Common Derivatives and Integrals

Derivative Rules:

1. *Constant Multiple Rule* $\frac{d}{dx}[cu] = cu'$, where c is a constant.

2. *Sum and Difference Rule* $\frac{d}{dx}[u \pm v] = u' \pm v'$

3. *Product Rule* $\frac{d}{dx}[uv] = uv' + vu'$

4. *Quotient Rule* $\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{vu' - uv'}{v^2}$

5. *Constant Rule,* $\frac{d}{dx}[c] = 0$

6. *Power Rule* $\frac{d}{dx}[u^n] = nu^{n-1}u'$

7. *Power Rule* $\frac{d}{dx}[x] = 1$

8. *Derivative Involving Absolute Value* $\frac{d}{dx}[|u|] = \frac{u}{|u|}(u'), u \neq 0$

9. *Derivative of the Natural Logarithmic Function* $\frac{d}{dx}[\ln u] = \frac{u'}{u}$

10. *Derivative of Natural Exponential Function* $\frac{d}{dx}[e^u] = e^u u'$

Example 1: Find the derivative of $f(x)=(4x-3x^2)(3+2x^2)$

Since there are two polynomials multiplied by each other, apply the third derivative rule, the *Product Rule*, to the problem.

This is the result of the *Product Rule*:

$$f'(x)=(4x-3x^2)\frac{d}{dx}[3+2x^2]+(3+2x^2)\frac{d}{dx}[4x-3x^2]$$

Now, take the derivative of each term inside of the brackets. Multiple derivative rules are used, including the *Sum and Difference Rule*, *Constant Rule*, *Constant Multiple Rule*, and *Power Rule*. When applied, the result is:

$$f'(x)=(4x-3x^2)(0+4x)+(3+2x^2)(4-6x)$$

Simplify:

$$f'(x)=(4x-3x^2)(4x)+(3+2x^2)(4-6x)$$

Multiply the polynomials by each other:

$$f'(x)=(16x^2-12x^3)+(12+8x^2-18x-12x^3)$$

$$f'(x)=16x^2-12x^3+12+8x^2-18x-12x^3$$

Combine like terms to get a simplified answer:

$$f'(x)=-24x^3+24x^2-18x+12$$

Integral Formulas:

Indefinite integrals have $+C$ as an arbitrary constant.

1. $\int kf(u)du=k\int f(u)du$, where k is a constant.

2. $\int[f(u)\pm g(u)]du=\int f(u)du\pm\int g(u)du$

3. $\int du=u+C$

4. $\int u^n du = \frac{u^{n+1}}{n+1} + C, n \neq -1$

5. $\int \frac{du}{u} = \ln|u| + C$

6. $\int e^u du = e^u + C$

Example 2: Evaluate $\int (4x^2 - 5x^3 + 12)dx$

To evaluate this problem, use the first four Integral Formulas. First, use formula 2 to make the large integral into three smaller integrals:

$$\int (4x^2 - 5x^3 + 12)dx = \int 4x^2 dx - \int 5x^3 dx + \int 12 dx$$

Second, pull out the constants by using formula 1:

$$= 4\int x^2 dx - 5\int x^3 dx + 12\int dx$$

Now find each integral using formulas 3 and 4:

$$= 4\left(\frac{x^{2+1}}{2+1}\right) - 5\left(\frac{x^{3+1}}{3+1}\right) + 12(x) + C$$

Although three integrals have been removed, only one constant *C* is needed because *C* represents all unknown constants. Therefore multiple *C*'s can be combined into just one *C*. To get the final answer, simplify the expression:

$$= 4\left(\frac{x^3}{3}\right) - 5\left(\frac{x^4}{4}\right) + 12x + C$$

$$= \frac{4}{3}x^3 - \frac{5}{4}x^4 + 12x + C$$

Derivatives Rules for Trigonometric Functions:

1. $\frac{d}{dx}[\sin(u)] = (\cos(u))u'$

2. $\frac{d}{dx}[\cos(u)] = -(\sin(u))u'$

3. $\frac{d}{dx}[\tan(u)] = (\sec^2(u))u'$

4. $\frac{d}{dx}[\cot(u)] = -(\csc^2(u))u'$

5. $\frac{d}{dx}[\sec(u)] = (\sec(u)\tan(u))u'$

6. $\frac{d}{dx}[\csc(u)] = -(\csc(u)\cot(u))u'$

Example 3: Find the derivative of $f(x) = \frac{\sin(x)}{\cos(x)}$

When finding the derivatives of trigonometric functions, non-trigonometric derivative rules are often incorporated, as well as trigonometric derivative rules. Looking at this function, one can see that the function is a quotient. Therefore, use derivative rule 4 on page 1, the *Quotient Rule*, to start this problem:

$$f'(x) = \frac{(\cos(x))\frac{d}{dx}[\sin(x)] - (\sin(x))\frac{d}{dx}[\cos(x)]}{\cos^2(x)}$$

Now use trigonometric derivative rules 1 and 2 to get:

$$f'(x) = \frac{(\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))}{\cos^2(x)}$$

Once the multiplication has been completed in the numerator of the fraction, the result is:

$$f'(x)=\frac{\left(\sin^2(x)+\cos^2(x)\right)}{\cos^2(x)}$$

Remember that $\sin^2(x)+\cos^2(x)=1$; therefore, substitute 1 for $\sin^2(x)+\cos^2(x)$ in the answer. The final result is:

$$f'(x)=\frac{1}{\cos^2(x)}$$

$$f'(x)=\sec^2(x)$$

Integrals of Trigonometric Functions:

1. $\int \sin(u)du=-\cos(u)+C$

2. $\int \cos(u)du=\sin(u)+C$

3 $\int \tan(u)du=-\ln|\cos(u)|+C$

4. $\int \cot(u)du=\ln|\sin(u)|+C$

5. $\int \sec(u)du=\ln|\sec(u)+\tan(u)|+C$

6. $\int \csc(u)du=-\ln|\csc(u)+\cot(u)|+C$

7. $\int \sec^2(u)du=\tan(u)+C$

8. $\int \csc^2(u)du=-\cot(u)+C$

9. $\int \sec(u)\tan(u)du=\sec(u)+C$

10. $\int \csc(u)\cot(u)du=-\csc(u)+C$

Example 4: Evaluate $\int(\cos(x)-\tan(x))dx$

Normal integration formulas are often used in addition to trigonometric formulas when doing trigonometric integration. For example, in this problem use integration formula 2:

$$\int(\cos(x)-\tan(x))dx=\int\cos(x)dx-\int\tan(x)dx$$

With the two smaller integrals, use trigonometric integration formulas 2 and 3 to find the solution:

$$=\sin(x)-(-\ln|\cos(x)|)+C$$

Simplify:

$$=\sin(x)+\ln|\cos(x)|+C$$

Special Differentiation Rules:

Chain Rule: In certain situations, there may be a differentiable function of *u*, such as $y=f(u)$, and $u=g(x)$, where $g(x)$ is a differentiable function of *x*. If this is the case, then $y=f(g(x))$ is a differentiable function of *x*.

To take the derivative of the composite function $y=f(g(x))$, use the formula:

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

This formula can be rewritten as:

$$\frac{d}{dx}[f(g(x))]=f'(g(x))g'(x)$$

Example 5: Find $\frac{dy}{dx}$ of the function $y=(3+x^4)^2$.

Use the formula $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ to find this derivative. Now $\left(3 + x^4\right) = u$, and the result is the function $y = u^2$.

Using the Chain Rule, we have

$\frac{dy}{dx} = 2\left(3 + x^4\right)\left(4x^3\right) = 8x^3\left(3 + x^4\right)$, where

$\frac{dy}{du} = 2\left(3 + x^4\right)$ And $\frac{du}{dx} = 4x^3$

Special Integration Formulas:

U-Substitution: Some integrals cannot be solved by using only the basic integration formulas. In some of these cases, one can use a process called u-substitution. This process helps simplify a problem before solving it.

Example 6: Solve $\int (2 + 3x)^{\frac{1}{2}}\, dx$

Referring to the given integral formulas, there are none that are able to solve this integral in its current form. When one comes to an integral in a form like this, it may be possible to simplify the integral to a form that is solvable by the given formulas. For this integral, u-substitution may be used. :

Using U- Substitution shows that:

u = 2 + 3x

$\frac{du}{dx} = \left(2 + 3x\right)' = 3$

$dx\left(\frac{du}{dx}\right) = 3dx$

$du = 3dx$

Having found that $2 + 3x = u$, and that $dx = \frac{1}{3}du$, substitute both into the original integral, $\int \sqrt{2+3x}dx$. The result is $\int \frac{1}{3}u^{\frac{1}{2}}du$, which can be solved by using integral formulas 1 and 4.

First, use formula 1, to find that:

$$= \int \frac{1}{3}u^{\frac{1}{2}}du = \frac{1}{3}\int u^{\frac{1}{2}}du$$

Integration formula 4 shows that:

$$= \frac{1}{3}\int u^{\frac{1}{2}}du$$

$$= \frac{1}{3}\left(\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right)+C$$

$$= \frac{1}{3}\left(\frac{u^{\frac{1}{2}+\frac{2}{2}}}{\frac{1}{2}+\frac{2}{2}}\right)+C$$

$$= \frac{1}{3}\left(\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right)+C$$

$$= \frac{1}{3}\left(\frac{2}{3}\right)u^{\frac{3}{2}}+C$$

$$= \frac{2}{9}u^{\frac{3}{2}}+C$$

After solving the integral for u, substitute u with $2 + 3x$ to get the final answer:

$$\frac{2}{9}u^{\frac{3}{2}}+C=\frac{2}{9}(2+3x)^{\frac{3}{2}}+C$$

Definite Integrals: Indefinite integrals always end with $+C$ to represent any unknown constants that might have been in the original equation. Variables are also contained in the answer. However, definite integrals provide a single number as an answer to an integration problem. This is done by first giving the integral upper and lower bounds, which represent the starting and ending points of the function on a graph. Once these bounds have been set, the integral is evaluated as usual, but without the $+C$ being placed at the end. Instead, after the integration has been done, a right bracket (]) is placed after the problem with the upper and lower bounds to the right of it ($\Big]_a^b$). Once the integral has been found, the upper and lower bounds are plugged into the variables in the equation, with the lower bound being subtracted from the upper one. Then the answer is left with a constant.

Example 7: Find $\int_1^3 (4x^2+5x+10)dx$. This problem is a definite integral with an upper bound of 3 and a lower bound of 1.

First, use integral formula 2 to break the integral up into three smaller integrals, which are easier to solve:

$$\int_1^3 (4x^2+5x+10)dx=\int_1^3 4x^2dx+\int_1^3 5xdx+\int_1^3 10dx$$

Second, use integration formula 1 to get:

$$=4\int_1^3 x^2dx+5\int_1^3 xdx+10\int_1^3 dx$$

Next, use formula 4:

$$=4\left(\frac{x^{2+1}}{2+1}\right)\Big]_1^3+5\left(\frac{x^{1+1}}{1+1}\right)\Big]_1^3+10x\Big]_1^3$$

$$=4\left(\frac{x^3}{3}\right)\Big]_1^3+5\left(\frac{x^2}{2}\right)\Big]_1^3+10x\Big]_1^3$$

$$= \frac{4}{3}x^3\Big]_1^3 + \frac{5}{2}x^2\Big]_1^3 + 10x\Big]_1^3$$

The problem is now integrated completely. Plug in the upper and lower bounds into the problem and simplify:

$$= \left(\frac{4}{3}\left(3^3\right) - \frac{4}{3}\left(1^3\right)\right) + \left(\frac{5}{2}\left(3^2\right) - \frac{5}{2}\left(1^2\right)\right) + \left(10(3) - 10(1)\right)$$

$$= \left(\frac{4}{3}(27) - \frac{4}{3}(1)\right) + \left(\frac{5}{2}(9) - \frac{5}{2}(1)\right) + \left(10(3) - 10(1)\right)$$

$$= \left(\frac{108}{3} - \frac{4}{3}\right) + \left(\frac{45}{2} - \frac{5}{2}\right) + \left(30 - 10\right)$$

$$= \frac{104}{3} + \frac{40}{2} + 20$$

$$= \frac{104}{3} + 20 + 20$$

$$= \frac{104}{3} + 40$$

$$= \frac{104}{3} + 40\left(\frac{3}{3}\right)$$

$$= \frac{104}{3} + \frac{120}{3}$$

$$= \frac{224}{3}$$

Integration by Parts: Some integrals can not be evaluated by using only the 16 basic integral formulas shown above. An example of an integral like this would be $\int xe^x dx$. There are formulas to find $\int xdx$ and $\int e^x dx$, but we do not have a formula in our list that can find $\int xe^x dx$. To evaluate an integral like this, use a method called "Integration by Parts". This

method is used to evaluate integrals where there are two separate functions of *x* contained in the integral, usually represented as *u* and *v*. An integral containing two separate functions of *x* usually follows the format $\int udv$. The answer to this integral is $\int udv = uv - \int vdu$. This formula can be used multiple times in a problem, depending on the values for *u*, *v*, *du*, and *dv*.

Example 8: Find $\int x^3 \ln x dx$:

With the two separate functions of x, x^3 and $\ln x$, choose which function is needed to be *u* and *dv*. It is usually best to assign simple functions to be *dv*. For this problem, make $u = \ln x$ and $dv = x^3$. Now find *v* and *du*. To find *v*, take the integral of *dv*. For *du*, take the derivative of *u*. Once this is done, the result is:

$$u = \ln x \qquad v = \frac{1}{4}x^4$$

$$du = \frac{1}{x} \qquad dv = x^3$$

With the new values of our variables, plug them into the formula $\int udv = uv - \int vdu$:

$$\int x^3 \ln x dx = \frac{1}{4}x^4 \ln x - \int \left(\frac{1}{4}x^4\right)\left(\frac{1}{x}\right)dx$$

Simplify:

$$= \frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^3 dx$$

After simplifying the integral, the simple integral is left and can be used to solve these formulas:

$$= \frac{1}{4}x^4 \ln x - \frac{1}{4}\left(\frac{x^{3+1}}{3+1}\right) + C$$

$$= \frac{1}{4}x^4 \ln x - \frac{1}{4}\left(\frac{1}{4}x^4\right) + C$$

$$= \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C$$

$$= \left(\frac{4}{4}\right)\frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C$$

$$= \frac{4}{16}x^4 \ln x - \frac{1}{16}x^4 + C$$

$$= \frac{x^4(4\ln x - 1)}{16} + C$$