the given identity.

1. $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$
2. $\tan\frac{1}{2}\theta = \csc\theta - \cot\theta$
3. $\dfrac{\sin 2\theta}{\sin\theta} - \dfrac{\cos 2\theta}{\cos\theta} = \sec\theta$
4. $\dfrac{\sin 3\theta}{\sin\theta} - \dfrac{\cos 3\theta}{\cos\theta} = 2$
5. $\tan 2\theta = \dfrac{2}{\cot\theta - \tan\theta}$
6. $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$
7. $\tan^2\frac{1}{2}\theta = \dfrac{\tan\theta - \sin\theta}{\tan\theta + \sin\theta}$
8. $\dfrac{\cos^2\psi}{\cos^2\theta} = \dfrac{1+\cos 2\psi}{1+\cos 2\theta}$

9. Some trigonometry textbooks used to claim incorrectly that $\sin\theta + \cos\theta = \sqrt{1+\sin 2\theta}$ was an identity. Give an example of a specific angle θ that would make that equation false. Is $\sin\theta + \cos\theta = \pm\sqrt{1+\sin 2\theta}$ an identity? Justify your answer.

10. Fill out the rest of the table below for the angles $0° < \theta < 720°$ in increments of 90°, showing θ, $\frac{1}{2}\theta$, and the signs (+ or −) of $\sin\theta$ and $\tan\frac{1}{2}\theta$.

θ	$\frac{1}{2}\theta$	$\sin\theta$	$\tan\frac{1}{2}\theta$
0° – 90°	0° – 45°	+	+
90° – 180°	45° – 90°		
180° – 270°	90° – 135°		
270° – 360°	135° – 180°		

θ	$\frac{1}{2}\theta$	$\sin\theta$	$\tan\frac{1}{2}\theta$
360° – 450°	180° – 225°		
450° – 540°	225° – 270°		
540° – 630°	270° – 315°		
630° – 720°	315° – 360°		

11. In general, what is the largest value that $\sin\theta\cos\theta$ can take? Justify your answer.

For Exercises 12-17, prove the given identity for any right triangle $\triangle ABC$ with $C = 90°$.

12. $\sin(A-B) = \cos 2B$
13. $\cos(A-B) = \sin 2A$
14. $\sin 2A = \dfrac{2ab}{c^2}$
15. $\cos 2A = \dfrac{b^2-a^2}{c^2}$
16. $\tan 2A = \dfrac{2ab}{b^2-a^2}$
17. $\tan\frac{1}{2}A = \dfrac{c-b}{a} = \dfrac{a}{c+b}$

1. 4. Solving trigonometric equations

A ***trigonometric equation*** is an involving a trigonometric function. If the equation is true for all values of the variable it is an ***identity***.

Trigonometric equations may be solved by the same methods used to solve other equations. For example, the use of algebraic techniques such as factoring and substitution may be helpful in solving trigonometric equation.

Example 1

1. ***Solve $2\sin^2 x - 3\sin x + 1 = 0$ for principal values of x.***

   $2\sin^2 x - 3\sin x + 1 = 0$

   $(2\sin x - 1)(\sin x - 1) = 0$ ***factor***

   $2\sin x - 1 = 0$ or $\sin x - 1 = 0$

   $\sin x = \frac{1}{2}$ $\sin x = 1$

   $x = \arcsin \frac{1}{2}$ $x = \arcsin 1$

   $x = 30^o$ $x = 90^o$

   The solutions are 30^o and 90^o

2. ***solve $2\tan x \sin x + 2\sin x = \tan x + 1$ for all values of x***

   $2\tan x \sin x + 2\sin x = \tan x + 1$

   $2\tan x \sin x + 2\sin x - \tan x - 1 = 0$

   $(\tan x + 1)(2\sin x - 1) = 0$

   $\tan x + 1 = 0$ or $2\sin x - 1 = 1$

   $\tan x = -1$ $\sin x = \frac{1}{2}$

   $x = \arctan(-1)$ $x = \arcsin \frac{1}{2}$

   $x = -45^o + n \cdot 180^o$ $x = 30^o + n \cdot 160^o$

   The solutions are $45^o + n \cdot 180^o$ $x = 30^o + n \cdot 160^o$ where n is any integer

3. ***Solve $\sin(x + 30^o) = \cos 2x$ if $0 \leq x \leq 90^o$.***

   $\sin(x + 30^o) = \cos 2x$

   $(x + 30^o) + 2x = 90^o$

   $3x = 60^o$ or $x = 20^o$

   If sinA =cosB, then $A + B = 90^o$ since the sine and the cosine are co-functions.

   The solution is 20^o

4. ***solve $2\sin^2 x - \cos x - 1 = 0$ for all values of x***

$2\sin^2 x - \cos x - 1 = 0$

$2(1 - \cos^2 x) - \cos x - 1 = 0$ **$\sin^2 x = 1 - \cos^2 x$**

$2\cos^2 x + \cos x - 1 = 0$

$(2\cos x - 1)(\cos x + 1) = 0$

$2\cos x - 1 = 0$ or $\cos x + 1 = 0$

$\cos x = \frac{1}{2}$ $\cos x = -1$

$x = \text{Arc}\cos\frac{1}{2}$ $x = \text{Arc}\cos(-1)$

$x = 60^o + n \cdot 360^o$ and $300^o + n \cdot 360^o$ and $180^o + n \cdot 60^o$ for any integer n.

5. ***Solve $2\sin 2x + 3\sin x - 2 = 0$ for principal values of x.***

$2\sin^2 x + 3\sin x - 2 = 0$

$(2\sin x - 1)(\sin x + 2) = 0$

$2\sin x - 1 = 0$ or $\sin x + 2 = 0$

$\sin x = \frac{1}{2}$ $\sin x = -2$

$x = \text{Arc}\sin\frac{1}{2}$ $x = \text{Arc}\sin(-2)$ there is no solution since sin x is in the interval $-1 \le \sin x \le 1$

The solution is 30^o.

Exercise 1.8.5.

Solve each equation for the principal values of x

1. $2\cos x - 1 = 0$
2. $2\sin x + 1 = 0$
3. $\sqrt{2}\sin x - 1 = 0$
4. $2\cos x - \sqrt{3} = 0$
5. $2\cos x + 1 = 0$

Each equation has how many solutions if $0^o \le Ø \le 360^o$?

1. $\sin Ø = 1$
2. $\cos Ø = \frac{\sqrt{3}}{2}$
3. $\tan Ø = -3$
4. $\sin 2Ø = ½$
5. $\sin 2Ø = \frac{\sqrt{3}}{2}$

Solve each question for general values of x.

1. $2\sin^2 x - \sin x - 1 = 0$
2. $\sin x + \sin x \cos x = 0$
3. $2\sin^2 x - 1 = 0$
4. $2\sin x + \sqrt{3} = 0$
5. $\tan^2 x - 1 = 0$

Solve each equation for principal values of x

1. $2\sin^2 x + \sin x = 0$
2. $\sqrt{2}\cos x - 1 = 0$
3. $4\sin^2 x - 3 = 0$
4. $\tan 2x = \cot x$
5. $2\cos^2 x = \sin x + 1$

SIMPLE TRIGONOMETRIC EQUATIONS

Basic trigonometry equation, of the form of $\sin\theta = k$, $\cos\theta = k$ and $\tan\theta = k$, where k is a constant are solved as follows.

1) By considering the sign of k, identity the possible quadrants in which θ lies.
2) Find the basic angle α. Remember that α is acute and $\sin\alpha$, $\cos\alpha$ and $\tan\alpha$ are all positive.
3) Find all values of θ mast be measured from the positive x-axis.

Example

Find all the values of θ such that

a) $\sin\theta = -0.5$, where $0^o < \theta < 360^o$

b) $\tan\theta = -1$, where $0^o < \theta < 360^o$

c) $\cos\theta = 0.46$, where $-360^o < \theta < 0^o$

Solution

a) $\sin\theta = -0.5$ $0^o < \theta < 360^o$, $\sin\theta < 0$ so θ is in the 3^{rd} or 4^{th} quadrant for the acute angle α $\sin\alpha > 0$ and so $\sin\alpha = 0.5$ $\alpha = 30$ therefore $\theta = 180^o + \alpha$, $360^o - \alpha$

$= 210^o, 330^o$

b) $\tan\theta = -1$, $0^o < \theta < 360^o$, $\tan\theta < 0$ so θ is in the 2^{nd} or 4^{th} quadrant

$\tan\alpha = -1$ so $\alpha = 45^o$. $\theta = 180^o - \alpha$, $360^o - \alpha = 135^o, 315^0$

c) $\cos\theta = 0.46$, $\cos\theta > 0$ so θ is in the 1^{st} or 4^{th} quadrant $\cos\alpha = 0.46 = \alpha = 62{,}6$

$\theta = -\alpha - (360^o - \alpha) = -62.6^0, -2974^0$

Exercise 1.7.C

1. Find all angles where $0^o < x < 360^o$ such that
   a) $\cos x = -0.17$
   b) $\tan x = 1.732$
   c) $\sin x = 0.866$
   d) $2\sin(-x) = 0.3$
   e) $\dfrac{8\cos x + 1}{2 - \cos x} = 3$
2. find the angles b/w -360^o and 180^o such that
   a) $\sin x = -\frac{1}{2}$
   b) $\cos x = \sqrt{3}/2$
   c) $\sin(-x) + 1 = 0$
   d) $\sqrt{2}\sin(90o - x) + 1 = 0$

1.5. Inverse of Trigonometric Functions.

You now need to define inverses of the trigonometric functions, starting with a brief review of the general concept of inverse functions. You first restate a few important facts about inverse functions.

Facts about inverse functions

For f a one-to-one function and f^{-1} its inverse:
1. If (a, b) is an element of f, then (b, a) is an element of f^{-1}, and conversely.
2 Range f = Domain of f^{-1}
3 Domain of f = Range of f^{-1}

The graph of f^{-1} is obtained by reflecting the graph of f in the line $y = x$.

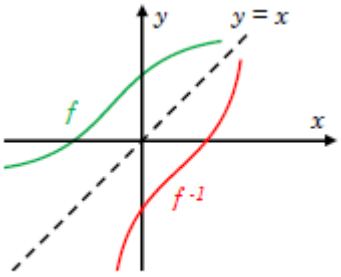
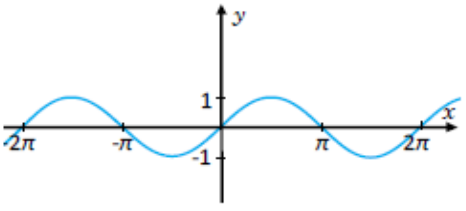


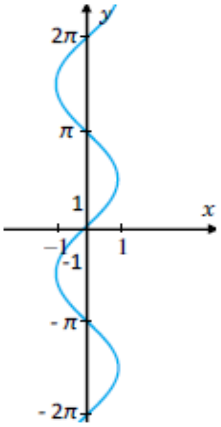
Figure 1.7.1

You know that a function f is invertible if it is one -to-one. All trigonometric functions are periodic; hence, each range value can be associated with infinitely many domain values.

As a result, no trigonometric function is one-to-one. So without restricting the domains, no trigonometric function has an inverse function as shown in Figure 1.7.2 below. To resolve this problem, you restrict the domain of each function so that it is one-to-one over the restricted domain. Thus, for this restricted domain, the function is invertible.



a Graph of $y = \sin x$



b Graph of $y = \sin^{-1} x$ on domain = $[-1, 1]$ and range $(-\infty, \infty)$

Figure 1.7.1

Inverse trigonometric functions are used in many applications and mathematical developments and they will be particularly useful to you when you solve trigonometric equations.

ACTIVITY 1.7.A

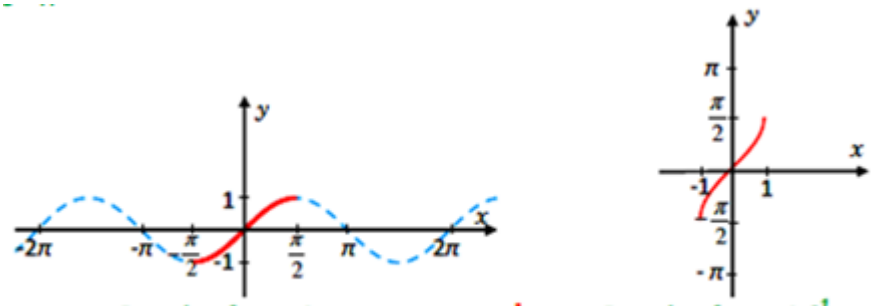
1 Find some intervals on which the sine function is one-to-one.

2 Draw the graph of $f(x) = \sin x$ when $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ and reflect it in the line $y = x$.

a. Inverse sine function

From Activity 1.7.A, you should have seen that the sine function is invertible on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Now, you can define the inverse sine function as follows.



a Graph of $y = \sin x$ b Graph of $y = \sin^{-1} x$

Figure 1.7.3

Definition 1.7.1 Inverse sine or Arcsine function

The inverse sine or arcsine function, denoted by $\sin^{-1}$ or arcsin, is defined by

$\sin^{-1} x = y$ or $\arcsin x = y$, if and only if $x = \sin y$ for $\frac{-\pi}{2} \le y \le \frac{\pi}{2}$

☑ Remark:

1 The inverse sine function is the function that assigns to each number x in $[-1, 1]$ the unique number y in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $x = \sin y$.

2 Domain of $\sin^{-1} x$ is $[-1, 1]$ and Range of $\sin^{-1} x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

3 From the definition, you have

$\sin(\sin^{-1} x) = x$ if $-1 \le x \le 1$ $\sin^{-1}(\sin x) = x$ if $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

☑ Caution:

$\sin^{-1} x$ is different from $(\sin x)^{-1}$ and $\sin x^{-1}$;

$(\sin x)^{-1} = \frac{1}{\sin x}$ and $\sin x^{-1} = \sin\left(\frac{1}{x}\right)$

Example1 Calculate $\sin^{-1} x$ for

a $x = 0$ **b** $x = 1$ **c** $x = \frac{\sqrt{3}}{2}$ **d** $x = -1$

Solution

a $\sin^{-1}(0) = 0$ since $\sin 0 = 0$ and $0 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

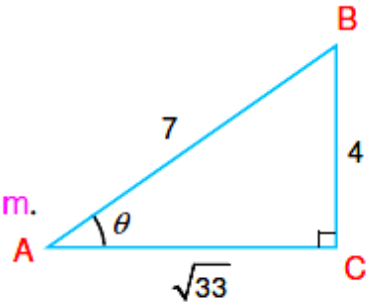
b $\sin^{-1}(1) = \frac{\pi}{2}$ since $\sin\left(\frac{\pi}{2}\right) = 1$ and $\frac{\pi}{2} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

c $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$ since $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

d $\sin^{-1}(-1) = -\frac{\pi}{2}$ since $\sin\left(-\frac{\pi}{2}\right) = -1$ and $-\frac{\pi}{2} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Example2 Compute $\cos\left(\sin^{-1}\left(\frac{4}{7}\right)\right)$.

Solution Let $\theta = \sin^{-1}\left(\frac{4}{7}\right)$. Then, $\sin\theta = \frac{4}{7}$ and drawing the reference triangle associated with θ, you have:



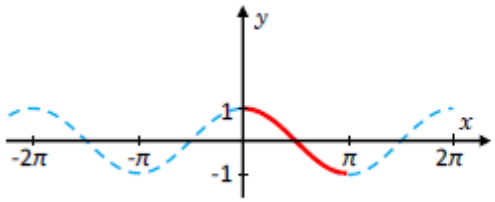
$\cos\theta = \frac{\sqrt{33}}{7}$

where $\sqrt{33}$ is calculated using Pythagoras' theorem.

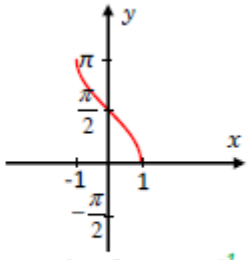
Therefore, $\cos\left(\sin^{-1}\left(\frac{4}{7}\right)\right) = \cos\theta = \frac{\sqrt{33}}{7}$

b. Inverse cosine function

You know that cos *x* is not one-to-one. Note, however, that cos *x* decreases from 1 to –1 in the interval $[0, \pi]$. Thus if $y = \cos x$ and *x* is restricted in the interval $[0, \pi]$, then for every *y* in [–1, 1], there is a unique *x* such that $\cos x = y$.



a Graph of $y = \cos x$



b Graph of $y = \cos^{-1} x$

Figure 1.7.4

Use this restricted cosine function to define the inverse cosine function. Reflecting the graph of $y = \cos x$ on $[0, \pi]$ in the line $y = x$, gives the graph of $f(x) = \cos^{-1} x$ as shown in Figures 1.7.4 and 1.7.5.

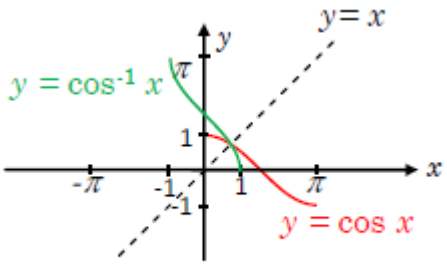


Figure 1.7.5

Definition 1.7.2

The inverse cosine or arccosine function, denoted by $\cos^{-1}$ or arccos, is defined by

$\cos^{-1} x = y$, if and only if $x = \cos y$, for $0 \le y \le \pi$.

☑ Remark:

1 Domain of $\cos^{-1} x$ is $[-1, 1]$ and Range of $\cos^{-1} x$ is $[0, \pi]$

2 From the definition, you have

$\cos(\cos^{-1} x) = x$, if $-1 \le x \le 1$.

$\cos^{-1}(\cos x) = x$, if $0 \le x \le \pi$.

Example 3 Calculate $y = \cos^{-1} x$ for

a $x = 0$ b $x = 1$ c $x = \frac{\sqrt{3}}{2}$ d $x = -1$

Solution

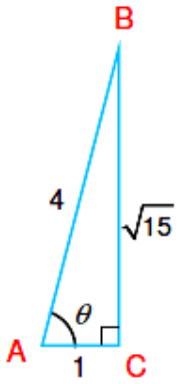
a $\cos^{-1}(0) = \frac{\pi}{2}$ since $\cos\frac{\pi}{2} = 0$ and $\frac{\pi}{2} \in [0, \pi]$

b $\cos^{-1}(1) = 0$ since $\cos 0 = 1$ and $0 \in [0, \pi]$

c $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$ since $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ and $\frac{\pi}{6} \in [0, \pi]$

d $\cos^{-1}(-1) = \pi$ since $\cos\pi = -1$ and $\pi \in [0, \pi]$

Example 4 Compute $\tan\left(\cos^{-1}\left(\frac{1}{4}\right)\right)$



Solution Let $\theta = \cos^{-1}\left(\frac{1}{4}\right)$, so that $\cos\theta = \frac{1}{4}$.

The opposite side is $\sqrt{4^2 - 1^2} = \sqrt{15}$

Thus, $\tan\left(\cos^{-1}\left(\frac{1}{4}\right)\right) = \tan\theta = \frac{\sqrt{15}}{1} = \sqrt{15}$

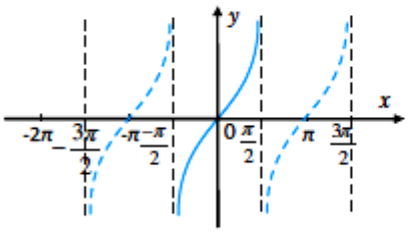
Example 5 Compute $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

Solution $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \pi - \cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = \pi - \frac{\pi}{4} = \frac{3}{4}\pi$

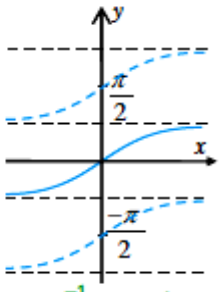
c. Inverse tangent function

The function $\tan x$ is not one-to-one on its domain as it can be seen from its graph.

To get a unique x for a given y, you restrict x to the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.



a *Graph of $y = \tan x$*



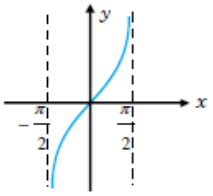
b *Graph of $y = \tan^{-1} x$ and range $= \{y \in \mathbb{R}: y \neq (2n-1)\frac{\pi}{2}; n \in \mathbb{N}\}$.*

Figure 1.7.6

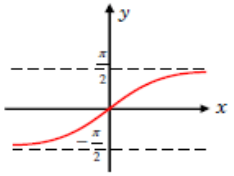
Definition 1.7.3

The inverse tangent function is a function denoted by $\tan^{-1}x$ or $\arctan x$ that assigns to each real number x the unique number y in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $x = \tan y$.

Reflecting the graph of $y = \tan x$ in the line $y = x$ gives the graph of $f(x) = \tan^{-1} x$ as shown in the Figures 1.7.6. and 1.7.7



a Graph of $y = \tan x$



b Graph of $y = \tan^{-1} x$

Figure 1.7.7

☑ Remark:

1 Domain of $\tan^{-1}x$ is $(-\infty, \infty)$ and Range of $\tan^{-1} x$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

You stress that $\frac{\pi}{2}$ is not in the range of $\tan^{-1}x$ because $\tan \frac{\pi}{2}$ is not defined.

2 From the above definition, you have,

$\tan(\tan^{-1} x) = x$ for all real x

$\tan^{-1}(\tan x) = x$, if $-\frac{\pi}{2} < x < \frac{\pi}{2}$

Example 6 Compute (in radians).

a $\tan^{-1}(0)$ b $\tan^{-1}\left(\sqrt{3}\right)$ c $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

Solution

a $\tan^{-1}(0) = 0$ because $\tan(0) = 0$ and $0 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

b $\tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$ because $\tan\frac{\pi}{3} = \sqrt{3}$ and $\frac{\pi}{3} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

c $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$ because $\tan -\frac{\pi}{6} = -\frac{1}{\sqrt{3}}$ and $-\frac{\pi}{6} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

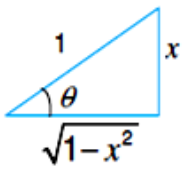
Example 7 Express $\tan(\sin^{-1} x)$ in terms of x.

Solution Here, you consider the following cases.

i Suppose $x = 0$, then $\tan(\sin^{-1}(0)) = \tan 0 = 0$.

ii Suppose $0 < x < 1$. Let $\theta = \sin^{-1} x$, then $\sin\theta = x$ and $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$

Look at the reference triangle given.



Hence, $\tan(\sin^{-1} x) = \tan\theta = \frac{x}{\sqrt{1-x^2}}$

iii If $-1 < x < 0$, then $\tan(\sin^{-1}x) = -\tan(\sin^{-1}(-x))$

$\Rightarrow \tan(\sin^{-1}(x)) = -\frac{-x}{\sqrt{1-x^2}} = \frac{x}{\sqrt{1-x^2}}$

$\therefore \tan(\sin^{-1}(x)) = \frac{x}{\sqrt{1-x^2}}$ for all $x \in (-1, 1)$

Inverse cotangent, secant, and cosecant functions

Here, the definitions of the inverse cotangent, secant and cosecant functions are given.

Whereas drawing the graphs is given as exercise.

Definition 1.7.4

i The inverse cotangent function $\cot^{-1}$ or arccot is defined by

$y = \cot^{-1} x$, if and only if $x = \cot y$ where $0 < y < \pi$ and $-\infty < x < \infty$.

ii The inverse secant function $\sec^{-1} x$ or arcsec x is defined by

$y = \sec^{-1} x$, if and only if $x = \sec y$ where $0 \le y \le \pi$, $y \ne \frac{\pi}{2}$, $|x| \ge 1$.

iii The inverse cosecant function $\csc^{-1} x$ or arccsc x is defined by

$y = \csc^{-1} x$, if and only if $x = \csc y$ where $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, $y \ne 0$, $|x| \ge 1$.

Example 9 Find the exact values of

a $\cot^{-1}\left(\sqrt{3}\right)$ b $\sec^{-1}(2)$ c $\csc^{-1}\left(-\frac{2}{\sqrt{3}}\right)$

Solution

a $y = \cot^{-1}\left(\sqrt{3}\right) \Rightarrow \cot y = \sqrt{3}$ and $0 < y < \pi \Rightarrow y = \frac{\pi}{6}$

b $\sec^{-1}(2) = \frac{\pi}{3}$ because $\sec\left(\frac{\pi}{3}\right) = 2$ and $0 < \frac{\pi}{3} < \frac{\pi}{2}$

c $\csc^{-1}\left(-\frac{2}{\sqrt{3}}\right) = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$

Exercise 1.7.D

1 Find the exact values of each of the following expressions without using a calculator or tables.

a $\sin^{-1}\left(-\frac{1}{2}\right)$ **b** $\cos^{-1}(3)$ **c** $\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$

d $\csc^{-1}\left(-\frac{2}{\sqrt{3}}\right)$ **e** $\sec^{-1}\left(\sqrt{2}\right)$ **f** $\cot^{-1}(-1)$

g $\cos\left(\sin^{-1}\left(\frac{12}{13}\right)\right)$ **h** $\sin^{-1}\left(\sin\left(\frac{\pi}{4}\right)\right)$ **i** $\sin^{-1}\left(\sin\frac{5\pi}{4}\right)$

j $\arccos\left(\cos\left(\frac{5\pi}{6}\right)\right)$ **k** $\cos\left(\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)$ **l** $\tan\left(\tan^{-1}\sqrt{3}\right)$

m $\tan\left(\arcsin\left(\frac{\sqrt{3}}{2}\right)\right)$ **n** $\cos^{-1}\left(\tan\left(\frac{-\pi}{4}\right)\right)$

2 Express each of the following expressions in terms of x.

a $y = \sin(\arctan x)$ **b** $y = \cos(\arcsin x)$ **c** $y = \tan(\arccos x)$

3 Prove each of the following identities.

a $\tan^{-1}(-x) = -\tan^{-1} x$ **b** $\operatorname{arcsec} x = \arccos\left(\frac{1}{x}\right)$ for $|x| \geq 1$

c $\sec^{-1} x = \sin^{-1}\left(\frac{1}{x}\right)$ for $|x| \geq 1$

4 Sketch the graph of:

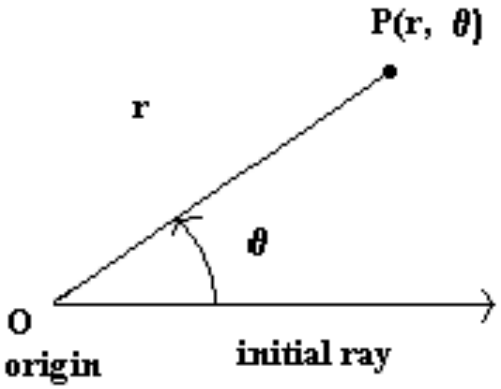
a $y = \operatorname{arccsc} x$ **b** $y = \operatorname{arcsec} x$ **c** $y = \operatorname{arccot} x$

5 Let $y = 3 + 2\arcsin(5x - 1)$. Express x in terms of y and determine the range of values of x and y.

1.6. Definition of Polar Coordinates.

Before we can start working with polar coordinates, we must define what we will be talking about. So let us first set us a diagram that will help us understand what we are talking about First, fix an origin (called the pole) and an initial ray from O.

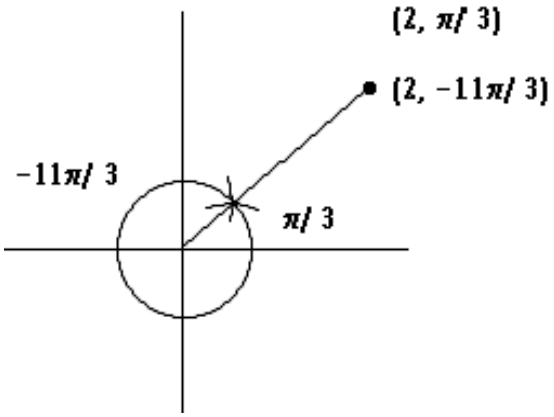
Each point P can be located by assigning to it a polar coordinate pair (r, θ) where r is the directed distance from O to P and θ is the directed angle from the initial ray to ray OP



When θ is positive, then the angle was measured counterclockwise, and when θ is negative, the angle was measured clockwise. By this fact, a given polar coordinate is not unique.

Example 1

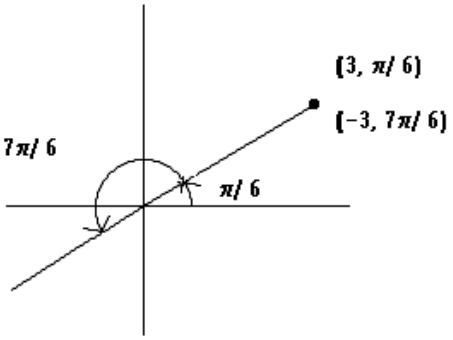
$P\left(3,\frac{\pi}{6}\right) = P\left(3,-\frac{11\pi}{6}\right)$ See diagram



Sometimes, there are occasions when we would like to allow r to be negative. For example, force on an object in a certain direction. This is why we would say that r is a directed distance

Example 2

$P\left(3,\frac{\pi}{6}\right) = P\left(-3,\frac{11\pi}{6}\right)$ See diagram

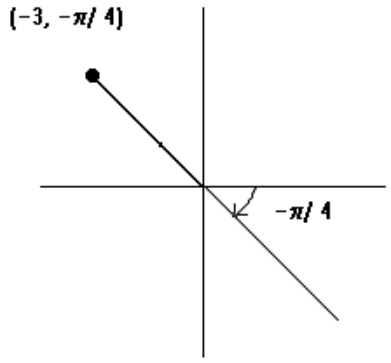


Example 3

First of all, let us plot the original point $\left(-3,-\frac{\pi}{4}\right)$

SOLUTION:

First of all, let us plot the original point



$r = -3 \quad \theta = -\frac{\pi}{4}, -\frac{\pi}{4} \pm 2\pi, -\frac{\pi}{4} \pm 4\pi, -\frac{\pi}{4} \pm 2k\pi$

(Remember we can go around the circle an infinite number of times in either direction, which is why I use +/- and multiples of 2π .). Now, if r is positive, what is the θ that gives the same point? What is the $\frac{\pi}{4}$ II? The answer is $\frac{3\pi}{4}$. Here are those related points.

$r = 3 \quad \theta = \frac{3\pi}{4}, \frac{3\pi}{4} \pm 2\pi, \frac{3\pi}{4} \pm 4\pi, \frac{3\pi}{4} \pm 2k\pi$

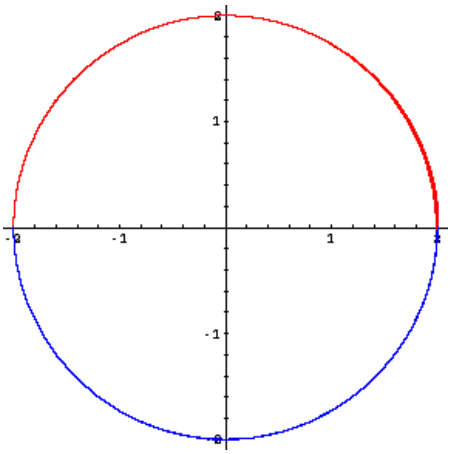
ELEMENTARY COORDINATE EQUATIONS AND INEQUALITIES

Some forms of basic polar equations that occur most often are the equations for a circle centered at the origin and the equation for the line through the origin

FACT: The equation of a circle with radius | a | centered at the origin is r = a.

Why did I say that the radius is | a |? Remember r is a directed distance, so the circle r = a and r = -a will be the same circle, but start in different places.

Example 4



To illustrate the above fact I will graph the half circles r = - 2 and r = 2 on the interval [0, π]. **r = - 2 is in the blue. r = 2 is in the red.** Notice that r = - 2 starts at π and goes to 2π , whereas r = 2 starts at 0 and goes to π

FACT:

The equation of a line through the origin making angle 0^0 with the initial ray is $\theta = 0^0$

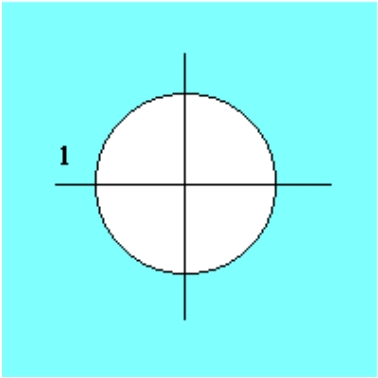
In my opinion, it is kind of a waste using a polar equation to represent the equation of a line, but it is possible to do.

Now, let us graph some polar equations and inequalities

Example 5: $r \geq 1$

SOLUTION:

The graph of this polar inequality will be the shaded region outside the circle of radius 1.

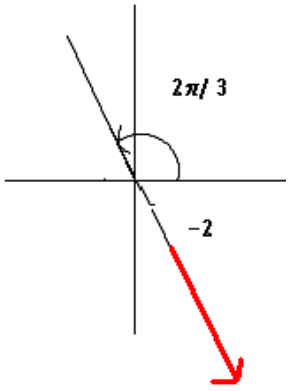


Example 6:

$\theta = \frac{2\pi}{3} \qquad r \leq -2$

SOLUTION:

This is the graph of the line that makes the angle $\frac{2\pi}{3}$ with the positive x-axis, but goes in the opposite direction starting at 2.



Example 7

$0 \leq \theta \leq \pi$, r = -1

SOLUTION:

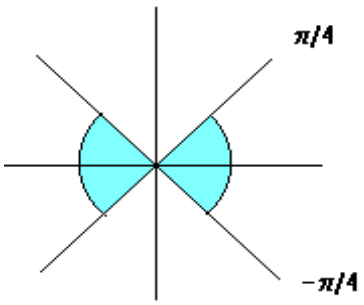
This is the half circle that starts at π and goes to 2 π . I know that the interval starts at 0, but r is negative. Therefore it goes in the opposite directions.

Example 8

$-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4} \quad -1 \leq r \leq 1$

SOLUTION:

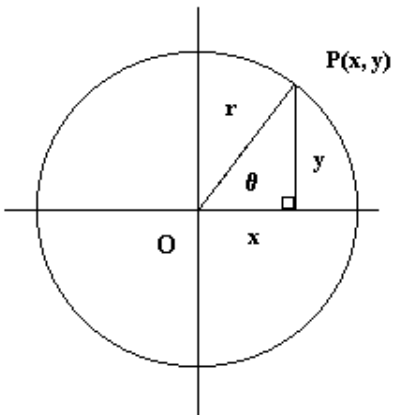
The graph of this set of inequalities is two wedges cut out of the circle with radii of -1 and 1, and all circles that are between those two values by the lines $\pm\frac{\pi}{4}$



Here are the basic equations that relate polar coordinates to Cartesian coordinates.

$x = r \cos \theta$, $y = r \sin \theta$, $x^2 + y^2 = r^2$, $\tan \theta = y/x$

Here is a diagram to help us understand where these equations came from.



CONVERTING FROM POLAR COORDINATES TO CARTESIAN COORDINATES

Example 9:

Convert $(0, \frac{\pi}{2})$ to Cartesian coordinates.

SOLUTION:

$x = 0 \cos(\frac{\pi}{2}) = 0$

$y = 0 \sin(\frac{\pi}{2}) = 0$

So $(0, \frac{\pi}{2})$ is equivalent to (0, 0) in Cartesian coordinates.

Example 10:

Convert the following polar coordinate to its equivalent Cartesian coordinate $\left(-\sqrt{2}, \frac{\pi}{4}\right)$

Solution:

$$x = -\sqrt{2}\cos\frac{\pi}{4} = -\sqrt{2}\left(\frac{1}{\sqrt{2}}\right) = -1$$

$$y = -\sqrt{2}\sin\frac{\pi}{4} = -\sqrt{2}\left(\frac{1}{\sqrt{2}}\right) = -1$$

So the equivalent Cartesian coordinates for the given polar coordinate is (-1, -1).

CONVERTING FROM A CARTESIAN EQUATION TO A POLAR EQUATION

Example 11:

Convert y = 10 into a polar equation.

Solution:

$$y = 10 \rightarrow r\sin\theta = 10 \rightarrow r = \frac{10}{\sin\theta} \rightarrow r = 10\csc\theta$$

This is a graph of a horizontal line with y-intercept at (0, 10).

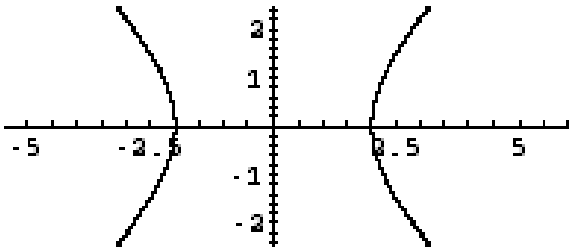
Example 12:

Convert $x^2 - y^2 = 4$ into a polar equation.

Solution:

$$x^2 - y^2 = 4 \rightarrow (r\cos\theta)^2 - (r\sin\theta)^2 = 4$$
$$\rightarrow r^2\cos^2\theta - r^2\sin^2\theta = 4$$
$$\rightarrow r^2(\cos^2\theta - \sin^2\theta) = 4$$
$$\rightarrow r^2\cos 2\theta = 4$$

This is an equation of a hyperbola, and here is its graph.



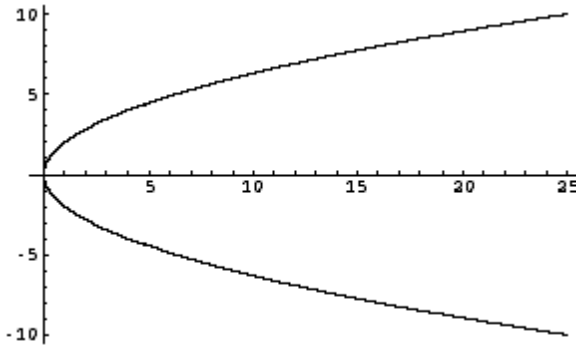
Example 13:

Convert $y^2 = 4x$ into a polar equation.

Solution:

$$y^2 = 4x \rightarrow (r\sin\theta)^2 = 4r\cos\theta$$
$$\rightarrow r^2\sin^2\theta = 4r\cos\theta$$
$$\rightarrow r\sin^2\theta = 4\cos\theta$$
$$\rightarrow r = \frac{4\cos\theta}{\sin^2\theta}$$
$$\rightarrow r = 4\cot\theta\csc\theta$$

This is an equation of a parabola, and here is its graph.



Example 14:

Convert (x + 2) 2 + (y - 4) 2 = 16 into a polar equation.

Solution:

First of all, I am going to multiply out the original equation

$(x + 2)^2 + (y - 4)^2 = 16 \Rightarrow x^2 + 4x + 4 + y^2 - 8y + 16 = 16$

$\Rightarrow x^2 + y^2 + 4x - 8y = -4$

Now convert this equation into its corresponding polar form.

$r^2 + 4r\cos\theta - 8r\sin\theta = -4$

This is an equation of a circle with center at (-2, 4) and radius 4.

***CONVERTING A POLAR EQUATION TO A CARTESIAN EQUATION* EXAMPLE 15:**

Convert $r\sin\theta = 4$ into its equivalent Cartesian equation

Solution:

$r\sin\theta = 4 \Rightarrow y = 4$

This is an equation of a horizontal line through the point (0, 4).

1. 9 Exercise

2. Convert the given point from polar coordinates to Cartesian coordinates.
   a. $(6, 210^\circ)$
   b. $(-4, 3\pi)$
   c. $(2, 11\pi/6)$
   d. $(6, 90^\circ)$
   e. $(-1, 405^\circ)$
3. Convert the given point from Cartesian coordinates to polar coordinates.
   a. $(3,1)$
   b. $(-1,-3)$
   c. $(0,2)$
   d. $(4,-2)$
   e. $(-2,0)$
4. Write the given equation in polar coordinates.
   a. $(x-3)^2 + y^2 = 9$
   b. $y = -x$
   c. $x^2 - y^2 = 1$
   d. $3x^2 + 4y^2 - 6x = 9$
5. Graph the function $r = 1+2\cos\theta$ in polar coordinates.

CHAPTER 2: CALCULUS

Objectives: By the end of this chapter students should be able to:

- define the main concept of calculus
- know the concept of limit
- calculate the limit of polynomial function, rational functions and trigonometric functions
- Differentiate algebraic functions and trigonometry functions.
- use the product, quotient and chain rule in derivative
- find first and second derivative
- know the properties of the first and second derivative
- sketch graph by using the derivative
- find a maximum and minimum (critical point)
- Find an inflection point.
- Integrate polynomial functions and simple rational functions.
- Integrate sine, cosine and tangent
- Find the area under the curve.
- Find out the area between two curves.
- Find volume.

What is Calculus?

Calculus is a mathematics of motion and change where there is motion or growth, where variable forces at work producing acceleration. Calculus is the right mathematics to apply this was true in the beginning of the subject and it is true today.

Calculus was the first created to meet the mathematical needs of scientists of the seventh century. Differential calculus deals with the problems of calculating rates of change. it enabled people to define slope of curves, to calculate the velocities , and accelerations of moving bodies , to find the firing . Integral calculus dealt with the problems of determining a function from information about its rates of change it enabled people to calculate the future location of a body from its present position and knowledge of the force acting on it , to find the areas of irregular regions in the plane to measure the lengths of curve, and to locate the centers of mass of arbitrary solids.

Calculus is fundamentally different from the mathematics that you have studied previously calculus is less static and more dynamic. it is concerned with change and motion , it deals with quantities for the reason it may be useful to have an overview of the subject before beginning intensive study . Here we give a glimpse of some of the main ideas of calculus by showing how limits arise when we attempt to solve a variety propels

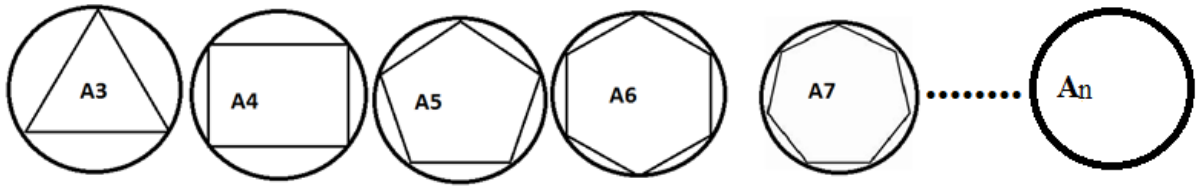
A circle with inscribed regular polygon let A_n , be the Area of the inscribed Polygon with n sides. As n increases, it appears that An becomes closer and closer to the area of the circle we say that the area of the circle is the limit of the areas of the inscribed polygons, and we write

$$A = \lim_{n \to \infty} An$$

2.1 Limit

Before we define a limit we give a glimpse by showing how limits arise when we attempt to solve a variety problem.

For example A circle with inscribed regular polygon



Let A_n, be the Area of the inscribed polygon with n sides. As n increase (the number of sides increases) it appears that A_n (the area of the circle) becomes closer and closer to the area of the circle, we say that that the area of circle is the limit of areas of the inscribed polygons, and we write $A = \lim_{n \to \infty} An$

Concept of limit

In this section, we develop the notation of limit using some language and illustrate the idea with some simple examples consider the functions

$g(x) = \frac{x^2-1}{x-1}$ and $f(x) = \frac{x^2-2}{x-1}$

What's the first thing that notice about these functions? You probably observed that both functions are undefined at$x = 1$, that means you cannot substitute 1 for x. It is helpful to compute some vales of both functions for x close to 1 as in the following tables:

x	$g(x)=\frac{x^2-1}{x-1}$	$f(x)=\frac{x^2-2}{x-1}$
	1.99	11.9
0.99	1.999	101.99
0.999	1.9999	1001.999
0.9999	1.99999	10001.9999
0.99999	1.999999	100001.99999
0.999999	1.999999	1000001.999999

In the first column of the table the x −values get closer to 1, but all less than 1. We use the notation$x \to \bar{1}$ to indicate that $x$approaches from the left side. Notice that with as $x$get closer & closer to 1$(x < 1)$, g(x) is getting closer and closer to 2.

We say that the limit of g(x) as$x$ approaches to 1 from the left is 2. Witten $\lim_{x \to \bar{1}} g(x) = 2$

On the other hand table indicate that as x gets closer and closer to 1$(withx < 1) f(x)$increase without bound (there is no number f(x) is approaching).

We say that limit of $f(x)$ as x approaches 1 from the left side does not exist, written.

$$\lim_{x \to \bar{1}} f(x) \text{ does not exist}$$

What happens to the function values for x close to 1 but larger than 1.

x	$g(x)=\frac{x^2-1}{x-1}$	$f(x)=\frac{x^2-2}{x-1}$
1.1	2.1	7.9
1.01	2.01	97.99
1.001	2.001	997.999
1.0001	2.0001	9997.9999
1.00001	2.00001	99997.99999
1.000001	2.000001	999997.999999
1.0000001	2	9999997.9999999

In the first column of the table, the x −values get closer to 1, but all greater than 1.

The table indicates that x approaches to 1 from the right side. As x gets closer and closer to 1.

We say that the limit of g(x) as x approaches 1 to from the right side is 2. Written

$\lim_{x \to 1^+} g(x) = 2$

On the other hand as x gets closer & closer to 1 $(withx > 1)$ f(x) decrease with out bound.

Similarly as the first.

$\lim_{x \to 1^+} g(x) = \lim_{x \to 1^-} g(x)$

One – sided limits, since the two

One-sided limits of g(x) as x approaches to 1 is 2.

Written $\lim_{x\to 1} g(x) = 2$

Another hand there is no common value for the one-sided limits of f(x). We say that the limit of f(x) as x approaches 1 does not exist

Written $\lim_{x\to 1} f(x) = 2$ does not exist

Theorem 1

For any constant c and a real number a

$$\lim_{x\to a} c = c$$

Theorem 2

For any real number a,

$$\lim_{x\to a} x = a$$

Theorem 3

Suppose lim f(x) and lim g(x) where both functions of x as x approaches to **a** both exist and c be any number constant then the following are true:

i. $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$

ii. $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$

iii. $\lim_{x\to a}[f(x) \bullet g(x)] = \lim_{x\to a} f(x)][\lim_{x\to a} g(x)]$

iv. $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, where $\lim_{x\to a} g(x) \neq 0$

Corollary

For any national number n, and any real number a,

$\lim_{x\to a} x^n = a^n$

Example 1

Find the limit of the following functions.

1. $\lim_{x\to 2} 4$
2. $\lim_{x\to 2}(x^2 + 4x - 8)$
3. $\lim_{x\to 3}(x^2 + 4)(x^3 + 2x + 1)$
4. $\lim_{x\to -2} x^3$
5. $\lim_{x\to 4}(5x^2 + 8x + 2)$

Solutions:

1. $\lim_{x\to 2} 4 = 4$
2. $\lim_{x\to 2} x^2 + 4x - 8$

$= \lim_{x\to 2} x^2 + 4 \lim_{x\to 2} x - \lim_{x\to 2} 8$

$= (2)^2 + 4(2) - 8 = 4 + 8 - 8 = 4$

3. $\lim_{x\to 3}(x^2 + 4)(x^3 + 2x + 1)$

$= \lim_{x\to 3}(x^2 + 4) \cdot \lim_{x\to 3}(x^3 + 2x + 1)$

$= [(3)^2 + 4] \cdot [(3)^3 + 2(3) + 1]$

$= (9 + 4) \cdot (27 + 6 + 1)$

$= (13) \cdot (34)$

$= 442$

Example 2

Find the limits of the following.

1. $\lim_{x\to 2} \frac{x^2-4}{x-2}$
2. $\lim_{x\to -3} \frac{3x+9}{x+3}$
3. $\lim_{x\to 2} \frac{x^2-5x+6}{x-2}$

Solution:

Find the limit of this function

1. $\lim_{x\to 2} \frac{x^2-4}{x-2}$

If we substitute the value x = 2, then the denominator and enumerator are (0/0), therefore we factorize:

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2} x + 2 = 2 + 2 = 4$$

2. $\lim_{x\to -3} \frac{3x+9}{x+3}$
3. $= \lim_{x\to -3} \frac{3(x+3)}{x+3}$

$= \lim_{x\to -3} 3$

$= 3$

4. $\lim_{x\to -3} \frac{x^2-5x+6}{x-2}$

$= \lim_{x\to -3} \frac{(x-3)(x-2)}{x-2}$

$= \lim_{x\to -3} (x-3)$

$= 2 - 3$

$= -1$

Exercise 1

Find the limit of the following functions.

1. $\lim_{x\to 3}\dfrac{3x+9}{x+3}$

2. $\lim_{x\to 3}\dfrac{x^2-5x+6}{x-3}$

3. $\lim_{x\to -2}x^3+4$

4. $\lim_{x\to -5}\left(x^2+4\right)\left(x^3+2x+1\right)$

5. $\lim_{x\to 3}x^2+4x-8$

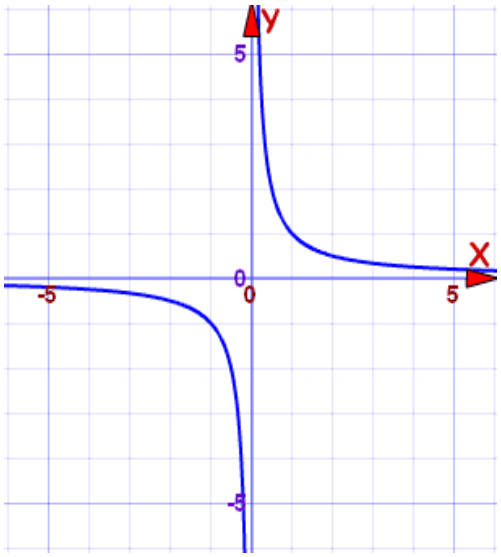
6. $\lim_{x\to 3}\dfrac{x^2-4}{x-2}$

7. $\lim_{x\to 3}\dfrac{2x^2-10x+12}{x-2}$

Limits of infinity

We consider the following cases:

1. $\lim_{x\to\infty}x^2=\infty$
2. $\lim_{x\to\infty}\dfrac{1}{x}=0$



Example 3

Find the limits

a) $\lim_{x\to\infty}\dfrac{3x^2}{4x^2}$

b) $\lim_{x\to\infty}\dfrac{4x^3+2x^2+1}{x^2-2x}$

c) $\lim_{x\to\infty}\dfrac{5x^4-3x^2+1}{x^5-2}$

Solution:

a) $\lim_{x\to\infty}\dfrac{3x^2}{4x^2}$ divide both sides by x^2

$$\lim_{x\to\infty}\frac{\frac{3x^2}{x^2}}{\frac{4x^2}{x^2}}=\lim_{x\to\infty}\frac{3}{4}=\frac{3}{4}$$

b) $\lim_{x\to\infty} \frac{4x^3+2x^2+1}{x^2-2x}$

$= \lim_{x\to\infty} \frac{\frac{4x^3}{x^3}+\frac{2x^2}{x^3}+\frac{1}{x^3}}{\frac{x^2}{x^3}-\frac{-2x^2}{x^3}}$

$= \lim_{x\to\infty} \frac{4+\frac{2}{x}+\frac{1}{x^3}}{\frac{1}{x}-\frac{2}{x^2}}$

$= \lim_{x\to\infty} \frac{4+\frac{2}{\infty}+\frac{1}{\infty}}{\frac{1}{\infty}-\frac{2}{\infty}}$

$= \frac{4+0+0}{0-0}$

$= \frac{4}{0} = \infty$

c) $\lim_{x\to\infty} \frac{5x^4-3x^2+1}{x^5-2}$ divide both sides by x^5

$= \lim_{x\to\infty} \frac{\frac{5x^4}{x^5}-\frac{3x^2}{x^5}+\frac{1}{x^5}}{\frac{x^5}{x^5}-\frac{2}{x^5}}$

$= \lim_{x\to\infty} \frac{\frac{5}{x}-\frac{3}{x^3}+\frac{1}{x^5}}{1-\frac{2}{x^5}}$

$= \frac{\frac{5}{\infty}-\frac{3}{\infty}+\frac{1}{\infty}}{1-\frac{2}{\infty}}$

$= \frac{0}{1-0}$

$= 0$

Exercise 2

1. $\lim_{x\to\infty} x^3+5x$
2. $\lim_{x\to\infty} \frac{4x^3+4x}{5x^3+3}$
3. $\lim_{x\to\infty} \frac{1}{x^3+3x}$

Continuity

İf you say that a machine had been in continuous operation for the past 48 hr .What do have in mind? that means that the machine has been in operation all of that time, without any

interreption all even a moment. Mathematicians mean very much the same thing when we say that a function is continuous on an interval means the graph can be dawn without interruption.

A function is said to be continuous on an interval if there is no disjoint on that interval

Definition

A function f in continuous of x = a when

I. F (a) is definded

II. $\lim_{x\to a} exist$

III. $\lim_{x\to a} f(x) = f(a)$

Otherwise f is said to be discontinuous at x = a

Example 4

Determine where the function is continuous or discontinuous at the gıven point .

$a)\ f(x) = \sqrt{x+2}\ at\ x = 2$ $\qquad b)\ f(x) = \dfrac{x}{x-1}\ at\ \ x = 1$

Solution

$a)\ (i) f(2) = \sqrt{2+2} = \sqrt{4} = 2$

$(ii) \lim_{x-a} f(x) = \sqrt{2+2} = 2$

$f(2) = \lim_{x\to 2} f(x)$ → then f(x) is continuous at x = 2

$b)\ (i) f(1) is not defined.$

$(ii) \lim_{x\to a} \dfrac{x}{x-1} does not exist is f(x) = \dfrac{x}{x-1} is not continuous\ at\ \ x = 1.$

Exercise 3

Determine the following functions are continuous or not at the given point.

$a)\ f(x) = \sqrt{x+3} at x = -4$

$b)\ f(x) = \dfrac{x^2-4}{x+2} at x = -2$

$c)\ f(x) = x^2 + 3 \qquad at x = 1$

$d)\ f(x) = x^3 + 2x + 2 \quad at x = 0$

$e)\ f(x) = cosx - x at x = 0$

$f)\ f(x) = \dfrac{2}{x} + 3 \quad at\ \ x = 0$

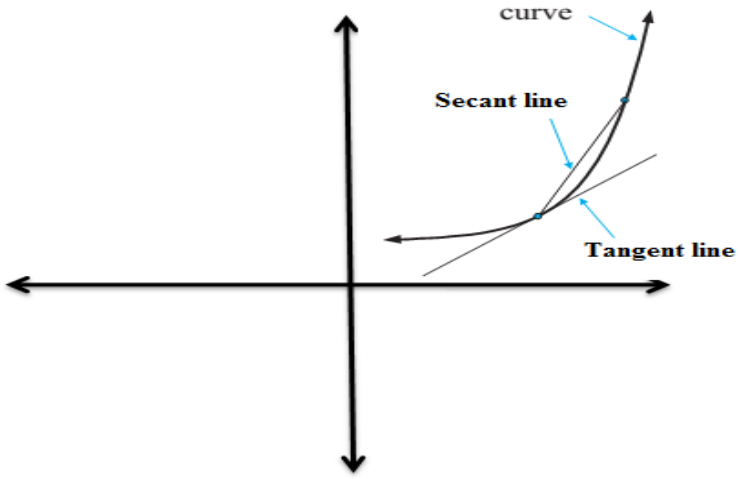
$h)\ f(x) = \begin{cases} x^2 & if x < 2 \\ 3x - 1 & if\ x \geq 2 \end{cases}$

$at\ x = 2$

$g)\ f(x) = \begin{cases} x^2 + 1 & if x < 1 \\ 2\csc 0^0 & if\ x \geq 1 \end{cases}$

$at\ x = 1$

2.2 DERIVATIVE



The slope of the secant line through the point (a,f(a)) and [(a+h) , f(a+h)] ıs gıven by

$M_{secant} = \frac{f(a+h) - f(a)}{(a+h)-a}$

Defınıtıon.

The slope M_{tan} of the tangent line to y = f(x) at x = a is given by

$M_{tan} = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$

Provided the limit exists.

Example5

Find the equation of the tangent line to y = x^2 + 2x +1 at x = 1

Solution

We compute the slope $M_{tan} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$

$M_{tan} = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$

$= \lim_{h\to 0} \frac{[(1+h)^2+2(h+1)+1\}-[1+2+1)]}{h}$

h---→0

Lım $\lim_{h\to 0}\frac{[1+2h+h^2-2h+2+1]-4}{h}$

$= \lim_{h\to 0}\frac{[h^2+4h+4-4]}{h}$

$= \lim_{h\to 0}\frac{h(2h+2)}{h}$

$= \lim_{h\to 0}(2h+2)$

= 2

Example 6

Find the equation of the tangent line$y = 3x + 1\ at\ x = 2$

Solution:

Slope of tangent = $M_{tan} = lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$

= Lım $\lim_{h\to 0}\frac{[3(2+h)+1-f(2)}{h}$

= Lım $\lim_{h\to 0}\frac{6+3h+1-7)}{h}$

= $\lim_{h\to 0}\frac{3h}{h}$

= 3

Definition

The slope M_{tan} of the tangent line to y = f(x) at x = a ıs given by

$M_{tan}=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$

Provided the limit exists. If the limit exist we say that f is differentiable at x = a

Example 7

$F(x) = 2x^3 + 4x - 5\ at\ x = 1$

Solution:

$\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$

$=\lim_{h\to 0}\frac{[2(1+h)^3+4(h+1)-5-[2+4-5]}{h}$

$= \lim_{h\to 0}\frac{[2(1+3h+3h^2+h^3)+4h+4-5-1]}{h}$

$= \lim_{h\to 0}\frac{[2+6h+6h^2+2h^3+4h+4-6]}{h}$

$= \lim_{h\to 0}\frac{2h^3+6h^2+10h]}{h}$

$= \lim_{h\to 0} \frac{h(2h^2+6h+10)}{h} = \lim_{h\to 0}(2h^2 + 6h + 10) = 10$

Definition 2

The derivative of f(x) is the function f'(x) is given by

$f'(x) = \lim_{h\to 0} \frac{(f+h)-f(x)}{h}$

Or $\lim_{h\to 0} \frac{f+\Delta x)-f(x)}{h}$

Theorem

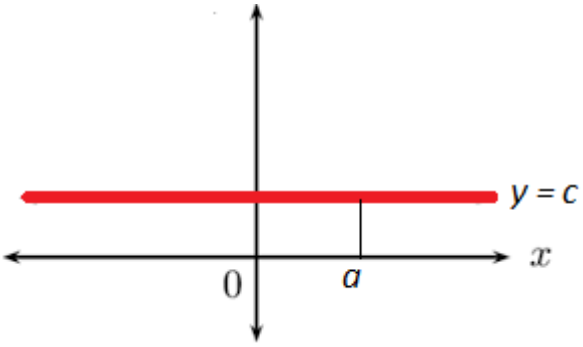
For any constant C $\frac{d}{dx}C = 0$

Proof

$\frac{d}{dx}C = f'(x)$

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$=\lim_{h\to 0} \frac{c-c}{h} = \frac{0}{h}$



Theorem 2

$\frac{d}{dx}x = 1$

Proof

$\frac{d}{dx}x = f'(x) = lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$f'(x) = lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$= lim_{h\to 0} \frac{(x+h)-x}{h}$

$= lim_{h\to 0} \frac{h}{h} = 1$

Theorem 3(powerrule)

For any ınteger n>0 $\frac{d}{dx}x^n = nx^{n-1}$

General Derivative Rules

If f(x) and g(x) are differentiable at x and c ıs any constant, then

I. $\frac{d}{dx}[f(x)+g(x)] = f'(x)+g'(x)$

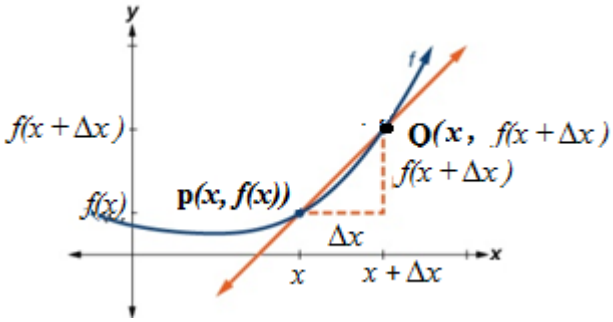
II. $\frac{d}{dx}[f(x)-g(x)] = f'(x)-g'(x)$

III. $\frac{d}{dx}[cf(x)] = cf'(x)$

Provided the limit exists .The process of computing a derivative ıs called dıfferentıatıon.

The slope of secant line

$$M_{\sec ant} = \frac{y_2 - y_1}{x_2 - x_1}$$



Slope of the secant = $\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{(x+\Delta x)-x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}$

Slope of tangent = $\lim_{x\to a}$ *slope of the secant*

$$= \lim_{x\to a}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

Slope of the curve = slope of Tangent to the curve at the point x = a

$$M_{\tan} = \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

Definition:

The derivative of a function is defined as

$$f(x)' = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Example 1

Differentiate the following using the first principle (difference of quotient)

a) $y = x^2 + 2$

b) $y = 2x^2 - 3x + 5$

c) $y = \frac{2}{x}$

Solution

From the above definition $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

a) $y = x^2 + 2$

Given

$f(x) = x^2 + 2$

$f(x+\Delta x) = (x+\Delta x)^2 + 2 \Rightarrow x^2 + 2x\Delta x + \Delta x^2 + 2$

There fore

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 + 2 - (x^2+2)}{\Delta x} \Rightarrow \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + \Delta x^2 + 2 - x^2 - 2}{\Delta x}$$

$$\Rightarrow \lim_{\Delta x \to 0} \frac{2x\Delta x + \Delta x^2}{\Delta x} \Rightarrow \lim_{\Delta x \to 0} \frac{\Delta x(2x + \Delta x)}{\Delta x} = \lim_{\Delta x \to 0} 2x + \Delta x$$

$$\therefore \frac{dy}{dx} = 2x$$

b) $y = 2x^2 - 3x + 5$

Given

$f(x) = 2x^2 - 3x + 5$

$f(x+\Delta x) = 2(x+\Delta x)^2 - 3(x+\Delta x) + 5 \Rightarrow 2x^2 + 4x\Delta x + 2\Delta x^2 - 3x - 3\Delta x + 5$

There fore

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{2x^2+4x\Delta x+2\Delta x^2-3x-3\Delta x+5-2x^2+3x-5}{\Delta x}$$

$$\Rightarrow \lim_{\Delta x\to 0}\frac{4x\Delta x+2\Delta x^2-3\Delta x}{\Delta x}\Rightarrow \lim_{\Delta x\to 0}\frac{\Delta x(4x+2\Delta x-3)}{\Delta x}=\lim_{\Delta x\to 0}4x+2\Delta x-3$$

$$\therefore \frac{dy}{dx}=4x-3$$

c) $y=\frac{2}{x}$

Given

$$f(x)=\frac{2}{x}$$

$$f(x+\Delta x)=\frac{2}{(x+\Delta x)}\Rightarrow \frac{2}{x+\Delta x}$$

Therefore

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\frac{2}{x+\Delta x}-\frac{2}{x}}{\Delta x}\Rightarrow \frac{\frac{2x-2(x+\Delta x)}{(x+\Delta x)x}}{\Delta x}$$

$$\Rightarrow \lim_{\Delta x\to 0}\frac{2x-2x-2\Delta x}{(x+\Delta x)x}\div \Delta x\Rightarrow \lim_{\Delta x\to 0}\frac{-2\Delta x}{x^2+x\Delta x}\times\frac{1}{\Delta x}$$

$$\Rightarrow \lim_{\Delta x\to 0}\frac{-2}{x^2+x\Delta x}\Rightarrow \frac{-2}{x^2+x(0)}$$

$$\therefore \frac{dy}{dx}=-\frac{2}{x^2}$$

Example 2

Find the gradient of the curve $y=3x^2-4x+6$ at x = 2, from the first principle of differentiation

Solution

Given

$$f(x)=3x^2-4x+6$$

$$f(x+\Delta x)=3(x+\Delta x)^2-4(x+\Delta x)+6\Rightarrow 3x^2+6x\Delta x+3\Delta x^2-4x-4\Delta x+6$$

There fore $\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{3x^2+6x\Delta x+3\Delta x^2-4x-4\Delta x+6-(3x^2-4x+6)}{\Delta x}$

$$\Rightarrow \lim_{\Delta x\to 0} \frac{3x^2+6x\Delta x+3\Delta x^2-4x-4\Delta x+6-3x^2+4x-6}{\Delta x}$$

$$\frac{dy}{dx}=\lim_{\Delta x\to 0} \frac{6x\Delta x+3\Delta x^2-4\Delta x}{\Delta x} \Rightarrow \lim_{\Delta x\to 0} \frac{\Delta x(6x+3\Delta x-4)}{\Delta x}$$

$$\Rightarrow \lim_{\Delta x\to 0} 6x+3\Delta x-4$$

$$\therefore \frac{dy}{dx}=6x-4$$

Therefore the gradient of the curve at x = 2 is 6(2) – 4 = 8

Exercise

1.Differentiate the following w.r.t. to x from the first principle

a. $y=x^2+5x-6$

b. $f(x)=\frac{3}{x}$

c. $y=x^3-2x$

d. $y=\sqrt{x}$

2.Find the gradient of the following curves at the indicated point using the first principle of differentiation

a. $y=2-x^2$

b. $y=x^3-2$

c. $y=\frac{1}{x}$

d. $y=4x^2-5x-7$

Differentiation by Rule

Derivative: is used to find the slope of a curve and the process used is called **differentiation**.

❖ **The derivative of the function $y = x^n$ is $\frac{dy}{dx} = nx^{n-1}$**

Example 8

Find $\frac{dy}{dx}$ for each of these functions

a) $y=x^3$

b) $y=7x^5$

c) $y = \frac{7}{x^3}$

d) $y = \frac{5}{x^7}$

e) $y = x^{\frac{2}{3}}$

f) $y = \frac{1}{\sqrt{x^5}}$

Solution

a) $y = x^3, \frac{dy}{dx} = 3x^{3-1} \Rightarrow \frac{dy}{dx} = 3x^2$

b) $y = 7x^5, \frac{dy}{dx} = 7(5)x^{5-1} \Rightarrow \frac{dy}{dx} = 35x^4$

c) $y = \frac{7}{x^3} = 7x^{-3}, \frac{dy}{dx} = 7(-3)x^{-3-1} \Rightarrow \frac{dy}{dx} = -21x^{-4} \therefore \frac{dy}{dx} = -\frac{21}{x^4}$

d) $y = \frac{5}{x^7}, \Rightarrow y = 5x^{-7}, \frac{dy}{dx} = 5(-7)x^{-7-1} \Rightarrow \frac{dy}{dx} = -35x^{-8} \therefore \frac{dy}{dx} = -\frac{35}{x^8}$

e) $y = x^{\frac{2}{3}}, \Rightarrow, \frac{dy}{dx} = \frac{2}{3}x^{\frac{2}{3}-1} \Rightarrow \frac{dy}{dx} = \frac{2}{3}x^{-\frac{1}{3}} \therefore \frac{dy}{dx} = \frac{2}{3x^{\frac{1}{3}}}$

f) $y = \frac{1}{\sqrt{x^5}}, \Rightarrow, y = \frac{1}{x^{\frac{5}{2}}} \Rightarrow y = x^{-\frac{5}{2}}, \frac{dy}{dx} = -\frac{5}{2}x^{-\frac{5}{2}-1}$

$= -\frac{1}{2}x^{-\frac{7}{2}} \Rightarrow -\frac{1}{2x^{\frac{7}{2}}} \therefore \frac{dy}{dx} = \frac{1}{2\sqrt{x^7}}$

Exercise 3

1. Find $f'(x)$ given that f(x) is:

a. x^4

b. $2x^{-5}$

c. $\frac{4}{3x^4}$

d. $6\sqrt{x}$

e. $\frac{2}{\sqrt[5]{x^3}}$

2. Find $\frac{dy}{dx}$ for :

a. $y = 3x^5$

b. $y = \sqrt{x^3}$

c. $y = -\frac{3}{5x^2}$

d. $y = \sqrt[5]{x}$

Rules of differentiation

1) Derivative of constant is zero.

If F(x) = c then F(x)' = 0

2) if f(x) = x then f(x)' = 1

3) Derivative of Sum or Difference of two Functions

To find the derivative of a function that has more than one term, you need to differentiate each function

$y = f(x) \pm g(x)$ then $\frac{dy}{dx} = \frac{d}{dx}\left(f(x) \pm g(x)\right)$

Example 9

If $y = 4x^3 - 2x^2$, find $\frac{dy}{dx}$ and interpret its meaning

Solution

As $y = 4x^3 - 2x^2$, $\frac{dy}{dx} = 12x^2 - 4x$

$\frac{dy}{dx}$ is

- the gradient function or derivative of $y = 4x^3 - 2x^2$ from which the gradient of the tangent at any point can be found
- the instantaneous rate of change of y with respect to x.

Example 10

Find $f'(x)$ if $f(x)$ is equal to:

a) F(x) = $6x^4 - 5x^2 + 3x - 3$ b) $8x + \frac{5}{x} - \frac{3}{x^2}$

Solution

a) $f(x) = 6x^4 - 5x^2 + 3x - 3$

$\therefore f'(x) = 6(4x^3) - 5(2x) + 3(1)$

$= 24x^3 - 10x + 3$

b) $f(x) = 8x + \frac{5}{x} - \frac{3}{x^2} \Rightarrow 8x + 5x^{-1} - 3x^{-2}$

$\therefore f'(x) = 8(1) + 5(-1x^{-2}) - 3(-2x^{-3})$

$= 8 - \frac{5}{x^2} + \frac{6}{x^3}$

Example 11

find the first and second derivative of the following

a) $y = x^3+4x^2+3x$

b) $y = x4+2x^3+6$

Solution

a) $y = x^3+4x^2+3x$

$\frac{dy}{dx} = 3x^2+8x+3$

$\frac{d^2y}{dx^2} = 6x+8$

b) $y = x^4+2x^3+6$

$y' = 4x^3+6x^2$

$y'' = 12x^2+12x$

Example 12

Find the gradient function of $y = x^3 - \frac{2}{x}$ and hence find the gradient of the tangent to the function at the point where x = 2.

Solution

$y = x^3 - \frac{2}{x}$

$= x^3 - 2x^{-1}$

$\therefore \frac{dy}{dx} = 3x^2 - 2(-1x^{-2}) = 3x^2 + \frac{2}{x^2}$

When $x = 2$, $\frac{dy}{dx} = 3(2)^2 + \frac{2}{(2)^2} = 12 + \frac{1}{2} = 12\frac{1}{2}$

When x=2, $\frac{dy}{dx} = 3(2)^2 + \frac{2}{2^2} = 12 + \frac{1}{2} = \frac{25}{2}$

So the tangent has a gradient of $\frac{25}{2}$

Example 13

Find $\frac{dy}{dx}$ for each of these functions

a) $y=(x-3)^2$

b) $y=\sqrt{x}\left(x^3-1\right)$

c) $y=x^3+\frac{6}{x^2}$

Solution

a) $y=(x-3)^2$ first expand the brackets

$y=(x-3)(x+3)$

$y=x^2-6x+9$

$\frac{dy}{dx}=2x-6$

b) $y=\sqrt{x}\left(x^3-1\right)$

$y=\sqrt{x}\left(x^3-1\right)=x^{\frac{1}{2}}\left(x^3-1\right)=x^{\frac{7}{2}}-x^{\frac{1}{2}}$

$\therefore \frac{dy}{dx}=\frac{7}{2}x^{\frac{7}{2}-1}-\frac{1}{2}x^{\frac{1}{2}-1}=\frac{7}{2}x^{\frac{5}{2}}-\frac{1}{2}x^{-\frac{1}{2}}$

$\Rightarrow \frac{dy}{dx}=\frac{7}{2}\sqrt{x^5}-\frac{1}{2\sqrt{x}}$

c) $y=x^3+\frac{6}{x^2}$

$y=x^3+\frac{6}{x^2}=x^3+6x^{-2}$

$\therefore \frac{dy}{dx}=3x^2+6(-2x^{-3})=3x^2-\frac{12}{x^3}$

Exercise 14

1. Find the first and second derivative if $f(x)$ is

a) $\frac{1}{2}x^4-6x^2$

b) $(2x-1)^2$

c) $(x+2)^3$

d) $\frac{2x-3}{x}$

e) $\frac{3x-6}{x^2}$

2. Differentiate with respect to x:

a) $\left(5-x^2\right)$

b) $\frac{6x^2-9x^4}{3x}$

c) $(x+1)(x-2)$

d) $x(x+1)(2x-5)$

3. Find the gradient of the tangent to:

a) $y=x^2$ at $x=2$

b) $y=\frac{8}{x^2}$ at $\left(9,\frac{8}{81}\right)$

c) $y=2x^2-3x+7$ at $x=1$

d) $y = \frac{2x^2 - 5}{x}$ at $x = 1$

e) $y = \frac{x^2 - 4}{x^2}$ at the point $\left(4, \frac{3}{4}\right)$

4. Find the gradient function of *f(x)* where *f(x)* is:

a) $4\sqrt{x} + x$

b) $\sqrt[3]{x}$

c) $\frac{5}{x^2\sqrt{x}}$

d) $2x - \frac{3}{x\sqrt{x}}$

e) $2x - \sqrt{x}$

f) $-\frac{2}{\sqrt{x}}$

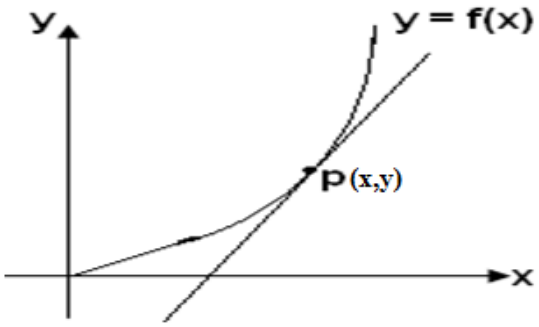
5. a) If $y = 4x - \frac{3}{x}$, find $\frac{dy}{dx}$ and interpret its meaning.

b) The position of a car moving along a straight road is given by $S = 2t^2 + 4t$ metres where t is the time in seconds. Find $\frac{dS}{dt}$ and interpret its meaning.

c) The cost of producing x toasters each week is given by $C = 1785 + 3x + 0{:}002x^2$ dollars. Find $\frac{dC}{dx}$ and interpret its meaning.

Gradient of the Curve

Since the gradient of a curve is continually changing, then the gradient of the curve at point p is equal to the gradient of the tangent line at point p.



Example 14

Find the gradient of the curve $y = x^3 - 4x^2 + 2$ at the point [1,-1]

Solution

$y = x^3 - 4x^2 + 2$,

$\frac{dy}{dx} = 3x^2 - 12x$

$when x = 1 \Rightarrow \frac{dy}{dx} = 3(1)^2 - 12(1)$

$= 3 - 12 \therefore \frac{dy}{dx} = -9$

Thus: the gradient of the curve at the point [1, -1] is -9.

Example 15

Find the gradient of the curve $f(x) = x^2 + \frac{1}{x}$ at the point p (1, 2)

Solution

To find the gradient of the curve first find $f'(x)$ or $\frac{dy}{dx}$

$f(x) = x^2 + \frac{1}{x} \Rightarrow f(x) = x^2 + x^{-1}$

$f'(x) = 2x^{2-1} + (-1)x^{-1-1} \Rightarrow 2x - x^{-2}$

$\therefore f'(x) = 2x - \frac{1}{x^2}$

The gradient of the curve at the point p (1, 2) is equal to $f'(x)$

$f'(x) = 2x - \frac{1}{x^2} \Rightarrow f'(x) = 2(1) - \frac{1}{(1)^2} = 2 - 1$

$\therefore f'(x) = 1$

The gradient of the curve at point p is 1.

Example 16

The gradient of the curve $y = 3x^2 + x - 3$, at the point p is 13 find the coordinates of point p.

Solution

When $y = 3x^2 + x - 3$, $\frac{dy}{dx} = 6x + 1$ $\frac{dy}{dx}$ = 13 at point p $\frac{dy}{dx} = 6x + 1 = 12 \Rightarrow x = 2$ .To find y coordinate substitute $x = 2$ into $y = 3x^2 + x - 3$, $y = 3(2)^2 + 2 - 3$

Therefore: the coordinates of the point p are (2, 11)

Example 17

A curve whose equation is $y = \frac{a}{x} + c$ passes the point (3, 9) with gradient 5. Find the values of the constants a and c ?

Solution

Rewriting $y = \frac{a}{x} + c \Rightarrow y = ax^{-1} + c$

$$\frac{dy}{dx} = -ax^{-2} \Rightarrow \frac{dy}{dx} = -\frac{a}{x^2}$$

Now, the point (3, 9) is on the curve

So $9 = \frac{a}{3} + c \Rightarrow 3c + a = 27$.....(1)

The gradient of the curve at x = 3 is 5

So $-\frac{a}{(3)^2} = 5 \Rightarrow -\frac{a}{9} = 5 \therefore a = -45$.....(2)

Then substituting $a = -45$ in equation (1)

$3c + (-45) = 27 \Rightarrow 3c = 27 + 45$

$\Rightarrow 3c = 72 \therefore c = 24$

$a = -45$ And $c = 24$

Exercise 5

1. Find the gradient of each of these curves at the point given

a. $y = x^2$ at (3,9)

b. $y = 2x^3$ at (2, 12)

c. $y = 5\text{-}x^2$ at (-2,1)

d. $y = 3 - \frac{2}{x}$ at $\left(5, \frac{5}{2}\right)$

2. Find the coordinates of any points on each of these curves where the gradient is as stated.

a. $y = x^3\text{-}x^2 + 3$, grad = 0

b. $y = \frac{4}{x}$, grad = -6

c. $y = x^2 + \frac{3}{2x^2}$, grad = 3

3.The curve C is defined by $y = ax^2 + b$, where a and b are constants. Given that the gradient of the curve at the point (2,-2) is 3. Find the values of a and b.

4. Given that the curve with the equation $y = Ax^2 + Bx$ has a gradient 7 at the point (6, 8), find the values of the constants A and B.

5. Find the coordinates of the points on the curve $y = x^2+2x-1$ at which the tangent line is parallel to the horizontal axis.

THE PRODUCT RULE

If $u(x)$ and $v(x)$ are two functions of x and $f(x) = u(x)\,v(x)$, then

$$f'(x) = u'(x)\,v(x) + u(x)\,v'(x).$$

Alternatively if $y = uv$ then $\dfrac{dy}{dx} = \dfrac{du}{dx}v + u\dfrac{dv}{dx}$.

Example 18

Find $\dfrac{dy}{dx}$ if:

a) $y = (x^2)\,(x^4)$

b) $y = x^2(x + 1)$

c) $y = x^2\left(x^2 + 2x\right)^4$

d) $y = \sqrt{x}\left(2x-1\right)^3$

Solution

a) $y = (x^2)\,(x^4)$

$$\frac{dy}{dx} = x^2\frac{dy}{dx}(x^4) + x^4 \cdot \frac{d}{dx}(x^2)$$

$$= x^2 \cdot 4x^3 + x^4 \cdot 2x$$

$$= 4x^5 + 2x^5$$

$$= 6x^5$$

b) $y = x^2(x + 1)$

$$\frac{dy}{dx} = x^2\frac{d}{dx}(x+1) + (x+1)\frac{d}{dx}(x^2)$$

$$= x^2 \cdot 1 + (x+1).2x$$

$$= x^2 + 2x^2 + 2x$$

$$= 3x^2 + 2x$$

c) $y = x^2\left(x^2 + 2x\right)^4$ is the product of $u = x^2$ and $v = \left(x^2 + 2x\right)^4$

$$\therefore u' = 2x \text{ and } v' = 4\left(x^2 + 2x\right)^3(2x+2) = \left(4x+8\right)\left(x^2 + 2x\right)^3$$

Now $\frac{dy}{dx} = u'v + uv'$ (product rule)

$2x\left(x^2+2x\right)^4 + x^2\left(4x+8\right)\left(x^2+2x\right)^3$

$= 2x\left(x^2+2x\right)^4 + \left(4x^3+8x^2\right)\left(x^2+2x\right)^3$

d) $y = \sqrt{x}\left(2x-1\right)^3$ is the product of $u = x^{\frac{1}{2}}$ and $v = \left(2x-3\right)^3$

$\therefore u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = 3\left(2x-3\right)^2(2)$...chain rule

$= 6\left(2x-3\right)^2$

Now $\frac{dy}{dx} = u'v + uv'$ (product rule)

$= \frac{1}{2}x^{-\frac{1}{2}}\left(\left(2x-3\right)^3\right) + x^{\frac{1}{2}}\left(6\left(2x-3\right)^2\right)$

$= \frac{1}{2}x^{-\frac{1}{2}}\left(2x-3\right)^3 + 6x^{\frac{1}{2}}\left(2x-3\right)^2$

Exercise 6

1 Use the product rule to differentiate:

a) $f(x) = x(x-1)$ **b)** $f(x) = 2x(x+1)$ **c)** $f(x) = x^2\sqrt{x+1}$

2 Find $\frac{dy}{dx}$ using the product rule:

a) $y = x^2(2x-1)$ **b)** $y = 4x(2x+1)^3$ **c)** $y = x^2\sqrt{3-x}$

d) $y = \sqrt{x}(x-3)^2$ **e)** $y = 5x^2(3x^2-1)^2$ **f)** $y = \sqrt{x}(x-x^2)^3$

3 Find the gradient of the tangent to:

a) $y = x^4(1-2x)^2$ at $x = -1$ **b)** $y = \sqrt{x}(x^2-x+1)^2$ at $x = 4$

c) $y = x\sqrt{1-2x}$ at $x = -4$ **d)** $y = x^3\sqrt{5-x^2}$ at $x = 1$

4 Consider $y = \sqrt{x}(3-x)^2$.

a) Show that $\frac{dy}{dx} = \frac{(3-x)(3-5x)}{2\sqrt{x}}$.

b) Find the x-coordinates of all points on $y = \sqrt{x}(3-x)^2$

where the tangent line is parallel to x-axis

5 find $\frac{dy}{dx}$ if

a) $y = (3x^2 - 1)(4x + 8)$

b) $y = \sqrt{3x}\,(4x - 8)$

c) $y = (x^2 + 5x)(x^3+1)$

THE QUOTIENT RULE

$Quotient\ functions\ have\ from\ Q(x) = \dfrac{u(x)}{v(x)}$

$Notice\ that\ u(x) = Q(x)v(x)\ and\ by\ the\ product\ rule,$

$u'(x) = Q'(x)v(x) + Q(x)v'(x)$

$$\therefore\ u'(x) - Q(x)v'(x) = Q'(x)v(x)$$

i.e, $Q'(x)v(x) = u'(x) - \frac{u(x)}{v(x)}v'(x)$

$$\therefore\ Q'(x)v(x) = \frac{u'(x)v(x) - u(x)v'(x)}{v(x)}$$

$$\therefore Q'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]} \quad and\ this\ formula\ is\ called\ the\ quotient\ rule.$$

So, if $Q(x) = \dfrac{u(x)}{v(x)}$ then $Q'(x) = \dfrac{u'(x)\,v(x) - u(x)\,v'(x)}{[v(x)]^2}$

or if $y = \dfrac{u}{v}$ where u and v are functions of x then $\dfrac{dy}{dx} = \dfrac{\dfrac{du}{dx}v - u\dfrac{dv}{dx}}{v^2}$.

Example1

$Use\ the\ quotient\ rule\ to\ find\ \dfrac{dy}{dx}\ \ if: a.\ y = \dfrac{1+3x}{x^2+1} \quad b.\ y = \dfrac{\sqrt{x}}{(1-2x)}$

Solution

$a.\ y = \dfrac{1+3x}{x^2+1}$ is a quotient with $u = 1 + 3x and v = x^2 + 1$

$Now\ \dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2} \quad quotient\ rule$

$= \dfrac{3(x^2+1) - (1+3x)2x}{(x^2+1)^2}$

$= \dfrac{3x^2 + 3 - 2x - 6x^2}{(x^2+1)^2}$

$= \dfrac{3 - 2x - 3x^2}{(x^2+1)^2}$

$b.\ y = \dfrac{\sqrt{x}}{(1-2x)}\ is\ a\ quotient$ Where $u = x^{\frac{1}{2}}$ and $1 - 2x)^2$

$$\therefore u' = \frac{1}{2}x^{\frac{1}{2}} \; and \; v' = 2(1-2x)^2\,(-2) \quad == \; -4(1-2x)$$

$$Now, \frac{dy}{dx} = \frac{u'v - uv'}{v^2}$$

$$= \frac{\frac{1}{2}x^{-\frac{1}{2}}\;(1-2x)^2 - x^{\frac{1}{2}}\;\mathrm{x}\; -4(1-2x)}{(1-2x)^4}$$

$$= \frac{\frac{1}{2}x^{-\frac{1}{2}}\;(1-2x)^2 \; + \; 4x^{\frac{1}{2}}(1-2x)}{(1-2x)^4}$$

$$= \frac{(1-2x)\left[\frac{1-2x}{\sqrt[2]{x}}\right] + \sqrt[4]{\mathrm{x}}}{(1-2x)^4} \qquad \boldsymbol{look\ for\ common\ factors}$$

$$= \frac{1-2x+8x}{2\sqrt{x}(1-2x)^3}$$

$$= \frac{6x+1}{2\sqrt{x}(1-2x)^3}$$

Example 2

$$f(x) = \frac{x^2+2}{x-3}$$

Solution

$$f(x) = \frac{f(x)}{g(x)} \qquad then,$$

$$f'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

$$= \frac{(x-3)2x-(x^2+2).1}{(x-3)^2} = \frac{2x^2-6x-x^2-2}{(x-3)^2}$$

$$= \frac{x^2-6x-2}{(x-3)^2}$$

Example 3

$$y = (\frac{1+3x}{1-2x})^3$$

Solution

$$y = (\frac{1+3x}{1-2x})^3$$

$$= \frac{(1+3x)^3[\frac{d}{dx}(1+3x)^3] - (1+3x)^3[\frac{d}{dx}(1-2x)^3}{[(1-2x)^3]}$$

$$= \frac{(1-2x)^3.3(1+3x)^2.3 - (1+3x)^3.3(1-2x)^2.(-2)}{(1-2x)^6}$$

$$= \frac{9(1-2x)^3(1+3x)^2 + 6(1+3x)^3(1-2x)^2}{(1-2x)^6}$$

$$= \frac{3(1-2x)^2(1+3x)^2[3(1-2x)+2(1+3x)]}{(1-2x)^6}$$

$$= \frac{15(1-2x)(1+3x)^2}{(1-2x)^6}$$

Exercise

1. Use the quotient rule to find $\frac{dy}{dx}$ if:

   a. $y = \frac{1+3x}{2-x}$ **b.** $y = \frac{x^2}{2x+1}$ **c.** $y = \frac{x}{x^2-3}$

   d. $y = \frac{\sqrt{x}}{1-2x}$ **e.** $y = \frac{x^2-3}{3x-x^2}$ **f.** $y = \frac{x}{\sqrt{1-3x}}$

2. Find the slope of the tangent to:

   a. $y = \frac{x}{1-2x}$ at $x = 1$ **b.** $y = \frac{x^3}{x^2+1}$ at $x = -1$

   c. $y = \frac{\sqrt{x}}{2x+1}$ at $x = 4$ **d.** $y = \frac{x^2}{\sqrt{x^2+5}}$ at $x = -2$

THE CHAIN RULE

If $y = g(u)$ where $u = f(x)$ then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.

This rule is extremely important and enables us to differentiate complicated functions much faster.

Example 2

Differentiate the following w.r.t.x

a. $y = (2x^3 - 2)^5$ b. $y = \sqrt{2-x^2}$

Solution

a. $y=\left(2x^3-2\right)^5$,

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$$

Let $u=2x^3-2$, then $y=u^5$

$\frac{du}{dx}=6x^2$, $\frac{dy}{du}=5u^4$, since u = $2x^3$-2

then $\frac{dy}{du}=(2x^3\text{-}2)^5$

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=5\left(2x^3-2\right)^4\times6x^2$$

$$\therefore\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=30x^2\left(2x^3-2\right)^4$$

For example, for any function *f(x):*

If $y=[f(x)]^n$ then $\frac{dy}{dx}=n[f(x)]^{n-1}\times f'(x)$.

Alternative method

$$y=\left(2x^3-2\right)^5$$

$$\frac{dy}{dx}=5\left(2x^3-2\right)^{5-1}\left(6x^2\right)\therefore\frac{dy}{dx}=30x^2\left(2x^3-2\right)^4$$

b. $y=\sqrt{2-x^2}$

$$y=\sqrt{2-x^2}\Rightarrow y=\left(2-x^2\right)^{\frac{1}{2}}$$

$$\frac{dy}{dx}=\frac{1}{2}\left(2-x^2\right)^{\frac{1}{2}-1}\Rightarrow\frac{1}{2}\left(2-x^2\right)^{-\frac{1}{2}}$$

$$\therefore\frac{dy}{dx}=\frac{1}{2\sqrt{\left(2-x^2\right)}}$$

Example 3

Find$\frac{dy}{dx}$.

a) $y=(x^2+4x)^4$ **b)** $y=(3x^4-2)^3$

Solution

a) $y = (x^2 + 4x)^4$

$\frac{dy}{dx} = 4((x^2 + 4x)^3(2x + 4)$

b) $y = (3x^4 - 2)^3$

$\frac{dy}{dx} = 3((3x^4 - 2)^2(12x^3) = 36x^3(3x^4 - 2)^2$

Exercise

1 Find the slope function $\frac{dy}{dx}$ for:

a) $y = (4x - 5)^2$ b) $y = \frac{1}{5 - 2x}$ c) $y = \sqrt{3x - x^2}$

d) $y = (1 - 3x)^4$ e) $y = 6(5 - x)^3$ f) $y = \sqrt[3]{2x^3 - x^2}$

g) $y = \frac{6}{(5x - 4)^2}$ h) $y = \frac{4}{3x - x^2}$ i) $y = 2\left(x^2 - \frac{2}{x}\right)^3$

2 Find the slope of the tangent to:

a) $y = \sqrt{1 - x^2}$ at $x = \frac{1}{2}$ b) $y = (3x + 2)^6$ at $x = -1$

c) $y = \frac{1}{(2x - 1)^4}$ at $x = 1$ d) $y = 6 \times \sqrt[3]{1 - 2x}$ at $x = 0$

e) $y = \frac{4}{x + 2\sqrt{x}}$ at $x = 4$ f) $y = \left(x + \frac{1}{x}\right)^3$ at $x = 1$

3 If $y = x^3$ then $x = y^{\frac{1}{3}}$.

a) Find $\frac{dy}{dx}$ and $\frac{dx}{dy}$ and hence show that $\frac{dy}{dx} \times \frac{dx}{dy} = 1$.

b) Explain why $\frac{dy}{dx} \times \frac{dx}{dy} = 1$ whenever these derivatives exist for any general function $y = f(x)$.

4 The gradient function of $f(x) = (2x - b)^a$ is $f'(x) = 24x^2 - 24x + 6$.
Find the constants a and b.

Exercise

1 Find the slope function $\frac{dy}{dx}$ for:

a) $y = (4x - 5)^2$ b) $y = \frac{1}{5 - 2x}$ c) $y = \sqrt{3x - x^2}$

d) $y = (1 - 3x)^4$ e) $y = 6(5 - x)^3$ f) $y = \sqrt[3]{2x^3 - x^2}$

g) $y = \frac{6}{(5x - 4)^2}$ h) $y = \frac{4}{3x - x^2}$ i) $y = 2\left(x^2 - \frac{2}{x}\right)^3$

2 Find the slope of the tangent to:

a) $y = \sqrt{1 - x^2}$ at $x = \frac{1}{2}$ b) $y = (3x + 2)^6$ at $x = -1$

c) $y = \frac{1}{(2x - 1)^4}$ at $x = 1$ d) $y = 6 \times \sqrt[3]{1 - 2x}$ at $x = 0$

e) $y = \frac{4}{x + 2\sqrt{x}}$ at $x = 4$ f) $y = \left(x + \frac{1}{x}\right)^3$ at $x = 1$

3 If $y = x^3$ then $x = y^{\frac{1}{3}}$.

a) Find $\frac{dy}{dx}$ and $\frac{dx}{dy}$ and hence show that $\frac{dy}{dx} \times \frac{dx}{dy} = 1$.

b) Explain why $\frac{dy}{dx} \times \frac{dx}{dy} = 1$ whenever these derivatives exist for any general function $y = f(x)$.

4 The gradient function of $f(x) = (2x - b)^a$ is $f'(x) = 24x^2 - 24x + 6$.
Find the constants a and b.

5 Supppose $y = \frac{a}{\sqrt{1 + bx}}$ where a and b are constants.

Find a and b given that when $x = 3$, $y = 1$ and $\frac{dy}{dx} = -\frac{1}{8}$.

Derivative of trigonometric functions

$$\frac{d}{dx}(sinx) = cosx$$

$$\frac{d}{dx}(cosx) = -sinx$$

$$\frac{d}{dx}(tanx) = \frac{1}{cosx^2}$$

Proof

$$\frac{d}{dx}(tanx) = \frac{d}{dx}\left(\frac{cosx}{sinx}\right)$$

$$= \frac{[\frac{d}{dx}(sinx).cosx] - \left[\frac{d}{dx}(cosx)\right].sinx}{cosx^2}$$

$$= \frac{cosx.cosx - (-sinx).snx}{cosx^2}$$

$$= \frac{cosx^2 + sinx^2}{cosx^2} = \frac{1}{cosx^2}$$

$$\frac{d}{dx}(tanx) = \frac{1}{cosx^2}$$

Example 1

a) $f(x) = x^2 cosx$

b) $f(x) = 2cosx$

c) $f(x) = cosx\ sinx$

d) $f(x) = \frac{cosx}{sinx}$

e) $f(x) = xcosx - xsinx$

Solution

a) $f'(x) = x^2 \frac{d}{dx}(cosx) + cosx\ \frac{d}{dx}(x^2)$

$= -x^2 sinx + 2xcosx$

$= x(2cosx - xsinx)$

b)$f'(x) = -2sin2x$

c) $f'(x) = cosx\ \frac{d}{dx}(sinx) + sinx \frac{d}{dx}(cosx)$

$= cosx.cosx + sinx(-sınx)$

$= cosx^2 - sinx^2$

$$f'(x) = \frac{sinx \frac{d}{dx}(cosx) - cosx \frac{d}{dx}(sinx)}{sinx^2}$$

$$= \frac{sinx\ (-sinx) - cosx.cosx}{sinx^2}$$

$$= \frac{-(sinx^2 + cosx)^2}{sinx^2}$$

$$= \frac{-1}{sinx^2}$$

=-cscx.

e)$f'(x) = x\frac{d}{dx}(cosx) + cosx\frac{d}{dx}(x) - [x\frac{d}{dx}(sinx) + sinx\frac{d}{dx}(x)]$

$= x(sinx) + cosx(1) - [xcosx + sinx(1)]$

$= -xsinx + cosx - xcosx$

$= -x[sinx + cosx] + cosx - sinx.$

Exercise

1) Find the derivative of each function

a) $f(x) = 6sinx$

b) $f(x) = sinx^2 + 4x^2$

c) $f(x) = sin3x - 2$

d) $f(x) = \frac{sin3x}{cosx\,3x}$

e) $f(x) = \frac{cosx - 3}{x^2}$

f) $f(x) = sinx.tanx$

g) $f(x) = 4xsin4x$

h) $f(t) = \frac{tant}{cost}$

i) $f(t) = \frac{1 - sin4t^2}{cos4t}$

j) $f(t) = cost.tant$

2) Find the equation of tangent line to $y = f(x)\ \ at\ x = a$

a) $f(x) = sinx$, $a = \frac{\pi}{2}$

b) $f(x) = cosx$, $a = \frac{\pi}{2}$

c) $f(x) = tanx$, $a = 2\pi$

d) $f(x) = cosx$, $a = \frac{\pi}{6}$

e) $f(x) = sinx$, $a = \frac{\pi}{3}$

3) use the position function to find the velocity at time t=a

a) $s(t) = t^2 + cost$ $a = \frac{\pi}{2}$

b) $s(t) = sint$ $a = 0$

c) $s(t) = \frac{sint}{t}$ $a = \pi$

TANGENTS AND NORMALS

Equation of a Tangent

Consider a curve y = f(x). If A is the point with x-coordinate a, then the slope of the tangent at this point is $f'(a) = m$.

The equation of the tangent is you can use differentiation to find the equation of the tangent line to a curve at particular point since the tangent is a straight, so its equation is given by:

$$y = mx + c \text{ Or } y - y_1 = m(x - x_1)$$

Example 1

Find the equation of the tangent to the curve $y = x^2$ at the point p (3, 9)

Solution

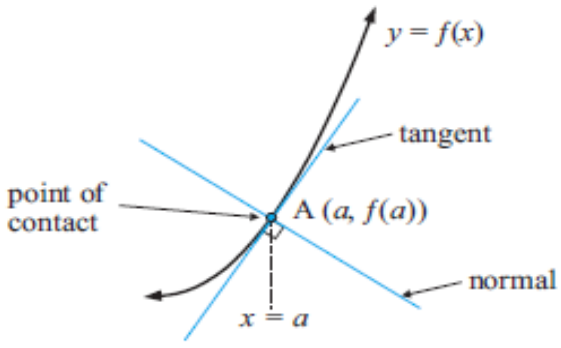
The gradient of the tangent to the curve at point p is given by $\frac{dy}{dx}$ evaluated when x = 3

$y = x^2$ so, $\frac{dy}{dx} = 2(3) = 6$

Therefore: $m = \frac{dy}{dx} = 6$.

The tangent passes through p (3, 9) Thus, 9 = 6(3) + c, C = -9

Therefore: the equation of the tangent is y = 6x-9



Alternatively:

The equation of the tangent line can be found by using $y - y_1 = m(x - x_1)$, where $m = \frac{dy}{dx} = 6$

And $y_1 = 9$, $x_1 = 3$ therefore $y - 9 = 6(x - 3)$ by simple substitution

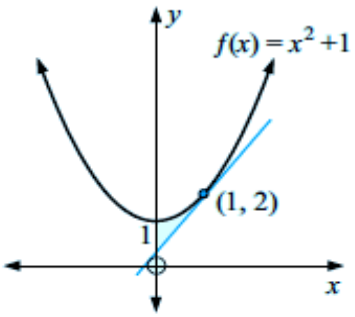
$\therefore$ The equation of the tangent line is $y = 6x - 9$

The **The Equation of the Normal line**normal to a curve at a point p is the straight line throu

Example 2

Find the equation of the tangent to $f(x) = x^2 + 1$ at the point where $x = 1$.

Solution



Since $f(1) = 1 + 1 = 2$, the point of contact is (1, 2).

Now $f'(x) = 2x \quad \therefore \quad f'(1) = 2$

$\therefore$ the tangent has equation $y - 2 = 2(x - 1)$

i.e., $y - 2 = 2x - 2$ or $y = 2x$

Normal line

The normal line at point P to the curve is a line perpendicular to the tangent at point of contact.Since tangent and normal are perpendicular to each other the product of their gradients is equal to -1

Therefore, the equation of the Normal line can be found the following formula:

y =$-\frac{1}{m}x + c$ *or* $y - y_1 = -\frac{1}{m}(x - x_1)$

When two lines are perpendicular (L_1L_2), the product of their sloops is equal to -1.

$m_1m_2 = -1$.

Example 3

Find the equation of the normal to the curve y = 3x^2+7x-2 at the point p where x = -1

Solution

When $x = -1, y = 3(-1)2 + 7(-1) - 2 = -6$

Therefore: *p has coordinates of* $(-1, -6)$

When y = 3x^2+7x-2, *then* $dy/dx = 6x + 7$ *at the point* $(-1, -6)$

$x = -1, dy/dx = 6(-1) + 7 = 1$

The gradient of the tangent line at p is 1.*Therefore, the gradient of the normal line at p is* $-1/1 = -1$. *The equation of the normal line at p is* $y = -x + c$

The normal line passes through p $(-1, -6)$, *therefore*: $= -6 = -(1) + c$

Then $C = -7$.*The equation of the Normal to the curve at point p* $(-1, -6)$ *is*:

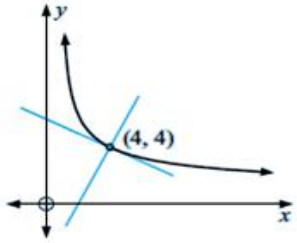
$y = -x - 7$

Example 4

Find the equation of the normal to $y = \dfrac{8}{\sqrt{x}}$ at the point where $x = 4$.

Solution

When $x = 4$, $y = \frac{8}{\sqrt{4}} = \frac{8}{2} = 4$, $\therefore$ the point of contact is $(4, 4)$.



Now as $y = 8x^{-\frac{1}{2}}$

$$\frac{dy}{dx} = -4x^{-\frac{3}{2}}$$

and when $x = 4$, $\dfrac{dy}{dx} = -4 \times 4^{-\frac{3}{2}} = -\frac{1}{2}$

$\therefore$ the normal at $(4, 4)$ has slope $\frac{2}{1}$.

So, the equation of the normal is

$$\frac{y-4}{x-4} = 2 \quad \text{i.e.,} \quad y - 4 = 2x - 8$$

$$\text{i.e.,} \quad y = 2x - 4.$$

Exercise

1 Find the equation of the tangent to:

a) $y = x - 2x^2 + 3$ at $x = 2$

b) $y = \sqrt{x} + 1$ at $x = 4$

c) $y = x^3 - 5x$ at $x = 1$

d) $y = \dfrac{4}{\sqrt{x}}$ at $(1, 4)$

e) $y = \dfrac{3}{x} - \dfrac{1}{x^2}$ at $(-1, -4)$

f) $y = 3x^2 - \dfrac{1}{x}$ at $x = -1$.

2 Find the equation of the normal to:

a) $y = x^2$ at the point $(3, 9)$

b) $y = x^3 - 5x + 2$ at $x = -2$

c) $y = \dfrac{5}{\sqrt{x}} - \sqrt{x}$ at the point $(1, 4)$

d) $y = 8\sqrt{x} - \dfrac{1}{x^2}$ at $x = 1$.

3 **a)** Find the equations of the horizontal tangents to $y = 2x^3 + 3x^2 - 12x + 1$.

b) Find the points of contact where horizontal tangents meet the curve $y = 2\sqrt{x} + \dfrac{1}{\sqrt{x}}$.

c) Find k if the tangent to $y = 2x^3 + kx^2 - 3$ at the point where $x = 2$ has gradient 4.

d) Find the equation of the other tangent to $y = 1 - 3x + 12x^2 - 8x^3$ which is parallel to the tangent at $(1, 2)$.

4 **a)** Consider the curve $y = x^2 + ax + b$ where a and b are constants. The tangent to this curve at the point where $x = 1$ is $2x + y = 6$. Find the values of a and b.

b) Consider the curve $y = a\sqrt{x} + \dfrac{b}{\sqrt{x}}$ where a and b are constants. The normal to this curve at the point where $x = 4$ is $4x + y = 22$. Find the values of a and b.

c) Show that the equation of the tangent to $y = 2x^2 - 1$ at the point where $x = a$, is $4ax - y = 2a^2 + 1$.

5 Find the equation of the tangent to:

a) $y = \sqrt{2x + 1}$ at $x = 4$

b) $y = \dfrac{1}{2 - x}$ at $x = -1$

c) $f(x) = \dfrac{x}{1 - 3x}$ at $(-1, -\frac{1}{4})$

d) $f(x) = \dfrac{x^2}{1 - x}$ at $(2, -4)$.

6 Find the equation of the normal to:

a) $y = \dfrac{1}{(x^2 + 1)^2}$ at $(1, \frac{1}{4})$

b) $y = \dfrac{1}{\sqrt{3 - 2x}}$ at $x = -3$

c) $f(x) = \sqrt{x}(1 - x)^2$ at $x = 4$

d) $f(x) = \dfrac{x^2 - 1}{2x + 3}$ at $x = -1$.

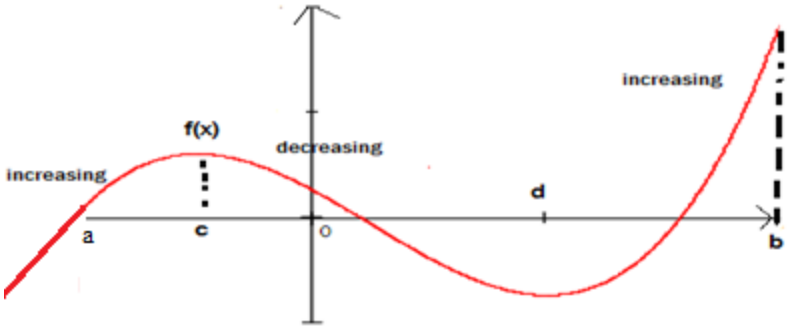
2.3 APPLICATIONS OF DERIVATIVES

Sketching the graph

if the function f(x) is continues on the interval $[a, b]$, then:

$f'(x) > 0$ *then* $f(x)$ *is increasing(rising)*

$f'(x) <$ *then* $f(x)$ *is decreasing(falling)*

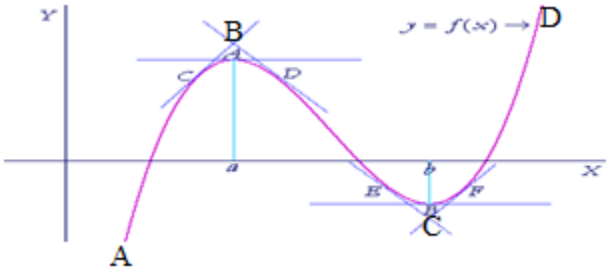


F(x) is increasing for $a \leq x, c$

F(x) is decreasing for $c < x < d$

F(x) is increasing for $d < x \leq b$

MAXIMA AND MINIMA



Consider the following graph

A is global minimum as it is the minimum value of y and occurs at an endpoint.

B is a local maximum as it is a turning point where the curve has shape and $F'(x) = 0$ at that point.

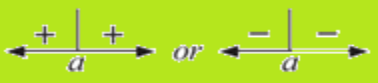
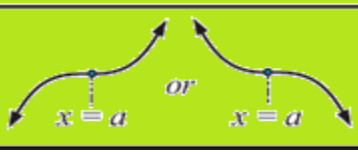


C is a local minimum as it is turning point where the curve has shape and $F'(x) = 0$ at that point.

D is a global maximum as it is the value of y and occurs at the end point of the domain.

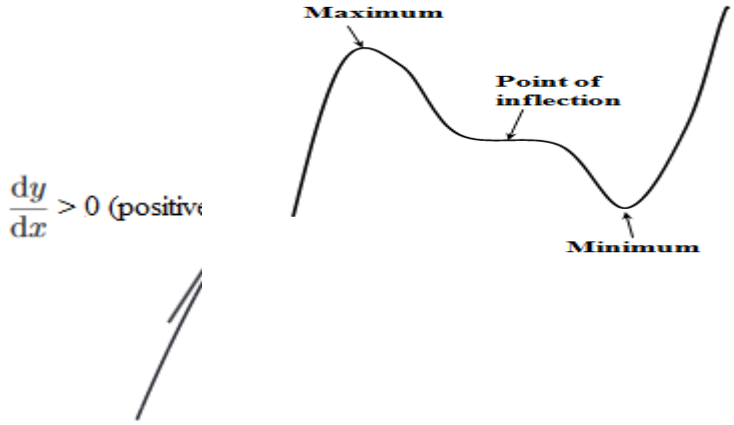
Note: For local maxima/minima , tangents at these points are horizontal and thus have a slope of 0, i.e d $F'(x) = 0$

SUMMARY

Stationary point	*Sign diagram of $f'(x)$ near $x = a$*	*Shape of curve near $x = a$*
local maximum		
local minimum		
horizontal inflection or stationary inflection		

Maximum point

At the maximum point, the gradient of the curve ispositiveat the left of the stationary point through zero at stationary point and negative at the right of the point.



Notice that:

⇒ At the right of the maximum point $\frac{dy}{dx} > 0$, and the gradient is positive.

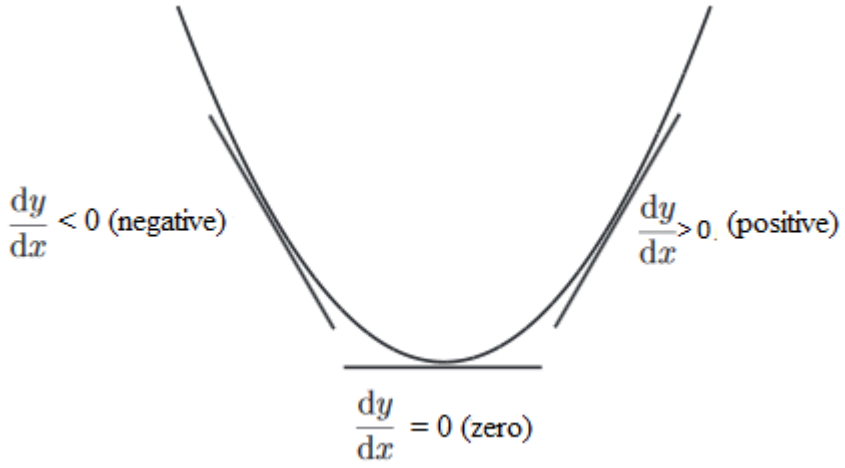
⇒ At the maximum point $\frac{dy}{dx} = 0$, and the gradient is zero.

⇒ At the right of the minimum point $\frac{dy}{dx} < 0$ and the gradient is negative.

At the maximum point, as x increases from left to right, the rate of change of the gradient with respect to x is non-positive.

Minimum point

For the minimum point, the gradient of the tangent goes from negative through zero to positive.



Notice that:

⇒ At the left of the minimum point $\frac{dy}{dx} < 0$ and the gradient is negative.

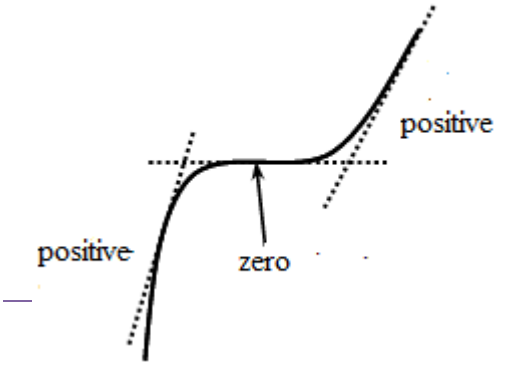
⇒ At the right of the minimum point $\frac{dy}{dx} > 0$ and the gradient is positive.

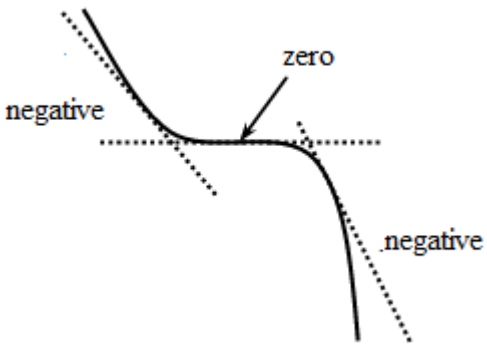
⇒ At the minimum point $\frac{dy}{dx} = 0$ and the gradient is zero.

In this case the gradient of the curve is increasing as x increases from left to right.

Points of inflexion

An inflection point *is a* point on a curve at which the sign ofcurve changes from Concave upward to concave downward or vice versa. The signs of the gradient at the point of inflection are same at both sides (left and right side) of the gradient. The two diagrams below are summarized this information.





Notice that:

⇒ At the left of the stationary point $\frac{dy}{dx}$>0 (positive)

⇒ At stationary point $\frac{dy}{dx}$ = 0

⇒ At the right of stationary point $\frac{dy}{dx}$> 0 (positive)

⇒ At the left of the stationary point $\frac{dy}{dx}$<0 (negative)

⇒ At stationary point $\frac{dy}{dx}$ = 0

⇒ At the right of stationary point $\frac{dy}{dx}$<0 (Negative

Be careful: Not all points of inflection are stationary points! The gradient of the curve must be zero at a stationary point.

In summary there are 4 types of behavior of the gradient in the stationary point.

No.	At left of stationary point	At stationary point	At right of stationary point	type
1	Positive	zero	Negative	maximum
2	Negative	zero	Positive	minimum
2	Positive	zero	Positive	inflection
3	Negative	zero	Negative	inflection

Finding the Stationary Point

You can find stationary points easily by following the following three steps

⇒ Make derivative of the given function

⇒ Put the result equal to zero

⇒ Solve the equation

Once we know the x-coordinates of a stationary point, we can find out its character by analyzing the behavior of the graph to the left and the right of the stationary point.

Example (1)

Find the coordinate of stationary point on the curve $y = x^2 - 8x+10$.

Solution

$y = x^2 - 8x+10$

$\frac{dy}{dx} = 2x-8$

$2x-8 = 0$

$2x = 8$

$X = 4$

Find y when x = 4

$y = x^2 - 8x+10$

$= 4^2 - 8(4) +10$

$= 16-32+10$

$=26-32$

$= -6$

Therefore, the coordinate of stationary point of the curve is (4, -6)

Example (2)

Find the coordinates of stationary points of the curve $y = 2x^3-3x^2-12x+5$. Find the nature of each coordinate and then sketch the curve .

Solution

y $= 2x^3-3x^2-12x+5$

$= 6x^2-6x-12$

$0 = 6x^2-6x-12$

$0 = x^2-x-2$ (by dividing each term by 6)

$(x-2)(x+1) = 0$

Thus, $x = 2$ or -1

When x = 2,

y $= 2x^3-3x^2-12x+5$

$= 2(2^3)- 3(2^2) - 12(2) + 5$

$= 16 - 12 - 24 + 5$

$= 21 -36$

$= -15$

When x = 1,

y $= 2x^3-3x^2-12x+5$

$= 2(-1^3)- 3(-1^2) - 12(-1) + 5$

$= -2 - 3 + 12 + 5$

$= -5 + 17$

$= 12$

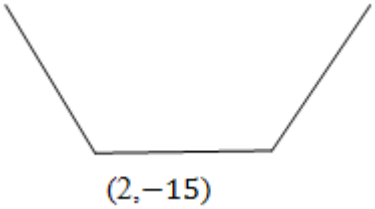
Therefore, the coordinates of stationary points are $(2, -15)$ and $(-1, 12)$.

To check the nature of the stationary points, test *x*-values around the stationary points to determine the sign of the gradient in the first derivative.

For the point $(2, -15)$

X	1	2	3
$\frac{dy}{dx}$	-12	0	5

So, the curve look like

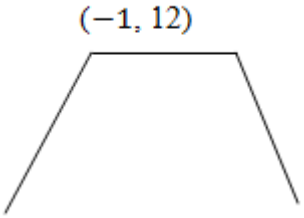


Therefore, the point $(2,-15)$ is a minimum

For the point $(-1, 12)$

X	-2	-1	0
$\frac{dy}{dx}$	24	0	-12

So, the curve is like



Therefore, the point $(-1, 12)$ is a maximum.

To illustrate these results in a diagram, plot the minimum and maximum points in a Cartesian plan.

Example (3)

Find the coordinates of stationary points for the curve $y= 2 + 3x^2–x^3$. Determine their nature and sketch the curve.

Solution

Step 1: Differentiate the function

$y= 2 + 3x^2–x^3$

$\frac{dy}{dx} = 6x-3x^2$

Step 2: put zero for the place of $\frac{dy}{dx}$

$0 = 6x-3x^2$

Step 3: solve the quadratic equation

Note $0 = 6x-3x^2$ is equal to $6x-3x^2 = 0$

$6x-3x^2 = 0$

$3x(2-x) = 0$

$3x = 0, 2 - x = 0$

$x = 0$ or 2

Find the value of y when x = 0 or 2

$y = 2 + 3x^2-x^3$

$= 2+3(0^2) -(0^3)$

$= 2$

$y = 2 + 3x^2-x^3$

$= 2+3(2^2) -(2^3)$

$= 2+12 - 8$

$= 14 - 8$

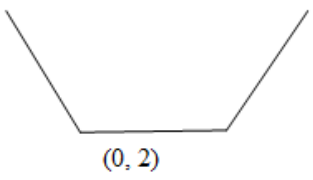
$= 6$

Therefore, the coordinates of stationary points are (0, 2) and (2, 6).

To check the nature of the stationary points, test *x*-values around the stationary points to determine the sign of the gradient in the first derivative.

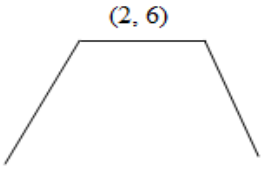
For the point (0, 2)

X	−1	0	1
$\frac{dy}{dx}$	−3	0	3

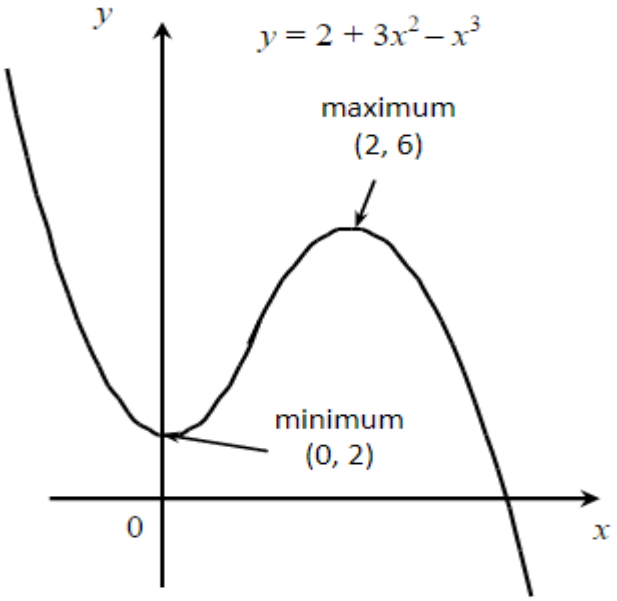


For the point (2, 6)

X	1	2	3
$\frac{dy}{dx}$	3	0	-9



To sketch the curve, use stationary values.



Stationary points and second derivative

To use the second derivative to distinguish stationary points, simply test the stationary point into the second derivative.

⇒ If the result is positive ($\frac{d^2y}{dx^2} > 0$), then the graph is concave up. Hence, the stationary point is a minimum.

⇒ If the result is negative ($\frac{d^2y}{dx^2} < 0$), then the graph is concave down. Hence the stationary point is a local maximum.

⇒ If the result is zero ($\frac{d^2y}{dx^2} = 0$), then the point is a horizontal point of inflection.

Example (4)

Find the coordinates of the stationary points for the curve $y = x^2+6x+10$. Determine the nature.

Solution

Step 1: differentiate the function

$y = x^2+6x+10$

$\frac{dy}{dx} = 2x$

Step 2: solve the equation

$0 = 2x$

$x = 0$

Find y when x = 0

$y = x^2+6x+10$

$= 2^2+6(2)+10$

$= 4+12+10$

$= 26$

Therefore, the coordinate is (0, 26)

Step 3: differentiate again

$$\frac{d^2y}{dx^2} = 2$$

Therefore:

$\frac{d^2y}{dx^2} > 2$ (minimum)

Example (5)

Find the coordinates of the stationary points for the curve $y = x^3+ 3x^2 –9x+ 5$. Determine the nature.

Solution

Step 1: differentiate the function

$y = x^3+ 3x^2 –9x+ 5$

$\frac{dy}{dx} = 3x^2 +6x – 9$

Step 2: solve the equation

$3x^2 +6x – 9 = 0$

$x^2 +2x – 3$

$(x+3)(x-1) =$

$x = -3$, or 1

Step 3: differentiate the function again

$\frac{d^2y}{dx^2}$ = 6x+6

Step 4: check the nature

$\frac{d^2y}{dx^2}$ = 6x+6

=6(-3)+6

= -18+6

= -12

$\frac{d^2y}{dx^2}$ < (maximum)

$\frac{d^2y}{dx^2}$ = 6x+6

=6(1)+6

= 6+6

= 12

$\frac{d^2y}{dx^2}$ > 0 (minimum)

Exercise

1. Find the value of x for which y has a stationary value

a) $y= 3x^2$

b) $y = x^2-6x+2$

c) $y = x^3-2x+9$

d) $y = 25x-5x^2$

e) $y = x^3-12x+8$

f) $y = x^2-10x+1$

g) $y = \frac{1}{x}$

h) $y = 6x^3+3x^2-24x+3$

2. Find the coordinates of stationary points of the following curves

a) $y = 2x^2$

b) $y = x^2 - 4x$

c) $y = x^2 - 6x + 5$

d) $y = 9x^3-25x$

e) $y = x^2 + 2x - 8$

f) $y = 16 - x^2$

g) $y = x+\frac{9}{x}$

h) $y = 6x - x^2$

i) $y = 1 - x - 2x^2$

j) $y = x^3 - 3x^2$

k) $y = x^3-3x - 5$

l) $y = 16 - x^4$

m) $y = x^3 - 3x$

n) $y = x^3 + 1$

3. Find the coordinates of the following curves and determine their nature.

a) $y = x^2-4x+1$

b) $y = x^2+6x+5$

c) $y = x^2(6-x)$

d) $y = 2x^3$

e) $y = x^3-64$

f) $y = x^3-2x^2$

4. Find the coordinates of stationary points of the following functions. Determine their nature using second derivative and sketch their graphs.

a) $f(x) = x^2+6x +1$

b) $f(x) = x^2 +8x+ 13$

c) $f(x) = x^3 - 18x+2$

d) $f(x) = 24 -x^3$

e) $f(x) = 8-4x +x^2$

f) $f(x) = 60x + 3x^2-4x^3$

5. Find the coordinates of stationary points of the following functions. Determine their nature and sketch their graphs.

a) $Y = x^2-8x$

b) $Y = x^2 -10x+ 3$

c) $Y = x2+ 2x- 7$

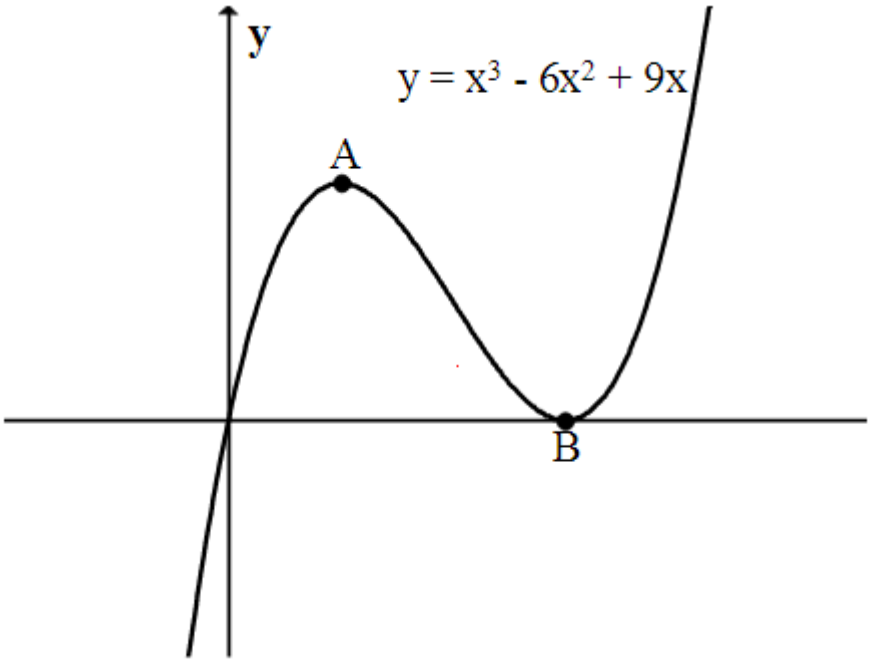
d) $Y = 25 -x^2$

Solve the following problems

1. Find the coordinates of the turning points of the curve with equation $y = x^3- 3x + 2$ and determine their nature.

2. The diagram below shows the curve with equation $y = x^3- 6x^2+ 9x$.

B has coordinates (3, 0). Find the coordinates of A.

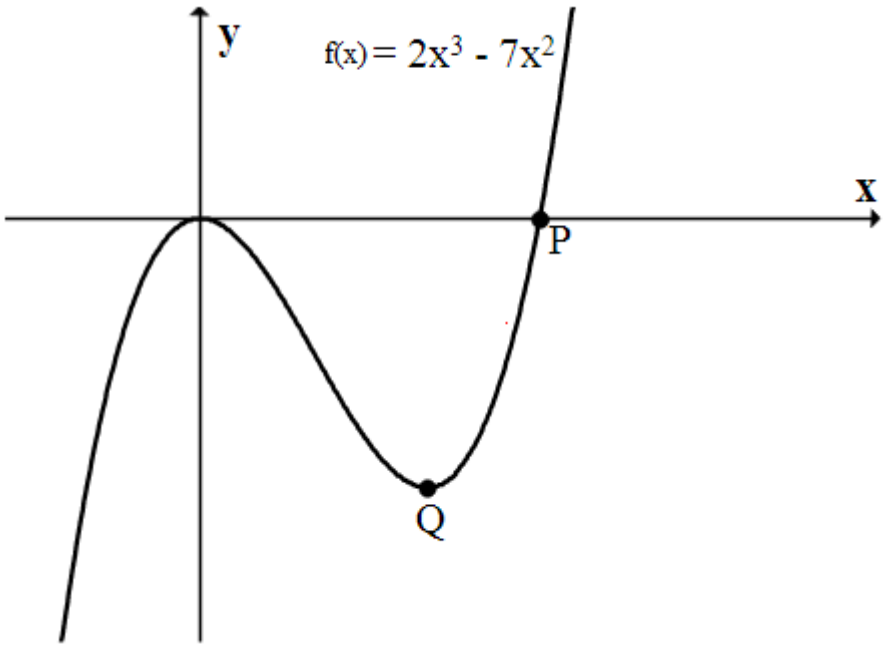


3. A curve has equation $f(x) = x^4- 32x$. Find the coordinates of the turning points of f(x) and determine their nature.

4. The graph below shows part of the curve f(x) = 2x3– 7x2.

a) Find the coordinates of P.

b) Find the x-coordinate of the minimum turning point Q.



5. A curve has equation $y = 2x^2 - x^4$.

a) Find the coordinates of the points where this curve cuts the x-axis.

b) Find the coordinates of the stationary points on this curve and determine their nature.

6. A curve has equation $y = x^3 - 6x^2 + 9x - 2$. Find the stationary points of the curve and determine the nature of each stationary point.

7. The function $f(x) = ax^3 + bx^2 + cx + d$ has turning points (0, 2), and (2, - 4). Find the constants *a*, *b*, *c*, and *d* and sketch the graph of *f* (x).

Gradient rate of change

Position, displacement, velocity and acceleration

Position is just the location of an object relative to some reference point. **Displacement,** Δx or Δy, is a change in position, not to be confused with distance. Displacements can be positive, negative, or zero. A displacement only depends on initial and final positions, not the motion in between. **Velocity** is the rate of change of displacement ($\frac{ds}{dt}$ = v). **Average velocity** is displacement divided by time, while average speed is distance divided by time. Speed is always positive or zero, but average velocity can be positive, negative, or zero. Acceleration (a) is the rate of change of velocity of a moving object. This means $\frac{dv}{dt}$ = a.

In general, $\frac{dy}{dx}$ gives the rate of change in y with respect to x.

Example 6

A particle is moving in a straight line and its distances, in metres, from a fixed point in the line after t seconds is given by the equation, $s = 12t-15t^2+4t^3$.

Find the:

a) velocity and acceleration of the particle after 3 seconds,

b) distance travelled between the two points when the velocity is instantaneously zero.

Solution

a) To get the velocity, differentiate the function and then solve.

$s = 12t-15t^2+4t^3$

$\frac{ds}{dt} = 12 -30t+12t^2$ (note $\frac{ds}{dt} = v$)

$V = 12 -30t+12t^2$

$v = 12-30(3) +12(3)^2$

$= 12 – 90 +108$

$= 120 – 90$

$= 30$

Therefore, the velocity is 30m/s.

To get the velocity, differentiate the function again and then solve.

$V = 12 -30t+12t^2$

$\frac{dv}{dt} = -30+24t$

$= -30+24(3)$

$= -30+72$

$= 42$

Therefore, the acceleration is $42m/s^2$

b) Velocity is zero when $\frac{ds}{dt} = 0$

$\frac{ds}{dt} = 12 -30t+12t^2$

$0 = 12 -30t+12t^2$

$2t^2 – 5t+2 =$

$(2t-1)(t - 2) =$

$2t = 1, t-2 =0$

$t = \frac{1}{2}$ or 2

Therefore,

$s = 12t-15t^2+4t^3$

$= 12(2) - 15(2)^2+4(2)^3$

$= -4$

$s = 12t-15t^2+4t^3$

$= 12(\frac{1}{2}) - 15(\frac{1}{2})^2 +4(\frac{1}{2})^3$

$= \frac{11}{4}$

Therefore, the distance when t = 2and when t = $\frac{1}{2}$

$S = \frac{11}{4} - (-4)$

$= \frac{11}{4} +4$

$= \frac{27}{4}$

Therefore, the distance travelled between the two points when the velocity is instantaneously zero is $\frac{27}{4}$ meter.

Example 7

If the position of a particle at time t is given by the equation below, find the velocity and acceleration of the particle at time, t = 2.

$S = t^2-11t+20$

Solution

$S = t^2-11t+20$

$\frac{ds}{dt} = 2t -11$

$V = 2t -11$

$= 2(2) -11$

$= 4 -11$

$= -7$

$V = 2t -11$

$\frac{dv}{dt} = 2$

$a = 2$

Exercise

1.Find the velocity, acceleration for the function $s = t^2+4t-5$ when t = 2 seconds.

2.The velocity of a particle moving on a line at time t is $v = t^2$. How many meters did the particle travel from t= 1 to t = 8?

3.A particle moves along the x-axis with acceleration at any time t given as $a(t) = 3t^2+4t+6$. If the particle's initial velocity is 10 and its initial position is 2, what is the position function?

4.If the position of a particle is given below, find the point at which the particle changes direction. S = t3 −12t+10.

5.A rock is tossed from a bridge 15 m above the water. The height of the rock, *h*, in metres above the water at t seconds can be modeled by the function $h(t) = -4.9t^2 + 12t + 15$.

a) Determine the instantaneous velocity at 1 s and at 2 s.

b) Determine the initial velocity of the rock.

c) When does the rock reach its maximum height? What is its maximum height?

d) When does it hit the water? With what velocity does it hit the water?

6.How far does a particle travel between zero and four seconds if its position is given by $s = t^3 - 8t^2$.

7. The position function of a cheetah moving across level ground in a straight line chasing after prey is given by the equation $s(t) = t^3 - 15t^2 + 63t$ where t is measured in seconds and s in metres.

a) What is the cheetah's velocity at 1s?

b) When the cheetah is momentarily stopped?

c) What are the positions of the cheetah in part b)?

d) When is the cheetah moving in the positive direction? When is it moving in the negative direction?

e) Find the position of the cheetah after 10 s.

f) Find the total distance traveled by the cheetah during the first 10 s.

8. A particle moves along a line. The particle's position, *s*, in metres at *t* seconds is modeled by $s(t) = 2t^3 - 15t^2 + 36t + 40$, where $t \geq 0$.

a) Determine the initial position of the particle.

b) What is the velocity at 1 s? at 5 s?

c) When is the particle stationary?

d) When is it moving away? Moving towards?

e) Determine the total distance traveled during the first five seconds.

9. A ladder which is 5 m long slips down a wall at a rate of 2 m/s. How fast, in m/s, is the base of the ladder moving away from the wall at the instant when its height above the ground is 3 metres?

10. A cone has radius 5 cm and height 15 cm. It is empty and is being filled with at a constant rate of $12\pi\, cm^3/s$. Find the rate of change of the radius, in cm/s, when the radius of the water is 2 cm.

11. A bus travels along a straight road for 600 m. It travels at a constant velocity for the whole journey, which takes 90 s. Sketch the displacement-time graph. What was the velocity of the bus?

Application of maximum and minimum

Example 8

A factory has x employees and makes a profit of P rand per week. The relationship between the profit and number of employees is expressed in the formula $P(x) = -2x^3 + 600x + 1000$, where x is the number of employees.

Solution

Step 1: find the derivative of P(x)

$P(x) = -2x^3 + 600x + 1000$

$p'(x) = -6x^2 + 600$

Step 2: solve the equation

$-6x^2 + 600 = 0$

$-6x^2 = 600$

$x^2 = 100$

$x = \pm 10$ (but, it is not possible that the factory to have a negative number of people)

So, to get the maximum profit of the factory; substitute x by 10:

$P(x) = -2x^3 + 600x + 1000$

$= -2(10^3) + 600(10) + 1000$

$= -2000 + 6000 + 1000$

$= 5000$

So, the maximum profit is 500 rand.

Example 9

A stone is dropped from the top of a tower. The distance, S metres, which it has fallen after t seconds is given by $S = 5t^2$. Calculate the average speed of the stone between t= 3 and t= 6 seconds.

Solution

$S= 5t^2$

$= 5(3^2)$

$= 45$

$S= 5(6^2)$

$= 180$

Therefore, the average speed $= \frac{s(6)-s(3)}{6-3}$

$= \frac{180-45}{6-3}$

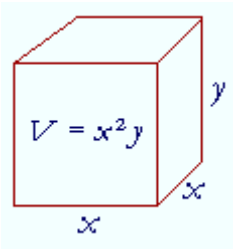
$= \frac{135}{3}$

$= 45\text{m/s}$

The average speed is 45m/s.

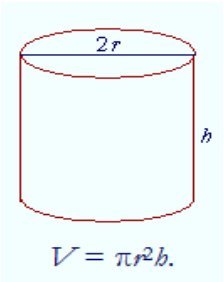
Exercise

1. A box having a square base and an open top is to contain 108 cubic feet. What should its dimensions be so that the material to make it will be a minimum? That is, what dimensions will cost the least?



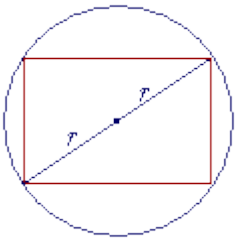
2. A farmer wishes to construct a rectangular storage bin with a volume of 1000 cubic feet. Both the top and the bottom of the bin are to be squares. Find the dimensions of the bin which will minimize its surface area.

3. A can is to be constructed in the form of a right circular cylinder. If it is to contain a given volume *V*, what dimensions will require the least amount of material? Show, in fact, that the height *h* of the can must equal its width, which is twice the radius *r*.



4. A metal can in the shape of a right circular cylinder without a top is to be made so that it holds 100 cubic centimeters. Find the dimensions of the can which minimize its surface area.

5. Find the dimensions of the rectangle of maximum area that can be inscribed in a circle of radius *r*. Show, in fact, that that area will be $2r^2$.



6. Farmer wishes to fence in a rectangular field with 600 yards of fencing. What should the dimensions of the field be in order to maximize the area of the field?

2.4 INTEGRATION

In many problems in calculus, we know the rate of change of one variable with respect to another , that is $\frac{dy}{dx}$, but if we have $\frac{dy}{dx}$ (the changes of y with respect to x) how we can get y in terms of x .The process of finding y from $\frac{dy}{dx}$ (or the function f(x) from $f'(x)$) is the reverse process of differentiation is called Integration . From our work on differential, we know if $y = 1/3x^3$, then

$dy/dx = 3/3\ x^2 = x^2$.

However, if $y = x^2 + 2$, $y = x^2 - 7$ and $y = x^2$, then the derivative all the functions $\frac{dy}{dx} = 2x$, then the integral is $x^2 + c$ and write $\int 2xdx = x^2 + c$, this reads " the integral of2 x with respect to x "

In general:

$$\int x^n d = x\ \frac{x^{n+1}}{n+1} + c, \quad \text{if } n \neq -1.$$

Example 1

a. $\int x^3 dx$ b. $\int x^2 dx$ c. $\int \frac{1}{x^2} dx$. d. $\int \sqrt{x}\, dx$

Solution.

a. $\int x^3 dx = \frac{x^4}{4} + c$

b. $\int x^2 dx = \frac{x^3}{3} + c$

c. $\int \sqrt{x dx} = \int x^{\frac{1}{2}} dx = \frac{3}{2} x^{\frac{3}{2}} + c$

d. $= \int \frac{1}{x^2} dx = -\frac{1}{x} + c$

RULES OF INTEGRATION

1. $\int c\, f(x)\, dx = c \int f(x)\, dx$, where c is a constant .

2. $\int [\, f(x) + g(x)\,]\, dx = \int f(x)\, dx + \int g(x)\, dx$

Example 2:

a. $\int 3x^2 dx$

b. $\int (\, x^3 + 4x\,) dx$

Solution:

a. $\int 3x^2 dx = 3\int x^2 dx = 3\, x^3/3 + c = x^3 + c$

b. $\int (\, x^3 + 4x\,) dx = \int (\, x^3\, dx + 4\int (\, x\,) dx$

$$= \frac{x^4}{4} + 2x^2 + c$$

Exercise 1

a $\int (x^4 - x^2 - x + 2)\, dx$ b $\int (\sqrt{x} + x\,)\, dx$ c $\int \left(3x - \frac{1}{x}\right) dx$

d $\int \left(x\sqrt{x} - \frac{2}{x}\right) dx$ e $\int \left(\frac{1}{x\sqrt{x}} + \frac{4}{x}\right) dx$ f $\int \left(\frac{1}{2}x^3 - x^4 + x^{\frac{1}{3}}\right) dx$

g $\int \left(x^2 + \frac{3}{x}\right) dx$ h $\int \left(\frac{1}{2x} + x^2 - x\right) dx$ i $\int \left(5 + \frac{1}{3}x^3 - \frac{4}{x}\right) dx$

Integral by substitution method

Some of the functions are not possible to integrate by rule $\int x^n\, d = x\, \frac{x^{n+1}}{n+1} + c$

Example

a) $\int x(x^2 + 3)^4\, dx$

$b)\ \int (x^2 + 3x)^4 (2x + 3)\, dx$

c) $\int \frac{4(x+1)}{(2x^2+4x+1)^2}\, dx$

$d)\ \int \frac{x^2}{(1+x^3)^4}\, dx$

Solution

a)

$\int x(x^2 + 3)^4\, dx$ Let $u = x^2 + 3$

$= \int u \frac{du}{2}$ du = 2xdx

$= \frac{1}{2}\int udu$ $\frac{du}{2} = xdx$

$= \frac{1}{2}\ \frac{u^2}{2} + c$

$= \frac{1}{4}(x^2 + 3) + c.$

$b) \int (x^2 + 3x - 2)^4 (2x + 3)\ dx$

lat $u = x^2 + 3x - 2$

$du = (2x + 3)dx$

$\int u^4\, du$

$\frac{u^5}{5} + c$

$\therefore\ \frac{(x^2+3x-2)^5}{5}\ dx$

$c)\ \int \frac{4(x+1)}{(2x^2+4x+1)^2}\ dx$

u: $2x^2 + 4x + 1$

$du = 4x + 4$

$\int \frac{du}{u^2} = \int u^{-2}\, du$

$= \frac{u^{-1}}{-1} + c$

$= -\frac{1}{u} + c$

$= -\frac{1}{2x^2+4x+1} + c$

$d) \int \frac{x^2}{(1+x^3)^4}\ dx$

$du = 3x^2 dx$

$\frac{du}{3} = x^2 dx$

$\int \frac{du}{\frac{3}{u^4}} = \frac{1}{3}\int u^{-4}\, du = \frac{1}{3}\ \frac{u^{-3}}{-3} + c$

$= -\frac{1}{9}u^{-3} + c$

$= -\frac{1}{9u^{-3}} + c$

$= -\frac{1}{9(1+x^3} + c$

a $\int (2x+5)^3\,dx$ b $\int \frac{1}{(3-2x)^2}\,dx$ c $\int \frac{4}{(2x-1)^4}\,dx$ d $\int (4x-3)^7\,dx$

e $\int \sqrt{3x-4}\,dx$ f $\int \frac{10}{\sqrt{1-5x}}$ g $\int 3(1-x)^4\,dx$ h $\int \frac{4}{\sqrt{3-4x}}\,dx$

i $\int 3(2x-1)^2\,dx$ j $\int (x^2-x)^2\,dx$ k $\int (1-3x)^3\,dx$

l $\int (1-x^2)^2\,dx$ m $\int 4\sqrt{5-x}\,dx$ n $\int (x^2+1)^3\,dx$

Exercise

Definite integrals

Definition: If f(x) is the integral of f(x), where f(x) is continuous on interval a ≤ x ≤ b denoted by

$\int_b^a f(x) = f(a) - f(b)$ then the integral is called definite integral .

Note: $\int_a^b f(x)dx$ reads " the integral of f(x) from x = a to x = b with respect to x ".

Example 1

Evaluate the following

$a)\ \int_1^3 (x^2+3)\,dx$ $b)\ \int_1^3 \frac{6x}{(2x^2+1)^4}\,dx$

solution

$= \frac{x^3}{3} + 3x \Big]_1^3$

$= \frac{(3)^3}{3} + 3(3) - [\frac{(1)^3}{3} + 3(1)]$

$= 9 + 9 - \frac{1}{3} - 3$

$= 15 - \frac{1}{3}$

$= \frac{44}{3}$

b) $\int_1^3 \frac{6x}{(2x^2+1)^4}\,dx$

$u = 2x^2 + 1$

$du = 4xdx$

$\frac{du}{4} = xdx$

$$\frac{6}{4}\int \frac{du}{u^4} = \int u^{-4}\, du$$

$$= \frac{6}{4}\ \frac{u^{-3}}{-3}$$

$$= -\frac{6}{12u^3}$$

$$= -\frac{1}{2(2x^2+1)^3}\Big]_0^2$$

$$= -\frac{1}{2(2(2)^2+1)^3} + \frac{1}{2(0+1)^3}$$

$$= -\frac{1}{1458} + \frac{1}{2}$$

$$= -\frac{1+729}{1458}$$

$$= \frac{728}{1458}$$

$$= -\frac{364}{729}$$

THE INTEGRAL OF TRIGONOMETRIC FUNCTION

Example: find the integral of the following:

$a)\ \int cos2xdx$

$b)\ \int sin4xdx$

$c)\ \int 4cos\frac{x}{2}dx$

$d)\ \int \sqrt{cosx}\, sinxdx$

$e)\ \int \frac{(tanx)^2}{cos^2x}dx$

Solution

$a)\ \int cos2xdx = \frac{1}{2}sin2x + c$

$b)\ \int sin4xdx = -\frac{1}{4}cos4x + c$

$c)\ 4\int cos\frac{x}{2}dx = \frac{8}{2}sin\frac{x}{2} + c$

Example

$a)\int (cosx)^3 sinxdx \qquad b)\int (sinx)^{\frac{3}{2}}cosxdx \qquad c)|\int (sin\frac{3}{2}x)^2 cos\frac{3}{2}xdx$

solution

$a)\int (cosx)^3 sinxdx \qquad let\ u = cosx \qquad -du = sinxdx$

$-\int u^3 du = -\frac{u^4}{4} + c \qquad = -\frac{1}{4}(cosx)^4 + c$

$b)\int (sinx)^{\frac{3}{2}}cosxdx$

$let\ u = sinx \qquad du = cosxdx$

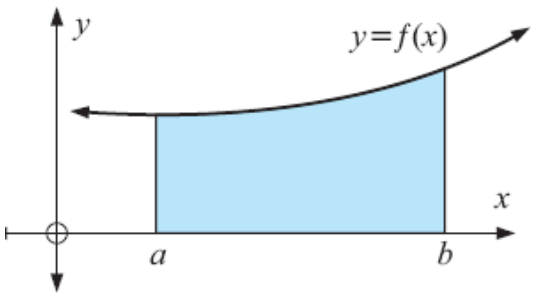
$\int u^{\frac{3}{2}}du \qquad = 2\frac{u^{\frac{5}{2}}}{5} + 5$

$= \frac{2}{5}(sinx)^{\frac{5}{2}} + c$

2.5 APPLICATION OF INTEGRAL CALCULUS.

The fundamental theorem of Calculus

The link between differential calculus and the definite integral. This link is called fundamental theorem of Calculus. We summarize " If f(x) is a continuous positive function on an interval [a, b], then the area under the curve between x = a and x = b is



The area between curve and x-axis

If f(x) is positive continuous function from [a, b], then the area bounded by f(x) = y and the x-axis is given by

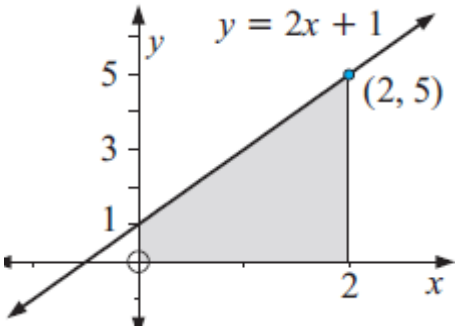
$\int_a^b f(x)dx$. or The area bound by $x = f(y)$ and $y - axis$ is given by $\int_a^b f(y)dy$

Example 1

Find the area between the line y = 2x +1 and x-axis for $0 \leq x \leq 2$

Solution

$\int_0^2 (2x + 1)dx = x^2 + 2x$

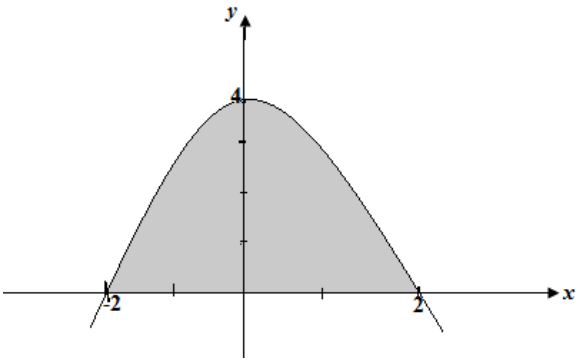


$2^2 + 2(2) - 0 = 8\ square\ unit$

Example

Find the area between the parabola $y = 4 - x^2$ and x-axis?

Solution



The points of intersection of the parabola and x-axis $\Rightarrow 4 - x^2 = 0$

$\therefore x^2 = 4 \Rightarrow x = \pm 2$

$$Area = \int_{-2}^{2} \left(4 - x^2\right) dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2}$$

$$\Rightarrow \left[4(2) - \frac{(2)^3}{3}\right] - \left[4(-2) - \frac{(-2)^3}{3}\right] \Rightarrow 8 - \frac{8}{3} - \left[-8 + \frac{8}{3}\right]$$

$$\Rightarrow 8-\frac{8}{3}+8-\frac{8}{3}=16-\frac{16}{3}=\frac{32}{3}\Rightarrow 10\frac{2}{3}\,sq..unit$$

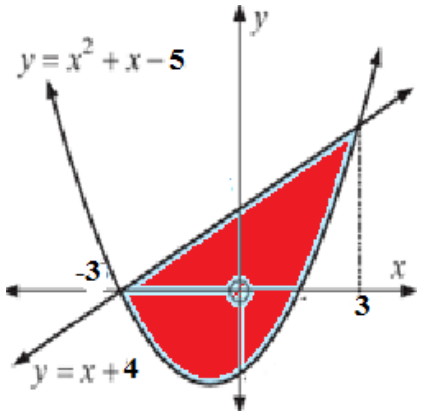
Area between two functions

If two functions f(x) and g(x) intersect at the points x = a and x = b and f(x) $\geq g(x)$ For all x in the interval [a, b] , then the area of the two functions is given by $\int_a^b [f(x)-g(x)]dx$.

Example:

Find the area of the region enclosed by the line y = x + 4 and the cuvre y = x^2 - 5

Solution



Intersection coordinate is

x +4 = x^2 + x – 5 = x + 4 - x^2 - x + 5 = 0

= x^2 - 9 = 0 =(x-3)(x+3)

= x = 3 or x = -3

$$\int_{-3}^{3}((x+4)-(x^2+x-5))dx$$

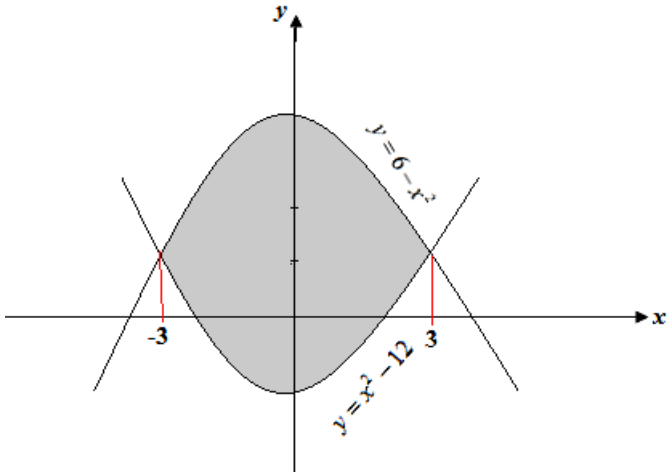
10 2/3 sq. Units .

Example 2

Find the area between the two curves $y=x^2-12$ and $y=6-x^2$

Solution

Sketching the curves



The coordinates of intersection points are

$$\left(x^2-12\right)=\left(6-x^2\right)$$

$$\Rightarrow 2x^2=18\Rightarrow x^2=9\therefore x=\pm 3$$

$$A=\int_{-3}^{3}\left[\left(6-x^2\right)-\left(x^2-12\right)\right]dx$$

$$\Rightarrow \int_{-3}^{3}\left(6-x^2-x^2+12\right)dx\Rightarrow \int_{-3}^{3}\left(18-2x^2\right)dx$$

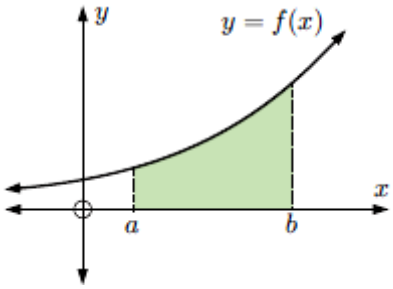
$$\Rightarrow \left[18x-\frac{2x^3}{3}\right]_{-3}^{3}=\left[18(3)-\frac{2(3)^3}{3}\right]-\left[18(-3)-\frac{2(-3)^3}{3}\right]$$

$$=54-18-\left[-54+18\right]\Rightarrow 54+54-18-18=108-36$$

$$\therefore A=72 sq.unit$$

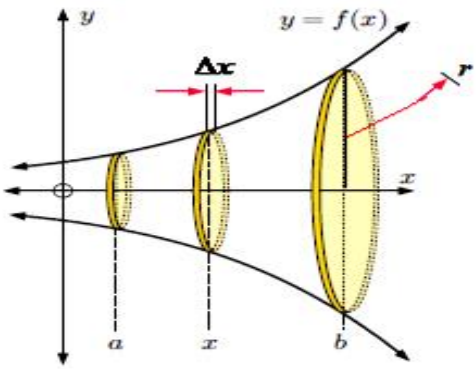
Volume of Revolution

Consider the curve $y = f(x)$ for a $a < x < b$. If the shaded part is **revolved about the** x**-axis** a 3-dimensional solid will be formed. This is called a **solid of revolution**.



We can use integration to find volumes of revolution between $x = a$ and $x = b$.

The solid can be thought to be made up of an infinite number of thin cylindrical discs.



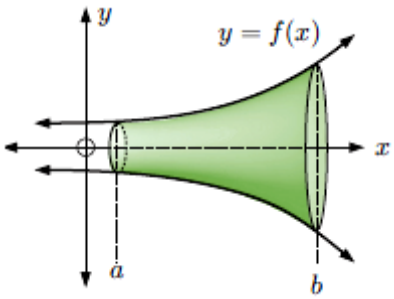
Volume = $\pi\left[f(x)\right]^2\Delta x$

Since the volume of the cylinder is $V=\pi r^2 h$

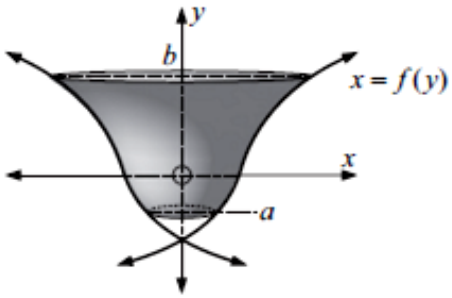
In volume of one disc is $\pi\left[f(x)\right]^2\Delta x$ there are infinite number of discs, letting $\Delta x\to 0$

We can summairize

$$\lim_{\Delta x\to 0}\sum_{x=a}^{x=b}\pi\left[f(x)\right]^2\Delta x=\int_a^b\pi\left[f(x)\right]^2\,dx=\pi\int_a^b y^2dx$$

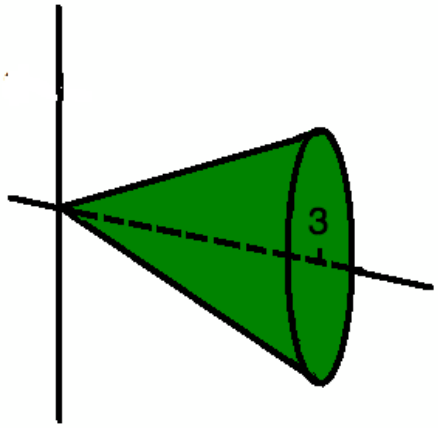


Similarly if part of the curve is rotated about the y-axis a solid of revolution will also be formed



Example 1

Find the volume of the solide formed the graph $y=\sqrt{x}\ for\ 0\le x\le 3$ is rolved about the x-axis .



Solution

$v=\pi\int y^2dx\ =\pi\int_0^3(\sqrt{x})^2\,dx$

$= \pi \int_0^3 x\, dx = \pi \frac{x^2}{2}]^3_0$

$= \frac{\pi}{2}(3)^2 - \frac{\pi}{2}(0)^2 = \frac{9}{2}\pi - 0 = \frac{9}{2}\pi$

Exercise

1. Find the volume of the solid formed when the following are revolved about the x-axis

$a)\ y = 2x\ for\ 0 \leq x \leq 2$

$b)\ y = \sqrt{x}\ for\ 0 \leq x \leq 5$

c) $y = x^2\ \ for\ \ 0 \leq x \leq 3$

$d)\ y = \sqrt{36 - x^2} for\ 0 \leq x \leq 6$

e)$y = x^3\ for\ 1 \leq x \leq 2$

CHAPTER 3: STATISTICS

Objectives: By the end of this lesson students should be able to.

- Understand why, how and where we use mean, median, and mode in discrete and continuous data.
- Draw and know how and where to use: pictograms, bar charts, pie charts, line charts, histograms with unequal intervals, frequency & cumulative frequency graphs.
- Understand and use dispersion statistics i.e. Mean deviation, standard deviation (in the normal distribution), and range, percentiles and quartiles in frequency and cumulative frequency distributions.
- Recognize that shapes are either symmetric or skewed.
- Identify different data presentations and interpret them.
- Compare related data as poor, medium and high correlation.

3.1 Measures of Central Tendency

The mean, median and mode of a data set are collectively known as measures of central tendency since these three measures focus on where the data is centered or clustered. We use statistics such as the mean, median and mode to obtain information about a population set of observed values.

The mean

The mean is useful for predicting future results when there are no extreme values in the data set. Many statistical analyses use the mean as a standard reference point. The mean income of a group of people might be of more interest to retailers and the tax office. The mean uses every value in the data and hence is a good representative of the data.

The median

The median may be more useful than the mean when there are extreme values in the data set as it is not affected by the extreme values. Use the median to describe an entire set of observations with a single value representing the center of the data. Half of the observations are above the median, half are below it. It is determined by ranking the data and finding observation number. The median is at the middle of an ordered (ranked) data set and is a useful measure for ordinal variables.

The mode

The mode is useful when the most common item, characteristic or value of a data set is required. The mode is the value that occurs most frequently in a set of observations. Minitab also displays how many data points equal the mode. The mode has applications in printing, manufacturing. For example, it is important to manufacture more of the most popular shoes; because manufacturing different shoes in equal numbers would cause a shortage of some shoes and an oversupply of others. It is the only measure of central tendency that can be used for data measured in a nominal scale.

3.2 Presentation of Data

Once data has been collected, the main tasks are the organization and presentation of the data in such a way that it becomes easily readable, understandable and interpretable. The data can be present as tables, charts, diagrams or graphs. The data in their original form (before it has been organized) is called raw data. Raw data is not very informative. Data must be turned into information through tables or charts

Tables

Tables are a standard method of presenting qualitative or categorical data, but they can also be used to summarize quantitative data. The table is a means of arranging data in rows and columns. The use of tables is pervasive throughout all aspects of academic life such as communication, research and data analysis etc. Tables are often used for reporting extensive numerical data in an organized manner.

For example:

On an exam consisting of 60 questions, 40 students received the following scores: 48, 20, 33, 23, 25, 49, 40, 27, 28, 41, 29, 47, 33, 30, 31, 44, 31, 36, 31, 36, 32, 33, 35, 41, 35, 36, 36, 37, 37, 40, 37, 40, 41, 43, 44, 40, 37, 39, 45, 37.

If you consider this exam results, it is not very informative. It needs to present in a readable and understandable way. When you present this exam results in a table as we do below, it will be very informative and understandable information.

Class Boundaries	Frequency
18 – 22	1
22 – 26	2
26– 30	4
30 – 34	7
34 – 38	11
38 – 42	8
42 – 46	4
46 – 50	3

Graphs

Graphs are a useful method to display quantitative data. It is a very common procedure to represent data in the form of a graph. There are many statistical graphs or charts, but the most common once are:

Pictogram

They are often used as part of advertising campaigns. For example: Senior students in the faculty of Education at Amoud University collected data on the number tortoises in certain field at different places and presented are presented in this frequency table.

Places	A	B	C	D	F
Frequency	200	900	600	300	100

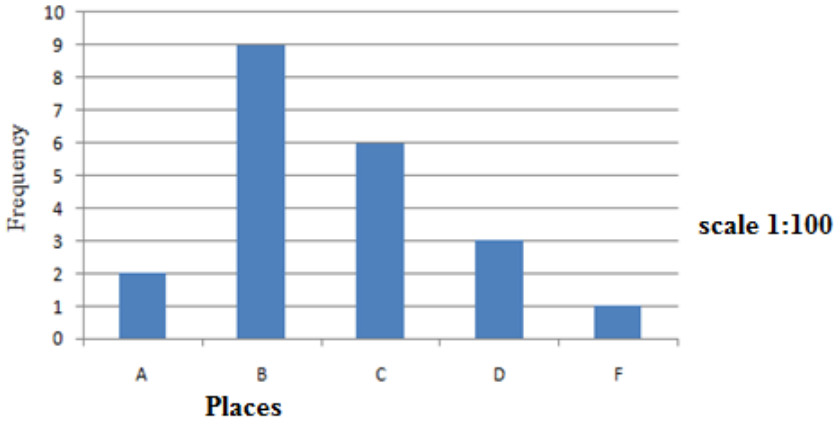
If we present the above data set on pictogram it will be like this:

Grade A	
Grade B	
Grade C	
Grade D	
Grade F	

 …..Represent 100 tortoises

Bar Charts

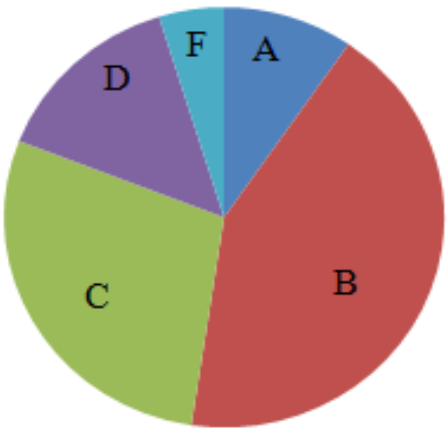
A bar chart is the most common type of diagram, and is frequently used. Bar charts display simple results clearly. They are not generally useful for large amounts of structured information. For example, if we present the information of above on bar chart, it looks like this:



Pie charts

A pie chart is a circular chart used to compare parts of the whole. It is divided into sectors that are equal in size to the quantity represented. A pie chart shows data in terms of proportions of a whole. The 'pie' is divided into segments that represent this proportion. This is done by dividing the angles at the centre. It is most useful where one or two results dominate the findings. Pie charts work best when a relatively small number of categories are used.

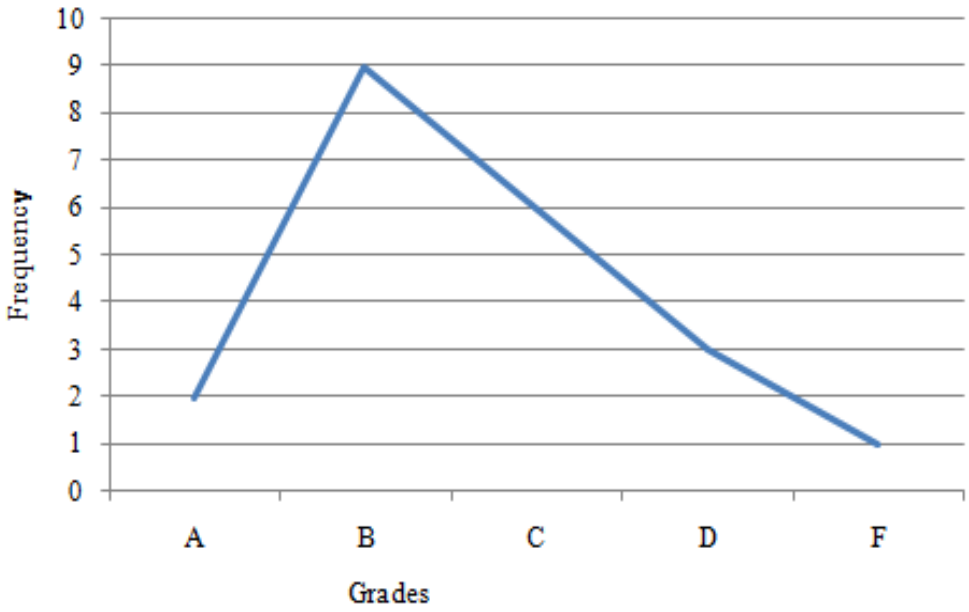
For example, if we present the information given of above on pie chart, it looks like this:



Line chart

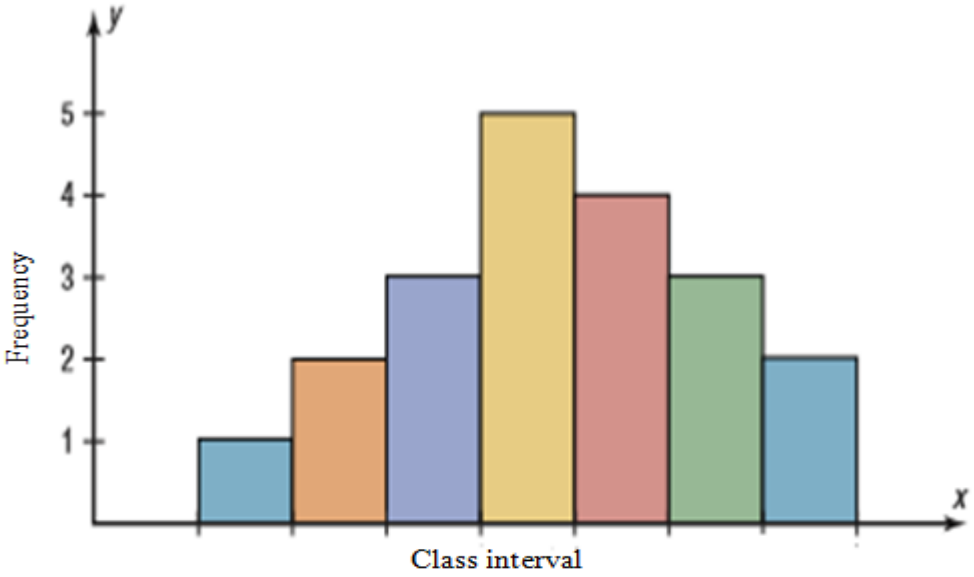
Line graphs are useful to display more than one relationship in the same picture.

For example, if we present the grades of above 21 students on line chart, it looks like this:



Histogram

A histogram looks similar to a bar chart but the bars of histogram are joined together while the bars of the bar charts are separate. For example, the graph below is histogram.



But, if you ask yourself, how this histogram is plotted, go back to chapter 3 (statistics and probability) in form 3 mathematics.

Frequency and Cumulative frequency graphs

Frequency is a number of times something happen. To construct the frequency graph, use the horizontal axis for mid-point or upper class bounder and vertical axis for frequency. Then plot points that represent the midpoint or upper class bounder and frequency of each class and connect the points in order to form a graph.

A cumulative frequency is also called ogive. An Ogive is a graph that represents cumulative frequencies of a data set. The cumulative frequency is the sum of the frequencies accumulated up to the upper boundary of a class in the distribution. To construct the cumulative frequency graph, use the horizontal axis for mid-point or upper class bounder and vertical axis for cumulative frequency. Then plot points that represent the midpoint or upper class bounder and cumulative frequency of each class and connect the points in order to form a graph.

Example

Draw for grouped data set below on:

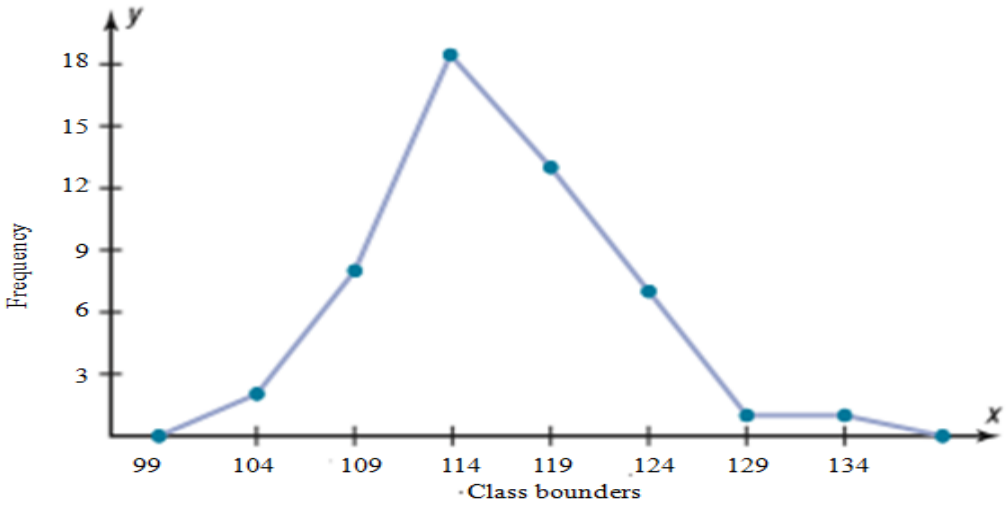
a) frequency graph

b) cumulative frequency graph

Class bounders	Frequency
99 – 104	2
104 – 109	8
109 – 114	18
114- 119	13
119 – 124	7
124 – 129	1
129 – 134	1

Solution

a) To represent this information on frequency graph, use upper class bounders and frequency directly as follows.



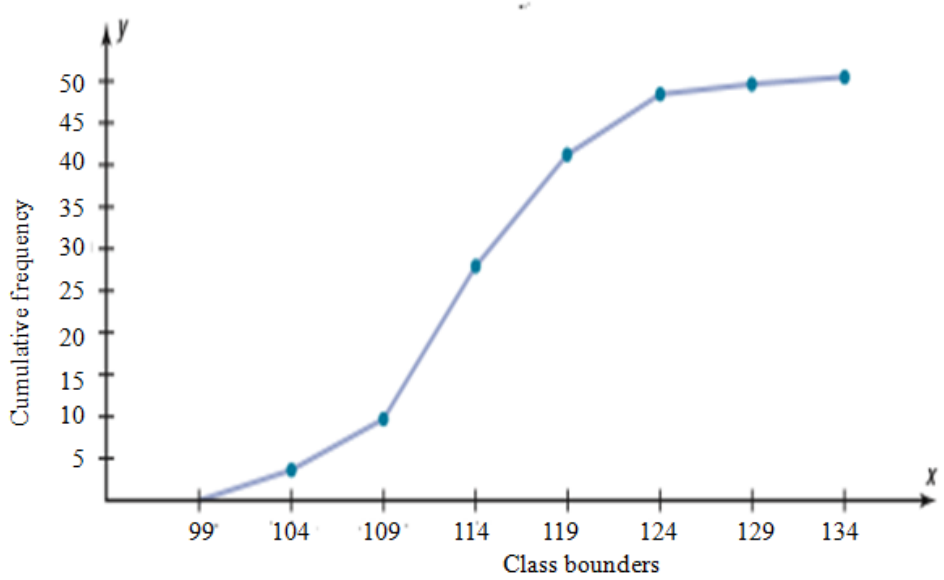
b) To represent this information on cumulative frequency graph, follow the following steps:

Step 1: Find the cumulative frequency for each class.

Step 2: connect the corresponding values of upper class bounders and cumulative frequencies.

Class bounders	Frequency	Cumulative frequency
99 – 104	2	2
104 – 109	8	10
109 – 114	18	28
114- 119	13	41
119 – 124	7	48
124 – 129	1	49
129 – 134	1	50

Thus the graph of cumulative frequency or ogive is as follows:



Exercise

1. A class of 35 students took a mathematics test. Their test marks are shown below:

   20 17 9 4 6 12 19 4 19 24 13 12 15 17 16 29 20 14 5 25 10

   11 28 17 8 14 12 18 27 23 30 23 18 8 22

   a) Complete the grouped frequency table below.

   b) Represent the information on histogram

   c) Draw frequency and cumulative frequency graph for this test scores

Test scores	Frequency
1 – 5	
6 – 10	
11 – 15	
16 – 20	
21 – 25	
26 – 30	

2. The grouped frequency table shows information about the heights of 46 students.

Heights (h) in meter	Frequency
$1.60 < h \leq 1.65$	3
$1.65 < h \leq 1.70$	9

$1.70 < h \leq 1.75$	17
$1.75 < h \leq 1.80$	10
$1.80 < h \leq 1.85$	5
$1.85 < h \leq 1.90$	2

a) Represent the above information on cumulative frequency graph.

b) Illustrate the heights of students on frequency graph

c) Draw a histogram, bar char and line chart for this data set.

3. Differentiate between the following:

a) Bar chart and histogram

b) Line graph and cumulative frequency graph

4. The total number of students in a school is 800. 571 of this total are boys while 229 of them are girls. Present this statistics on pie chart.

5. 50 students took exams in a class and exam scores are as follows:

70, 53, 43, 85, 54, 68, 55, 97, 63, 91, 60, 89, 75, 62, 87, 66, 10, 68, 69, 70, 71, 88, 81, 77, 57, 85, 72, 41, 77, 94, 78, 36, 79, 86, 53, 88, 89, 95, 96, 45, 99, 30, 66, 75, 69, 83, 39, 58, 49,78.

a) Copy and complete the table below using examination scores this 50 students

b) Present the information on:

i) Frequency graph

ii) Histogram

iii) Cumulative frequency graph

iv) Bar chart

Examination scores (s)	Frequency (f)
$30 < s \leq 40$	
$40 < s \leq 50$	

6. The table shows the number of tyres fitted for the following months by a local garage.

Month	number of tyres
May	22
June	35
July	45
August	20

Present the data set on the table above on:

a) bar chart

b) Pie chart

3.3 Measures of Dispersion

Dispersion is also called spread, variability or dispersion. It refers to how spread out a set of data around the mean or median is.

Mean Deviation

The mean deviation is the first measure of dispersion. It is the average of absolute differences between each value in a set of value, and the average of all the values of that set. The mean deviation is sometimes called the Mean Absolute Deviation (MAD) because it is the mean of the absolute deviations. The Mean Absolute Deviation (MAD) of a set of data is the average distance between each data value and the mean.

The formula of mean deviation is:

Mean deviation = $\frac{\Sigma|x-\bar{x}|}{N}$

Example (1)

Calculate the Mean Deviation of 3, 4, 6, 7, 8.

Solution

To calculate the mean deviation, you must following the following steps

- Find the mean of the data set, $\bar{x}$

- Find the difference between each data value and the mean (deviation), $x - \bar{x}$
- Find the absolute value of the deviation, $|x - \bar{x}|$
- Find the sum of the absolute value , $\Sigma|x - \bar{x}|$.
- Divide the sum by the number of terms of the data set

$\bar{X} = \frac{\Sigma x}{N} = \frac{3+4+6+7+8}{5} = \frac{28}{5} = 5.6$

X	$\bar{X}$	$X - \bar{X}$	$\lvert X - \bar{X}\rvert$
3	5.6	-2.6	2.6
4	5.6	-1.6	1.6
6	5.6	0.4	0.4
7	5.6	1.4	1.4
8	5.6	2.4	2.4

$\Sigma|x - \bar{x}| = 8.4$

Mean deviation = $\frac{\Sigma|x-\bar{x}|}{N}$

$= \frac{8.4}{5} = 1.68$

The Absolute mean deviation of grouped data is obtained as $\frac{\Sigma f|x-\bar{x}|}{\Sigma f}$ where x is the mid-point, $\bar{x}$ is the mean and f is the frequency of the data set.

Example (2)

The following data shows the mass of 40 items of Asma's shop. Calculate the mean absolute deviation of the data.

Mass (kg)	Frequency (f)
20-24	2
25-29	6
30-34	7
35-39	12
40-44	8
45-49	4
50-54	1

Solution

To calculate mean deviation of grouped data, you should follow the following steps

- Find the mid-point of the data, x
- Find the product of frequency and mid-point
- Find the deviation, $x - \bar{x}$
- Find the absolute value of the deviation, $|x - \bar{x}|$
- Find the product of frequency and abosulute deviation value, $f|x - \bar{x}|$

Mass (kg)	Frequency (f)	Mid-point (x)	Fx	$\bar{x}$	$x - \bar{x}$	$\|x - \bar{x}\|$	$f\|x - \bar{x}\|$
20-24	2	22	44	36.25	−14.25	14.25	28.5
25-29	6	27	162	36.25	−9.25	9.25	55.5
30-34	7	32	224	36.25	−4.25	4.25	29.75
35-39	12	37	444	36.25	0.75	0.75	9
40-44	8	42	336	36.25	5.75	5.75	46
45-49	4	47	188	36.25	10.75	10.75	43
50-54	1	52	52	36.25	15.75	15.75	15.75
	$\sum f = 40$		$\sum fx = 1450$				$\sum f\|x - \bar{x}\| = 227.5$

$\bar{x} = \frac{\sum x}{\sum f} = \frac{1450}{40} = 36.25$

Absolute mean deviation = $\frac{\sum f|x - \bar{x}|}{\sum f} = \frac{227.5}{40}$

Variance and standard deviation

Variance

Variance is the measures of how spread out a distribution is. In other words, it is the measures of variability. Variance is symbolized by σ^2 or S^2. The variance is computed as the average squared deviation of each number from its mean. The variance of ungrouped data is obtained by:

$$\sigma^2 = \frac{\sum(X-\bar{X})^2}{\sum N}$$

Example (1)

Find the variance of: 1, 2, 3, 4, 5.

Solution

To calculate the variance, you must follow the following steps

- Calculate mean of the given data, $\bar{x}$.
- Subtract the mean from each value in turn, $x - \bar{x}$ (deviation).
- Square each value you get from deviation, $(x - \bar{x})^2$.
- Add up all squared values, $\sum(x - \bar{x})^2$.
- Divide the total by n.

$$\bar{x} = \frac{1+2+3+4+5}{5} = \frac{15}{5} = 3$$

X	$\bar{x}$	$x - \bar{x}$	$(x - \bar{x})^2$
1	3	-2	4
2	3	-1	1
3	3	0	0
4	3	1	1
5	3	2	4
			$\sum 18$

$$\sigma^2 = \frac{\sum(X-\bar{X})^2}{\sum N} = \frac{18}{5} = 3.6$$

Standard deviation

The standard deviation is a measure that summarizes the amount by which every value within a dataset varies from the mean. Effectively it indicates how tightly the values in the dataset are bunched around the mean value. The standard deviation formula of ungrouped data is very simple: it is the square root of the variance.

$$\sigma = \sqrt{\frac{(X-\bar{X})^2}{N}}$$

Example (1)

Find the standard deviation of: 1, 2, 3, 4, 5.

Solution

$$\bar{x} = \frac{1+2+3+4+5}{5} = \frac{15}{5} = 3$$

X	$\bar{x}$	$x - \bar{x}$	$(x - \bar{x})^2$
1	3	-2	4
2	3	-1	1
3	3	0	0
4	3	1	1
5	3	2	4
			$\sum 18$

$$\sigma = \sqrt{\frac{(X-\bar{X})^2}{N}} = \sqrt{\frac{18}{5}} = = \sqrt{3.6} = = 1.90$$

Example (2)

Find the variance and standard deviation of 8, 10, 12, 14, 15, 16, 19, and 20.

Solution

$$\bar{X} = \frac{8+10+12+14+15+16+19+20}{8} = \frac{114}{8} = 14.25$$

X	$\bar{x}$	$x-\bar{x}$	$(x-\bar{x})^2$
8	14.25	−6.25	39.0625
10	14.25	−4.25	18.0625
12	14.25	−2.25	5.0625
14	14.25	−0.25	0.0625
15	14.25	0.75	0.5625
16	14.25	1.75	3.0625
19	14.25	4.75	22.5625
20	14.25	5.75	33.0625
			∑121.5

$\sigma^2 = \frac{\sum(X-\bar{X})^2}{\sum N}$ $\sigma = \sqrt{\frac{(X-\bar{X})^2}{N}}$

$= \frac{121.5}{8}$ $= \sqrt{\frac{121.5}{8}}$

= 15.1875(4d.p) $= \sqrt{15.1875}$

= 3.8971

The variance of a grouped data is obtained as $\sigma^2 = \frac{\sum f(X-\bar{X})^2}{\sum N}$ while the standard deviation of a grouped data is obtained as $\sigma = \sqrt{\frac{\sum f(X-\bar{X})^2}{\sum f}}$, where x is the mid-point, $\bar{x}$ is the mean and f is the frequency of the data set.

Example (3)

The table below shows the distance travelled to school by a class of 20 students in a certain school. Calculate the variance and standard deviation of this data set.

Solution

To calculate variance and standard deviation of the grouped data, you should follow the following steps:

- Find the mid-point of the data, x

- Find the product of frequency and mid-point
- Find the deviation of the data, $x - \bar{x}$
- Find the square of the deviation, $(x - \bar{x})^2$
- Find the product of frequency and result of squared deviation, $f(x - \bar{x})^2$.

Distance (km)	Frequency (f)	Mid-point (x)	Fx	$\bar{x}$	$x - \bar{x}$	$(x - \bar{x})^2$	$f(x - \bar{x})^2$
$0 \leq d < 1$	7	0.5	3.5	1.5		-1	1
$1 \leq d < 2$	8	1.5	12	1.5		0	0
$2 \leq d < 3$	4	2.5	10	1.5		1	1
$4 \leq d < 5$	1	4.5	4.5	1.5		3	9
	$\sum f = 20$		$\sum fx = 30$				$\sum f(x - \bar{x})^2 = 11$

$\bar{x} = \frac{\sum x}{\sum f} = \frac{30}{20} = 1.5$

$\sigma = \sqrt{\frac{\sum f(X - \bar{X})^2}{\sum f}} = \sqrt{\frac{11}{20}} = \sqrt{0.55} = 0.74$ (2 d.p)

Properties of standard deviation

When using standard deviation keep in mind the following properties.

- Standard deviation is only used to measure spread or dispersion around the mean of a data set.
- Standard deviation is never negative.
- Standard deviation is sensitive to outliers.
- A single outlier can raise the standard deviation and in turn, distort the picture of spread.
- For data with approximately the same mean, the greater the spread, the greater the standard deviation.
- If all values of a data set are the same, the standard deviation is zero because each value is equal to the mean.

Exercise

1. What does variance measure?
2. How do we compute a variance?
3. What is the difference between variance and standard deviation?
4. State whether the following statistic is a measure of center or a measure of spread.

   a) mean
   b) range
   c) variance
   d) Inter-quartile range
   e) standard deviation
   f) median

5. Calculate the Mean Deviation of :

   a) 3, 6, 6, 7, 8, 11, 15, 16.
   b) 89, 110, 100, 99, 102, 40, 75,

6. Find the standard deviation of each of the following

   a) 10, 10, 10, 10
   b) 5, 6, 7, 8, 9, 10
   c) 0, 15, 15, 30
   d) 45, 24, 30, 35, 31
   e) 13, 15, 17,19, 25
   f) 90, 40, 83, 60, 69, 100

7. Find the mean deviation, variance and standard deviation of the following set of numbers.

   a) 1,3,5,7,9
   b) 12, 15, 20, 40
   c) 8, 9, 10, 11, 13, 14, 15
   d) 2, 5, 7, 9,12, 20, 15, 9
   e) 9, 12, 10, 5, 8, 15, 9
   f) 10, 20, 30, 40, 50, 60, 70, 80, 90, 100

8. Find the mean deviation of: 92, 83, 88, 94, 91, 85, 89, 90
9. Calculate the variance and standard deviation of the: 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10.
10. Consider two more data sets: data Set x: 2, 4, 3, 4, 5, 6 and data Set y: 3, 2, 4, 10, 5, 4,12

    Calculate the mean deviation variance and standard deviation for data sets x and y.

11. The Biology examination results of 28 students are summarized in a grouped frequency table below. Calculate absolute mean deviation, variance and standard deviation.

Examination results	Frequency
0 – 4	1
4 – 8	3
8 – 12	8
12 – 16	14
16 – 20	2

12. A population consists of four observations: {1, 3, 5, 7}. What is the variance?

13. This table shows the subject selected by 40 students in an educational institution. Calculate variance and standard deviation.

Subject	Frequency
Biology	5
Mathematics	7
Information Technology	9
History	5
Physics	11
English	3

14. A student took 8 exams in a class and had scores of 84, 93, 65, 80, 48, 90, 96 and 78. Find the:

    a) Mean for her test score
    b) mean deviation for her test scores
    c) Variance for her test scores
    d) Standard deviation for her test scores

15. Calculate the mean deviation of the following data: 29, 30, 25, 48, 30, 50, 78, 99, 54, 26.

16. Calculate absolute mean deviation, variance and standard deviation of the following data set:

Interval	Frequency
$10 \leq x < 20$	2
$20 \leq x < 30$	5
$30 \leq x < 40$	10
$40 \leq x < 50$	12
$60 \leq x < 70$	9
$70 \leq x < 80$	6
$80 \leq x < 90$	4
$90 \leq x < 100$	3

17. This frequency table gives information about the ages of 54 teachers. Calculate the mean deviation, variance and standard deviation of the data set.

Age (A) in years	**Frequency**
$20 \leq A < 30$	10
$30 \leq A < 40$	22
$40 \leq A < 50$	13
$50 \leq A < 60$	7
$60 \leq A < 70$	2

18. A simple random sample consists of four observations: {1, 3, 5, 7}. Based on these sample observations, what is the best estimate of the standard deviation of the population?

19. The grouped frequency table below shows information about the weekly wages of 85 factory workers. Calculate the mean deviation, variance and standard deviation of the data set.

Weekly wage (W) in Somaliland Shillings	Frequency
$50\,000 < W \leq 60\,000$	5
$60\,000 < W \leq 70\,000$	9
$70\,000 < W \leq 80\,000$	25
$90\,000 < W \leq 100\,000$	18
$110\,000 < W \leq 120\,000$	12
$120\,000 < W \leq 130\,000$	9
$130\,000 < W \leq 140\,000$	5
$140\,000 < W \leq 150\,000$	2

Range, inter-quartile range and percentile

Range

The range of a data set is the difference between the largest and smallest data values. It is the simplest measure of variability. It is very sensitive to the smallest and largest data values. The range is simple to compute and is useful when you wish to evaluate the whole of a dataset. The range is useful for showing the spread within a dataset and for comparing the spread between similar datasets.

Example

Find the range for the data set: 38, 55, 75, 40, 58, 51, 64, 53, 69, 57, 68, 59, 63.

Solution

Range = 75 – 40 = 35

Quartiles and Inter-quartile Range (IQR)

Quartiles:

Quartiles are the values that divide a list of numbers into quarters. First put the list of numbers in order; then cut the list into four equal parts. The first quartile (lower quartile), is denoted by Q1, and is the median of the lower half of the data set (25%). The second quartile (middle quartile) is denoted by Q2, and is the median of whole data set (50%). The third quartile (upper quartile) is the median of the upper half of the data set (75%).

For *n* numbers listed in order:

- the median is the $\frac{1}{2}(n+1)$ th number,
- the lower quartile is the $\frac{1}{4}(n+1)$th number,
- The upper quartile is the $\frac{3}{4}(n+1)$th number.

Example (1)

Find Q1, Q2 and Q3 for the numbers: 9, 12, 15,20, 25, 30, 32

Solution

Lower quartile (Q1) = 12

Middle quartile (Q2), which is also the Median, = 20

Upper quartile (Q3) = 30

Example (2)

Find the lower, middle and upper quartile for these set of numbers: 3, 4, 5, 6, 7, 8, 9, 10, 11

Solution

Quartile 1 (Q1) = $\frac{4+5}{2} = \frac{9}{2} = 4.5$

Quartile 2 (Q2) = 7

Quartile 3 (Q3) = $\frac{9+10}{2} = \frac{19}{2} = 9.5$

Inter-quartile Range (IQR):

The inter-quartile range is a measure that indicates the extent to which the central 50% of values within the dataset are dispersed. It is based upon, and related to, the median. The inter-quartile range of a data set is the difference between the third quartile and the first quartile.

Inter-quartile range = upper quartile −lower quartile,

Thus, IQR = Q3 − Q1

Example (1)

Find the inter-quartile range for the date set of: 4, 6, 9, 10, 17, 19, and 21.

Solution

4, 6, 9, 10, 17, 19, 21



Q1 Q2 Q3

IQR = Q3 – Q1 = 19 – 6 = 13

Example (2)

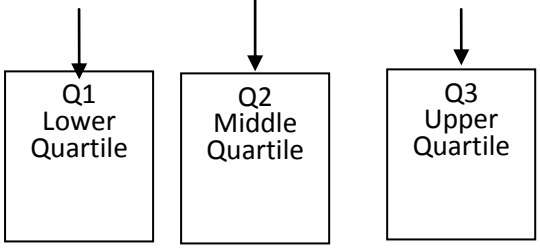
Find the lower quartile, upper quartile middle quartile and inter-quartile range for the data set below:

2, 5, 9, 11, 3, 16, 8, 1, 7, 20, 15, 12, 17, 4

Solution

Put the numbers in order from least to greatest first

1, 2, 3, 4, 5, 7, 8, 9, 11, 12, 15, 16, 17, 20



Thus,

Q1= 4

Q2 = $\frac{8+9}{2} = \frac{17}{2}$ = 8.5

Q3 = 15

IQR = Q3 – Q1

= 15– 4

= 11

Grouped data and Quartiles

To find the quartiles of grouped data, you use the following formulae:

Lower quartile (Q1) =l+ $\frac{\frac{n}{4}-cf}{f}$ × c

Middle quartile or median (Q2) = l+ $\frac{\frac{n}{2}-cf}{f}$ × c

Upper quartile (Q3) = l+ $\frac{\frac{4n}{3}-cf}{f}$ × c

Where l= lower class bounder

cf = cumulative frequency of the preceding class

f = frequency of the class

c = class size

n = total number of observation in the data set

Example (3)

The table below shows the mass of sugar sold by 50 shopkeepers in a town.

Mass (m) in kilogram	**Frequency**
$10 < m \leq 20$	3
$20 < m \leq 30$	6
$30 < m \leq 40$	7
$40 < m \leq 50$	8
$50 < m \leq 60$	10
$60 < m \leq 70$	6
$70 < m \leq 80$	5
$80 < m \leq 90$	3
$90 < m \leq 30$	2

Calculate:

a) Lower quartile of the data set

b) Middle quartile of the data set

c) Upper quartile of the data set

Solution

To calculate lower, middle and upper quartiles of the data set:

- Cumulative frequency of the data set
- Identify the class interval within which the class required (Q1, Q2, and Q3) located.

Mass (m) in kilogram	**Frequency (f)**	**Cumulative frequency (cf)**	
$10 < m \leq 20$	3	3	
$20 < m \leq 30$	6	9	
$30 < m \leq 40$	7	16	← Q1
$40 < m \leq 50$	8	24	
$50 < m \leq 60$	10	34	← Q2
$60 < m \leq 70$	6	40	← Q3
$70 < m \leq 80$	5	45	
$80 < m \leq 90$	3	48	
$90 < m \leq 30$	2	50	

a) Given: l = 30, cf = 9, f = 7 and c = 10

Lower quartile (Q1) =$l+\frac{\frac{n}{4}-cf}{f}\times c$

$= 30+\frac{\frac{50}{4}-9}{7}\times 10$

$= 30+\frac{\mathbf{12.5-9}}{\mathbf{7}}\times 10$

$= 30+\frac{3.5}{7}\times 10$

$= 30+\mathbf{5}$

$= 35$

b) Given: l = 50, cf = 24, f = 10 and c = 10

Middle quartile or median (Q2) = $l+\frac{\frac{n}{2}-cf}{f}\times c$

$= 50+\frac{\frac{50}{2}-24}{10}\times 10$

$= 50+\frac{\mathbf{25-24}}{\mathbf{10}}\times 10$

$= 50+\frac{1}{7}\times 10$ $= 50+ 1.43$

$= 51.43$

c) Given: l = 600, cf = 34, f = 6 and c = 10

d) Upper quartile (Q3) = $l+\frac{\frac{4n}{3}-cf}{f}\times c$

$= 60+\frac{\frac{3(50)}{4}-34}{6}\times 10$

$= 60+\frac{\mathbf{37.5-34}}{\mathbf{6}}\times 10$

$= 60+\frac{3.5}{6}\times 10 = 60+ 5.8\overline{3}$

$= 65.8\overline{3}$

Percentiles

Percentiles are like quartiles, except that percentiles divide the set of data into 100 equal parts while quartiles divide the set of data into 4 equal parts. Percentiles are often used in education and health-related fields to indicate how one individual compares with others in a group. Percentiles measure position from the bottom. Percentiles are most often used for determining the relative standing of an individual in a population or the rank position of the individual. A percentile provides information about how the data are spread over the interval from the smallest value to the largest value.

To calculate the k^{th} percentile of the data set where k is any number between zero and one hundred, do the following steps:

1. Order all the values in the data set from smallest to largest.
2. Multiply k percent by the total number of values, n.
3. If the index obtained in Step 2 is not a whole number, round it up to the nearest whole number and Count the values in your data set from left to right (from the smallest to the largest value) until you reach the number you obtained.
4. If the index obtained in Step 2 is a whole number, Count the values in your data set from left to right until you reach the number indicated by Step 2. The k^{th} percentile is the average of that corresponding value in your data set and the value that directly follows it.

Example (1)

A 25 students took exams in a class and exam scores is order from lowest to highest as follows: 43, 54, 55, 60, 62, 66, 68, 68, 69, 70, 71, 72, 77, 78, 79, 86, 87, 88, 89, 94, 95, 96, 97, 99, 99. Find the:

a) 90th percentile for these scores.

b) 20th percentile for these scores.

Solutions

a) 90^{th} percentile = 90% off 25

$$= \frac{90}{100} \times 25 = 22.5 \cong 23$$

Counting from left to right, you go until you find the 23 value in the data set. That value is 97, and it's the 90th percentile for this data set.

b) 20^{th} percentile = 20% off 25

$$=\frac{20}{100} \times 25 = 5$$

This is a whole number; the 20th percentile is the average of the 5th and 6th values in the ordered data set (62 and 66).

Therefore the 20th percentile comes to $\frac{62+66}{2} = 64$.

Percentile Ranking

A percentile rank is the percentage of scores that fall at or below a given score. The formula of percentile ranking of a score, x, out of a set of n scores, where x is included is: $\frac{B+0.5E}{n} \times 100$ where B is the number of scores below x and E is the number of scores equal to x and n is the number of scores.

Example (1)

A student took 8 exams in a class and had scores of 84, 93, 65, 80, 48, 90, 96 and 78. Find the percentile ranking 78 for the 8 test scores.

Solution

Order all the values in the data set from smallest to largest: 48, 65, 78, 80, 84, 90, 93, 96

Percentile ranking= $\frac{B+0.5E}{n} \times 100$

$$= \frac{2+0.5(1)}{8} \times 100$$

$$= \frac{2+0.5}{8} \times 100$$

$$= \frac{2.5}{8} \times 100 \ = \frac{250}{8}$$

$$=31.25$$

The score of 78 is at the 31.25th percentile for this test.

Example (2)

The math test scores were: 50, 65, 70, 72, 72, 78, 80, 82, 84, 84, 85, 86, 88, 88, 90, 94, 96, 98, 98, 99.

Find the percentile rank for a score of 82 and 88on this test.

Solution

Percentile ranking of 82 = $\frac{B+0.5E}{n} \times 100$

$= \frac{7+0.5(1)}{20} \times 100$

$= \frac{7+0.5}{20} \times 100$

$= \frac{7.5}{20} \times 100$

$= \frac{750}{20}$

$=37.5$

The score of 82 is at the 37.5th percentile for this test.

Percentile ranking 88= $\frac{B+0.5E}{n} \times 100$

$= \frac{12+0.5(2)}{20} \times 100$

$= \frac{12+1}{20} \times 100$

$= \frac{13}{20} \times 100$

$= \frac{1300}{20}$

$=65$

The score of 88 is at the 65th percentile for this test.

Exercise

1. Find the range each of the following numbers.

a) 1, 2, 3, 4, 5, 6, 7

b) 7, 10, 14, 17

c) 11, 5, 8, 3, 8

d) 12, 7, 9, 13, 17, 5, 14, 8

2. Find the inter-quartile range of:

a) 2, 5, 7, 8, 9

b) 10, 15, 9, 27, 3, 5, 18.

c) 12, 10, 6, 8, 13, 19, 20, 16

d) 12, 14, 2, 11, 10, 13, 2, 1, 8, 3

3. Find the range and inter-quartile range in each of the following:

a) 6, 8, 9, 10, 20

b) 7, 8, 3, 7, 12, 19, 10, 3, 20

c) 5, 8, 12, 14, 1, 0, 19, 6, 17, 18

d) 10, 25, 30, 40, 23, 8, 11, 34, 55

4. For the scores 3, 5, 7, 9, 12, 21, 25, 30, calculate the 25th percentile

5. The history test scores were: 45, 55, 58 60, 65, 66, 68, 70, 70, 73, 77, 78, 81, 81, 81, 86, 86, 89, 90, 92, 93, 95, 96, 96, 97, 98, 99. Find the percentile rank for a score of 66 and 81 on this test.

6. The marks of 30 students geography are given below:
   46, 79, 66, 66, 68, 47, 45, 44, 48, 58, 92, 79, 70, 81, 68, 97, 96, 85, 79, 71, 32, 63, 66, 79, 88, 88, 94, 77, 59, 80. Find the:

   a) Percentile rank for 58, 79 and 94 for exam scores.

   b) 45th percentile and 95th percentile

7. The Final Exam test scores were: 62, 66, 71, 75, 75, 78, 81, 83, 84, 85, 85, 87, 89, 89, 91, 92, 93, 94, 95, 99. Find the:

   a) Percentile rank for a score of 85 on this test.

   b) 25th and 91th percentile

8. Muna recorded the heights of 12 girls in her class in centimeters. She put the heights in order.
   144, 150, 152, 160, 162, 162, 167,167, 170, 172, 177, 181. Find:

   a) The range.

   b) The lower quartile.

   c) The upper quartile.

d) The inter-quartile range

9. The heights of students in inches in Block 3 math class are 55, 59, 59, 60, 61, 63, 64, 64, 65, 68, 68, 69, 72, 74. Find the percentile rank for a height of 61 inches.

10. The following data are exam grades of 10 students in a math class.

Exam score interval	Frequency
69-76	1
77-84	4
85-92	3
93-100	2

a) Which interval contains the first (lower) quartile?

b) Which interval contains the third (upper) quartile?

c) If students who received at least an 85% on this exam received a "math star" pencil, what percent of the students received a pencil?

11. Consider grouped date below.

Interval	Frequency
20-24	1
25-29	2
30-34	4
35-39	3
40-44	2
45-49	5
50-54	7
55-59	3
60-64	2
65-69	1

a) Calculate the 25th percentile for the exam scores

b) Find the 50th percentile for the exam scores.

c) Determine the 70th percentile for the exam scores.

12. The teacher recorded the times some girls to do a 50 questions exam paper.

Time (t) in Minutes	Frequency
$30 < t \leq 40$	1
$40 < t \leq 50$	3
$50 < t \leq 60$	7
$60 < t \leq 70$	10
$70 < t \leq 80$	6
$80 < t \leq 90$	2

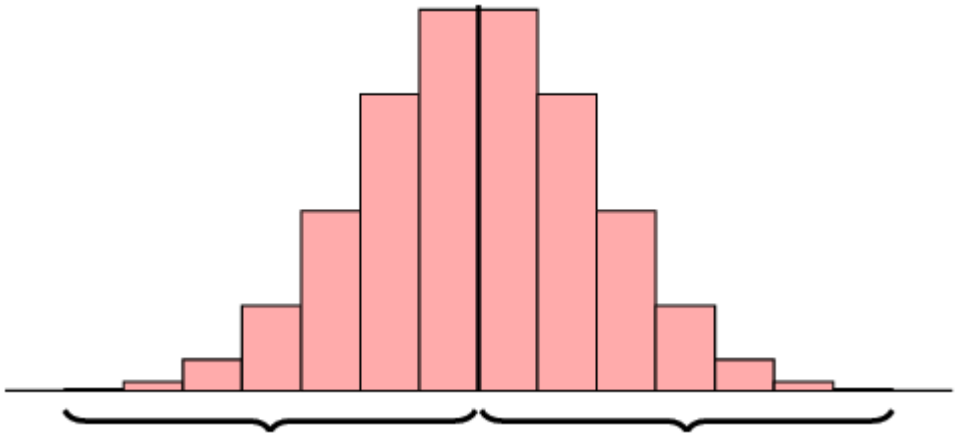
a) Calculate the 25th percentile for the exam scores
b) Find the 50th percentile for the exam scores.
c) Determine the 70th percentile for the exam scores.

3.4 Distributions and Shapes

The data sets can be classifying into three basic categories that describes the shape of the data in the distribution: symmetric, left skewed, right skewed. These classifications can be used for any data set,

Symmetric Distributions

A symmetric distribution is also known as normal, or bell shape. A symmetric distribution is one where the left and right hand sides of the distribution are roughly equally balanced around the mean. The histogram below shows a typical symmetric distribution



The mean of the distribution is almost same as the median of the distribution. This means, the right and left tails of the distribution are balanced.

In general, for symmetric distributions, the mean is approximately equal to the median. The tails of the distribution are the parts to the left and to the right, away from the mean. The tail is the part where the counts in the histogram become smaller. For a symmetric distribution, the left and right tails are equally balanced, meaning that they have about the same length.

The figure below shows the box and whisker diagram for a typical symmetric data set.



The median is in the middle of first and third quartiles.

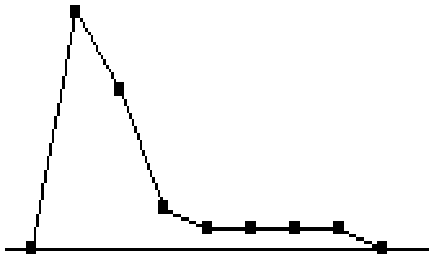
Another major property of a symmetric distribution is that its median (second quartile) lies in the middle of its first and third quartiles.

Skewed Distribution

Skewed distributions are asymmetrical and have data that clusters toward one end either to the left or to the right. This means, a distribution is skewed if one of its tails is longer than the other.

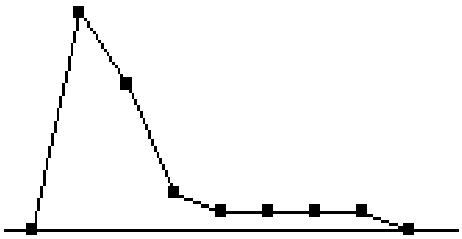
There are two major types of skewness, positive and negative skewness.

Positive skew



This distribution is positive skewed. This means that the distribution has a long tail in the positive direction of the distribution.

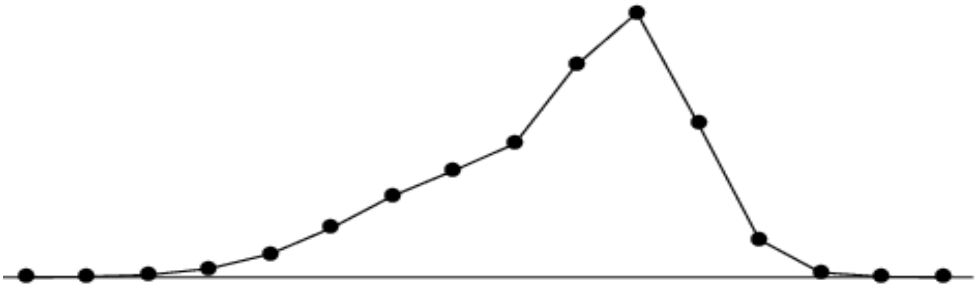
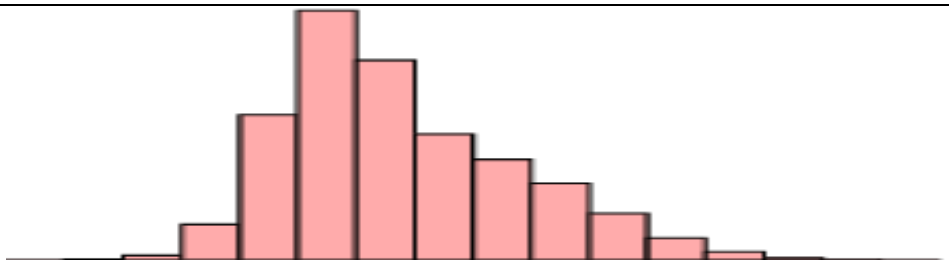
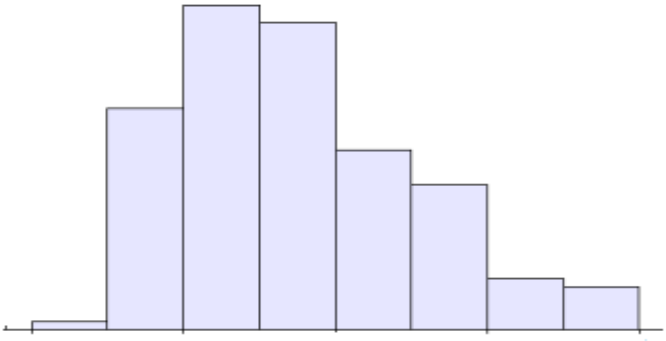
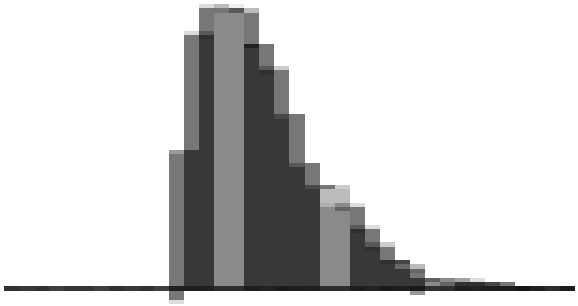
Negative skew

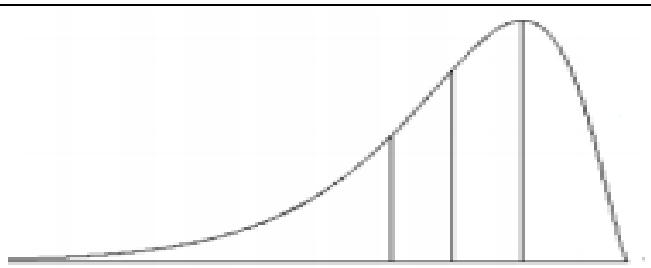
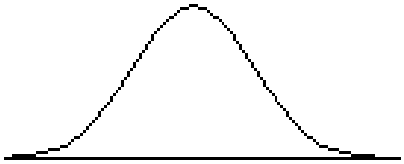
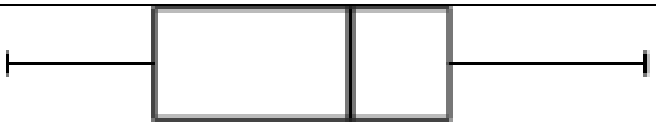
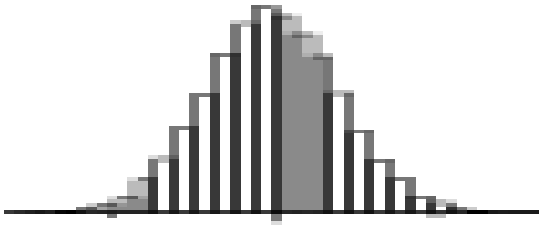
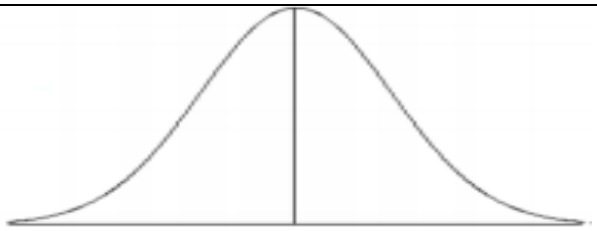
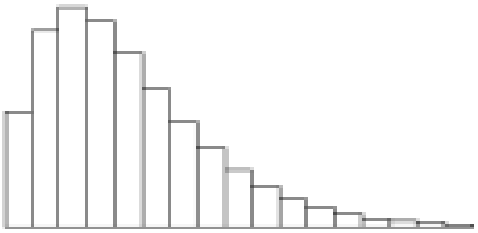


This distribution has a negative skew since it has a long tail in the negative direction of the distribution.

Exercise

State whether each of the following distributions is symmetric, skewed right or skewed left

No	Distribution
1	
2	
3	
4	

5	
6	
7	
8	
9	
10	
11	

3.5 Interpretation of Data

Data interpretation is part of daily life for most people. Interpretation is the process of making sense of numerical data that has been collected, analyzed, and presented. For example, what does it mean that 25% of Somalilanders rated the textbook a 5 and 75% rated it a 4. Interpretation is the process of attaching meaning to the data. Data interpretation usually need to basic steps. First, you have to read tables, charts or graphs in order to obtain certain information. Secondly, you have to manipulate the information in order to get an answer.

Example (1)

Consider the Table below. Suppose this is information collecting in certain city in Somaliland.

Year	Number of births
2012	62
2013	67
2014	71
2015	83

Here, you see the year as the category and the number of births as the numerical count.

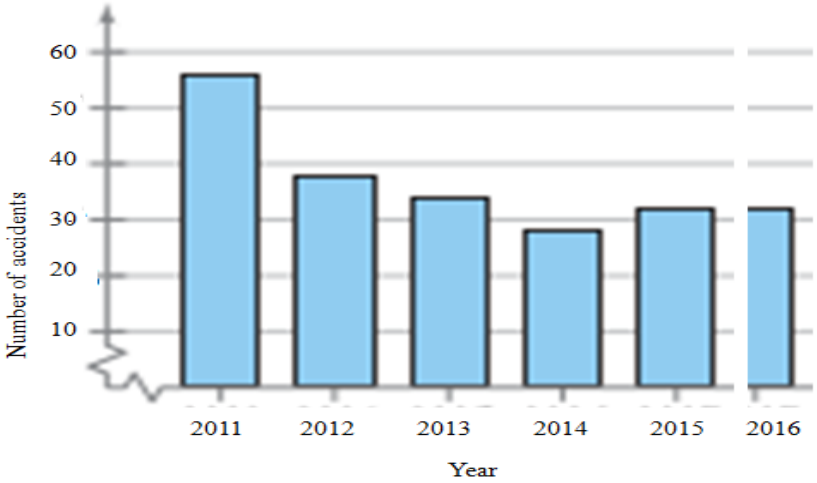
What should be added to this table to provide the reader with more information?

Solution

This table showed the number of birth of the city in specific years. Also the table showed that the number of birth increase year after year in that city.

Example (2)

Look at graph below. Suppose this is information reported in capital city of Somaliland.



You see the year as the category and the number of accidents as the numerical count.

What should be added to this chart to provide the reader with more information?

Solution

This bar chart concerned the accidents occur in Somaliland between 2011 and 2016. Additionally, this graph showed that the most accidents happened in 2011 and least accidents happen in 2014.

Exercise

1. Table below shows information taken from 25 students in a class. Interpret the information in the table.

Class	Frequency	Percent
A	5	20
B	7	28
O	9	36
AB	4	16

2. The national examination in Somaliland was set by 7850 students nearly in 2015; the average results are summarized in the table below. Interpret the examination results.

Subject of the examination	**Average grade of each subject in percentage.**
Islamic	95
Biology	82
Mathematics	76
Physics	79
Arabic	69
Chemistry	75
English	83

3. Sixty-two workers were asked how they travelled to work. Their results are shown in the table below. Interpret the information in the table.

Way they travel	Frequency
Walk	23
Car	11
Train	3
Bicycle	10
Bus	15

4. Wajale's average monthly rainfall, in millimeters, in the first six months of the 2014 is shown in the table.

Month	January	February	March	April	May
Average monthly rainfall (mm)	30	23	54	47	36

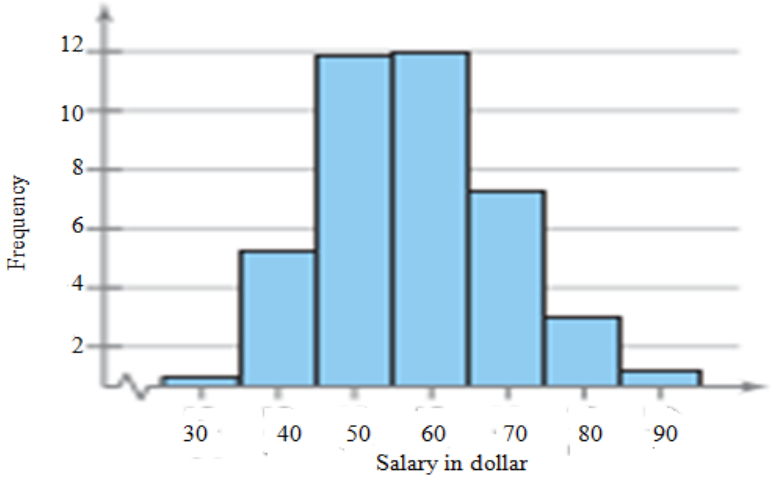
5. Here are the numbers of days of 41 workers were absent from work in a year.

Interval of absents	**Number of days of absents**
0 – 4	5
5 – 9	17
10 – 14	9
15 – 19	3
20 – 24	4
25 – 29	2
30 – 34	1

6. The numbers of televisions in 100 houses are shown in the table. Interpret information in the table.

Number of televisions	Number of houses
0	5
2	35
3	39
More than 3	24

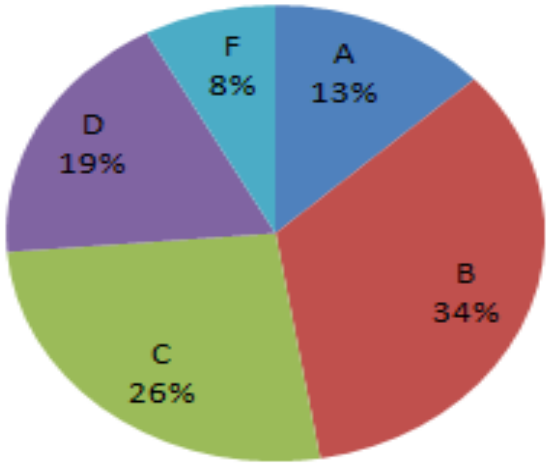
7. This graph showed the employee salary in a certain company. Interpret information in the graph.



8. The table showed the lengths of phone calls made by a salesperson in one week. Interpret the data set in the table.

Length (L) of calls in minutes	Frequency
$1< L \leq 10$	8
$11< L \leq 20$	11
$21< L \leq 30$	15
$31< L \leq 40$	31
$41< L \leq 50$	9
$51< L \leq 60$	3

9. The pie chart below demonstrates the grades of group of students in a academic institution. Interpret the information in the chart.



3.7 Correlation

Correlation is the relationship between two variables. Moreover, correlation is a statistical measure that indicates the extent to which two or more variables fluctuate together. Correlation is a statistical technique that can show whether and how strongly pairs of variables are related. For example, height and weight are related. When two sets of data are strongly linked together we say they have a High Correlation and vice verse.3

Types of Correlation
There are many different types of correlation. The most common once are:

1. **A positive correlation:** the positive correlation indicates the extent to which those variables increase or decrease in parallel. This means, an increase in one variable increases the value in another and vice verse. For example: temperature with ice cream sales. As temperature goes up, on average ice cream sales go up.
2. **A negative correlation:** the negative correlation indicates the extent to which one variable increases as the other decreases. In this case, an increase in one variable decreases the value of another. For example: hours spent watching TV with grades in class. The more hours a student spends watching TV per week, typically the lower his/her average grade.
3. **Zero Correlation**: If the change in values of one set doesn't affect the values of the other, then the variables are said to have no correlation or zero correlation. In this case, the variables are not related to each other and there is no apparent relationship between the variables. For example, the race and colour of the person has nothing to do with Allas' reward.

CHAPTER 4: BINOMIAL AND PROBABILITY

Objectives: By the end of this chapter students should be able to

- Use the concept of fundamental counting principles
- Differentiate permutation and combination
- Solve problems about permutations and combinations
- Use the Binomial Theorem and Pascal's triangle in expansions e.g. $(a \pm b)^n$ where n is a positive integer.
- Use of Combinations in the Binomial Theorem.
- Work out problems on binomial probability
- Work probability tree and associated application

4.1 INTRODUCTION

In Form Three, you have studied basic concepts of probability. In this section you will revise some definitions before we proceed to the next section.

- ✓ An Experiment is an activity (measurement or observation) that generates results.
- ✓ An Outcome (Sample point) is any result obtained in an experiment.
- ✓ A Sample Space (S) is a set that contains all possible outcomes of an experiment.
- ✓ An Event is any subset of a sample space.

Example 1:

When a "fair" coin is tossed, the possible results are either head (H) or tail (T). Consider an experiment of tossing a fair coin twice.

a) What are the possible outcomes?

b) Give the sample space.

c) Give the event of H appearing on the second throw.

Give the event of at least one T appearing.

Solution:

a) HH, HT, TH, TT

b) S = {HH, HT, TH, TT}

c) A = {HH, TH}

d) B = {HT, TH, TT}

> **Note:**
> In tossing a coin, if the coin is fair, the two possible outcomes have an equal chance of occurring. In this case, we say that the outcomes are equally likely.

Probability of an event (E)

If an event E can happen in *m* ways out of *n* equally likely possibilities, the probability of the occurrence of an event E is given by

$$P(E) = \frac{\text{number of favourable outcomes}}{\text{total number of possible outcomes}} = \frac{n(E)}{n(S)} = \frac{m}{n}$$

Example 2:

A box contains 4 red and 5 black balls. If one ball is drawn at random, what is the probability of getting a

a) Red ball?

b) Black ball?

Solution

Let event R = a red ball appears and event B = a black ball appears. Then,

a) P(R) = $\frac{n(R)}{n(S)} = \frac{4}{9}$

b) P(B) = $\frac{n(B)}{n(S)} = \frac{5}{9}$

Example 3:

If a number is to be selected at random from the integers 1to10, what is the probability that the number is

a) Odd?

b) Divisible by 3?

Solution

S = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

a) Odd is the event E = {1, 3, 5, 7, 9} = P(odd) = $\frac{number\ of\ odds}{total\ number} = \frac{5}{10} = \frac{1}{2}$

b) Divisible by 3 is the event E = {3, 6, 9} = P(divisible by 3) = $\frac{3}{10}$

Example 4

The numbers 1 to 20 are each written on a card. The 20 cards are mixed together. One card is chosen at random from the pack. Find the probability that the number on the card is:

a) Even

b) A factor of 24

c) Prime

d) A composite number

Solution

a) Required outcomes are 2,4,6,8,10,12,14,16,18,20.

∴P(even) = $\frac{10}{20} = {}^{1}/_{2}$

b) Favorable outcomes are 1,2,3,4,6,8,12,24.

P(factor of 24) = ${}^{8}/_{20}$=2/5.

c) Favorable outcomes are 2,3,5,7,11,13,17,19

∴P (Prime) = ${}^{8}/_{20} = {}^{2}/_{5}$

d) Composite numbers are 4,6,8,9,10,12,14,15,16,18,20

∴P(composite) = $\frac{11}{20}$

Exercises 4.1.

6. A letter is chosen at random from the alphabet. Find the probability that it is:
   a) F
   b) F or T
   c) the letters of the word 'frequency'
   d) Not one of the letters of the word" table''.
7. If 250 tickets are sold for a raffle, what is the probability of winning when you buy:
   a) 1 ticket
   b) 5 tickets
   c) 250 tickets
   d) 0 tickets
8. A dice is rolled. Find the probability that the score is:
   a) A composite number
   b) a score of 5
   c) Less than 6
   d) greater than 2
9. Tiles, each lettered with one different letter of the alphabet are put into a bag. If one tile is drawn out at random, calculate the probability that it is:
   a) An A or a P
   b) A vowel
   c) A consonant
   d) An X, a Y or a Z
   e) A letter in your first name
10. A playing card is drawn from a standard deck of 52 cards. What is the probability that the card is:
   a) a king?
   b) a heart?
   c) the king of hearts?
   d) either a king or a heart?
   e) black?
   f) honor?
   g) black honor?
   h) either black or an honor?
   i) honor or picture card?
   j) a red card?
   k) a queen?
   l) a number 6 card?
   m) a face card?
   n) a black card or a face card?
   o) a red card or number 6 card?
11. In a bag are 100 discs numbered from 1 to 100. A disc is selected at random from the bag. Find the probability that the number on the selected disc is:
   a) even
   b) a multiple of 5
   c) a multiple of 10
   d) a multiple of 5 or even
   e) multiple of 3 or 12

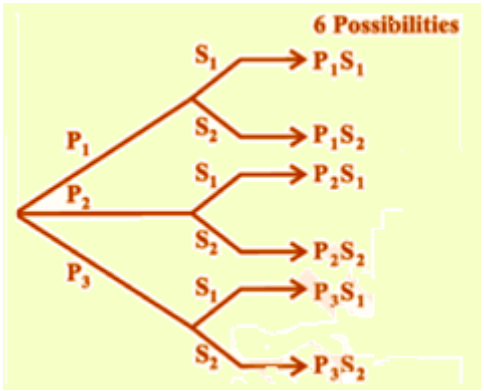
4.2 Permutation and Combination

Introduction

Suppose you have a suitcase with a number lock. The number lock has 4 wheels each labeled with 10 digits from 0 to 9. The lock can be opened if 4 specific digits are arranged in a particular sequence with no repetition. Somehow, you have forgotten this specific sequence of digits. You remember only the first digit which is 7. In order to open the lock, how many sequences of 3-digits you may have to check with? To answer this question, you may, immediately, start listing all possible arrangements of 9 remaining digits taken 3 at a time. But, this method will be tedious, because the number of possible sequences may be large. Here, in this Chapter, we shall learn this question, you may, immediately, start listing all possible arrangements of 9 remaining digits taken 3 at a time. But, this method will be tedious, because the number of possible sequences may be large. Here, in this Chapter, we shall learn some basic counting techniques which will enable us to answer this question without actually listing 3-digit arrangements. In fact, these techniques will be useful in determining the number of different ways of arranging and selecting objects without actually listing them. As a first step, we shall examine a principle which is most fundamental to the learning of these techniques.

Fundamental Principle of Counting

Let us consider the following problem. Amiin has 3 pants (trousers) and 2 shirts. How many different pairs of a pant and a shirt, can he dress up with? There are 3 ways in which a pant can be chosen, because there are 3 pants available. Similarly, a shirt can be chosen in 2 ways. For every choice of a pant, there are 2 choices of a shirt. Therefore, there are $3 \times 2 = 6$ pairs of a pant and a shirt. Let us name the three pants as P1, P2, P3 and the two shirts as S1, S2. Then these six possibilities can be illustrated in the Figure right



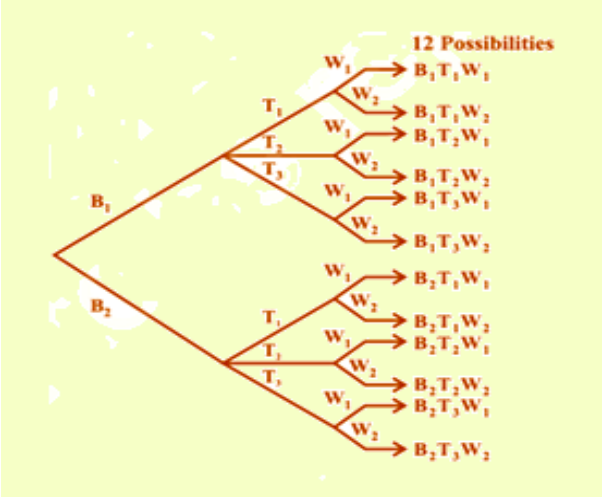
Let us consider another problem

Let us consider another problem

Of the same type Sareedo has 2 school bags, 3 tiffin boxes and 2 water bottles. In how many ways

can she carry these items (choosing one each). A school bag can be chosen in 2 different ways. After a school bag is chosen, a tiffin box can be chosen in 3 different ways. Hence, there are 2 × 3 = 6 pairs of school bag and a tiffi box. For each of these pairs a water bottle can be chosen in 2 different ways Hence, there are 6 × 2 = 12 different ways in which, Sareedo can carry these items to school. If we name the 2 school bags as B1, B2, the three tiffin boxes as T1, T2, T3 and

the two water bottles as W1, W2, these possibilities can be illustrated in the Figure below



In fact, the problems of the above types are solved by applying the following principle known as the fundamental principle of counting, or, simply, the multiplication principle, which states that

*"*If an event can occur in m different ways, following which another event can occur in n different ways, then the total number of occurrence of the events in the given order is m×n*."*

The above principle can be generalised for any finite number of events. For example, for 3 events, the principle is as follows: 'If an event can occur in *m* different ways, following which another event can occur in *n* different ways, following which a third event can occur in *p* different ways, then the total number of occurrence to 'the events in the given order is *m* × *n* × *p*." In the first problem, the required number of ways of wearing a pant and a shirt was the number of different ways of the occurrence of the following events in succession:

(i) the event of choosing a pant

(ii) the event of choosing a shirt.

In the second problem, the required number of ways was the number of different ways of the occurrence of the following events in succession:

(i) the event of choosing a school bag

(ii) the event of choosing a tiffin box

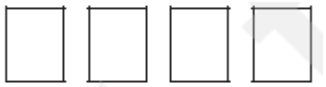
(iii) the event of choosing a water bottle.

Here, in both the cases, the events in each problem could occur in various possible orders. But, we have to choose any one of the possible orders and count the number of different ways of the occurence of the events in this chosen order.

Example 1

Find the number of 4 letter words, with or without meaning, which can be formed out of the letters of the word ROSE, where the repetition of the letters is not allowed.

Solution

There are as many words as there are ways of filling in 4 vacant places  by the 4 letters, keeping in mind that the repetition is not allowed. The first place can be filled in 4 different ways by anyone of the 4 letters R, O, S, E. Following which, the second place can be filled in by anyone of the remaining 3 letters in 3 different ways, following which the third place can be filled in 2 different ways; following which, the fourth place can be filled in 1 way. Thus, the number of ways in which the 4 places can be filled, by the multiplication principle, is $4 \times 3 \times 2 \times 1 = 24$. Hence, the required number of words is 24.

If the repetition of the letters was allowed, how many words can be formed?

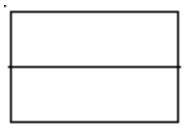
One can easily understand that each of the 4 vacant places can be filled in succession in 4 different ways. Hence, the required number of words = $4 \times 4 \times 4 \times 4 = 256$

Example2

Given 4 flags of different colours, how many different signals can be generated, if a signal requires the use of 2 flags one below the other?

Solution

There will be as many signals as there are ways of filling in 2 vacant places



in succession by the 4 flags of different colours. The upper vacant place can be filled in 4 different ways by anyone of the 4 flags; following which, the lower vacant place can be filled in 3 different ways by anyone of the remaining 3 different flags. Hence, by the multiplication principle, the required number of signals = $4 \times 3 = 12$.

Example 3

How many 2 digit even numbers can be formed from the digits 1, 2, 3, 4, 5 if the digits can be repeated?

Solution

There will be as many ways as there are ways of filling 2 vacant places



in succession by the five given digits. Here, in this case, we start filling in unit's place, because the options for this place are 2 and 4 only and this can be done in 2 ways; following which the ten's place can be filled by any of the 5 digits in 5 different ways as the digits can be repeated. Therefore, by the multiplication principle, the required number of two digits even numbers is 2 × 5, i.e., 10.

Example 4

Find the number of different signals that can be generated by arranging at least 2 flags in order (one below the other) on a vertical staff, if five different flags are available.

Solution

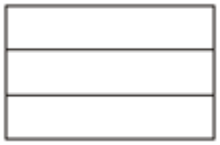
A signal can consist of either 2 flags, 3 flags, 4 flags or 5 flags. Now, let us count the possible number of signals consisting of 2 flags, 3 flags, 4 flags and 5 flags separately and then add the respective numbers.

There will be as many 2 flag signals as there are ways of filling in 2 vacant places



in succession by the 5 flags available. By Multiplication rule, the number of ways is 5 × 4 = 20.

Similarly, there will be as many 3 flag signals as there are ways of filling in 3 vacant places



in succession by the 5 flags. The number of ways is 5 × 4 × 3 = 60.

Continuing the same way, we find that The number of 4 flag signals = 5 × 4 × 3 × 2 = 120 and the number of 5 flag signals = 5 × 4 × 3 × 2 × 1 = 120 Therefore, the required no of signals = 20 + 60 + 120 + 120 = 320.

Exercise 4.2.1. .

1) How many 3-digit numbers can be formed from the digits 1, 2, 3, 4 and 5 assuming that

i. repetition of the digits is allowed?

ii. repetition of the digits is not allowed?

2) How many 3-digit even numbers can be formed from the digits 1, 2, 3, 4, 5, 6 if the digits can be repeated?

3) How many 4-letter code can be formed using the first 10 letters of the English alphabet, if no letter can be repeated?

4) How many 5-digit telephone numbers can be constructed using the digits 0 to 9 if each number starts with 67 and no digit appears more than once?

5) A coin is tossed 3 times and the outcomes are recorded. How many possible outcomes are there?

6) Given 5 flags of different colours, how many different signals can be generated if each signal requires the use of 2 flags, one below the other?

Permutations

In example 1 of the previous, we are actually counting the different possible arrangements of the letters such as rose, reos, ..., etc. Here, in this list, each arrangement is different from other. In other words, the order of writing the letters is important. Each arrangement is called a permutation of 4 different letters taken all at a time. Now, if we have to determine the number of 3-letter words, with or without meaning, which can be formed out of the letters of the word number, where the repetition of the letters is not allowed, we need to count the arrangements NUM, NMU, MUN, NUB, ..., etc. Here, we are counting the permutations of 6 different letters taken 3 at a time. The required number of words = $6 \times 5 \times 4 = 120$ (by using multiplication principle).

If the repetition of the letters was allowed, the required number of words would be $6 \times 6 \times 6 = 216$.

Definition A permutation is an arrangement in a definite order of a number of objects taken some or all at a time. In the following sub Section, we shall obtain the formula needed to answer these questions immediately.

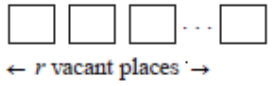
Permutations when all the objects are distinct

Theorem

The number of permutations of n different objects taken r at a time, where $0 < r \leq n$ and the objects do not repeat is $n(n-1)(n-2)\ldots(n-r+1)$, which is denoted by ${}^{n}P_{r}$.

Proof

There will be as many permutations as there are ways of filling in r vacant

Places 

by the n objects. The first place can be filled in n ways; following which, the second place can be filled in $(n-1)$ ways, following which the third place can be filled in $(n-2)$ ways,..., the rth place can be filled in $(n-(r-1))$ ways. Therefore, the number of ways of filling in r vacant places in succession is $n(n-1)(n-2)\ldots(n-(r-1))$ or $n(n-1)(n-2)\ldots(n-r+1)$

This expression for ${}^nP_r$ is cumbersome and we need a notation which will help to reduce the size of this expression. The symbol $n!$ (read as factorial n or n factorial) comes to our rescue. In the following text we will learn what actually $n!$ means.

Factorial notation

The notation $n!$ represents the product of first n natural numbers, i.e., the product $1 \times 2 \times 3 \times \ldots \times (n-1) \times n$ is denoted as $n!$. We read this symbol as 'n factorial'.

Thus, $1 \times 2 \times 3 \times 4 \ldots \times (n-1) \times n = n!$

$1 = 1!$

$1 \times 2 = 2!$

$1 \times 2 \times 3 = 3!$

$1 \times 2 \times 3 \times 4 = 4!$ and so on.

We define $0! = 1$

We can write $5! = 5 \times 4! = 5 \times 4 \times 3! = 5 \times 4 \times 3 \times 2!$

$= 5 \times 4 \times 3 \times 2 \times 1!$

Clearly, for a natural number n

$n! = n(n-1)!$

$= n(n-1)(n-2)!$ [provided $(n \geq 2)$]

$= n(n-1)(n-2)(n-3)!$ [Provided $(n \geq 3)$] and so on.

Example 5

Evaluate

(i) $5!$ (ii) $7!$ (iii) $7! - 5!$

Solution

(i) $5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$

(ii) $7! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 = 5040$ and

(iii) $7! - 5! = 5040 - 120 = 4920$.

Example 6

Compute

(i) $\frac{7!}{5!}$

(ii) $\frac{12!}{(10!)(2!)}$

Solution

(i) $\frac{7!}{5!} = \frac{7\times 6\times 5!}{5!} = 7\times 6 = 42$

(ii) $\frac{12!}{(10!)(2!)} = \frac{12\times 11\times 10!}{(10!)\times 2\times 1} = 6\times 11 = 66$

Example 7

Evaluate $\frac{n!}{r!(n-r)!}$, when *n*== *5, r= 2.*

Solution

we have to evaluate $\frac{5!}{2!(5-2)!}$(since *n*== *5, r= 2)*

We have $\frac{5!}{2!(5-2)!} = \frac{5!}{2!\times 3!} = \frac{4\times 5}{2} = 10$

Example 8

If $\frac{1}{8!} + \frac{1}{9\times 8!} = \frac{x}{10\times 9\times 8!}$

Solution

We have $\frac{1}{8!} + \frac{1}{9\times 8!} = \frac{x}{10\times 9\times 8!}$

Therefore $1+\frac{1}{9} = \frac{x}{10\times 9}$ or $\frac{10}{9} = \frac{x}{10\times 9}$

so x = 100.

Exercise 4.2.2.

1. Evaluate each of the following
   a) 8!
   b) 4!-3!
   c) $\frac{10!}{7!}$

d) $\frac{5!\times 12!}{6!\times 9!}$

2. Is 3! +4! = 7!?

3. If $\frac{1}{6!}+\frac{1}{7!}=\frac{x}{8!}$, find x

4. Evaluate $\frac{n!}{(n-r)!}$

a) $n=6, r=2$ b) $n=8, r=5$ c) $n=10, r=2$

Derivation of the formula for ${}^nP_r$

${}^nP_r=\frac{n!}{(n-r)!},\ 0\le r\le n$

Let us now go back to the stage where we had determined the following formula:

${}^nP_r=n(n-1)\ (n-2)\ \ldots\ (n-r+1)$

Multiplying numerator and denominator by $(n-r)\ (n-r-1)\ \ldots\ 3\times 2\times 1$, we get

${}^nP_r=\frac{n(n-1)\ (n-2)\ \ldots\ (n-r+1)(n-r)(n-r-1)\ldots 3\times 2\times 1}{(n-r)(n-r-1)\ldots 3\times 2\times 1}=\frac{n!}{(n-r)!}$

Thus ${}^nP_r=\frac{n!}{(n-r)!}$, where $0< r\le n$

This is a much more convenient expression for ${}^nP_r$ than the previous one.

In particular, when $r=n,\ {}^nP_r=\frac{n!}{0!}=n!$

Counting permutations is merely counting the number of ways in which some or all objects at a time are rearranged. Arranging no object at all is the same as leaving behind all the objects and we know that there is only one way of doing so. Thus, we can have

${}^nP_0=1=\frac{n!}{n!}=n!$ ...(1)

Therefore, the formula (1) is applicable for $r = 0$ also.

Thus ${}^nP_r=\frac{n!}{(n-r)!}$, $0\le r\le n$.

Theorem 2

The number of permutations of *n* different objects taken *r* at a time, where repetition is allowed, is n^r.

Proof is very similar to that of Theorem 1 and is left for the reader to arrive at.

Here, we are solving some of the problems of the pervious Section using the for ${}^nP_r$ to illustrate its usefulness.

In Example 1, the required number of words = ${}^4P_4 = 4! = 24$. Here repetition is not allowed. If repetition is allowed, the required number of words would be $4^4 = 256$.

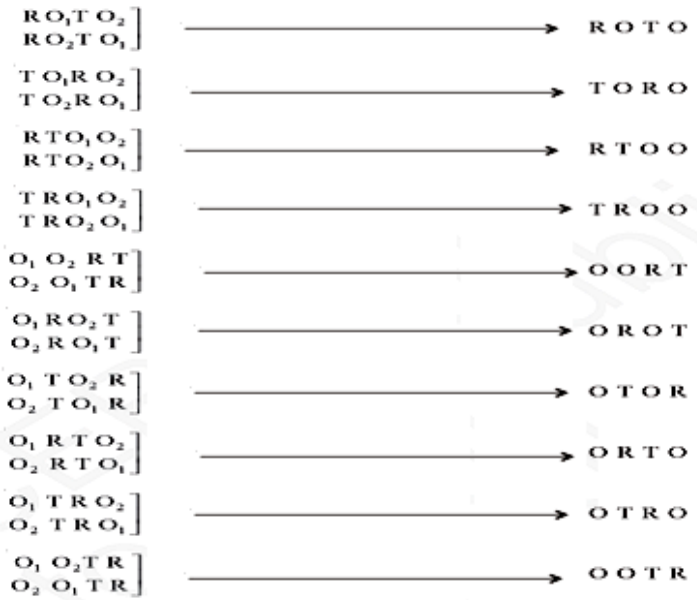
The number of 3-letter words which can be formed by the letters of the word NUMBER = ${}^6P_3 = \frac{6!}{3!} = 4 \times 5 \times 6 = 120$. Here, in this case also, the repetition is not allowed. If the repetition is allowed, the required number of words would be $6^3 = 216$.

The number of ways in which a Chairman and a Vice-Chairman can be chosen from amongst a group of 12 persons assuming that one person cannot hold more than one position, clearly = ${}^{12}P_2 = \frac{12!}{10!} = 11 \times 12 = 132$.

Permutations when all the objects are not distinct objects

Suppose we have to find the number of ways of rearranging the letters of the word **root**. In this case, the letters of the word are not all different. There are 2 Os, which are of the same kind. Let us treat, temporarily, the 2 Os as different, say, O1 and O2. The number of permutations of 4-different letters, in this case, taken all at a time is 4!. Consider one of these permutations say, RO1O2TCorresponding to this permutation, we have 2 ! permutations RO1O2T and RO2O1T which will be exactly the same permutation if O1 and O2 are not treated as different, i.e., if O1 and O2 are the same O at both places. Therefore, the required number of permutations = $\frac{4!}{2!} = 3 \times 4 = 12$.

Permutations when O_1, O_2, are different	Permutations when O_1, O_2, are the same O
RO_1, O_2T RO_2, O_2T	ROOT
TO_1, O_2R TO_2, O_2R	TOOR



Let us now find the number of ways of rearranging the letters of the word **institute**. In this case there are 9 letters, in which I appears 2 times and T appears 3 times. Temporarily, let us treat these letters different and name them as I1, I2, T1, T2, T3. The number of permutations of 9 different letters, in this case, taken all at a time is 9 !. Consider one such permutation, say, I1 NT1 SI2 T2 U E T3. Here if I1, I2 are not same and T1, T2, T3 are not same, then I1, I2 can be arranged in 2! ways and T1, T2, T3 can be arranged in 3! ways. Therefore, 2! × 3! permutations will be just the same permutation corresponding to this chosen permutation I1NT1SI2T2UET3. Hence, total number of different permutations will be $\frac{9!}{2!3!}$ We can state (without proof) the following theorems:

Theorem 3

The number of permutations of n objects, where p objects are of the same kind and rest are all different = $\frac{n!}{p!}$

In fact, we have a more general theorem.

Theorem 4

The number of permutations of n objects, where p_1 objects are of one kind, p_2 are of second kind, ..., p_k are of k^{th} kind and the rest, if any, are of different kind is

$$\frac{n!}{p_1!\,p_2!\ldots p_k!}$$

Example 9

Find the number of permutations of the letters in the word **haabhaabad.**

Solution

Here, there are 10 objects (letters) of which there are 5A's, 2 B's and 2 H's .

Therefore, the required number of arrangements = $\frac{10!}{5!2!2!} = \frac{10\times 9\times 8\times 7\times 6}{2\times 2} = 7560$

Example

10 How many 4-digit numbers can be formed by using the digits 1 to 9 if repetition of digits is not allowed?

Solution

Here order matters for example 1234 and 1324 are two different numbers.

Therefore, there will be as many 4 digit numbers as there are permutations of 9 different digits taken 4 at a time.

Therefore, the required 4 digit numbers = ${}^{9}P_4 = \frac{9!}{(9-4)!} = \frac{9!}{5!} = 9\times 8\times 7 = 3024.$

Example 11

How many numbers lying between 100 and 1000 can be formed with the digits 0, 1, 2, 3, 4, 5, if the repetition of the digits is not allowed?

Solution

Every number between 100 and 1000 is a 3-digit number. We, first, have to count the permutations of 6 digits taken 3 at a time. This number would be ${}^{6}P_3$. But, these permutations will include those also where 0 is at the 100's place. For example, 092, 042, . . ., etc are such numbers which are actually 2-digit numbers and hence the number of such numbers has to be subtracted from ${}^{6}P_3$ to get the required number. To get the number of such numbers, we fix 0 at the 100's place and rearrange the remaining 5 digits taking 2 at a time. This number is ${}^{5}P_2$. So

The required number = ${}^{6}P_3 - {}^{5}P_2 = \frac{6!}{3!} - \frac{5!}{3!}$

$= 4 \times 5 \times 6 - 4 \times 5 = 100$

Example 12

Find the value of n such that

(i) ${}^{n}P_5 = 42\ {}^{n}P_3,\ n > 4$ (ii) $\frac{{}^{n}P_4}{{}^{n-1}P_4} = \frac{5}{3},\ n > 4$

Solution

(i) Given that

$^{n}P_{5} = 42\,^{n}P_{3}$

or $n\,(n-1)\,(n-2)\,(n-3)\,(n-4) = 42\,n(n-1)\,(n-2)$

Since $n > 4$ so $n(n-1)\,(n-2) \neq 0$

Therefore, by dividing both sides by $n(n-1)\,(n-2)$, we get

$(n-3\,(n-4) = 42$

or $n^2 - 7n - 30 = 0$

or $n^2 - 10n + 3n - 30$

or $(n-10)\,(n+3) = 0$

or $n - 10 = 0$ or $n + 3 = 0$

or $n = 10$ or $n = -3$

As n cannot be negative, so $n = 10$.

(ii) Given that $\frac{^{n}P_{4}}{^{n-1}P_{4}} = \frac{5}{3}$

Therefore $3n\,(n-1)\,(n-2)\,(n-3) = 5(n-1)\,(n-2)\,(n-3)\,(n-4)$ or $3n = 5\,(n-4)$ [as $(n-1)\,(n-2)\,(n-3) \neq 0$, $n > 4$] or $n = 10$.

Example 13

Find the number of different 8-letter arrangements that can be made from the letters of the word DAUGHTER so that

(i) all vowels occur together (ii) all vowels do not occur together.

Solution

(i) There are 8 different letters in the word DAUGHTER, in which there are 3 vowels, namely, A, U and E. Since the vowels have to occur together, we can for the time being, assume them as a single object (AUE). This single object together with 5 remaining letters (objects) will be counted as 6 objects. Then we count permutations of these 6 objects taken all at a time. This number would be 6P6 = 6!. Corresponding to each of these permutations, we shall have 3! permutations of the three vowels A, U, E taken all at a time . Hence, by the multiplication principle the required number of permutations = 6 ! × 3 ! = 4320.

(ii) If we have to count those permutations in which all vowels are never together, we first have to find all possible arrangments of 8 letters taken all at a time, which

can be done in 8! ways. Then, we have to subtract from this number, the number of permutations in which the vowels are always together.

Therefore, the required number $8! - 6! \times 3! = 6!(7\times 8 - 6)$

$= 2 \times 6!(28 - 3)$

$= 50 \times 6! = 50 \times 720 = 36000$

Example 14

In how many ways can 4 red, 3 yellow and 2 green discs be arranged in a row if the discs of the same colour are indistinguishable ?

Solution

Total number of discs are 4 + 3 + 2 = 9. Out of 9 discs, 4 are of the first kind (red), 3 are of the second kind (yellow) and 2 are of the third kind (green).

Therefore, the number of arrangements $\frac{9!}{4!3!2!} = 1260.$

Example 15

Find the number of arrangements of the letters of the word

Independence. In how many of these arrangements,

(i) do the words start with P

(ii) do all the vowels always occur together

(iii) do the vowels never occur together

(iv) do the words begin with I and end in P?

Solution

There are 12 letters, of which N appears 3 times, E appears 4 times and D appears 2 times and the rest are all different. Therefore

The required number of arrangements = $\frac{12!}{3!4!2!} = 1663200.$

(i) Let us fix P at the extreme left position, we, then, count the arrangements of the remaining 11 letters. Therefore, the required number of words starting with P are(i) Let us fix P at the extreme left position, we, then, count the arrangements of the remaining 11 letters. Therefore, the required number of words starting with P are

$\frac{11!}{3!2!4!} = 138600.$

(ii) There are 5 vowels in the given word, which are 4 Es and 1 I. Since, they have to always occur together, we treat them as a single object EEEEI for the time being.

This single object together with 7 remaining objects will account for 8 objects. These 8 objects, in which there are 3Ns and 2 Ds, can be rearranged in

$\frac{8!}{3!2!}$ ways. Corresponding to each of these arrangements, the 5 vowels E, E, E,

E and I can be rearranged in $\frac{5!}{4!}$ ways. Therefore, by multiplication principle the required number of arrangements

$\frac{8!}{3!2!} \times \frac{5!}{4!} = 16800.$

(iii) The required number of arrangements

= the total number of arrangements (without any restriction) – the number of arrangements where all the vowels occur together.

= 1663200 – 16800 = 1646400

(iv) Let us fix I and P at the extreme ends (I at the left end and P at the right end).

We are left with 10 letters.

Hence, the required number of arrangements

$= \frac{10!}{3!2!4!} = 12600.$

Exercise 4.2.2.

1. How many 3-digit numbers can be formed by using the digits 1 to 9 if no digit is repeated?
2. How many 4-digit numbers are there with no digit repeated?
3. How many 3-digit even numbers can be made using the digits 1, 2, 3, 4, 6, 7, if no digit is repeated?
4. Find the number of 4-digit numbers that can be formed using the digits 1, 2, 3, 4, 5 if no digit is repeated. How many of these will be even?
5. From a committee of 8 persons, in how many ways can we choose a chairman and a vice chairman assuming one person cannot hold more than one position?
6. Find n if ${}^{n-1}P_3 : {}^{n}P_4 = 1:9.$
7. Find r if (i) ${}^{5}P_r = 2^6 P_{r-1}$ (ii) ${}^{5}P_r = {}^{6}P_{r-1}$
8. How many words, with or without meaning, can be formed using all the letters of the word EQUATION, using each letter exactly once?

9. How many words, with or without meaning can be made from the letters of the word MONDAY, assuming that no letter is repeated, if.

(i) 4 letters are used at a time, (ii) all letters are used at a time,

(iii) all letters are used but first letter is a vowel?

10. In how many of the distinct permutations of the letters in MISSISSIPPI do the four I's not come together?
11. In how many ways can the letters of the word PERMUTATIONS be arranged if the

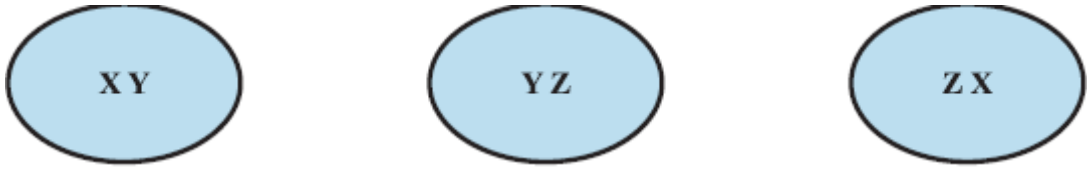
    (i) words start with P and end with S, (ii) vowels are all together,

    (iii) there are always 4 letters between P and S?

Combinations

Let us now assume that there is a group of 3 lawn tennis players X, Y, Z. A team consisting of 2 players is to be formed. In how many ways can we do so? Is the team of X and Y different from the team of Y and X ? Here, order is not important.

In fact, there are only 3 possible ways in which the team could be constructed.



These are XY, YZ and ZX (Figure above).

Here, each selection is called a combination of 3 different objects taken 2 at a time.

In a combination, the order is not important.

Now consider some more illustrations.

Twelve persons meet in a room and each shakes hand with all the others. How do we determine the number of hand-shake. X shaking hands with Y and Y with X will not be two different handshakes. Here, order is not important. There will be as many handshakes as there are combinations of 12 different things taken 2 at a time.

Seven points lie on a circle. How many chords can be drawn by joining these points pair-wise? There will be as many chords as there are combinations of 7 different things taken 2 at a time. Now, we obtain the formula for finding the number of combinations of *n* different objects taken *r* at a time, denoted by ${}^{n}C_{r}$.. Suppose we

have 4 different objects A, B, C and D. Taking 2 at a time, if we have to make combinations, these will be AB, AC, AD, BC, BD, CD. Here, AB and BA are the same combination as order does not alter the combination. This is why we have not included BA, CA, DA, CB, DB and DC in this list. There are as many as 6 combinations of 4 different objects taken 2 at a time, i.e., ${}^4C_2 = 6$.

Corresponding to each combination in the list, we can arrive at 2! permutations as 2 objects in each combination can be rearranged in 2! ways. Hence, the number of permutations = ${}^4C_2 \times 2!$

On the other hand, the number of permutations of 4 different things taken 2 at a time = ${}^4P_2$.

Therefore ${}^4P_2 = {}^4C_2 \times 2!$ or $\frac{4!}{(4-2)!2!} = {}^4C_2$

Now, let us suppose that we have 5 different objects A, B, C, D, E. Taking 3 at a time, if we have to make combinations, these will be ABC, ABD, ABE, BCD, BCE, CDE, ACE, ACD, ADE, BDE. Corresponding to each of these 5C3 combinations, there are 3! permutations, because, the three objects in each combination can be rearranged in 3 ! ways. Therefore, the total of permutations = ${}^5C_3 \times 3!$

Therefore ${}^5P_3 = {}^5C_3 \times 3!$ or $\frac{5!}{(5-3)!3!} = {}^5C_3$

These examples suggest the following theorem showing relationship between permutation and combination:

Theorem 5

${}^nP_r = {}^nC_r \times r! \quad 0 < r \leq n$

Proof

Corresponding to each combination of ${}^nC_r$ we have r ! permutations, because r objects in every combination can be rearranged in r ! ways.

Hence, the total number of permutations of n different things taken r at a time is ${}^nC_r \times r!$.On the other hand, it is ${}^nP_r$ .Thus ${}^nP_r = {}^nC_r \times r! \quad 0 < r \leq n$

Remarks

1. From above $\frac{n!}{(n-r)!} = {}^nC_r \times r!$ i.e.${}^nC_r = \frac{n!}{r!(n-r)!}$.

In particular, if $r = n$, ${}^{n}C_{n} = \frac{n!}{n!0!} = 1$

2. We define ${}^{n}C_{0} = 1$, i.e., the number of combinations of *n* different things taken nothing at all is considered to be 1. Counting combinations is merely counting the number of ways in which some or all objects at a time are selected. Selecting nothing at all is the same as leaving behind all the objects and we know that there is only one way of doing so. This way we define ${}^{n}C_{0} = 1$.

3. As $\frac{n!}{0!(n-r)!} = 1 = {}^{n}C_{0} = 1$, the formula ${}^{n}C_{r} = \frac{n!}{r!(n-r)!}$ is applicable for $r = 0$ also.

Hence ${}^{n}C_{r} = \frac{n!}{r!(n-r)!}$, $0 < r \leq n$

4. ${}^{n}C_{n-r} = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{(n-r)!r!} = {}^{n}C$

i.e., selecting *r* objects out of *n* objects is same as rejecting $(n - r)$ objects.

5. ${}^{n}C_{a} = {}^{n}C_{b} \Rightarrow a = b$ or $a = n - b$ o i.e., $n = a + b$

Theorem6

${}^{n}C_{r} + {}^{n}C_{r-1} = {}^{n+1}C_{r}$

Proof

We have ${}^{n}C_{r} + {}^{n}C_{r-1} = \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!(n-r)!}$

$$= \frac{n!}{r \times (r-1)!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)(n-r)!}$$

$$= \frac{n!}{(r-1)!(n-r)!} + \frac{1}{r} + \frac{1}{n-r+1}$$

$$= \frac{n!}{r!(n-r)!} + \frac{n-r+1+r}{r(n-r+1)}$$

$$= \frac{(n+1)!}{r!(n+1-r)!}$$

$$= {}^{n+1}C_{r}$$

Example 1

If ${}^{n}C_{9} = {}^{n}C_{8}$, find ${}^{n}C_{17}$.

Solution

We have ${}^{n}C_{9} = {}^{n}C_{8}$

i.e., $\frac{n!}{9!(n-9)!}=\frac{n!}{(n-8)!8!}$

Or $\frac{1}{9}=\frac{1}{n-8}$ or n-8 = 9 or n = 17

Therefore nC17 = ${}^{17}C_{17} = 1$

Example 2

A committee of 3 persons is to be constituted from a group of 2 men and 3 women. In how many ways can this be done? How many of these committees would consist of 1 man and 2 women?

Solution

Here, order does not matter. Therefore, we need to count combinations.

There will be as many committees as there are combinations of 5 different persons taken 3 at a time. Hence, the required number of ways = ${}^{5}C_{3}=\frac{5!}{3!2!}=\frac{4\times5}{2}$ =10

Now, 1 man can be selected from 2 men in ${}^{2}C_{1}$ ways and 2 women can be selected from 3 women in ${}^{3}C_{2}$ ways. Therefore, the required number of committees

${}^{2}C_{1}\times{}^{3}C_{2}=\frac{2!}{1!1!}\times\frac{3!}{2!1!}=6$

Example 3

What is the number of ways of choosing 4 cards from a pack of 52 playing cards? In how many of these

(i) four cards are of the same suit,

(ii) four cards belong to four different suits,

(iii) are face cards,

(iv) two are red cards and two are black cards,

(v) cards are of the same colour?

Solution

There will be as many ways of choosing 4 cards from 52 cards as there are combinations of 52 different things, taken 4 at a time. Therefore

The required number of ways = ${}^{52}C_{4}=\frac{52!}{4!48!}=\frac{49\times50\times51\times52}{2\times3\times4}$ =270725

(i) There are four suits: diamond, club, spade, heart and there are 13 cards of each suit. Therefore, there are ${}^{13}C_{4}$ ways of choosing 4 diamonds. Similarly, there are

${}^{13}C_4$ ways of choosing 4 clubs, ${}^{13}C_4$ ways of choosing 4 spades and ${}^{13}C_4$ ways of choosing 4 hearts. Therefore

The required number of ways $= {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4$

$$= 4 \times \frac{13!}{4!9!} = 2860$$

(ii) There are13 cards in each suit.

Therefore, there are ${}^{13}C_1$ ways of choosing 1 card from 13 cards of diamond, ${}^{13}C_1$ ways of choosing 1 card from 13 cards of hearts, ${}^{13}C_1$ ways of choosing 1 card from 13 cards of clubs, ${}^{13}C_1$ ways of choosing 1 card from 13 cards of spades. Hence, by multiplication principle, the required number of ways

$$= {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 = 13^4$$

(iii) There are 12 face cards and 4 are to be selected out of these 12 cards. This can be done in ${}^{12}C_4$ ways. Therefore, the required number of ways $= {}^{12}C_4 = \frac{12!}{4!8!} = 495$

(iv) There are 26 red cards and 26 black cards. Therefore, the required number of ways = ${}^{26}C_2 \times {}^{26}C_2$

$$= \left(\frac{26!}{2!24!}\right) = (325)^2 = 105625$$

(v) 4 red cards can be selected out of 26 red cards in ${}^{26}C_4$ ways.

4 black cards can be selected out of 26 black cards in ${}^{26}C_4$ ways.

Therefore, the required number of ways = ${}^{26}C_4 + {}^{26}C_4$

$$= 2 \times \frac{26!}{4!22!} = 29900$$

Exercise 4.2.3.

1. If ${}^{n}C_8 = {}^{n}C_2$, find ${}^{n}C_2$.
2. Determine *n* if

   (i) ${}^{2n}C_3 : {}^{n}C_3 = 12:1$ (ii) ${}^{2n}C_3 : {}^{n}C_3 = 1 = 11:1$
3. How many chords can be drawn through 21 points on a circle?
4. In how many ways can a team of 3 boys and 3 girls be selected from 5 boys and 4 girls?

5. Find the number of ways of selecting 9 balls from 6 red balls, 5 white balls and 5 blue balls if each selection consists of 3 balls of each colour.
6. Determine the number of 5 card combinations out of a deck of 52 cards if ther is exactly one ace in each combination.
7. In how many ways can one select a cricket team of eleven from 17 players I which only 5 players can bowl if each cricket team of 11 must include exactly 4 bowlers?
8. A bag contains 5 black and 6 red balls. Determine the number of ways in which 2 black and 3 red balls can be selected.
9. In how many ways can a student choose a programme of 5 courses if 9 courses are available and 2 specific courses are compulsory for every student?

4.2. Random Experiments and Their Outcomes

4.3.1 Introduction

At the beginning of this section, you saw the basic definitions of experiment, event and sample space. In this section, you will use these terms again and also see additional concepts.

Definition

A random experiment is an experiment (activity) which produces some well defined results. If the experiment is repeated under identical conditions it does not necessarily produce the same results.

Example 1

Give the outcome for each of the following experiments

a) Tossing a coin
b) Tossing a pair of coins
c) Rolling a dice
d) Rolling a pair of dice

Solution

a) (H, T)
b) (HH, HT, TH, TT)
c) (1, 2, 3, 4, 5, 6)
d) (1, 1) (1, 2) (1, 3) (1, 4) (1, 5) (1, 6)
(2, 1) (2, 2) (2, 3) (2, 4) (2, 5) (2, 6)
(3, 1) (3, 2) (3, 3) (3, 4) (3, 5) (3, 6)
(4, 1) (4, 2) (4, 3) (4, 4) (4, 5) (4, 6)

(5, 1) (5, 2) (5, 3) (5, 4) (5, 5) (5, 6)

(6, 1) (6, 2) (6, 3) (6, 4) (6, 5) (6, 6)

Example 2 If a fair die is thrown, any one of the outcomes 1, 2, 3, 4, 5, 6 has an equal chance of appearing at the top. Therefore, they are considered as equally likely.

Note:
Outcome of a random experiment are to be said to be equally likely when there is no reason to expect any one of the outcomes in preference to another. That is, each element has equal chance of being chosen.

Example 3

a) A fair die is thrown. How many favorable outcomes are there for getting an even number?

b) In picking a playing card from a pack of 52 cards, what is the number of favorable outcomes to getting a picture card?

Solution:

a) There are 3 favorable outcomes. These are 2, 4 and 6.

b) There are 12 favorable outcomes - 4 Jacks, 4 Queens and 4 Kings.



Four Jacks Four Queens Four Kings

Figure 5

Events

Recall that any subset of a sample space is called an event and is usually denoted by *E*. An event is a collection of sample points.

Example 4

The four faces of a regular tetrahedron are numbered 1, 2, 3 and 4. If it is thrown and the number on the bottom face (on which it stands) is registered, then list the events of this experiment.

Solution

The sample space = {1, 2, 3, 4}.

The possible events are {1}, {2}, {3} and {4}.

ACTIVITY

List some events of the following experiments.

a. Tossing a coin three times.

b. Inspecting produced items.

c. Selecting a number at random from integers 1 through to 12.

d. Drawing a ball from a bag containing 4 red and 6 white balls.

e. A married couple expecting a child.

Types of events

1. **Simple Event (Elementary Event:** is an event containing exactly one sample point.

For example: In a toss of one coin, the occurrence of tail is a simple event.

2. **Compound Event:** When two or more events occur simultaneously, their joint occurrence is known as a compound event, an event that has more than one sample point.

Example 6

When a die is rolled, if you are interested in the event "getting even number", then the event will be a compound event, i.e. { 2, 4, 6}. We can determine the possible number of events that can be associated with an experiment whose sample space is S. As events are subsets of a sample space, and any set with m elements has 2m subsets, the number of events associated with a sample space with m elements is 2m. (Sometimes this is called the exhaustive number of events).

Example 7

Suppose our experiment is tossing a fair coin. The sample space for this experiment is S = {H, T}. Thus, this sample space has a total of four possible events that are subsets of S. The list of the possible events is { }, {H}, {T}, and {H, T}.

Occurrence or Non-occurrence of an event

During a certain experiment, there are two possibilities associated with an event, namely, occurrence or non-occurrence of the event.

Example 8

If a die is thrown, then S = {1, 2, 3, 4, 5, 6}. Let *E* be the event of getting odd number, then *E* = {1, 3, 5}. When we throw the die, if the outcome is 3, as 3Î*E*, then

we say that *E* has occurred. If in another trial, the outcome is 4, then as 4ÏE, we say that *E* has not occurred (not *E*).

3. **Complement of an Event:** *E*, denoted by *E* ' (not *E*) consists of all events in the sample space that are not in *E*.

Example 9

Let a die be rolled once. Let *E* be the event of a prime number appearing at the top i.e. *E* = {2, 3, 5}. Give the complement of the event.

Solution:

E ' = {1, 4, 6}.

Note:

$E' = S - E = \{w : w \in Sandw \notin E\}$

ACTIVITY (ALGEBRA OF EVENTS)

Discuss the following:

a. Union and intersection of two events:

b. State properties of union and intersection.

c. What are exhaustive and mutually exclusive events?

d. When are two events called independent?

Note:

Since events are sets (subsets of the sample space) one can form union, intersection and complement of them. The operations obey algebra of sets, like commutativity, distributivity, De Morgan's laws and so on.

Exhaustive Events:

Are events where at least one of them must necessarily occur every time the experiment is performed?

Example 10:

If a die is thrown give instances of exhaustive events. Solution: The sample space is S = {1, 2, 3, 4, 5, 6}. From this, the events {1}, {2}, {3}, {4}, {5}, {6} are exhaustive events. The events {1, 2}, {3, 4}, {4, 5, 6} are also exhaustive events for this experiment. More generally, events E_1, E_2 , …, E_n form a set of exhaustive events of a sample space S where E_1, E_2,,..., E_n are subsets of S and $E_1 \cup E_2 \cup ... E_n = S$.

Mutually Exclusive Events: are events that cannot happen at the same

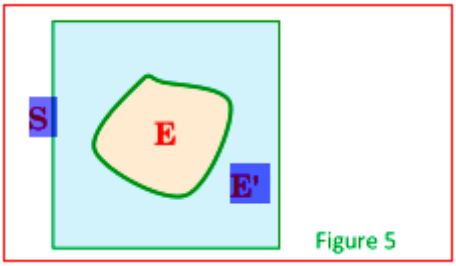


Figure 5

Example 11:

Say whether or not the following are mutually exclusive events.

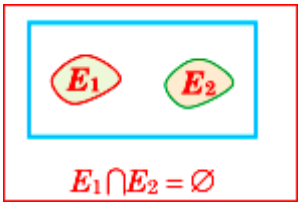
I. When a coin is tossed once, the events {H} and {T}.

II. When a die is rolled, E_1 = getting an even number

E_2 = getting a prime number

Solution:

I. Either we get head or tail but we cannot get both at the same time. Thus, {H} and {T} are mutually exclusive events.



II. E_1 and E_2 are not mutually exclusive because 2 is even and prime at the same time.

Exhaustive and Mutually Exclusive Events: If S is a sample space associated with a random experiment and if E_1, E_2, ..., E_n are subsets of S such that

i $E_i \cap E_j = \varnothing$ for $i \neq j$ and,

ii $E_1 \cup E_2 \cup ... \cup E_n = S$, then the collection of the events E_1, E_2,..., E_n forms a mutually exclusive and exhaustive set of events.

Example 12:

If a die is thrown, the events {1}, {2}, {3}, {4}, {5}, {6} are mutually exclusive and exhaustive events. But, the events {1, 2}, {3, 4}, {4, 5, 6} are not because {3, 4} $\cap$ {4, 5, 6} $\neq \emptyset$.

Independent Events: Two events are said to be independent, if the occurrence or non occurrence of one event does not affect the occurrence or non-occurrence of the other.

Example 13 In a simultaneous throw of two coins, the event of getting a tail on the first coin and the event of getting a tail on the second coin are independent.

Example 14

If a card is drawn from a well shuffled pack of cards and is replaced before drawing a second card, then the result from drawing the second card is independent of the result of the first drawn card.

Dependent Events Two events: are said to be dependent, if the occurrence or non occurrence of one event affects the occurrence or non-occurrence of the other.

Example 14

If a card is drawn from a well shuffled pack of cards and the card is not replaced, then the result of drawing a second card is dependent on the first draw.

Probability of an Event

In form three, you dealt with an experimental approach to probability. You also discussed the definition of theoretical probability of an event. Probability can be measured by three different approaches.

a. The classical (mathematical) approach.

b. The empirical (relative frequency) approach.

c. The axiomatic approach.

d. The classical approach

This is the kind of probability that you discussed in form three. In the classical approach to probability, if all the outcomes of a random experiment are equally likely and mutually exclusive, then the probability of an event E is

$$P(E) = \frac{n(E)}{n(S)} = \frac{\text{number of outcomes favoring } E}{\text{number of all possible outcomes}}$$

Example 15 A fair die is tossed once. What is the probability that an even number appears?

Solution: E = an even number shows up = {2, 4, 6}. Then, $P(E) = \frac{n(E)}{n(S)} = \frac{3}{6} = \frac{1}{2}$.

The empirical approach

This approach is based on the relative frequency of an event (or outcome) when an experiment is repeated a large number of times. Here, the probability of an event E is

the proportion of outcomes favourable to E in the experiment. Thus,

$$P(E) = \frac{\text{frequency of } E}{\text{total number of observations}} = \frac{f_E}{N}$$

Example 16

If records show that 60 out of 100, 000 bulbs produced are defective (D), then the probability of a newly produced bulb being defective is given by

$$P(D) = \frac{f_D}{N} = \frac{60}{100{,}000} = 0.0006$$

The axiomatic approach

In this approach, the probability of an event is given as a function that satisfies the following definition:

Let *S* be the sample space of a random experiment. With each event *E* we associate a real number called the probability of *E*, denoted by *P*(*E*), that satisfies the following properties called axioms (or postulates) of probability.

1. $0 \le P(E) \le 1$
2. $P(S) = 1$, S is the sample space (the sure event)
3. If E_1 and E_2 are mutually exclusive events, then
   $P(E_1 \cup E_2) = P(E_1) + P(E_2)$

Note:

P is a function with domain the set of subsets of S (Sample space) and its range is the set of real numbers between 0 and 1 (both inclusive). Thus we note the following:

a. The probability of an event is always between 0 and 1.

b. If E = Ø (the impossible event), then P (Ø) = 0, and if E = S (the certain event), then P(S) = 1.

c. If E ∪ E' = S then P (E ∪ E') = P(S) = 1, and P (E ') = 1 – P (E), where E' = S \ E (not E).

Example 17:

A box contains 6 red balls. One ball is drawn at random. Find the probability of getting

i. a red ball

ii. a white ball

Solution

i. the box contains all red balls. Hence we are sure that red will occur. Then the probability of getting a red is one. That is $P(R) = \frac{n(R)}{n(S)} = \frac{6}{6} = 1$

ii. The box contains no white balls. The chance of getting white ball is impossible, and the probability is zero. That is, $P(W) = \frac{n(W)}{n(S)} = \frac{0}{6} = 0$

Example 18

A bag contains 3 red, 5 black, and 4 white marbles. One marble is drawn at random. What is the probability that the marble is:

a) Black

b) Not black

Solution

a) $P(B) = \frac{5}{12}$

b) P(not black) = 1-P(black) ……. Complementary events.

$$= 1 - \frac{5}{12}$$

$$= \frac{7}{12}$$

Therefore, P(B) +P(not black) = $\frac{5}{12} + \frac{7}{12} = \frac{12}{12} = 1$

Example 19 Which of the following cannot be valid assignments of probabilities for outcomes of sample space $S = \{w_1, w_2, w_3\}$ where $w_i \cap w_j = \varnothing$, if $i \neq j$.

	w_1	w_2	w_3
a.	0.3	0.6	0.2
b.	0.2	0.5	0.3
c.	0.3	-0.2	0.9

Solution

a. is not valid assignment because the sum of the probabilities is not 1.

b. is valid; all the properties in the axiom above are satisfied.

c. is not valid because probability cannot be negative.

Odds in favour of and odds against an event

If *m* and *n* are probabilities of the occurrence and non-occurrence of an event respectively, then the ratio *m: n* is called the odds in favour of the event. The ratio *n: m* is called the odds against the event.

Example 20 The odds against a certain event are 5 : 7. Find the probability of its occurrence.

Solution Let E be the event. Then, we are given that number (not E) = 5 and number (E) = 7.

$n(S) = n(\text{not } E) + n(E) = 5 + 7 = 12$

$\therefore P(E) = \frac{7}{12}$.

Example 21 The odds in favour of an event are 3 : 8. Find the probability of its occurrence.

Solution $n(E) = 3$, $n(\text{not } E) = 8$. Thus $n(S) = 3 + 8 = 11$.

$\therefore P(E) = \frac{3}{11}$.

Rules of probability

In the last section, you have seen different types of events and approaches to probability. We will now discuss some essential rules for probability and probabilities of the different types of events.

ACTIVITY

For two events E_1 and E_2 discuss what conditions apply to the following rules.

a. $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

b. $P(E_1 \cup E_2) = P(E_1) + P(E_2)$

c. Illustrate each of the above by using a Venn diagram

In your previous discussions, you saw how to determine probabilities of events.

Example 22

Find the probability of obtaining a 6 or 4 in one roll of a die.

Solution

In one roll of a die, the sample space is S = {1, 2, 3, 4, 5, 6}.

Obtaining 6 or 4 gives the event E = {4, 6}.

$$\text{Thus } P(4 \text{ or } 6) = P(E) = \frac{\text{number of outcomes favouring } E}{\text{number of all possible outcomes}} = \frac{2}{6} = \frac{1}{3}.$$

Trying to calculate probabilities by listing all outcomes and favourable outcomes may not always be convenient. For more complex situations, there are rules we can use to help us calculate probabilities.

Addition rule of probability

From previous discussions, recall that, if $E_1 \ldots E_2, E_n$ form a set of exhaustive events of a sample space S, then $E_1 \cup E_2 \cup \ldots \cup E_n = S$. Moreover, the probability of an event E, i.e. P (E) is given by

P(E) =

$$P(E) = \frac{\text{number of outcomes favoring E}}{\text{total number of outcomes in the sample space}} = \frac{n(E)}{n(S)}$$

With these we can easily calculate probabilities of compound events by making use of the addition rule stated below.

Addition rule

If E_1 and E_2 are any two events, then,

$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$ and

If the events are mutually exclusive, (i.e $E_1 \cap E_2 = \emptyset$) then $P(E_1 \cap E_2) = 0$, so, that

$P(E_1 \cup E_2) = P(E_1) + P(E_2)$.

Example 23

a. Find the probability of obtaining a 6 or 4 in one roll of a die.

b. Find the probability of getting Head or Tail in tossing a coin once.

c. A die is rolled once. Find the probability that it is even or it is divisible by 3.

Solution

a) Let E_1 be event of getting 6. E_2 be event of getting 4. Then E_1 and E_2 are mutually exclusive events.

Thus, $P(E_1 \cup E_2) = P(E_1) + P(E_2) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$

b) The even are mutually exclusive.

Thus, P(H or T)= P(H) +P(T) $= \frac{1}{2} + \frac{1}{2} = \frac{2}{2} = 1$

c) S = {1, 2, 3, 4, 5, 6}

Let E_1 = getting even = {2, 4, 6}

E_2 = getting a number a divisible by 3 = {3, 6}.

Then E_1 and E_2 are not mutually exclusive, because $P(E_1 \cap E_2)$= {6}

Thus, P(even or divisible by 3) = P(even) +P(divisible by 3) – P(even and divisible by 3).

$$= \frac{3}{6} + \frac{2}{6} - \frac{1}{6} = \frac{1}{2} + \frac{1}{3} - \frac{1}{6} = \frac{2}{3}$$

This shows the addition rule of probability with two events. What do you think the rule will be for three or more events? The rule can be extended for a finite number of events, but becomes increasingly complicated. For example, for three events it becomes:

Note:

$P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_1 \cap E_3) - P(E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_3)$

Multiplication rule of probability

This rule is useful for determining the probability of the joint occurrence of events. It is based on the concepts of independence or dependence of events, discussed earlier. Let us take a brief revision of independent and dependent events. When the occurrence of the first event affects the occurrence of the second event in such a way that the probability is changed, the events are said to be dependent.

Example 24

A bag contains 3 black and 2 white balls. We draw two balls one after the other with replacement (the second is draw after the first is replacement). Find the probability that the first ball is black and the second ball is also black.

Solution

Let event A be the first ball is black.

Let event B the second ball is black.

Then P(A) = $\frac{3}{5}$ and P(B) = $\frac{3}{5}$(since the ball is replaced, the sample space is not affected.

Example 25

Suppose we repeat the experiment in the example 23, but this time the first ball is not replaced. This time:

P(A) = P(first ball is black) = $\frac{3}{5}$

If the first ball is black P(B) = $\frac{2}{4}$ (one black ball has been removed).

If the first ball was not black P(B) = $\frac{3}{4}$

Recognizing dependence or independence is of paramount importance in using the multiplication rule of probability. When occurrence of one event depends on the occurrence of another event, we say the second event is conditioned by the first event. This leads into what is called conditional probability.

Conditional probability

If E_1 and E_2 are two events, the probability that E_2 occurs given that E_1 has already occurred is denoted by $P(E_2 \mid E_1)$ and is called the conditional probability of E_2 given

that E_1 has already occurred. If the occurrence or non occurrence of E_1 does not affect the probability of E_2, or if E_1 and E_2 are independent, then $P(E_2 \mid E_1) = P(E_2)$. This is often called the multiplication rule of probability.

Multiplication rule of probability

If E_1 and E_2 are any two events, the probability that both events occur, denoted by $P(E_1 \text{ and } E_2) = P(E_1 \cap E_2) = P(E_1E_2)$ is given by $P(E_1 \cap E_2) = P(E_1) \times P(E_2 \mid E_1)$. Wherever $P(E_2) \neq 0$

Note:

If E_1 are independent, then $P(E_2 \mid E_1) = P(E_2)$. Hence $P(E_1 \cap E_2) = P(E_1) \times P(E_2)$ for independent event E_1 and E_2

Example 26

a) A box contains 3 red and 2 black balls. One ball is drawn at random, is not replaced, and a second ball is drawn. Find the probability that the first ball is red and the second is black.

b) A die is rolled and a coin is tossed. Find the probability of getting 3 on the die and a tail in the coin.

c) A bag contains 3 red, 4 blue and 3 white balls. Three balls are drawn one after the other. Find the probability of getting a red ball on the first draw, a blue ball on the second draw and a white ball on the third draw if

i. Each ball is drawn, but then is replaced back before the next draw.

ii. The balls are drawn without replacement.

Solution

a. Let E_1 = getting red in the first draw.

E_2 = getting black in the second draw.

$$P(E_1 \cap E_2) = P(E_1) \times P(E_1 \mid E_2) = \frac{3}{5} \times \frac{2}{4} = \frac{6}{20} = \frac{3}{10}$$

b. Let E_1 = getting 3 on the die and E_2 = getting tail on the coin. Since the two events are independent,

$$P(E_1 \cap E_2) = P(E_1) \times P(E_2) = \frac{1}{6} \times \frac{1}{2} = \frac{1}{12}$$

c. Let E_1 = getting red, in the first draw, E_2 = getting blue in the second draw, E_3 = getting white in the third draw.

i. The balls are replaced after each draw. The events are independent.

$P(E_1 \cap E_2 \cap E_3) = P(E_1) \times P(E_2) \times P(E_3) = \frac{3}{10} \times \frac{4}{10} \times \frac{3}{10} = \frac{36}{1000} = \frac{9}{250}$

ii. The balls are not replaced, so events are dependent

$P(E_1 \cap E_2 \cap E_3) = P(E_1) \times P(E_2 \mid E_1) \times P(E_3 \mid E_1 \text{ and } E_2) = \frac{3}{10} \times \frac{4}{9} = \frac{1}{20}$

Exercise 4.3.A.

1. A die is rolled. What is the probability of scoring

a. 4?

b. 3 or 5?

2. In throwing a die, consider the following events.

E_1 = the number that shows up is even

E_2 = the number that shows up is prime

E_3 = the number that shows up is more than 3

a. Determine the event $E_2 \cap E_3$

b. Determine the number of elements in $E_1 \cap E_2$

c. Determine the number of elements in $P(E_1 \cap E_2 \cap E_3)$

d. Determine the $P(E_1 \cap E_2)$

e. Determine the $P(E_1 \cup E_2)$

f. Determine $P(E_1 \cup E_2 \cup E_3)$

3. From a pack of 52 playing cards, one card is drawn. Find the probability that it is

a. either a King or a Jack; b either

b. Queen or red.

4. A die is thrown twice. What is the probability of scoring a 3, followed by a 4?

5. A red ball and 4 white balls are in a box. If two balls are drawn without replacement, what is the probability of

a. getting a red ball on the first draw and a white ball on the second?

b. getting two white balls?

6. Two cards are drawn from a pack of 52 cards. What is the probability that the first is an Ace and the second is a King,

c. if the first card was replaced before the second was drawn? B

d. if the cards were drawn without replacement?

7. A box contains 24 pens, 10 of which are red. A pen is picked at random. What is the probability that the pen is not red?

8. The following table gives assignments of probabilities for outcomes from a sample

9. space

	w_1	w_2	w_3	w_4	w_5	w_6	w_7
a	0.1	0.001	0.05	0.03	0.01	0.2	0.6
b	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$
c	0.1	0.2	0.3	0.4	0.5	0.6	0.7
d	-0.1	0.2	0.3	0.4	-0.2	0.1	0.3
e	$\frac{1}{14}$	$\frac{2}{14}$	$\frac{3}{14}$	$\frac{4}{14}$	$\frac{5}{14}$	$\frac{6}{14}$	$\frac{13}{14}$

a. Which of the probabilities are invalid assignments? Why?

b. Why is (b) a valid assignment of probabilities.

10. In throwing a die what is the probability of getting even or prime number?

11. Two students are selected from a class of 28 girls and 22 boys one after the other. What is the probability that the second student selected is a boy given that the first was a girl?

Tree diagram probability

You have seen how to determine probability by using either of the product rules (for independent or dependent events). It is also possible to show joint events using tree diagrams and tables, and calculate probabilities from these.

Example 27

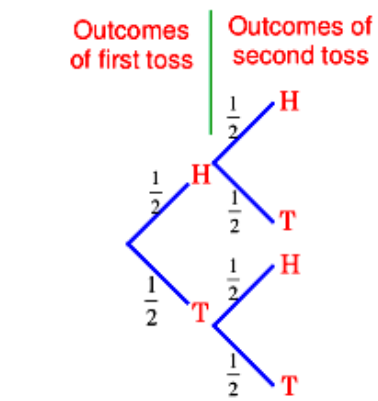
A fair coin is tossed twice. Find the probability that both outcomes will be Heads.

Solution:

From the multiplication rule, P (HH) = P (H) × P (H) = $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

You can use a tree diagram and/or table to portray the possible outcomes.

Using tree diagram



Joint event	Probability of joint event
HH	$\frac{1}{4}$
HT	$\frac{1}{4}$
TH	$\frac{1}{4}$
TT	$\frac{1}{4}$

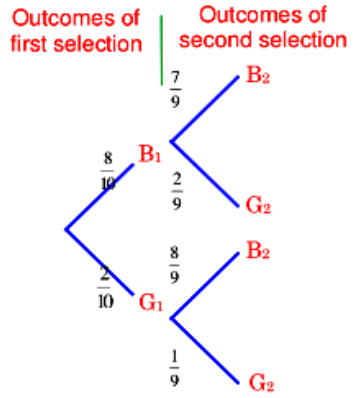
Therefore, the probability that both outcomes are heads is $\frac{1}{4}$

Example 28

Suppose that a group of 10 students contain eight boys (B) and two girls (G). If two students are chosen randomly without replacement, find the probability that the two students chosen are both boys.

Solution: $P(B_1 \text{ and } B_2) = P(B_1) \times P(B_2/B_1) = \frac{8}{10} \times \frac{7}{9} = \frac{56}{90} = \frac{28}{45}$.

Hence, the probability that the two students chosen are both boys is $\frac{56}{90}$.



Joint Event	Probability of joint event
B_1 and B_2	$\frac{56}{90}$
B_1 and G_2	$\frac{16}{90}$
G_1 and B_2	$\frac{16}{90}$
G_1 and G_2	$\frac{2}{90}$

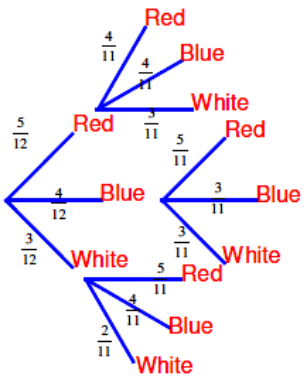
Example 29

A bag contains 5 red balls, 4 blue balls, and 3 white balls. Two balls are drawn one after the other, without replacement.

a. Find the probability that both are red.

b. Draw tree diagram representing the experiment.

Solution: $P(R \text{ and } R) = \frac{5}{12} \times \frac{4}{11} = \frac{20}{132} = \frac{5}{33}$.



Joint Event	Probability of joint Event
R and R	$\frac{5}{12}\times\frac{4}{11}$
R and B	$\frac{5}{12}\times\frac{4}{11}$
R and W	$\frac{5}{12}\times\frac{3}{11}$
B and R	$\frac{4}{12}\times\frac{5}{11}$
B and B	$\frac{4}{12}\times\frac{3}{11}$
B and W	$\frac{4}{12}\times\frac{3}{11}$
W and R	$\frac{3}{12}\times\frac{5}{11}$
W and B	$\frac{3}{12}\times\frac{4}{11}$
W and W	$\frac{3}{12}\times\frac{2}{11}$

Example 30

Two dice are thrown simultaneously. Find the probability that the sum of the numbers scored is

a. 7

b. greater than 9

c. less than 4

Solution:

		Second die					
		1	2	3	4	5	6
First die	1	(1 , 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
	2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
	3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
	4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
	5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
	6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

From the table above $n(S) = 36$.

a Let E = the sum of numbers at the top is 7. Then $n(E) = 6$.

$\therefore P(E) = \frac{6}{36} = \frac{1}{6}$.

b Let E = sum of the numbers at the top is greater than 9 (i.e., 10 or 11 or 12)

$\therefore P(E) = \frac{6}{36} = \frac{1}{6}$.

c Let E = sum is less than 4 (i.e. 2 or 3). Then, $n(E) = 3$

$\therefore P(E) = \frac{3}{36} = \frac{1}{12}$.

Exercise 4.3.4.

1. A box contains 5 red and 6 white balls. If one ball is drawn at random, find the probability that it will be

a. red or white?

b. not red?

c. yellow?

2. From a pack of 52 playing cards, three cards are drawn one after the other. What is the probability that all are Kings if

a. drawing is made with replacement?

b. drawing is made without replacement?

3. Use the table in Example 45, to find the probability that

a. the sum of the top numbers is 12.

b. the sum of the top numbers is 13.

c. the sum of the numbers is greater than 10.

4. There are 4 black, 2 red and 4 white balls in a box. If three balls are selected at random what is the probability that

a. all the balls selected are black?

b. at least one ball is white?

c. all the balls are of different colour?

5. Two lamps are to be chosen from a pack of 12 lamps where four are defective and the rest are non defective. What is the probability that

a. both are defective?

b. One is defective?

c. at most one is defective?

4.3.5. COMBINATIONATORIAL PROBABILITY

Permutations and combinations can sometimes be used to find probabilities of various events particularly when large sample sizes occur.It is useful to remember that:

$$\text{P[an event]} = \frac{\text{number with the required properties of the event}}{\text{total number of unrestricted possibilities}}.$$

For example, if we select at random a team of 4 boys and 3 girls from a squad of 8 boys and 7 girls, the total number of unrestricted possibilities is ${}_{15}C_7$ since we want any 7 of the 15 available for selection. The number with the required property of '4 boys **and** 3 girls' is ${}_8C_4 \times {}_7C_3$ as from the 8 boys we want any 4 of them **and** from the girls we want any 3 of them.

$$\therefore \quad \text{P[4 boys and 3 girls]} = \frac{{}_8C_4 \times {}_7C_3}{{}_{15}C_7}.$$

The biggest hurdle in probability problems involving permutations or combinations seems to be in sorting out which to use.

Remember:

- **permutations** involve orderings of objects or things, whereas
- **Combinations** involve selections (such as committees or teams).

Example 31

From a squad of 13 which includes 4 brothers, a team of 7 is randomly selected by drawing names from a hat. Determine the probability that the team contains

a. all the brothers

b. at least 2 of the brothers.

Solution

There are ${}_{13}C_7$ different teams of 7 that can be chosen from 13.

a. Of these teams contain all 4 brothers and any 3 others.

$$\therefore p[\text{contains all the brothers}] = \frac{{}_4C_4 \times_9 C_3}{{}_{13}C_7} = 0.04895$$

b. P[at least 2 brothers]=P[2 brothers or 3 brothers or 4 brothers

$$= \frac{{}_4C_2 \times_9 C_5}{{}_{13}C_7} + \frac{{}_4C_3 \times_9 C_4}{{}_{13}C_7} + \frac{{}_4C_4 \times_9 C_3}{{}_{13}C_7} = 0.7832$$

Example 32

5 letters U, S, T, I, N are placed at random in a row. What is the probability that the word UNITS is spelled out?

Solution

There are 5! different permutations of the letters, one of which spells UNITS.

$\therefore$ P(UNITS is spelled) $= \frac{1}{5!} = \frac{1}{120}$

Notice that counting permutations is not essential here. We could have used:

$$P(\text{UNITS}) = \frac{1}{5} \times \frac{1}{4} \times \frac{1}{3} \times \frac{1}{2} \times \frac{1}{1} = \frac{1}{120}$$

5 to choose from and we want only U

now 4 are left and we want N

etc.

Exercise 4.3.5

1. A committee of 4 is chosen from 11 people by random selection. What is the chance that sisters X and Y are on the committee?
2. 4 alphabet blocks; D, A, I and S are placed at random in a row. What is the likelihood that they spell out either AIDS or SAID?
3. A team of 7 is randomly chosen from a squad of 12. Determine the probability that both the captain and vice-captain are chosen.
4. 4 people are killed in an air crash and 3 professional golfers were aboard. Determine the chance that none of the golfers was killed if 22 people were aboard.
5. 5 boys sit at random on 5 seats in a row. Determine the probability that two of them (Kayse and Ahmed, say)

a. sit at the ends of the row

b. Sit together.

4.3 BINOMIAL PROPABABILITY

BINOMIAL EXPANSIONS

The sum a + b is called a **binomial** as it contains two terms.

Any expression of the form $(a + b)^n$ is called a **power of a binomial**.

All binomials raised to a power can be expanded using the same general principles. In this chapter, therefore, we consider the expansion of the general expression $(a + b)^n$ where $n \in N$.

Consider the following algebraic expansions:

$$(a+b)^1 = a + b$$
$$(a+b)^2 = a^2 + 2ab + b^2$$

$$\begin{aligned}(a+b)^3 &= (a+b)(a+b)^2 \\ &= (a+b)(a^2 + 2ab + b^2) \\ &= a^3 + 2a^2b + ab^2 + a^2b + 2ab^2 + b^3 \\ &= a^3 + 3a^2b + 3ab^2 + b^3\end{aligned}$$

The binomial expansion of $(a + b)^2$ is $a^2+2ab+b^2$.

The binomial expansion of $(a + b)^3$ is $a^3+3a^2b+3ab^2+b^3$.

From the **Investigation** we obtained $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

$= a^4 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + b^4$

Notice that:

⇒ As we look from left to right across the expansion, the powers of a decrease by 1, while the powers of b increase by 1.

⇒ The sum of the powers of *a* and *b* in each term of the expansion is 4.

- The number of terms in the expansion is 4 +1 = 5.
- The coefficients of the terms are row 4 of Pascal's triangle.

For the expansion of $(a+b)^n$ where $n \in \mathbb{N}$:

- As we look from left to right across the expansion, the powers of *a* *decrease* by 1, while the powers of *b* *increase* by 1.
- The sum of the powers of *a* and *b* in each term of the expansion is *n*.
- The number of terms in the expansion is $n + 1$.
- The coefficients of the terms are row *n* of Pascal's triangle.

In the following examples we see how the general binomial expansion $(a + b)^n$ may be put to use.

Example 1

Using $(a+b)^3 = a^3 + 3a^2b^1 + 3a^1b^2 + b^3$, find the binomial expansion of:

a $(2x+3)^3$ b $(x-5)^3$

Solution

a In the expansion of $(a+b)^3$ we substitute $a=(2x)$ and b = (3).

$\therefore (2x+3)^3 = (2x)^3 + 3(2x)^2(3) + 3(2x)^1(3)^2 + (3)^3$

$= 8x^3 + 36x^2 + 54x + 27$

b We substitute $a = (x)$ and b = (− 5)

$\therefore (x - 5)^3 = (x)^3 + 3(x)^2(-5) + 3(x)(-5)^2 + (-5)^3$

$= x^3 - 15x^2 + 75x - 125$

Example 2

Find the:

a 5th row of Pascal's triangle b binomial expansion of $\left(x - \frac{2}{x}\right)^5$.

Solution

a
1 ← the 0th row, for $(a+b)^0$
1 1 ← the 1st row, for $(a+b)^1$
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1 ← the 5th row, for $(a+b)^5$

b Using the coefficients obtained in a, $(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

Letting $a = (x)$ and $b = \left(\frac{-2}{x}\right)$, we find

$$\left(x - \frac{2}{x}\right)^5 = (x)^5 + 5(x)^4\left(\frac{-2}{x}\right) + 10(x)^3\left(\frac{-2}{x}\right)^2 + 10(x)^2\left(\frac{-2}{x}\right)^3 + 5(x)\left(\frac{-2}{x}\right)^4 + \left(\frac{-2}{x}\right)^5$$

$$= x^5 - 10x^3 + 40x - \frac{80}{x} + \frac{80}{x^3} - \frac{32}{x^5}$$

Exercise 4.3

1. Use the binomial expansion of $(a+b)^3$ to expand and simplify :

a) $(p+q)^3$ d) $(2x-3)^3$ g) $\left(3x+\frac{1}{3}\right)^3$

b) $(x+1)^3$ e) $(2x-1)^3$ h) $(2x+5)^3$

c) $(2x-3)^3$ f) $(2a-b)^3$ i) $(3x+7)^3$

2. Use $(a+b)^4 = a^4+4ab^3+6a^2b^2+4a^3b+b^4$ to expand and simplify:

a) $(1+x)^4$ c) $(3-x)^4$ e) $(p-q)^4$

b) $(2z+b)^4$ d) $(1+2x)^4$ f) $(5x+y)^4$

3. Expand and simplify

a) $(x+2)^5$ c) $(2x+y)^8$ e) $(2p+1)^5$

b) $(x-2y)^6$ d) $(a+b)^5$ f) $(2x-1)^6$

The binomial coefficient $\binom{n}{r}$

The numbers in the Pascal's triangle are called binomial confidents because of their importance in the binomial expansion.

To calculate the other binomial coefficients we can use a formula. It involves the product of consecutive positive integers, so to keep it simple we introduce **factorial notation**.

Factorial notation

For $n \geqslant 1$, $n!$ is the product of the first n positive integers.

$$n! = n(n-1)(n-2)(n-3)\ldots \times 3 \times 2 \times 1$$

$n!$ is read "n factorial".

For example, the product $6 \times 5 \times 4 \times 3 \times 2 \times 1$ can be written as $6!$.

Notice that $8 \times 7 \times 6$ can be written using factorial numbers only as

$$8 \times 7 \times 6 = \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1} = \frac{8!}{5!}$$

The **factorial rule** is $n! = n \times (n-1)!$ for $n \geqslant 1$

which can be extended to $n! = n(n-1)(n-2)!$ and so on.

Using the factorial rule with $n = 1$, we have $1! = 1 \times 0!$

We hence define $0! = 1$

Example 3

What integer is equal to each of the following:

a) 4! c) $\frac{5!}{3!}$

b) 8! d) $\frac{7!}{4! \times 3!}$

Solutions

a) $4! = 4 \times 3 \times 2 \times 1 = 24$

b) $8! = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40320$

c) $\frac{5!}{3!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} = \frac{120}{6} = 20$

d) $\frac{7!}{4! \times 3!} = \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1 \times 3 \times 2 \times 1} = 35$

THE BINOMIAL COEFFICIENT

The binomial coefficient is defined by

The binomial coefficient is defined by

$$\binom{n}{r} = \underbrace{\frac{n(n-1)(n-2)\ldots(n-r+2)(n-r+1)}{r(r-1)(r-2)\ldots 2\times 1}}_{\text{factor form}} = \underbrace{\frac{n!}{r!(n-r)!}}_{\text{factorial form}}$$

The binomial coefficient is sometimes written: ${}^{n}_{r}C$ or C^{n}_{r}

Example 4

Use the formula $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ to evaluate:

a) $\binom{5}{2}$ b) $\binom{11}{7}$

Solutions

a) $\binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{5\times4\times3\times2\times1}{2\times1\times3\times2\times1} = 10$

b) $\binom{11}{7} = \frac{11!}{7!(11-7)!}\ \frac{11\times10\times9\times8\times7\times6\times5\times4\times3\times2\times1}{7\times6\times5\times4\times3\times2\times1\times4\times3\times2\times1} = 330$

Exercise 4.5.

1 Find $n!$ for $n = 0, 1, 2, 3, \ldots, 10$.

2 Simplify without using a calculator:

a $\frac{6!}{4!}$ b $\frac{100!}{99!}$ c $\frac{7!}{5!\times 2!}$

3 Simplify: a $\frac{n!}{(n-1)!}$ b $\frac{(n+2)!}{n!}$

4 Use the formula $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ to evaluate:

a $\binom{3}{1}$ b $\binom{4}{2}$ c $\binom{7}{3}$ d $\binom{10}{4}$

5 a Use the formula $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ to evaluate:

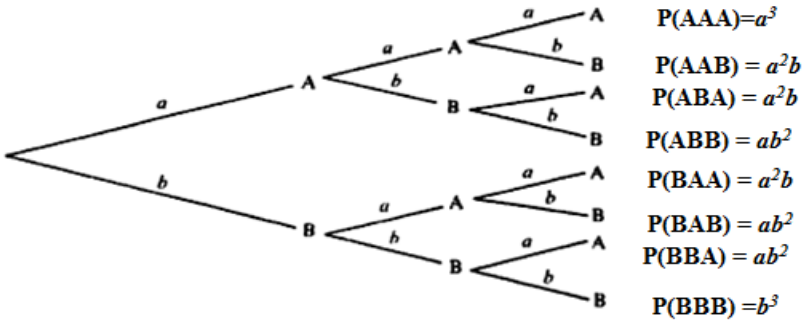
i $\binom{8}{2}$ ii $\binom{8}{6}$

b Can you explain these results using Pascal's triangle?

Binomial probability

Suppose that a trail is repeated a number of times, say 3, and that in each trail there are two possible outcomes A and B. if P (A) = a and P (B) = b.

And the outcome of each trail is independent of the previous trails, the tree diagram would be:



The probability of obtaining outcome B on each occasion = P (BBB) = b^3

Or of obtaining outcome A exactly twice = P(AAB) + P(ABA) + P(BAA)= $3a^3b$

Clearly for more trails, say 5, the tree diagram would become very large and so an alternative method is needed. If the trails are carried out five times the various outcomes would be:

5 A's, 4 A's, 1B 3A's, 2 B's 2A's, 3 B's 1A, 4 B's 5 B's

The 5 A's occur in five trials in $^5C_5$ = 1 way (AAAAA)
The 4 A's occur in five trials in $^5C_4$ = 5 ways (BAAAA, ABAAA, AABAA, AAABA, AAAAB)
The 3 A's occur in five trials in $^5C_3$ = 10 ways etc.

Thus the probabilities of the various combinations of A's and B,s would be

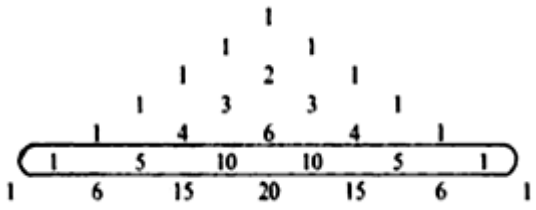
$^5C_5a^5$ $^5C_4a^4b$ $^5C_3a^3b^2$ $^5C_2a^2b^3$ $^5C_1ab^4$ $^5C_0b^5$

or

$1a^5$ $5a^4b$ $10a^3b^2$ $10a^2b^3$ $5ab^4$ $1b^5$

Notice that these numbers could also be obtained from the sixth row of Pascal's Triangle :

Similarly the seventh row would give the coefficients for trails and son on.



Example

Acoin is biased so that $P(head)=\frac{2}{3}$ and $P(tail)=\frac{1}{3}$. If the coin is tossed 6 times find the probability of obtaining

a) 6 heads

b) Exactly 5 heads

c) At least 5 heads

d) At least 1 tail

e) 3 heads and a tail with the heads occurring on successive tosses of the coin.

Solution

First list the various combinations of probabilities with $h=P(head)=\frac{2}{3}$, and $t=P(tail)=\frac{1}{3}$.

These are

${}^6C_6h^6 \quad {}^6C_5h^5t \quad {}^6C_4h^4t^2 \quad {}^6C_3h^3t^3 \quad {}^6C_2h^2t^4 \quad {}^6C_1ht^5 \quad {}^6C_0t^6$

or

$1h^6 \quad 6h^5t \quad 15h^4t^2 \quad 21h^3t^3 \quad 15h^2t^4 \quad 6ht^5 \quad 1t^6$

a) $P(6head)=1h^6=\left(\frac{2}{3}\right)^6=\frac{64}{729}$

b) $P(5head)= \quad 6h^5t^1=6\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)=\frac{64}{243}$

c) $\mathrm{P}(\text{at least 5 heads}) = \mathrm{P}(\text{exactly 5 heads}) + \mathrm{P}(5 \text{ heads})$

$=\frac{64}{243}+\frac{64}{729}=\frac{256}{729}$

d) $\mathrm{P}(\text{at least 1 tail}) = 1-\mathrm{P}(\text{no tails}) = 1\text{-}\mathrm{P}(6\text{heads})$

$= 1\text{-}\frac{64}{729} \Rightarrow \frac{665}{729}$

e) $\mathrm{P}(HHHTTT) = \left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^3=\frac{2^3}{3^6}$

However, there are $\frac{4!}{3!}$ ways of arranging 3 heads and 3 tails with the three heads keeping together

HHHTTT, THHHTT, TTHHHT, TTTHHH.

Thus the required probability is $2 \times \frac{2^3}{3^6} = \frac{32}{729}$

Exercise 4.6

1. There are only 2 boys in a group of 6 Pupils. A group of 6 pupils is to selected. Find the probability that both boys are in the group

2. A bag contains 10 discs of which 3 red and 7 are blue. If 6 discs are selected at random, find the probability that all red discs are selected?

3. Five unbiased coins are tossed.

a. Find the probability of getting three heads and two tails

b. The five coins are tossed eight times. Find the probability of getting three heads and two tails exactly four times. Give your answer correct to three decimal places.

4. In a large consignment of trails, 8% are defective. Show that the probability of a random sample of 10 articles containing at least two defectives is approximately 0.19?

5. Four letters are to be chosen from a, b, c, d e, f. find the probability that both vowels are in the group chosen.

6. There are 5b green, 4 yellow and 3 blue discs in a bag. from which 4 discs are chosen at random. Find the probability that 4 discs selected will contain

a) Exactly 3 blue discs

b) Exactly 3 yellow discs

c) At least one green disc

CHAPTER 5: COMPLEX NUMBERS

Objectives: By the end of this chapter students should be able to

- define the complex numbers
- Show that the real and complex number is a field.
- Know that the complex number is not an order field.
- display different parts of complex numbers
- to add, subtract, multiply and divide complex numbers in rectangular form
- to simplify complex numbers
- To solve quadratic equations using the quadratic formula especially imaginary roots
- To change complex numbers from rectangular to polar form and vice versa
- To find product and quotient of complex number in polar form
- To prove de moivres theorem
- To find powers and roots of complex numbers in polar form using de moivres theorem

5.1 Introductory Remark

The Pythagoreans (500-275 B.C) found that the simple equation

$x^2 + 2$ .....(1)

has no real numbers as its solutions . If equation (1) were to have a solution, then a new kind of numbers had to be invented called irrational numbers.

The irrational numbers $\pm\sqrt{2}$ are both solutions to equation (1).

Irrational numbers were not put on a firm mathematical foundation until the nineteenth century.

The rational and irrational numbers together constitute the real number system.

Is there any need to consider another number system? Yes.

If we want to solve the simple quadratic equationx^2+1= 0 there is no solution because there no real number x that can be squared to produce (-1). Once again a new number must be invented.

It was not until the nineteenth century that the irrational numbers were placed on a firm mathematical foundation.

5.2 Complex Number System

DEFİNİTİON:

A complex number is a number of the form a +bi (standard form) where a and b are real numbers and ***i*** is called the imaginary unit.

Names for particular kinds of Complex numbers include:

i Imaginary unıt

$a + bi$ Complex number (where a and b are real number)

$a + bi = bi\,,\ b \neq 0$ Pure imagınary number

$a + bi\ = a$ Real number

$a - b\,i$ Conjugate of $a + bi$

Equality of complex numbers

Two complex numbers $a+ bi$ and $c\ + di$ are equal if and only if $a = c$ and $b = d$

5.3 Operation of Complex numbers in Rectangular form.

Operations with complex numbers

For any complex numbers $a + bi$ and $c + di$ the following are true:

I. $(a + bi) + (c + di) = (a + c) + (b + d)\,i$

II. $(a + bi) - (c + di) = (a - c) + (b - d)i$

III. $(a + bi)\,x\,(c + di) = (ac - bd) + (ad + bc)\,i$

IV. $\frac{a+bi}{c+di} = \frac{(ac+bd)+(bc-ad)i}{c^2+d^2}$

Example l

1. i^{27}

2. $(3 + 4i) + (5 + 2i)$

3. $(8 + 10i) - (6 + 4i)$

4. $(2 + 3i)(4 + 5i)$

5. $\frac{2+7i}{4+3i}$

Solution

1. $i^{27} = i^{24} . i^3 = i^{4(6)} i^3 \, 1^6 . i^3 = i \cdot$
2. $(3 + 4i) + (5 + 2i) = 8 + 6i$
3. $(8 + 10i) - (6 + 4i) = (8 + 10i) + (-6 - 4i) = 2 + 6i$
4. $(2 + 3i)(4 + 5i)$

   = (8-15) + $(10 + 12)i$

   $=-7 + 22i$

5. $\frac{2+7i}{4+3i} == \frac{8+21 + (28-6)i}{16+9}$

   $=\frac{29+22i}{25}=$

   $=\frac{29}{25} + \frac{22}{25} i$

Example 2

Simplify

1) $(3 - 2i)(4 + 3i)$

2) $\frac{4+3i}{12+5i}$

Solution

1)$(3 - 2i)(4 + 3i)$

$=(12 + 6) + (9 - 8)i$

$= 18 + i$

2) $\frac{4-3i}{12+5i}$

$= \frac{(48-15)+(20-36)i}{12^2+5^2}$

$= \frac{33-56i}{169}$

$= \frac{33}{169} - \frac{56}{169} i$

Again we use in(2) the conjugate of $a + bi$ is $a - bi$ and multiply the conjugate

$\frac{4-3i}{12+5i} \cdot \frac{12-5i}{12-5i}$

$=\frac{(48-15)\ +\ (-20-36)i}{12^2+5^2}$

$=\frac{33-5i}{13x13}$

$=\frac{33}{169}$ -- $\frac{56}{169}i$

They are similar.

Theorem:

Product of a complex number and its conjugate is a real number.

$(a+bi)(a-bi)\ =\ a^2\ +b^2$

Example 3

Write in standard form

(1) $\frac{1}{3+4i}$ (2) $\frac{5-3i}{2+i}$

Solution:

1.Multiply numerator and denominator or by the conjugate of denominator

$\frac{1}{3+4i}\cdot\frac{3-4i}{3-4i}$

$=\frac{3-4i}{3^2\ +4^2}$

$=\frac{3-4i}{25}$

$=\frac{3}{25}-\ \frac{4}{25}i$

checking: multiply $(\ 3+4i)$ by the answer

$3+4i\ (\frac{3}{25}-\ \frac{4}{25}i)$

$=\left(\frac{9}{25}+\ \frac{16}{25}+\frac{12}{25}i-\frac{12}{25}i\right)$

$=\frac{25}{25}=1$

2.$\frac{5+3i}{2+i}\cdot\frac{2-i}{2-i}$

$=\frac{10-3\ +(-5-6)i}{4+1}\frac{7-11i}{5}$

$=\ \frac{7}{5}-\ \frac{11}{5}i$

Check:

$(2+i)(\frac{7}{5}-\frac{11}{5}i)$

$= \left(\frac{14}{5}+\frac{11}{5}\right)+\left(\frac{7}{5}-\frac{22}{5}i\right) = \frac{25}{5}-\frac{15}{5}i = 5-3i$

Exercise 1

1) Simplify the following complex numbers then write in standard form.
   a) $(2+4i)^2 \ -3(2+4i)+15$
   b) $\frac{4-5i}{3i}$
   c) $\frac{1}{1-\sqrt{-9}}$
2) Simplify the following
   a) $i^{15}+i^{25}$
   b) 1) $(8+3i)-(1-6i)$
   c) $(i-6)^2$
   d) $\frac{1+i}{2+2i}$
   e) $(4+5i)+(3+2i)$
   f) $(4+3i)(2-i)$
   g) $(-3-10i)(1+5\sqrt{3}i$
   h) $(6+4i)(-3-4i)$
   i) $\frac{3-i}{2+i}$
   j) $\frac{2+4i}{3-2i}$

5.4 Solving Quadratic Equations.

Example 5

Solve the following equations

a)$x^2 - 4x + 13 = 0$

b)$x^2 - 2x + 4 = 0$

c) $x^2 - 2x + 5 = 0$

Solution

a)$x^2 + 9 = 0 \rightarrow x^2 - 9i^2 = 0$ Factarize

$= (x + 3i)(x - 3i)$

$x = -3i \; or \; x = 3i$

b) $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \; substitute$

$= \frac{-4 \pm \sqrt{16 - 4(1)(13)}}{2.1} = \frac{-4 \pm \sqrt{36i}}{2a}$

$= \frac{4 \pm 6}{2} \; i$

$x = 2 + 3i \;\; or \;\; 2 - 3i$

c) $x = \frac{2 \pm \sqrt{4 - 4(1)(5)}}{10}$

$= \frac{2 \pm \sqrt{-16}}{10}$

$= \frac{2 \pm 4i}{10} = \frac{1}{5} + \frac{2}{5}i \; or \; \frac{1}{5} - \frac{2}{5}i$

Example 6

Find the value of x and y that make the statement true

(a)$3x - 5yi = 12 - 20i$

(b) $(x + yi)(2x + yi) = \; -7 + 9i$

Solution

(a) $3x = 12$ and $-5y = 20$

$x = 4$ $y = -4$

(b) $(x + yi)(2x + yi) = \; -7 + 9i$

$2x^2 - \; y^2 + 3xyi = \; -7 + 9i$ then

$2x^2 - \left(\frac{3}{x}\right)^2 = \; -7$

$2x^2 - \frac{9}{x^2} = -7$

$2x^4 - 9 = -7x^2$

$2x^4 + 7x^2 - 9 = 0$

$2x^4 - 2x^2 + 9x^2 - 9 = 0$

$2x^2(x^2 - 1) + 9(x^2 - 1)$

$(2x^2 + 9)(x^2) = 0$

$x^2 = 1 \leftrightarrow x = \pm 1$

Or

$2x^2 + 9 = 0$

$2x^2 = -9$ Imaginary…

The value x=± 1

When x=1 $\rightarrow y = \frac{3}{1} = 3$

When x=-1 $\rightarrow y = \frac{3}{-1} = -3$

$x = \pm 1$ *and* $y = \pm 3$

Exercise 2

1. Solve each equation.
   a) $x^2 + x + 1 = 0$
   b) $x^2 + 4x + 29 = 0$
2. Find the value of x and y that make each sentence true

a) $3x - 5yi = 15 - 20i$

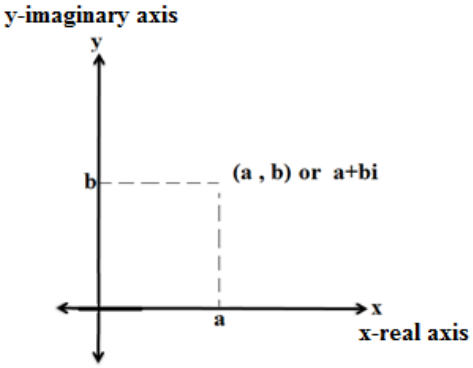
b) $\sqrt{3}x + 7yi = 6 - 2i$

c) $(x - y) + (2x + y)i = -3 + 9i$

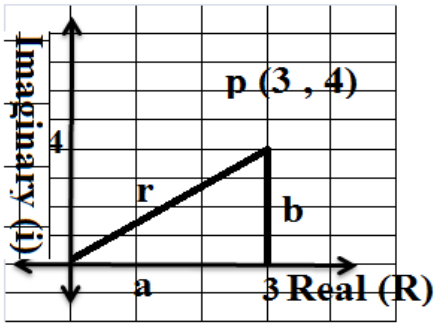
d) $(2x - y) + (x + y)i = -4 - 5i$

Rectangular form of Complex number:

The Complex number can expressed in the form $+bi$, where a and b the real numbers, the complex number is said to be in rectangular form of the complex number sometimes the rectangular form is written as ordered pair (a, b).



To graph the complex number is rectangular form, the horizontal axis is real axis and the vertical axis is the imaginary axis. The point P on the graph at the right is the graph of the complex number $3 + 4i$



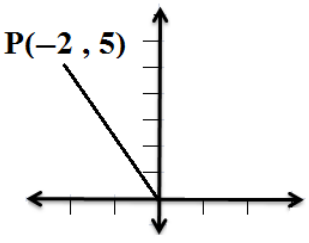
Definition

The absolute value (length of the hypotenuse) of the complex number

$z = a + bi$ is $|a + bi| = \sqrt{a^2 + b^2}$

Example 6

Find the length (r) of complex number $z = -2 + 5i$ and plot.



$r = \sqrt{a^2 + b^2}$

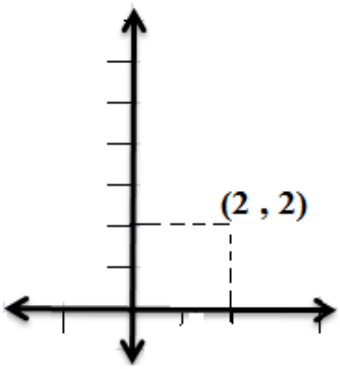
$r = \sqrt{(-2)^2 + (5)^2}$

$r = \sqrt{4 + 25}$

$r = \sqrt{29}$

5.5 Complex Numbers in a Polar form.

The polar form of complex number $z = a + bi$



Is $z = r(cos\theta + i\sin\theta)$ where, $a = rcos\theta$, $b = r\,sin\theta\; r = \sqrt{a^2 + b^2}$ and $tan\theta = \frac{b}{a}$ .or $cos\,\theta = \frac{a}{\sqrt{a^2+b^2}}$ the number r *is* the modulus of z and θ is called argument of z

Example 7:

Convert $Z = 2 + 2i$ in polar form,

Solution

$r = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$

$cos\theta = \frac{x}{r} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}}$

Since the points is the first quadrant $\theta = \frac{\pi}{4}$, the polar form is

$Z = r[cos\theta + i\,sin\theta]$

Then the complex number written in polar form is.

$Z = 2\sqrt{2}\;\left[cos\frac{\pi}{4} + i\,sin\frac{\pi}{4}\right]$

Example 8

Convert $2\sqrt{3} - 2i$ in to polar form.

Solution

$r = \sqrt{x^2 + y^2}$

$r = \sqrt{(2\sqrt{3})^2 + (2)^2}$

$r = \sqrt{4(3) + 4}$

$r = \sqrt{16}\;\; r = 4$

$Cos\theta = \frac{2\sqrt{3}}{4} \quad = \frac{\sqrt{3}}{2}$

$\theta = \frac{\pi}{3}$ sin fourth quadrant this $\theta = 2\pi - \frac{\pi}{3}$

$\theta = \frac{5\pi}{3}$

$z = r[cos\theta + i\,sin\theta]$

$z = r\left[cos\frac{5\pi}{3} + i\,sin\frac{5\pi}{3}\right]$

Example 9

Convert $z = 10[cos\pi + i\,sin\pi]$ into a rectangular form

Solution

we want to find the coordinate $x\;and\;y$

$x = rcos\theta \rightarrow x = 10\,cos\pi = 10$

$y = rsin\theta \rightarrow y = 10\ sin\pi = 0$, then the complex number written in rectangular form is .

$z = 10 + 0i$

$z = 10$

Example 10

Write the complex number $\mathbb{Z} = -2 - 2\sqrt{3}$ in polar form.

Solution

$$r = \left|-2 - 2\sqrt{3\,i}\right| = \sqrt{(-2)^2 + \left(2\sqrt{3}\right)^2}$$

$r = \sqrt{4 + 12}$

$r = \sqrt{16}$

$r = 4$

$$\cos\theta = \frac{-2}{4}$$

$$\cos\theta = \frac{-1}{2} \qquad , \theta = \left(\frac{\pi}{3}\right)$$

And $\mathbb{Z} = -2 - 2\sqrt{3}\,i$ Lies in the third quadrant

You choose $\theta\ to\ \ be\ \theta = \pi + \frac{\pi}{3}$

$$\theta = \frac{4\pi}{3}$$

so, the polar form is

$\mathbb{Z} = r[\cos\theta + i\sin\theta\,]$

$$\mathbb{Z} = 4\left[cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right]$$

Example 11

Write the complex $z = \sqrt{8}\left[cos\left(\frac{-\pi}{3}\right) + i\ sin\left(\frac{-\pi}{3}\right)\right]$ in rectanglur form. (standard form)

Solution

$$z = \sqrt{8}\left[cos\left(\frac{-\pi}{3}\right) + i\ sin\left(\frac{-\pi}{3}\right)\right]$$

$$= \sqrt{8}\left[\left(\frac{1}{2}\right) + i\left(\frac{-\sqrt{3}}{2}\right)\right]$$

$$= 2\sqrt{2}\left[\frac{1}{2} + \left(\frac{-\sqrt{3}}{2}\right)i\right]$$

$= \sqrt{2} - \sqrt{3}\sqrt{2}\,i$

$= \sqrt{2} - \sqrt{6}\,i$

5.6 Product and Quotient of Complex numbers in a Polar form.

When two complex numbers are written in polar form, the product or quotient can easily be computed.

Let $Z_1 = r_1(cos\theta_1 + isin\theta_1)$ and

$Z_2 = r_2(cos\theta_2 + isin\theta_2)$

$$Z_1 . Z_2 = r_1(cos\theta_1 + isin\theta_1) . r_2(cos\theta_2 + isin\theta_2)$$

$$= r_1r_2[cos\theta_1 cos\theta_2 + icos\theta_1 isin\theta_2 + isin\theta_1 cos\theta_2 _ sin\theta_1 sin\theta_2]$$

$$= r_1r_2\big[[cos\theta_1 cos\theta_2 - sin\theta_1 sin\theta_2] + i[cos\theta_1 sin\theta_2 + sin\theta_1 cos\theta_2]\big]$$

$$= \big[[cos(\theta_1 + \theta_2] + i[cos\theta_1 sin\theta_2 + sin\theta_1 cos\theta_2]\big]$$

$$= r_1r_2[\cos(\theta_1 + \theta_2) + i\, sin(\theta_1 + \theta_2)]$$

> The products of two complex numbers
> $Z_1 = r_1(cos\theta_1 + isin\theta_1)$and
> $Z_2 = r_2(cos\theta_2 + isin\theta_2)$ is
> $Z_1 . Z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\, sin(\theta_1 + \theta_2)]$

Example12: Find the product of the two complex number $(Z_1 . Z_2)$

$Z_1 = 2\left[cos\frac{2\pi}{3} + i\, sin\frac{2\pi}{3}\right]$ *and* $Z_2 = 8\left[cos\frac{11\pi}{6} + i\, sin\frac{11\pi}{6}\right]$

Solution

$$Z_1 . Z_2 = 2\left[cos\frac{2\pi}{3} + i\, sin\frac{2\pi}{3}\right]8\left[cos\frac{11\pi}{6} + i\, sin\frac{11\pi}{6}\right]$$

$$= 16\left[cos\left(\frac{2\pi}{3} + \frac{11\pi}{6}\right) + i\, sin\left(\frac{2\pi}{3} + \frac{11\pi}{6}\right)\right]$$

$$= 16\left[cos\frac{15\pi}{6} + i\, sin\frac{15\pi}{6}\right]$$

$$= 16\left[cos\frac{5\pi}{3} + i\, sin\frac{5\pi}{3}\right]$$

$$= 16\left[cos\frac{\pi}{2} + i\, sin\frac{\pi}{2}\right]$$

$$= 16[0 + i\,(1)]$$

$$= 16i$$

Checking

If we write $Z_1$and Z_2 in standard form.

$Z_1 = -1 + \sqrt{3}i$ *and* $Z_2 = 4\sqrt{3} - 4i$

Thais multiplying algebraically we get

$$Z_1.Z_2 = (-1+\sqrt{3}i)(4\sqrt{3}-4i)$$
$$= -4\sqrt{3}i + 4\sqrt{3i} + 4i + 12i$$
$$= 16i$$

The quotient of two complex numbers

Let $Z_1 = r_1(cos\theta_1 + isin\theta_1)$

$Z_2 = r_2(cos\theta_2 + isin\theta_2)$

$$\frac{Z_1}{Z_2} = \frac{r_1(cos\theta_1 + isin\theta_1)}{r_2(cos\theta_2 + isin\theta_2)}$$

Rationalizing the denominator we get

$$\frac{r_1}{r_2}\frac{(cos\theta_1 + isin\theta_1)}{(cos\theta_2 + isin\theta_2)} \cdot \frac{(cos\theta_1 - isin\theta_1)}{(cos\theta_2 - isin\theta_2)}$$

$$\frac{Z_1}{Z_2} = \frac{r_1}{r_2}\left[\frac{cos\theta_1 cos\theta_2 + sin\theta_1 sin\theta_2 + i[sin\theta_1 cos\theta_2 - cos\theta_1 sin\theta_2]}{cos^2\theta_2 + sin^2\theta_2}\right]$$

$$\frac{Z_1}{Z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\ sin(\theta_1 - \theta_2)]$$

The Quotient of two complex number

$Z_1 = r_1(cos\theta_1 + isin\theta_1)$and

$Z_2 = r_2(cos\theta_2 + isin\theta_2)$ is

$$\frac{Z_1}{Z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\ sin(\theta_1 - \theta_2)]$$

$$\frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\ sin(\theta_1 - \theta_2)]$$

Example 13

$Z_1 = 4\left[cos\frac{\pi}{6} + i\ sin\frac{\pi}{6}\right]$ *and* $Z_2 = 2\left[cos\frac{\pi}{3} + i\ sin\frac{\pi}{3}\right]$

Find $\frac{Z_1}{Z_2}$

Solution:

$$\frac{Z_1}{Z_2} = \left[cos\left(\frac{\pi}{6}-\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{6}-\frac{\pi}{3}\right)\right]$$

$$= 2\left[cos\left(\frac{-\pi}{6}\right)+i\sin\left(\frac{-\pi}{6}\right)\right]$$

$$= 2\left[cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right]$$

$$= 2\left[\frac{\sqrt{3}}{2} - \frac{1}{2}i\right]$$

$$= \sqrt{3}-i$$

Example 14

$Z_1 = 24[cos300^0 + i\ sin300^0]$ *and* $Z_2 = 8[cos75^0 + i\ sin75^0]$

Find $\frac{Z_1}{Z_2}$

Solution

$$\frac{Z_1}{Z_2} = \frac{24}{8}[\cos(300^0 - 75^0) + i\ sin(300^0 - 75^0)]$$

$$= 3[\cos 225^0 + i\ sin225^0]$$

$$= 3\left[\frac{-\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right]$$

$$= \frac{-3\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}\ i$$

Exercise 3

Q_2 find each product or quotient. Then write the result in rectangular form.

a) $3(cos30^\circ + isin30^\circ . 2(cos60^\circ + isın\ 60^\circ)$

b) $2\left(cos\frac{3\pi}{4}+isin\frac{3\pi}{4}\right) \div \sqrt{2}\,(cos\frac{2\pi}{3}+isin\frac{2\pi}{3})$

c) $6\left(cos\frac{\pi}{6}+isin\frac{\pi}{6}\right).2(cos\frac{2\pi}{3}+isin\frac{2\pi}{3})$

d) $3\left(cos\frac{7\pi}{6}+isin\frac{7\pi}{6}\right) \div 6(cos\frac{\pi}{6}+isin\frac{\pi}{6})$

e) $3\sqrt{2}\left(cos\frac{\pi}{4}+isin\frac{\pi}{4}\right) \div \sqrt{2}\,(cos\frac{\pi}{2}+isin\frac{\pi}{2})$

f) $3(cos\pi + isin\pi) \div 2(cos\frac{\pi}{6}+isin\frac{\pi}{6})$

g) $8\left(cos\frac{3\pi}{2}+isin\frac{3\pi}{2}\right) \div \frac{4}{5}(cos\frac{\pi}{2}+isin\frac{\pi}{2})$

5.7 De moivers theorem

Abraham Demoivre $(1667-1754)$ was a mathematician. He worked the study of probability, also worked extensively in number theory and finding power of a complex number written in polar form. Which bears his name, in this lesson.

If $Z = r[cos\theta + i\, sin\theta]$

$Z^n = r^n[conr\theta + i\, sinn\theta]$

Proof

Let$Z = r[cos\theta + i\, sin\theta]$

$Z^2 = r[con\theta + i\, sin\theta].r[con\theta + i\, sin\theta]$

$r^2[con2\theta + i\, sin2\theta]$we have seen in product of two complex numbers.

$Z^3 = Z^2.Z = r^2[cos2\theta + i\, sin2\theta]\,.r[cos\theta + i\, sin\theta]$

$= r^3\big[cos2\theta con\theta - sin2\theta sin\theta + i[\, cos2\theta sin\theta + sin2\theta cos\theta]\big]$

$=r^3[cos(2\theta + \theta) + i\, sin(2\theta + \theta)]$

$=r^3[cos3\theta + i\, sin3\theta]$

Similarly

$Z^4 = r^4[cos4\theta + i\, sin4\theta]$

Then:

$Z^n = r^n[conr\theta + i\, sinn\theta]$, for n is any positive integer

5.8 Powers and Roots of Complex numbers.

Power and roots of complex numbers by using the De moivre's theorem

Example15:

$Z = (1-i)$ find Z^4 in

a) rectangular form b) polar form

Solution

a. rectangular

$Z^4 = (1-i)((1-i)((1-i)((1-i)$if we multiple

$Z^2 = (1-i)((1-i)$

$= 1-1-2i$

$= -2i$

$Z^4 = Z^2\,.Z^2$

$= (-2i)(-2i)$

$= -4$

b. polar form

$r = \sqrt{(1)^2 + (-1)^2}$

$r = \sqrt{2}$

$cos\theta = \frac{1}{\sqrt{2}}$

$cos\theta = \frac{\sqrt{2}}{2}$

$\theta = \frac{\pi}{4}$in the 4^{th} quadrant.

$\theta = 2\pi - \frac{\pi}{4}$

$= \frac{7\pi}{4}$

$Z = \sqrt{2}\left[cos\frac{7\pi}{4} + i\ sin\frac{7\pi}{4}\right]$

$Z^4 = \left(\sqrt{2}\right)^4\left[cos\frac{(4)\ 7\pi}{4} + i\ sin\frac{(4)\ 7\pi}{4}\right]$

$Z^4 = 4[cos7\pi + i\ sin7\pi]$

In rectangular from

$Z^4 = 4[cos7\pi + i\ sin7\pi]$

$Z^4 = 4(-1) + 0$

$Z^4 = -4$

Example 16

Let Z= $(-2 + 2\sqrt{3}i\ )$ find Z^6 ?

Solution

Rectangular from is difficult because we multiple 6 times the complex number $(-2 + 2\sqrt{3}i\ )$.

We use in polar form

$r = \sqrt{(-2)^2 + (2\sqrt{3})^2}$

$r = \sqrt{4 + 12}$

$r = 4$

$cos\theta = \frac{-2}{4} = \frac{-1}{2}$

$\theta = \frac{\pi}{3}$in second quadrant that means

$$\theta = \pi - \frac{\pi}{3}$$

$$\theta = \frac{2\pi}{3}$$

The polar form is

$$Z = 4\left[cos\frac{2\pi}{3} + sin\frac{2\pi}{3}\right]$$

$$Z^6 = 4^6\left[cos\left(6.\frac{2\pi}{3}\right) + i\ sin\left(6.\frac{2\pi}{3}\right)\right]$$

$$= 4096[\cos(4\pi) + i\ \sin(4\pi)]$$

$$= 4096[(1) + i\ 0]$$

$$= 4096$$

Roots of a complex number

n^{th}Roots of a complex number for a positive integer n, the complex number $Z = r[cos\theta + i\sin\theta]$ has a exactly a distinct n^{th} roots given by

$$Z^{\frac{1}{n}} = \left[soc\left(\frac{\theta + 2\pi k}{n}\right) + i\ sin\left(\frac{\theta + 2\pi k}{n}\right)\right]$$

Where $k = 0,1,2,3,4,5, \ldots \ldots \ldots \ldots \ldots ,, n-1$

Let $Z = 2 - 2\sqrt{3}i$ find square root of Z?

Solution

$z^2 = 2 - 2\sqrt{3}i$In rectangular from

$Z = a + bi$

$Z^2 = a^2 - b^2 + 2abi$

$(a-b) + 2abi = 2 - 2\sqrt{3}i$Equationof complex number

$a^2 - b^2 = 2 \ldots \ldots \ldots \ldots \ldots \ldots \ldots (1)$

$2ab = -2\sqrt{3} \ldots \ldots \ldots \ldots \ldots \ldots . . (2)$

Solving (b) in equation (2)

$$b = \frac{-\sqrt{3}}{a}$$

Substitution the value of b in equation (1)

$$a^2 - \left(\frac{-\sqrt{3}}{a}\right)^2 = 2$$

$$a^2 - \frac{\sqrt{3}}{a^2} = 2$$

$$a^4 - 2a^2 - 3 = 0$$

$$(a^2 + 1)(a^2 - 3) = 0$$

$$a^2 + 1 = 0 \quad or \quad a^2 - 3 = 0$$

$$a^2 = -1 \quad or \quad a^2 = 3$$

$$a = \sqrt{-1} \ or \ a = \pm\sqrt{3}$$

Since a is real number we take $a = \pm\sqrt{3}$

When $a = \sqrt{3} \rightrightarrows \quad b = \frac{-\sqrt{3}}{\sqrt{3}} \quad = 1$

$$a = -\sqrt{3} \ \rightrightarrows \quad b = \frac{-\sqrt{3}}{-\sqrt{3}} \quad = 1$$

The square of $(Z - 2\sqrt{3}i)$ is

$$Z_1 = \sqrt{3} - i \quad or \ Z_2 = -\sqrt{3} + i$$

The simplest way is in a polar form.

The above Example find square root by using in polar form.

if $Z = 2 - 2\sqrt{3}i$

$$Z^{\frac{1}{2}} = ?$$

$$r = \sqrt{(2)^2 + \left(-2\sqrt{3}\right)^2}$$

$$= \sqrt{4 + 12}$$

$$= 4$$

$$Z^{\frac{1}{2}} = r^{\frac{1}{2}}\left[cos\left(\frac{\theta + 2\pi k}{2}\right) + i\ sin\left(\frac{\theta + 2\pi k}{2}\right)\right]$$

$$r = \sqrt{4}$$

$$= 2$$

$$\theta = cos^{-1}\left(\frac{1}{2}\right)$$

$= \frac{\pi}{3}$Infourth quadrant.

$$\theta = 2\pi - \frac{\pi}{3}$$

$= \frac{5\pi}{3}$

$$Z = 2\left[cos\left(\frac{\frac{5\pi}{3} + 2\pi k}{2}\right) + i\, sin\left(\frac{\frac{5\pi}{3} + 2\pi k}{2}\right)\right]$$

Where $k = 0, 1$

$Z_1 = 2\left[cos\frac{5\pi}{6} + i\, sin\frac{5\pi}{6}\right]$ *or* $Z_2 = 2\left[cos\frac{11\pi}{6} + i\, sin\frac{11\pi}{6}\right]$

$= 2\left[-\frac{\sqrt{3}}{2} + \frac{\sqrt{1}}{2}\right]$ $= 2\left[\frac{1}{2} - \frac{\sqrt{3}}{2}i\right]$

$= -\sqrt{3} + i$ $= 1 - \sqrt{3}\, i$

It's similar to the first.

Example 17:

Find the cube root of -8

Solution

$Z = -8 + 0i$

$Z^{\frac{1}{2}} = ?$

$r = 8$

$r^3 = 8$

$\Rightarrow r = 2$

$cos\theta = \frac{-8}{8} = -1$

$= \cos(-1)$

$= \pi$

$$Z^{\frac{1}{3}} = 2\left[cos\left(\frac{\pi + 2\pi k}{3}\right) + i \sin\left(\frac{\pi + 2\pi k}{3}\right)\right]$$

Where $k = 0, 1, 2$

$Z_1 = 2\left[cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right)\right]$, $Z_2 = 2[cos\pi + i \sin\pi]$, $Z_3 = 2\left[cos\frac{5\pi}{3} + i\ \sin\frac{5\pi}{3}\right]$

$= 1 + \sqrt{3}$ $= -2$ $= 1 - \sqrt{3}\, i$

Exercise 4

1. if s_i and s_2 roots of a quadratic equation $ax^2+bx+c=0$ then $s_1+s_2=-\frac{b}{a}$ and $s_1.s_2=\frac{c}{a}$ verify that $4\sqrt{2}\,(cos\frac{\pi}{4} + isin\frac{\pi}{4})$ and $4\sqrt{2}\,\{(\cos\left(-\frac{\pi}{4}\right) + isin\left(-\frac{\pi}{4}\right)\}$ are the polar form of the roots of $x^2 - 8x + 32 = 0$.

2. Find each power Express the vesult is rectangular form.
   a) $(2+2i)^8$
   b) $(-1+i)^4$
   c) $\left(-2-2\sqrt{3}\right)^3$
   d) $(\sqrt{3}-i)^7$

3. Solve each quation then
   a) $x^3+1=0$
   b) $x^3+8=0$
   c) $x^3-8=0$

4. find each power roots Express the form $a+bi$.
   a) $\left(-\sqrt{3}+i\right)^4$
   b) $\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}\right)^6$
   c) $(-1+i)^4$
   d) $\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^8$
   e) $[(\cos 40+i\sin 40)]^4$
   f) $(1-i)^6$
   g) $\left(\sqrt{3}-i\right)^6$
   h) $\left(-\frac{\sqrt{3}+i}{2}\right)^{20}$
   i) $\left(1-i\sqrt{3}\right)^8$

3. Find the roots of Z. write the answer in form a+bi
   a) $Z^3 = 8$
   b) $Z^4 = 1$
   c) $Z^4 = -16$
   d) $Z^5 = -i$
   e) Z^6

REFERENCE LISTS

Sandler, A.J. & Thorning, D.W.(2004).Understanding pure mathematics. Uk:Oxford university press.

Federal Democratic Republic of Ethiopia, Ministry of Education (2010). *Mathematics student textbook*(grades). ADDIS ABABA, ETHIOPIA: Star educational books distributors' pvt. Ltd.

Urban. P, Owen. J. , Martin, D., Haese, R. Haese, S. & Bruce, M.(2004). *Mathematics for the international student Mathematics HL (Core).* International Baccalaureate Diploma Programme. Australia: Haese & Harris Publications