Chapter 4

Derivatives and Application of derivatives

Tangent Line:

Definition *The tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$is the line through with slope*

$$m = lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

Example 1 Find an equation of the tangent line to the parabola $y = x^2$ at the point(1,1) .

If we write $= a + h$, then $h = x - a$. Then h approaches 0 if and only if x approaches a. Therefore, an equivalent way of stating the definition of the derivative, as we saw in finding tangent lines:

$$f'(x) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

The tangent line to y=f(x) at (a, f(a)) is the line through (a, f(a)) whose slope is equal to f'(a), then the derivative of f at a is defined as:

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

Example 2: Find the derivative of the function $f(x) = x^2 - 8x + 9$ at the number a.

Definitions of Derivative

Definition 1: The **derivative of a function *f* at** a point a , denoted by $f'(a)$, is given by:

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(h)}{h}$$

Definition 2: The **derivative of a function *f* at** a variable *x*, denoted by $f'(x)$, is given by:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(h)}{h}$$

Given any number *x* for which this limit exists, we assign to *x* the number$f'(x)$. So we can regard as a new function, called the **derivative of*f***.

Example 3 Use the definitions of derivatives find the derivative of f if $f(x) = \sqrt{x-1}$, State the domain of f'.

Solution: $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h-1}+\sqrt{x-1}}{h} = \lim_{h\to 0} \frac{\sqrt{x+h-1}-\sqrt{x-1}}{h} \cdot \frac{\sqrt{x+h-1}+\sqrt{x-1}}{\sqrt{x+h-1}+\sqrt{x-1}}$

$$= \lim_{h\to 0} \frac{(x+h-1)-(x-1)}{h(\sqrt{x+h-1}+\sqrt{x-1})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h-1}+\sqrt{x-1}} = \frac{1}{\sqrt{x-1}+\sqrt{x-1}} = \frac{1}{2\sqrt{x-1}}$$

We see that $f'(x)$ exists if $x > 1$, so the domain of f' is $(1, \infty)$.

Other Notations

If $y = f(x)$ to indicate that the independent variable is x and the dependent variable is y, then some common alternative notations for the derivative are as follows:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = Df(x) = D_x f(x)$$

If we want to indicate the value of a derivative d/dx in Leibniz notation at a specific number, we use the notation

$$\frac{dy}{dx}\Big|_{x=a} \qquad or \qquad \frac{dy}{dx}\Big]_{x=a}$$

which is a synonym for $f'(a)$.

Definition A function f is said to be **differentiable at *a*** if $f'(a)$ exists. It is **differentiable on an open interval** (a, b)[or (a, ∞) or $(-\infty, a)$or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

EXAMPLE 4 Where is the function $f(x) = |x|$ differentiable?

SOLUTION If $x > 0$, then

$$f'(x) = \lim_{h\to 0} \frac{|x+h|-|x|}{h} = \lim_{h\to 0} \frac{(x+h)-x}{h} = \lim_{h\to 0} \frac{h}{h} = \lim_{h\to 0} 1 = 1$$

and so f is differentiable for any $x > 0$.

Similarly, for $x < 0$ we have $|x| = -x$. Then

$$f'(x) = \lim_{h\to 0} \frac{|x+h|-|x|}{h} = \lim_{h\to 0} \frac{-(x+h)-(-x)}{h} = \lim_{h\to 0} \frac{-h}{h} = \lim_{h\to 0}(-1) = -1$$

and so f is differentiable for any $x < 0$.

$f'(0)$ does not exist. (Why?)

Theorem : If f is differentiable at a, then f is continuous at a.(The converse is not true)

Example 5 : $f(x) = |x|$

Derivatives of Polynomials and Exponential Functions

Constant Functions

Derivative of a Constant Function $\frac{d}{dx}(c) = 0$

Power Functions

The Power Rule: If *n* is any real number, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

EXAMPLE 6

(a) If $f(x) = x^6$, then$f'(x) = 6x^5$. (b) If $y = x^{1000}$,then $y' = 1000x^{999}$

(c) If $y = t^4$, then $\frac{dy}{dt} = 4t^3$. (d) $\frac{d}{dr}(r^3) = 3r^2$.

$(e)\ f(x) = \frac{1}{x^2} \Longrightarrow \frac{dy}{dx} = -\frac{2}{x^3}$ *(f)* $y = \sqrt[3]{x^2}$ exercise

The Constant Multiple Rule. If *c* is a constant and is a differentiable function, then

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$$

EXAMPLE 7

(a) $\frac{d}{dx}(3x^4) = 3\frac{d}{dx}(x^{4)} = 3(4x^3) = 12x^3$.

The Sum/Difference Rule If *f* and g are both differentiable, then

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$$

EXAMPLE 8 : $\frac{d}{dx}(x^8 + 12x^5 - 4x^4 + 10x^3 - 6x + 5)$

$=8x^7 + 60x^4 - 16x^3 + 30x^2 - 6$

The Product Rule If *f* and g are both differentiable, then

$$\frac{d}{dx}[f(x).g(x)] = f(x)\frac{d}{dx}\big(g(x)\big) + g(x)\frac{d}{dx}(f(x))$$

EXAMPLE 9

Find the derivative of$f(x) = (x^2 - 1)(10x^{3/2})$

The Quotient Rule If *f* and g are both differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f(x)\frac{d}{dx}(g(x)) - g(x)\frac{d}{dx}(f(x))}{[g(x)]^2},$$

Provided that $g(x) \neq 0$.

EXAMPLE 10 : Find the derivative *of* $f(x) = \frac{x^2}{x^4+1}$

Exponential Functions

The Derivative of the exponential function $f(x) = a^x$:

$$f'(x) = a^x ln a$$

Example 11 : $\frac{d}{dx}(2^x) = 2^x ln 2$.

Definition of the Number *e*

e i s the number such that $\lim_{h\to 0} \frac{e^h - 1}{h} = 1$

Note: $e \approx 2.71828$

Derivative of the Natural Exponential Function

$$\frac{d}{dx}(e^x) = e^x$$

EXAMPLE 12 If $f(x) = e^x - x$, find f'.

Derivatives of a Logarithmic Function

$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$ and $\frac{d}{dx}(\ln x) = \frac{1}{x}$

EXAMPLE 13 Differentiate $y = \ln(x^3 + 1)$.

Derivatives of Trigonometric Functions

Example 14 Use definitions of derivatives show that the derivatives of $f(x) = sinx$ is

$$f'(x) = cosx$$

Derivatives of Trigonometric Functions

$$\frac{d}{dx}(sinx) = cosx \qquad \frac{d}{dx}(cscx) = -cscxcotx$$

$$\frac{d}{dx}(cosx) = -sinx \qquad \frac{d}{dx}(secx) = secxtanx$$

$$\frac{d}{dx}(tanx) = sec^2 x \qquad \frac{d}{dx}(cotx) = -csc^2 x$$

EXAMPLE 15 **Differentiate** $f(x) = \frac{secx}{1+tanx}$.

Solution: $\boldsymbol{f'(x) = \frac{secx(tanx-1)}{(1+tanx)^2}}$

The Chain Rule

The Chain Rule If f and g are both differentiable and $F = fog$ is the composite function f *and* g where $(fog)(x) = f(g(x))$, then F is differentiable, denoted by F' is defined as:

$$F'(x) = f'(g(x))g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

EXAMPLE 16 : Find $F'(x)$ if $F(x) = \sqrt{x^2+1}$.

SOLUTION

We expressed F as $F(X) = (fog)(x) = f\big(g(x)\big)$ *where* $f(u) = \sqrt{u}$ *and* $g(x) = x^2 + 1$. Since

$$f'(u) = \frac{1}{2}u^{-1/2} = \frac{1}{2\sqrt{u}} \text{ and } g'(x) = 2x$$

Then $$F'(x) = f'\big(g(x)\big)g'(x) = \frac{1}{2\sqrt{x^2+1}}.2x = \frac{x}{\sqrt{x^2+1}}$$

The Power Rule Combined with the Chain Rule

If n is any real number and $u = g(x)$ is differentiable, then

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

Alternatively, $$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1}.g'(x)$$

EXAMPLE 17 : Differentiate $y = (x^3 - 1)^{100}$.

Implicit Differentiation

The functions that we have met so far can be described by expressing one variable explicitly in terms of another variable—for example,

$$y = \sqrt{x^3+1} \quad \text{or} \quad y = \sin x$$

Or, in general, $= f(x)$. Some functions are defined implicitly by a relation between $x$and y such as

$$x^2 + y^2 = 25 \quad \text{or} \quad x^3 + y^3 = 6xy$$

Implicit differentiation consists of differentiating both sides of the equation with respect to x and then solving the resulting equation for y'.

EXAMPLE 18

If $x^2 + y^2 = 25$, then find $\frac{dy}{dx}$.

Solution

Differentiate both sides of the equation $x^2 + y^2 = 25$:

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$

Since $$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2)\frac{dy}{dx} = 2y\frac{dy}{dx}$$

Thus $$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

Derivatives of Inverse Trigonometric Functions

Arcsine Function:

Recall the definition of the arcsine function:

$$y = sin^{-1}x \quad \Rightarrow \quad \sin y = x, \text{ where } -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

Differentiating $\sin y = x$ implicitly with respect to x, we obtain

$$\cos y \frac{dy}{dx} = 1 \qquad or \qquad \frac{dy}{dx} = \frac{1}{\cos y}$$

Now $\cos y \ge 0, since\ {}^{-\pi}/_{2} \le y \le {}^{\pi}/_{2}, so$

$$\cos y = \sqrt{1 - sin^2 y} = \sqrt{1 - x^2}$$

Thus $\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$

$$\boxed{\frac{d}{dx}(sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}}$$

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}(sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(csc^{-1}x) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(tan^{-1}x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(cot^{-1}x) = -\frac{1}{1+x^2}$$

EXAMPLE 19 Differentiate (a) $y = \frac{1}{sin^{-1}x}$ and (b)$f(x) = xarctan\sqrt{x}$.

Solution:

(a) $\frac{dy}{dx} = \frac{d}{dx}(sin^{-1}x)^{-1} = -(sin^{-1}x)^{-2}\frac{d}{dx}(sin^{-1}x)$

$$= -\frac{1}{(sin^{-1}x)^2\sqrt{1-x^2}}$$

(b) Exercise

Higher Derivatives

If f is a differentiable function, then its derivative f' is also a function, so f' may have a derivative of its own, denoted by $(f')' = f''$. This new function f'' is called the **second derivative** of f because it is the derivative of f'.Using Leibniz notation; we write the second derivative of $y = f(x)$ as

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

Another notation is $f''(x) = D^2f(x)$.

The **third derivative** $f'''$is the derivative of the second derivative: $f''' = (f'')'$.

$$y''' = f'''(x) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} = D^3f(x)$$

The process can be continued. The fourth derivative f'''' is usually denoted by $f^{(4)}$. In general, the n^{th} derivative of f is denoted by $f^{(n)}$ and is obtained$f$ from by differentiating n times. If $y = f(x)$, then we write

$$y^{(n)} = f^{(n)}(x) = \frac{d^n y}{dx^n} = D^n f(x)$$

EXAMPLE 20

If $f(x) = x\cos x$, then find $f''(x)$.

Hyperbolic Functions

Definition of the Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$

Hyperbolic Identities

$$\sinh(-x) = -\sinh x \qquad \cosh(-x) = \cosh x$$

$$\cosh^2 x - \sinh^2 x = 1 \qquad 1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$

EXAMPLE 21 Prove (a) $\cosh^2 x - \sinh^2 x = 1$ (b) $1 - \tanh^2 x = \operatorname{sech}^2 x$

Solution

(a) $\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} + 2 + e^{-2x}}{4} = \frac{4}{4} = 1$

(b) Exercise

Derivatives of Hyperbolic Functions

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x$$

We list the differentiation formula for the hyperbolic functions in the following Table.

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

$$\frac{d}{dx}(\cosh x) = \sinh x \qquad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

Example 22 $\frac{d}{dx}(\cosh\sqrt{x}) = \sinh\sqrt{x}.\frac{d}{dx}\sqrt{x} = \frac{\sinh\sqrt{x}}{2\sqrt{x}}$

Inverse Hyperbolic Functions

$$y = sinh^{-1}x \Leftrightarrow \sin hy = x$$
$$y = cosh^{-1}x \Leftrightarrow \cosh y = x \ \ and\ y \geq 0$$
$$y = tanh^{-1}x \Leftrightarrow \tanh y = x$$

Inverse hyperbolic functions can be expressed in terms of logarithms.

In particular, we have:

$\cosh^{-1} x = \ln(x+\sqrt{x^2+1})$ $\quad x \in R$

$\sinh^{-1} x = \ln(x+\sqrt{x^2+1})$ $\quad x \geq 1$

$\tanh^{-1} x = \frac{1}{2}\ln(\frac{1+x}{1-x})$ $\quad -1 < x < 1$

EXAMPLE 23 Show that $sinh^{-1}x = \ln\left(x+\sqrt{x^2+1}\right)$.

Solution Let $y = sinh^{-1}x$.Then

$$x = \sinh y = \frac{e^y - e^{-y}}{2}$$

So $\quad e^y - 2x - e^{-y} = 0$

Or, multiplying by e^y,

$$e^{2y} - 2xe^y - 1 = 0$$
$$(e^y)^2 - 2x(e^y) - 1 = 0$$
$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}$$

Note that $e^y > 0$, but $x - \sqrt{x^2+1} < 0$. Thus, the minus sign is inadmissible and we have

$$e^y = x + \sqrt{x^2+1}$$

Therefore $\quad y = \ln(e^y) = \ln(x+\sqrt{x^2+1})$

Derivatives of Inverse Hyperbolic Functions

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}} \qquad \frac{d}{dx}(\operatorname{csch}^{-1}x) = -\frac{1}{|x|\sqrt{x^2+1}}$$

$$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}} \qquad \frac{d}{dx}(\operatorname{sech}^{-1}x) = -\frac{1}{x\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2} \qquad \frac{d}{dx}(\coth^{-1} x) = -\frac{1}{1-x^2}$$

Exercise 24 : Prove that $\frac{d}{dx}(sinh^{-1}x) = \frac{1}{\sqrt{1+x^2}}$.

Applications of Differentiation



Maximum and Minimum Values

Definition A function f has an **absolute maximum** (or **global maximum**) at c if $f(c) \geq f(x)$ for all x in D, where D is the domain of f . The number $f(c)$ is called the **maximum value** of $f$on D. Similarly,$f$has an **absolute minimum** at c if $f(c) \leq f(x)$for all x in$D$ and the number $f(c)$ is called the **minimum value** of $f$on D. Themaximum and minimum values of$f$ are called the **extreme values** of f.

Figure 1 shows the graph of a function f with absolute maximum at d and absolute minimum at a. Note that$(d, f(d))$ is the highest point on the graph and$(a, f(a))$ is the lowestpoint.

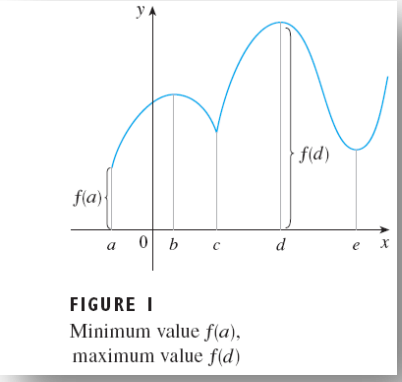


FIGURE 1
Minimum value $f(a)$,
maximum value $f(d)$

Definition A function $f$has a **local maximum** (or **relative maximum**) at c if $f(c) \leq f(x)$when x is near c. [This means $f(c) \geq f(x)$ that for all x in some open interval containing c.] Similarly,f has a **local minimum** at c if $f(c) \leq f(x)$ when $x$is near c.

EXAMPLE 1 The function $f(x) = \cos x$ takes on its (local and absolute) maximum value of 1 infinitely many times, since$\cos 2n\pi = 1$ for any integer$n$ and$-1 \leq \cos x \leq 1$ forall$x$. Likewise,$\cos(2n + 1)\,\pi = -1$ is its minimumvalue, where$n$ is any integer.

EXAMPLE 2 If$f(x) = x^2$, then$f(x) \geq f(0)$ because$x^2 \geq 0$ for all. Therefore,$f(0) = 0$is the absolute (and local) minimum value of$f$. This corresponds to the fact that theorigin is the lowest point on the parabola$= x^2$. However, there is no highest point on the parabola and so this function has no maximum value.

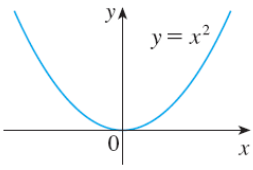
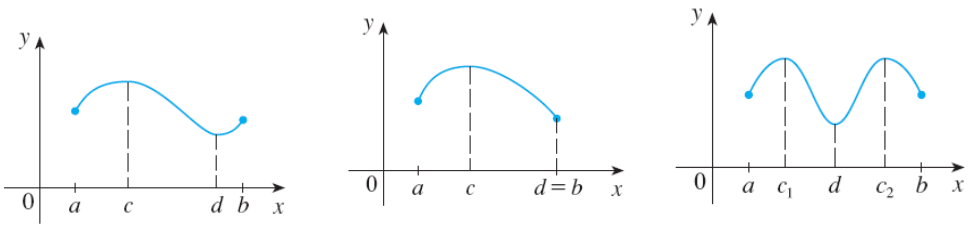


FIGURE 2
Minimum value 0, no maximum

The Extreme Value Theorem If f is continuous on a closed interval $[a, b]$, then $f$attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

The Extreme Value Theorem is illustrated in Figure 5.



Figures 6 and 7 show that a function need not possess extreme values if either hypothesis (continuity or closed interval) is omitted from the Extreme Value Theorem.

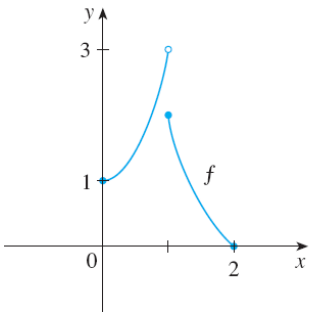


FIGURE 6
This function has minimum value $f(2) = 0$, but no maximum value.

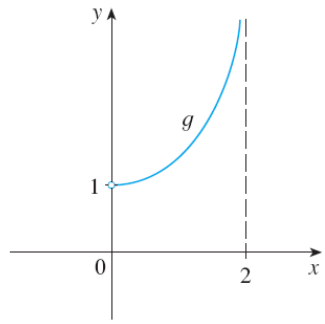


FIGURE 7
This continuous function g has no maximum or minimum.

Fermat's Theorem If f has a local maximum or minimum at c, and if $f'(c)$exists, then $f'(c) = 0$.

Definition A **critical number** of a function f is a number c in the domain of $f$such that either $f'(0) = 0$ or $f'(c)$does not exist.

EXAMPLE 7 Find the critical numbers of$f(x) = x^{3/5}(4 - x)$.

SOLUTION The Product Rule gives

$$f'(x) = \frac{3}{5}x^{-2/5}(4 - x) + x^{3/5}(-1) = \frac{3(4-x)}{5x^{2/5}} - x^{3/5}$$

$$= \frac{3(4-x)-5x}{5x^{2/5}} = \frac{12-8x}{5x^{2/5}}$$

[The same result could be obtained by first writing$(x) = 4x^{3/5} - x^{8/5}$.] Therefore,$f'(x) = 0$ if $12 - 8x = 0$, that is$x = \frac{3}{2}$, and$f'(x)$ does not exist when$x = 0$. Thus, thecritical numbers are$\frac{3}{2}$ and 0.

If f has a local maximum or minimum at c, then c is a critical number of f.

To find an absolute maximum or minimum of a continuous function on a closed interval:

The Closed Interval Method To find the *absolute* maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).

2. Find the values of f at the endpoints of the interval.

3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value

EXAMPLE 8 Find the absolute maximum and minimum values of the function

$$f(x) = x^3 - 3x^2 + 1 \qquad -1/2 \le x \le 4$$

SOLUTION Since f is continuous on $[-\frac{1}{2}, 4]$, we can use the Closed Interval Method:

$$f(x) = x^3 - 3x^2 + 1$$

$$f'(x) = 3x^2 - 6x = 3x(x - 2)$$

Since $f'(x)$ exists for all x, the only critical numbers of f occur when $f'(x) = 0$, that is, $x = 0$ or $x = 2$ .Notice that each of these critical numbers lies in the interval $(-\frac{1}{2}, 4)$.

The values of f at these critical numbers are

$$f(0) = 1 \qquad f(2) = -3$$

The values of f at the endpoints of the interval are

$$f\left(-\tfrac{1}{2}\right) = \tfrac{1}{8} \qquad f(4) = 17$$

Comparing these four numbers, we see that the absolute maximum value is $f(4) = 17$ and the absolute minimum value is $(2) = -3$.

Note that in this example the absolute maximum occurs at an endpoint, whereas the absolute minimum occurs at a critical number.

The Mean Value Theorem

To arrive at the Mean Value Theorem we first need the following result.

Rolle's Theorem Let f be a function that satisfies the following three hypotheses:

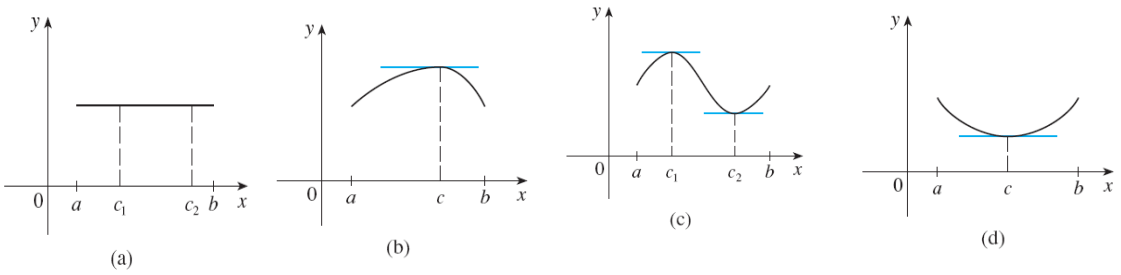
1. $\boldsymbol{f}$ is continuous on the closed interval $[a, b]$.

2. $\boldsymbol{f}$ is differentiable on the open interval (a, b).

3. $\boldsymbol{f(a) = f(b)}$

Then there is a number c in (a, b) such that $f'(c) = 0$.

Figure 1 shows the graphs of four such functions. In each case it appears that there is at least one point $(c, f(c))$ on the graph where the tangent is horizontal and therefore $f'(c) = 0$. Thus, Rolle's Theorem is plausible.



EXAMPLE Prove that the equation $x^3 + x - 1 = 0$has exactly one real root.

SOLUTION First we use the Intermediate Value Theorem to show that a root exists.

Let $(x) = x^3 + x - 1$. Then $f(0) = -1 < 0$and $f(1) = 1 > 0$. Since f is a polynomial, it is continuous, so the Intermediate Value Theorem states that there is a number $c$between 0 and 1 such that $(c) = 0$. Thus, the given equation has a root.

To show that the equation has no other real root, we use Rolle's Theorem and argue by contradiction. Suppose that it had two roots $a$and b. Then $f(a) = 0 = f(b)$ and, since $f$is a polynomial, it is differentiable on (a, b) and continuous on $[a, b]$. Thus, by Rolle's Theorem, there is a number c between a and b such that $f'(c) = 0$. But

$$f'(x) = 3x^2 + 1 \geq 1 \qquad for\ all\ x$$

(since$x^2 \geq 0$) so$f'(x)$ can never be 0. This gives a contradiction. Therefore, the equationcan't have two real roots.

The Mean Value Theorem Let $f$be a function that satisfies the following hypotheses:

1. $f$is continuous on the closed interval $[a, b]$.

2. f is differentiable on the open interval (a, b).

Then there is a number$c$ in (a, b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

or, equivalently,

$$f(b) - f(a) = f'(c)(b - a)$$

EXAMPLE To illustrate the Mean Value Theorem with a specific function, let's consider $f(x) = x^3 - x, a = 0, b = 2$. Since f is a polynomial, it is continuous and differentiable for all , so it is certainly continuous on $[0,2]$ anddifferentiable on$(0,2)$. Therefore, bythe Mean Value Theorem, there is a number$c$ in$(0,2)$ such that

$$f(2) - f(0) = f'(c)(2 - 0)$$

Now$(2) = 6, f(0) = 0$, and$f'(x) = 3x^2 - 1$, so this equation becomes

$$6 = (3c^2 - 1)2 = 6c^2 - 2$$

Which gives$c^2 = \frac{4}{3}$, that is, $c = \pm 2/\sqrt{3}$. But$c$ must lie in$(0,2)$, so $c = 2/\sqrt{3}$.

How Derivatives Affect the Shape of a Graph

What Does f' Say about f?

Increasing/Decreasing Test

(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

EXAMPLE 1 Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing.

SOLUTION $f'(x) = 12x^3 - 12x^2 - 24x = 12x(x-2)(x+1)$

To use the I/D Test we have to know where $f'(x) > 0$ and where $f'(x) < 0$. This depends on the signs of the three factors of $f'(x)$, namely, $12x$,$x-2$, and $x+1$. We divide the real line into intervals whose endpoints are the critical numbers -1,0 , and 2and arrange our work in a chart. A plus sign indicates that the given expression is positive, and a minus sign indicates that it is negative. The last column of the chart gives the conclusion based on the I/D Test. For instance,$f'(x) < 0$for $0 < x < 2$, so f is decreasing on (0, 2). (It would also be true to say that f is decreasing on the closed interval $[0,2]$.)

Interval	$12x$	$x-2$	$x+1$	$f'(x)$	f
$x < -1$	$-$	$-$	$-$	$-$	$decreasing\ on\ (-\infty,-1)$
$-1 < x < 0$	$-$	$-$	$+$	$+$	$increasing\ on\ (-1,0)$
$0 < x < 2$	$+$	$-$	$+$	$-$	$decreasing\ on\ (0,2)$
$x > 2$	$+$	$+$	$+$	$+$	$increasing\ on\ (2,\infty)$

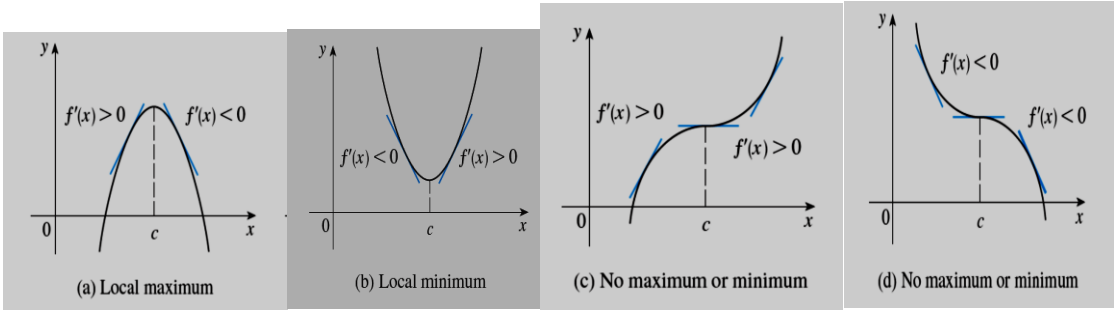
The First Derivative Test Suppose that c is a critical number of a continuous function f.

(a) If f'changes from positive to negative at c , then f has a local maximum at c.

(b) If f'changes from negative to positive at c, then f has a local minimum at c.

(c) If f'does not change sign at c (for example, if f' is positive on both sides of c or negative on both sides), then f has no local maximum or minimum at c.

It is easy to remember the First Derivative Test by visualizing diagrams such as those in Figure 3.



EXAMPLE 2 Find the local minimum and maximum values of the function $f$in Example 1 above.

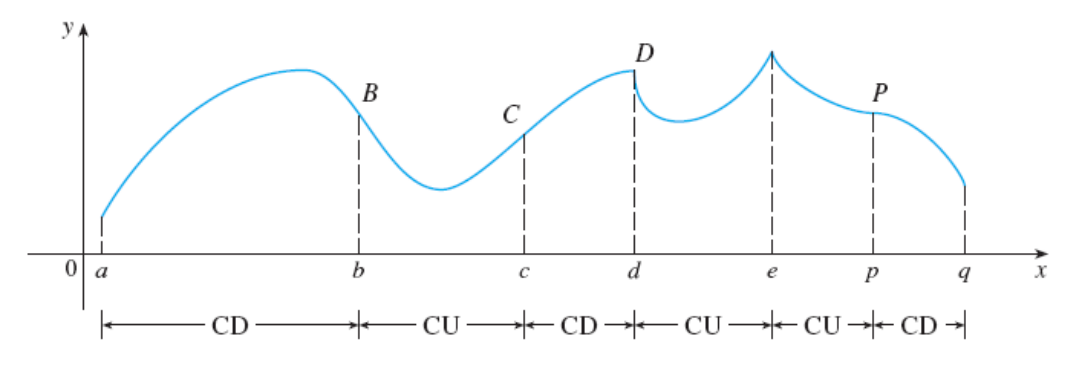
SOLUTION From the chart in the solution to Example 1 we see that$f'(x)$ changes fromnegative to positive at -1, so$f(-1) = 0$ is a local minimum value by the First DerivativeTest. Similarly,f' changes

from negative to positive at 2, so $f(2) = -27$is also alocal minimum value. As previously noted,$f(0) = 5$ is a local maximum value because$f'(x)$changes from positive to negative at 0.

What Does f'' Say about f?

Definition If the graph of f lies above all of its tangents on an interval I, then it is called **concave upward** on I. If the graph of f lies below all of its tangents on I, it is called **concave downward** on I.

Figure 7 shows the graph of a function that is concave upward (abbreviated CU) on the intervals (b, c),(d, e), and(e, p) and concave downward (CD) on the intervals(a, b),(c, d),and(p, q).



Concavity Test

(a) If $f''(x) > 0$ for all x in I , then the graph of f is concave upward on I.

(b) If $f''(x) < 0$ for all $x$in , then the graph of f is concave downward on I.

Definition A point P on a curve $y = f(x)$is called an **inflection point** if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

Another application of the second derivative is the following test for maximum andminimum values. It is aconsequence of the Concavity Test.

The Second Derivative Test Suppose f'' is continuous near c.

(a) If $f'(c) = 0$and $f''(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

EXAMPLE 6 Discuss the curve $y = x^4 - 4x^3$ with respect to concavity, points of inflection, and local maxima and minima. Use this information to sketch the curve.

SOLUTION If $f(x) = x^4 - 4x^3$, then

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$$

$$f''(x) = 12x^2 - 24x = 12x(x - 2)$$

To find the critical numbers we set $f'(x) = 0$ and obtain $x = 0$ and $x = 3$. To use the Second Derivative Test we evaluate f'' at these critical numbers:

$$f''(0) = 0 \qquad f''(3) = 36 > 0$$

Since $f'(3) = 0$ and $f''(3) > 0$, $f(3) = -27$ is a local minimum. Since $f''(0) = 0$, the Second Derivative Test gives no information about the critical number 0. But since $f'(x) < 0$ for $x < 0$ and also for $0 < x < 3$, the First Derivative Test tells us that f does not have a local maximum or minimum at 0. [In fact, the expression for $f'(x)$ shows that f decreases to the left of 3 and increases to the right of 3.]

Since $f''(x) = 0$ when $x = 0$ or 2, we divide the real line into intervals with these numbers as endpoints and complete the following chart.

Interval	$f''(x) = 12x(x-2)$	concavity
$(-\infty, 0)$	$+$	Upward
$(0,2)$	$-$	Downward
$(2, \infty)$	$+$	Upward

The point $(0,0)$ is an inflection point since the curve changes from concave upward to concave downward there. Also $(2, -16)$ is an inflection point since the curve changes from concave downward to concave upward there.

Using the local minimum, the intervals of concavity, and the inflection points, we sketch the curve in Figure 11.

Indeterminate Forms and L'Hospital's Rule

Suppose we are trying to analyze the behavior of the function

$$F(x) = \frac{\ln x}{x-1}$$

Although F is not defined when $= 1$, we need to know how F behaves *near* 1. In particular, we would like to know the value of the limit

$$\lim_{x \to 1} \frac{lnx}{x-1}$$

In computing this limit we can't apply Law of the limit of a quotient, because the limit of the denominator is 0. In fact, although the limit in the above function exists, its value is not obvious because both numerator and denominator approach 0 and $\frac{0}{0}$ is not defined.

In general, if we have a limit of the form

$$\lim_{x\to a}\frac{f(x)}{g(x)}$$

where both $f(x)\to 0$ and $g(x)\to 0$ as $x\to 0$, then this limit may or may not exist and is called an **indeterminate form of type $\frac{0}{0}$**.

L'Hospital's Rule Suppose f and g are differentiable and $g'(x)\neq 0$ near a (except possibly at). Suppose that

$$\lim_{x\to a} f(x)=0 \quad \text{and} \quad \lim_{x\to a} g(x)=0$$

or that $\lim_{x\to a} f(x)=\pm\infty$ and $\lim_{x\to a} g(x)=\pm\infty$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.) Then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

NOTE L'Hospital's Rule is also valid for one-sided limits and for limits at infinity or negative infinity; that is, "$x\to a$ " can be replaced by any of the symbols $\to a^+, x\to a^-, x\to\infty$, or $x\to-\infty$.

EXAMPLE 1 Find $\lim_{x\to 1}\frac{lnx}{x-1}$.

SOLUTION Since

$$\lim_{x\to 1}\ln x=\ln 1=0 \qquad and \qquad \lim_{x\to 1}(x-1)=0$$

we can apply l'Hospital's Rule:

$$\lim_{x\to 1}\frac{lnx}{x-1}=\lim_{x\to 1}\frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x-1)}=\lim_{x\to 1}\frac{1/x}{1}$$

$$=\lim_{x\to 1}\frac{1}{x}=1$$

EXAMPLE 2 Calculate $\lim_{x\to\infty}\frac{e^x}{x^2}$.

SOLUTION We have $\lim_{x\to\infty} e^x=\infty$ and $\lim_{x\to\infty} x^2=\infty$, so l'Hospital's Rule gives

$$\lim_{x\to\infty}\frac{e^x}{x^2}=\lim_{x\to\infty}\frac{\frac{d}{dx}(e^x)}{\frac{d}{dx}(x^2)}=\lim_{x\to\infty}\frac{e^x}{2x}$$

Since $e^x\to\infty$ and $2x\to\infty$ as $x\to\infty$, the limit on the right side is also indeterminate, but a second application of l'Hospital's Rule gives

$$\lim_{x\to\infty}\frac{e^x}{x^2}=\lim_{x\to\infty}\frac{e^x}{2x}=\lim_{x\to\infty}\frac{e^x}{2}=\infty$$

Indeterminate Products

If $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=\infty$ (or $-\infty$), then it isn't clear what the value of $\lim_{x\to a} f(x)g(x)$, if any, will be. There is a struggle between f and g. If f wins, the answer will be 0 ; if g wins, the answer will be ∞ (or $-\infty$). Or there may be a compromise where the answer is a finite nonzero number. This kind of limit is called an **indeterminate form of type $0\cdot\infty$**. We can deal with it by writing the product fg as a quotient:

$$fg = \frac{f}{1/g} \text{ or } \qquad fg = \frac{g}{1/f}$$

This converts the given limit into an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ so that we canuse l'Hospital's Rule.

EXAMPLE Evaluate $\lim_{x\to 0^+} x\ln x$.

SOLUTION The given limit is indeterminate because, as $\to 0^+$, the first factor $f(x)$approaches 0 while the second factor $(\ln x)$ approaches $-\infty$. Writing $x = 1/(1/x)$, we have$1/x \to \infty$ as$x \to 0^+$, so l'Hospital's Rule gives

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+}(-x) = 0.$$

Indeterminate Differences

If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then the limit

$$\lim_{x\to a}[f(x) - g(x)]$$

is called an **indeterminate form of type**$\infty - \infty$.

To find out the limit, we try to convert the difference into a quotient (for instance, by using a common denominator, or rationalization, or factoring out a common factor) so that we have an indeterminate form of type $\frac{0}{0}$or $\frac{\infty}{\infty}$.

EXAMPLE Compute $\lim_{x\to(\frac{\pi}{2})^-}(\sec x - \tan x)$.

SOLUTION First notice that $\sec x \to \infty$ and $\tan x \to \infty$as $\to (\frac{\pi}{2})^-$, so the limit is indeterminate.Here we use a common denominator:

$$\lim_{x\to(\frac{\pi}{2})^-}(\sec x - \tan x) = \lim_{x\to(\pi/2)^-}(\frac{1}{\cos x} - \frac{\sin x}{\cos x})$$

$$= \lim_{x\to(\pi/2)^-} \frac{1-\sin x}{\cos x} = \lim_{x\to(\pi/2)^-} \frac{-\cos x}{-\sin x} = 0$$

Note that the use of l'Hospital's Rule is justified because$1 - \sin x \to 0$ and$\cos x \to 0$as$\to (\pi/2)^-$.

Indeterminate Powers

Several indeterminate forms arise from the limit

$$\lim_{x\to a}[f(x)]^{g(x)}$$

1. $\lim_{x\to a} f(x) = 0$	and	$\lim_{x\to a} g(x) = 0$	type0^0
2. $\lim_{x\to a} f(x) = 0$	and	$\lim_{x\to a} g(x) = 0$	type∞^0
3. $\lim_{x\to a} f(x) = 1$	and	$\lim_{x\to a} g(x) = \pm\infty$	type1^∞

Each of these three cases can be treated either by taking the natural logarithm:

Let $y = [f(x)]^{g(x)}$, then $\ln y = g(x)lnf(x)$

or by writing the function as an exponential:

$$[f(x)]^{g(x)} = e^{g(x)lnf(x)}.$$

EXAMPLE 8 Calculate $\lim_{x\to 0^+}(1 + \sin 4x)^{\cot x}$.

SOLUTION First notice that as $x \to 0^+$, we have $1 + \sin 4x \to 1$ and $\cot x \to \infty$, so thegiven limit is indeterminate. Let

$$y = (1 + \sin 4x)^{\cot x}$$

Then $\ln y = \ln[(1 + \sin 4x)^{\cot x}] = \cot\ x ln(1 + \sin 4x)$

sol'Hospital's Rule gives

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln(1+\sin 4x)}{\tan x} = \lim_{x\to 0^+} \frac{\frac{4\cos 4x}{1+\sin 4x}}{sec^2 x} = 4$$

So far we have computed the limit of ln$y$, but what we want is the limit of y. To find this we use the fact that :

$y = e^{\ln y}$:

$\lim_{x\to 0^+}(1 + \sin 4x)^{\cot x} = \lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y} = e^4$

Curve Sketching

Guidelines for Sketching a Curve

The following checklist is intended as a guide to sketching a curve$y = f(x)$ by hand. Notevery item is relevant to every function. (For instance, a given curve might not have anasymptote or possess symmetry.) But the guidelines provide all the information you needto make a sketch that displays the most important aspects of the function.

A. Domain It's often useful to start by determining the domain$D$of$f$, that is, the set of valuesof x for which $f(x)$is defined.

B. Intercepts

C. Symmetry

(i) If$f(-x) = f(x)$ for all$x$in$D$,the function is an **even function** and the curve is symmetric aboutthe -axis.

(ii) If $f(-x) = -f(x)$for all$x$in , then$f$ is an **odd function** and the curve issymmetric about the origin.

iii) If$f(x + p) = f(x)$ for all$x$ inD , where$p$ is a positive constant, then$f$ is calleda **periodic function** and the smallest such number is called the **period.** For instance,$y = \sin x$has period2π.

D. Asymptotes

(i) *Horizontal Asymptotes.* if either$\lim_{x\to\infty} f(x) = L$or $\lim_{x\to-\infty} f(x) = L$, then the line $y = L$is a horizontalasymptote of the curve$y = f(x)$.If it turns out that$\lim_{x\to\infty} f(x) = \infty(-\infty)$, then we do not have anasymptoteto the right.

(ii) *Vertical Asymptotes.* The line$x = a$ is a verticalasymptote if at least one of the following statements is true:

$\lim_{x\to a^+} f(x) = \infty$ $\lim_{x\to a^-} f(x) = \infty$

$\lim_{x\to a^+} f(x) = -\infty$ $\lim_{x\to a^-} f(x) = -\infty$

(For rational functions you can locate the vertical asymptotes by equating the denominatorto 0 after canceling any common factors. But for other functions this method doesnot apply.)

(iii) *Oblique Asymptotes.* Some curves have asymptotes that are *oblique*, that is, neither horizontal nor vertical. If

$$\lim_{x\to\infty}[f(x)-(mx+b)]=0$$

then the line$y=mx+b$ is called an **oblique/slant asymptote.** For rational functions, slant asymptotes occurwhen the degree of the numerator is one more than the degree of the denominator. In sucha case the equation of the oblique asymptotecan be found by long division.

E. Intervals of Increase or Decrease Use the I/D Test. Compute$f'(x)$ and find the intervalson which$f'(x)$ ispositive (f is increasing) and the intervals on which $f'(x)$is negative(f is decreasing).

F. Local Maximum and Minimum Values

G. Concavity and Points of Inflection Compute$f''(x)$ and use the Concavity Test.

H. Sketch the Curve Using the information in items A–G, draw the graph. Sketch the asymptotes as dashed lines.

EXAMPLE 1 Use the guidelines to sketch the curve $y=\frac{2x^2}{x^2-1}$.

A. The domain is

$$\{x|x^2-1\neq 0\}=\{x|x\neq\pm1\}=(-\infty,-1)\cup(-1,1)\cup(1,\infty)$$

B. The x- and y-intercepts are both 0.

C. Since$(-x)=f(x)$, the function$f$ is even. The curve is symmetric about the -axis.

D. $\boldsymbol{\lim_{x\to\pm\infty}\frac{2x^2}{x^2-1}=\lim_{x\to\pm\infty}\frac{2}{1-1/x^2}=2}$

Therefore, the line $y=2$ is a horizontal asymptote.

Since the denominator is 0 when $=\pm1$, we compute the following limits:

$\lim_{x\to1^+}\frac{2x^2}{x^2-1}=\infty$ $\lim_{x\to1^-}\frac{2x^2}{x^2-1}=-\infty$

$\lim_{x\to1^+}\frac{2x^2}{x^2-1}=-\infty$ $\lim_{x\to1^-}\frac{2x^2}{x^2-1}=\infty$

Therefore, the lines and are vertical asymptotes.

E. $\boldsymbol{f'(x)=\frac{4x(x^2-1)-2x^2.2x}{(x^2-1)^2}=\frac{-4x}{(x^2-1)^2}}$

Since$f'(x)>0$ when$x<0(x\neq-1)$ and$f'(x)<0$when $>0(x\neq1)$,f isincreasing on$(-\infty,-1)$ and$(-1,0)$ and$(-1,0)$and decreasing on$(0,1)$ and$(1,\infty)$.

F. The only critical number is $x=0$. Sincef' changes from positive to negative at $0,f(0)=0$is a local maximum by the First Derivative Test.

G. $\boldsymbol{f''(x)=\frac{-4(x^2-1)-2x^2.2x}{(x^2-1)^2}=\frac{-4x}{(x^2-1)^2}}$

Since$12x^2+4>0$ for all , we have

$f''(x) > 0 \Leftrightarrow x^2 - 1 > 0 \Leftrightarrow |x| > 1$

And $f''(x) < 0 \Leftrightarrow |x| < 1$. Thus, the curve is concave upward on the intervals $-\infty, -1)$ and $1, \infty)$ and concave downward on $(-1,1)$. It has no point of inflectionsince 1 and -1 are not in the domain of f.

H. Using the information in E–G, we finish the sketch in Figure.

Other Applications of Derivatives

Rate of Change

The first interpretation/application of a derivative is rate of change. If $f(x)$ represents a quantity at any x then the derivative $f'(a)$ represents the instantaneous rate of change of $f(x)$ at $x = a$.

Example *Suppose* that the amount of water in a holding tank at t minutes is given by $V(t) = 2t^2 - 16t + 35$. Determine each of the following.

(a) Is the volume of water in the tank increasing or decreasing at $t = 1$ minute?

(b) Is the volume of water in the tank increasing or decreasing at $t = 5$ minutes?

(c) Is the volume of water in the tank changing faster at $t = 1$ or $t = 5$ minutes?

(d) Is the volume of water in the tank ever not changing? If so, when?

Solution

We are going to need the rate of change of the volume to answer these questions. This means that we will need the derivative of this function since that will give us a formula for the rate of change at any time t. The derivative is.

$V'(t) = 4t - 16$ OR $\frac{dV}{dt} = 4t - 16$

if the rate of change is positive then the quantity will be increasing and if the rate of change is negative then the quantity will be decreasing.

(a) Is the volume of water in the tank increasing or decreasing at $t = 1$ minute?

In this case all that we need is the rate of change of the volume at $t = 1$or,

$V' = -12$ $\quad$ $\frac{dV}{dt}|_{t=1} = -12$

So, at $t = 1$ the rate of change is negative and so the volume must be decreasing at this time.

(b) Is the volume of water in the tank increasing or decreasing at $t = 5$ minutes?(exercise)

(c) Is the volume of water in the tank changing faster at $t = 1$ or $t = 5$ minutes?

To answer this question all that we look at is the size of the rate of change and we don't worry about the sign of the rate of change. All that we need to know here is that the larger the number the faster the rate of change.

(d) Is the volume of water in the tank ever not changing? If so, when?

The volume will not be changing if it has a rate of change of zero. In order to have a rate of change of zero this means that the derivative must be zero. So, to answer this question we will then need to solve

$V'(t) = 0$ OR $\frac{dV}{dt} = 0$

This is easy enough to do.

$4t - 16 = 0$ $t = 4$

So at $t = 4$ the volume isn't changing. Note that all this is saying is that for a brief instant thevolume isn't changing. It doesn't say that at this point the volume will quit changing permanently.

Velocity

Definition If $f(t)$ is the position of a particle moving on a coordinate line, then the instantaneous velocity of the particle at time t is defined by

$$v(t) = v'(t) = \frac{dv}{dt}$$

Example *Suppose* that the position of an object after *t* hours is given by,

$$g(t) = \frac{t}{t+1}$$

Answer both of the following about this object.

(a) Is the object moving to the right or the left at $t = 10$ hours?

(b) Does the object ever stop moving?

Solution

The derivative is,

$$g'(t) = \frac{1}{(t+1)^2}$$

(a) Is the object moving to the right or the left at $t = 10$ hours?

To determine if the object is moving to the right (velocity is positive) or left (velocity is negative) we need the derivative at $t = 10$.

$$g'(10) = \frac{1}{121}$$

So the velocity at $t = 10$ is positive and so the object is moving to the right at $t = 10$.

(b) Does the object ever stop moving?

The object will stop moving if the velocity is ever zero. However, note that the only way arational expression will ever be zero is if the numerator is zero. Since the numerator of thederivative (and hence the speed) is a constant it can't be zero.

Therefore, the velocity will never stop moving.

In fact, we can say a little more here. The object will always be moving to the right since the velocity is always positive.

Acceleration

Definition If $f(t)$ is the position function of a particle moving on a coordinate line, then the instantaneous acceleration of the particle at time t is defined by

$$a(t) = v'(t) = \frac{dv}{dt}$$

Or alternatively, since $v(t) = f'(t)$,

$$a(t) = f''(t) = \frac{d^2f}{dt^2}$$

Example Let $f(t) = t^3 - 6t^2$ be the position function of a particle moving along a $f -$ axis,where$f$ is in meters and t is in seconds.Find the instantaneous acceleration $a(t)$.

Solution $a(t) = \frac{dv}{dt} = 6t - 12.$

Interpreting the sign of acceleration.*Aparticle in rectilinear motion is speeding up when its velocity and acceleration have the same sign and slowing down when they have opposite signs.*