DESSIE SPECIAL BOARDING SCHOOL

MATHEMATICS MODULE FOR GRADE 11TH

THIS MODULE IS PREPARED FOR GRADE 11th STUDENTS TO STUDY AT THEIR DORMITARY

THE MODULE CONTAINS SHORT NOTES, EXAMPLES, EXERCISES, PRACTICE QUESTIONS AND THEIR SOLUTIONS BASED ON GRADE 11th MATHEMATICS NEW CURRICULUM

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CHAPTER ONE

RELATIONS AND FUNCTIONS

After completing this section, the student should be able to:

- define Cartesian product of two sets
- understand the notion of relation and function
- know the difference between relation and function
- determine the domain and range of relations and functions
- find the inverse of a relation

The student is familiar with the phrase ordered pair. In the ordered pair $(2,3),(-2,4)$ and (a,b); $2,-2$ and a are the first coordinates while $3,4$ and b are the second coordinates.

- **CARTESIAN PRODUCT**

Given sets $A=\{3,4\}$ and $B=\{4,5,9\}$. Then, the set $\{(3,4),(3,5),(3,9),(4,4),(4,5),(4,9)\}$ is the Cartesian product of A and B, and it is denoted by $A\times B$.

Definition : Suppose A and B are sets. The Cartesian product of A and B, denoted by $A\times B$, is the set which contains every ordered pair whose first coordinate is an element of A and second coordinate is an element of B, i.e.

$$A\times B=\{(a,b):a\in A \text{ and } b\in B\}.$$

Example : For $A=\{2,4\}$ and $A=\{-1,3\}$, we have

a) $A\times B=\{(2,-1),(2,3),(4,-1),(4,3)\}$, and

b) $B\times A=\{(-1,2),(-1,4),(3,2),(3,4)\}$.

From this example, we can see that $A\times B$ and $B\times A$ are not equal. Recall that two sets are equal if one is a subset of the other and vice versa. To check equality of Cartesian products we need to define equality of ordered pairs.

Definition : (Equality of ordered Pairs)

Two ordered pairs (a,b) and (c,d) are equal if and only if $a=c$ and $b=d$.

Example : Let $A=\{1,2,3\}$ and $B=\{a,b,c\}$. Then,

$$A\times B=\{(1,a),(1,b),(1,c),(2,a),(2,b),(2,c),(3,a),(3,b),(3,c)\}.$$

Definition : (Relation)

If A and B are sets, any subset of $A\times B$ is called a relation from A into B.

Suppose R is a relation from a set A to a set B. Then, $R \subseteq A \times B$ and hence for each $(a,b) \in A \times B$, we have either $(a,b) \in R$ or $(a,b) \notin R$. If $(a,b) \in R$, we say "a is R-related (or simply related) to b", and write aRb. If $(a,b) \notin R$, we say that "a is not related to b".
In particular if R is a relation from a set A to itself, then we say that R is a relation on A.

Example :

1. Let $A = \{1,3,5,7\}$ and $B = \{6,8\}$. Let R be the relation "less than" from A to B. Then, $R = \{(1,6),(1,8),((3,6),(3,8),(5,6),(5,8),(7,8)\}$.
2. Let $A = \{1,2,3,4,5\}$ and $B = \{a,b,c\}$.
   a) The following are relations from A into B;
      i) $R_1 = \{(1,a)\}$
      ii) $R_2 = \{(2,b),(3,b),(4,c),(5,a)\}$
      iii) $R_3 = \{(1,a),((2,b),(3,c)\}$
   b) The following are relations from B to A;
      i) $R_4 = \{(a,3),(b,1)\}$
      ii) $R_5 = \{(b,2),(c,4),(a,2),(b,3)\}$
      iii) $R_6 = \{(b,5)\}$

Definition : Let R be a relation from A into B. Then,
a) the domain of R, denoted by $Dom(R)$, is the set of first coordinates of the elements of R, i.e
$$Dom(R) = \{a \in A : (a,b) \in R\}$$
b) the range of R, denoted by $Range(R)$, is the set of second coordinates of elements of R, i.e
$$Range(R) = \{b \in B : (a,b) \in R\}$$

Remark: If R is a relation form the set A to the set B, then the set B is called the codomain of the relation R. The range of relation is always a subset of the codomain.

Example :

1. The set $R = \{(4,7),(5,8),(6,10)\}$ is a relation from the set $A = \{1,2,3,4,5,6\}$ to the set $B = \{6,7,8,9,10]$. The domain of R is $\{4,5,6\}$, the range of R is $\{7,8,10\}$ and the codomain of R is $\{6,7,8,9,10\}$.
2. The set of ordered pairs $R = \{(8,2),(6,-3),(5,7),(5,-3)\}$ is a relation between the sets $\{5,6,8\}$ and $\{2,-3,7\}$, where $\{5,6,7\}$ is the domain and $\{2,-3,7\}$ is the range.

Remark:

1. If $(a,b) \in R$ for a relation R, we say a is related to (or paired with) b. Note that a may also be paired with an element different from b. In any case, b is called the image of a while a is called the pre-image of b.
2. If the domain and/or range of a relation is infinite, we cannot list each element assignment, so instead we use set builder notation to describe the relation. The situation we will encounter most frequently is that of a relation defined by an equation or formula. For example,

   $$R = \{(x,y) : y = 2x - 3, x, y \in IR\}$$

   is a relation for which the range value is 3 less than twice the domain value. Hence, $(0,-3),(0.5,-2)$ and $(-2,-7)$ are examples of ordered pairs that are of the assignment.

Example

1. Let $A = \{1,2,3,4,6\}$

   Let R be the relation on A defined by $R = \{(a,b) : a,b \in A, a \text{ is a factor of } b\}$. Find the domain and range of R.

Solution: We have

$R = \{(1,1),(1,2),(1,3),(1,4),(1,6),(2,2),(2,4),(2,6),(3,3),(3,6),(4,4),(6,6)\}$.

Then, $Dom(R) = \{1,2,3,4,6\}$ and $Range(R) = \{1,2,3,4,6\}$.

2. Let $A = \{1,2,3,4,5\}$ and $B = \{1,2,3,\cdots,67\}$.

   Let $R = \{(x,y) \in A \times B : x \text{ is cube root of } y\}$. Find a) R b) $Dom(R)$ c) $Range(R)$

Solution: We have $1 = \sqrt[3]{1}, 2 = \sqrt[3]{8}, 3 = \sqrt[3]{27}, 4 = \sqrt[3]{64}, 5 = \sqrt[3]{125}$ and 1,8,27 and 64 are in B whereas 125 is not in B. Thus, $R = \{(1,1),(2,8),(3,27),(4,64)\}$, $Dom(R) = \{1,2,3,4\}$ and $R = \{1,8,27,64\}$.

Remark:

1. A relation R on a set A is called
   i) a universal relation if $R = A \times A$
   ii) identity relation if $R = \{(a,a) : a \in A\}$
   iii) void or empty relation if $R = \phi$
2. If R is a relation from A to B, then the inverse relation of R, denoted by R^{-1}, is a relation from B to A and is defined as:

   $$R^{-1} = \{(y,x) : (x,y) \in R\}.$$

Observe that $Dom(R) = Range(R^{-1})$ and $Range(R) = Dom(R^{-1})$. For instance, if $R = \{(1,4),(9,15),(10,2)\}$ is a relation on a set $A = \{1,2,3,\cdots,20\}$, then $R^{-1} = \{(4,1),(15,9),(2,10)\}$

Example : Let R be a relation defined on IN by $R=\{(a,b): a,b\in IN,\ a+2b=11\}$.

Find a) R b) $Dom(R)$ c) $Range(R)$ d) R^{-1}

Solution: The smallest natural number is 1.

$b=1 \Rightarrow a+2(1)=11 \Rightarrow a=9$

$b=2 \Rightarrow a+2(2)=11 \Rightarrow a=7$

$b=3 \Rightarrow a+2(3)=11 \Rightarrow a=5$

$b=4 \Rightarrow a+2(4)=11 \Rightarrow a=3$

$b=5 \Rightarrow a+2(5)=11 \Rightarrow a=1$

$b=6 \Rightarrow a+2(6)=11 \Rightarrow a=-1\notin IN$

Therefore, $R=\{(9,1),(7,2),(5,3),(3,4),(1,5)\}$, $Dom(R)=\{1,3,5,7,9\}$, $Range(R)=\{1,2,3,4,5\}$ and $R^{-1}=\{(1,9),(2,7),(3,5),(4,3),(5,1)\}$.

- **FUNCTIONS**

Mathematically, it is important for us to distinguish among the relations that assign a unique range element to each domain element and those that do not.

Definition : (Function)
A function is a relation in which each element of the domain corresponds to exactly one element of the range.

Example : Determine whether the following relations are functions.

a) $R=\{(5,-2),(3,5),(3,7)\}$ b) $\{(2,4),(3,4),(6,-4)\}$

Solution:

a) Since the domain element 3 is assigned to two different values in the range, 5 and 7, it is not a function.

b) Each element in the domain, $\{2,3,6\}$, is assigned no more than one value in the range, 2 is assigned only 4, 3 is assigned only 4, and 6 is assigned only – 4. Therefore, it is a function.

Remark: Map or mapping, transformation and correspondence are synonyms for the word function. If f is a function and $(x,y)\in f$, we say x is mapped to y.

Definition : A relation f from A into B is called a function from A into B, denoted by

$$f : A \to B \quad \text{or} \quad A \xrightarrow{f} B$$

if and only if

(i) $Dom(f) = A$

(ii) No element of A is mapped by f to more than one element in B, i.e. if $(x, y) \in f$ and $(x, z) \in f$, then $y = z$.

Remark:

1. If to the element x of A corresponds $y (\in B)$ under the function f, then we write $f(x) = y$ and y is called the image of x under y and x is called a pre-image of y under f
2. The symbol $f(x)$ is read as "f of x" but not "f times x".
3. In order to show that a relation f from A into B is a function, we first show that the domain of f is A and next we show that f well defined or single-valued, i.e. if $x = y$ in A, then $f(x) = f(y)$ in B for all $x, y \in A$.

Example :

1. Let $A = \{1,2,3,4\}$ and $B = \{1,6,8,11,15\}$. Which of the following are functions from A to B.
   a) f defined by $f(1) = 1, f(2) = 6, f(3) = 8, f(4) = 8$
   b) f defined by $f(1) = 1, f(2) = 6, f(3) = 15$
   c) f defined by $f(1) = 6, f(2) = 6, f(3) = 6, f(4) = 6$
   d) f defined by $f(1) = 1, f(2) = 6, f(2) = 8, f(3) = 8, f(4) = 11$
   e) f defined by $f(1) = 1, f(2) = 8, f(3) = 11, f(4) = 15$

Solution:

a) f is a function because to each element of A there corresponds exactly one element of B

b) f is not a function because there is no element of B which correspond to $4 (\in A)$.

c) f is a function because to each element of A there corresponds exactly one element of B. In the given function, the images of all element of A are the same.

d) f is not a function because there are two elements of B which are corresponding to 2. In other words, the image of 2 is not unique.

e) f is a function because to each element of A there corresponds exactly one element of B.

As with relations, we can describe a function with an equation. For example, $y=2x+1$ is a function, since each x will produce only one y.

2. Let $f=\{(x,y): y=x^2\}$. Then, f maps:

1 to 1	-1 to 1
2 to 4	-2 to 4
3 to 9	-3 to 9

More generally any real number x is mapped to its square. As the square of a number is unique, f maps every real number to a unique number. Thus, f is a function from $\Re$ into $\Re$.
We will find it useful to use the following vocabulary: The independent variable refers to the variable representing possible values in the domain, and the dependent variable refers to the variable representing possible values in the range. Thus, in our usual ordered pair notation (x,y), x is the independent variable and y is the dependent variable.

- **DOMAIN, CODOMAIN AND RANGE OF A FUNCTION**

For the function $f: A \to B$

(i) The set A is called the domain of f

(ii) The set B is called the codomain of f

(iii) The set $\{f(x): x \in A\}$ of all image of elements of A is called the range of f

Example :

1. Let $A=\{1,2,3\}$ and $B=\{1,2,3,\cdots,10\}$. Let $f: A \to B$ be the correspondence which assigns to each element in A, its square. Thus, we have $f(1)=1, f(2)=4, f(3)=9$. Therefore, f is a function and $Dom(f)=\{1,2,3\}$, $Range(f)=\{1,4,9\}$ and codomain of f is $\{1,2,3,\cdots,10\}$.

2. Let $A=\{2,4,6,7,9\}, B=IN$. Let x and y represent the elements in the sets A and B, respectively. Let $f: A \to B$ be a function defined by $f(x)=15x+17, x \in A$.

   The variable x can take values 2, 4, 6, 7, 9. Thus, we have
   $f(2)=15(2)+17=47, f(4)=77, f(6)=107, f(7)=122, f(9)=152$.
   This implies that $Dom(f)=\{2,4,6,7,9\}$, $Range(f)=\{47,77,107,122,152\}$ and codomain of f is IN.

3. Let f be the subset of $Q \times Z$ defined by $f=\left\{\left(\frac{p}{q},p\right): p,q \in Z, q \neq 0\right\}$. Is f a function?

   Solution: First we note that $Dom(f)=Q$. Then, f satisfies condition (i) in the definition of a function. Now, $\left(\frac{2}{3},2\right) \in f$, $\left(\frac{4}{6},4\right) \in f$ and $\frac{2}{3}=\frac{4}{6}$ but $f\left(\frac{2}{3}\right)=2 \neq 4=f\left(\frac{4}{6}\right)$. Thus f is not well defined. Hence, f is not a function from Q to Z.

4. Let f be the subset of $Z \times Z$ defined by $f = \{(mn, m+n) : m,n \in Z\}$. Is f a function?

Solution: First we show that f satisfies condition (i) in the definition. Let x be any element of Z. Then, $x = x \cdot 1$. Hence, $(x, x+1) = (x \cdot 1, x+1) \in f$. This implies that $x \in Dom(f)$. Thus, $Z \subseteq Dom(f)$. However, $Dom(f) \subseteq Z$ and so $Dom(f) = Z$. Now, $4 \in Z$ and $4 = 4 \cdot 1 = 2 \cdot 2$. Thus, $(4 \cdot 1, 4+1)$ and $(2 \cdot 2, 2+2)$ are in f. Hence we find that $4 \cdot 1 = 2 \cdot 2$ and $f(4 \cdot 1) = 5 \neq 4 = f(2 \cdot 2)$. This implies that f is not well defined, i.e, f does not satisfy condition (ii). Hence, f is not a function from Z to Z.

5. Determine whether the following equations determine y as a function of x, if so, find the domain.

a) $y = -3x + 5$ b) $y = \dfrac{2x}{3x-5}$ c) $y^2 = x$

Solution:

a) To determine whether $y = -3x + 5$ gives y as a function of x, we need to know whether each *x-value* uniquely determines a *y-value*. Looking at the equation $y = -3x + 5$, we can see that once x is chosen we multiply it by – 3 and then add 5. Thus, for each *x* there is a unique y. Therefore, $y = -3x + 5$ is a function.

b) Looking at the equation $y = \dfrac{2x}{3x-5}$ carefully, we can see that each *x-value* uniquely determines a *y-value* (one *x-value* can not produce two different *y-values*). Therefore, $y = \dfrac{2x}{3x-5}$ is a function.

As for its domain, we ask ourselves. Are there any values of x that must be excluded? Since $y = \dfrac{2x}{3x-5}$ is a fractional expression, we must exclude any value of x that makes the denominator equal to zero. We must have

$$3x - 5 \neq 0 \quad \Leftrightarrow \quad x \neq \frac{5}{3}$$

Therefore, the domain consists of all real numbers except for $\dfrac{5}{3}$. Thus, $Dom(f) = \{x : x \neq \dfrac{5}{3}\}$.

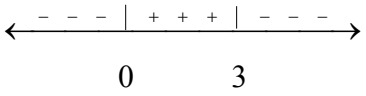
c) For the equation $y^2 = x$, if we choose $x = 9$ we get $y^2 = 9$, which gives $y = \pm 3$. In other words, there are two $y-$values associated with $x = 9$. Therefore, $y^2 = x$ is not a function.

6. Find the domain of the function $y=\sqrt{3x-x^2}$.

Solution: Since y is defined and real when the expression under the radical is non-negative, we need x to satisfy the inequality $3x-x^2 \geq 0 \quad \Leftrightarrow \quad x(3-x) \geq 0$

This is a quadratic inequality, which can be solved by analyzing signs:

Sign of $3x-x^2$



Since we want $3x-x^2=x(3-x)$ to be non-negative, the sign analysis shows us that the domain is $\{x: 0 \leq x \leq 3\}$ or $[0,3]$.

Exercise

1. Let R be a relation on the set $A=\{1,2,3,4,5,6\}$ defined by $R=\{(a,b): a+b \leq 9\}$.
   i) List the elements of R
   ii) Is $R=R^{-1}$
2. Let R be a relation on the set $A=\{1,2,3,4,5,6,7\}$ defined by $R=\{(a,b): 4 \text{ divides } a-b\}$.
   i) List the elements of R
   ii) Find $Dom(R)$ & $Range(R)$
   iii) Find the elements of R^{-1}
   iv) Find $Dom(R^{-1})$ & $Range(R^{-1})$
3. Let $A=\{1,2,3,4,5,6\}$. Define a relation on A by $R=\{(x,y): y=x+1\}$. Write down the domain, codomain and range of R. Find R^{-1}.
4. Find the domain and range of the relation $\{(x,y): |x|+y \geq 2\}$.
5. Let $A=\{1,2,3\}$ and $B=\{3,5,6,8\}$. Which of the following are functions from A to B?
   a) $f=\{(1,3),(2,3),(3,3)\}$
   b) $f=\{(1,3),(2,5),(1,6)\}$
   c) $f=\{(1,8),(2,5)\}$
   d) $f=\{(1,6),(2,5),(3,3)\}$
6. Determine the domain and range of the given relation. Is the relation a function?
   a) $\{(-4,-3),(2,-5),(4,6),(2,0)\}$
   b) $\{(8,-2),(6,-\frac{3}{2}),(-1,5)\}$
   c) $\{(-\sqrt{3},3),(-1,1),(0,0),(1,1),(\sqrt{3},3)\}$
   d) $\{(-\frac{1}{2},\frac{1}{6}),(-1,1),(\frac{1}{3},\frac{1}{8})\}$
   e) $\{(0,5),(1,5),(2,5),(3,5),(4,5),(5,5)\}$
   f) $\{(5,0),(5,1),(5,2),(5,3),(5,4),(5,5)\}$
7. Find the domain and range of the following functions.
   a) $f(x)=1+8x-2x^2$
   b) $f(x)=\dfrac{1}{x^2-5x+6}$
   c) $f(x)=\sqrt{x^2-6x+8}$
   d) $f(x)=\begin{cases}3x+4, & -1 \leq x<2\\ 1+x, & 2 \leq x \leq 5\end{cases}$
8. Given $f(x)=\begin{cases}3x-5, & x<1\\ x^2-1, & x \geq 1\end{cases}$.

   Find a) $f(-3)$ b) $f(1)$ c) $f(6)$

REAL VALUED FUNCTIONS AND THEIR PROPERTIES

After completing this section, the student should be able to:

- perform the four fundamental operations on polynomials
- compose functions to get a new function
- determine the domain of the sum, difference, product and quotient of two functions
- define equality of two functions

Let f be a function from set A to set B. If B is a subset of real number system $\Re$, then f is called a real valued function, and in particular if A is also a subset of $\Re$, then $f: A \to B$ is called a real function.

Example: 1. The function $f: \Re \to \Re$ defined by $f(x) = x^2 + 3x + 7$, $x \in \Re$ is a real function.

2. The function $f: \Re \to \Re$ defined as $f(x) = |x|$ is also a real valued function.

- **OPERATIONS ON FUNCTIONS**

Functions are not numbers. But just as two numbers a and b can be added to produce a new number $a + b$, so two functions f and g can be added to produce a new function $f + g$. This is just one of the several operations on functions that we will describe in this section.

Consider functions f and g with formulas $f(x) = \dfrac{x-3}{2}$, $g(x) = \sqrt{x}$. We can make a new function $f + g$ by having it assign to x the value $\dfrac{x-3}{2} + \sqrt{x}$, that is,

$$(f+g)(x) = f(x) + g(x) = \frac{x-3}{2} + \sqrt{x} \ .$$

Definition : Sum, Difference, Product and Quotient of two functions

Let $f(x)$ and $g(x)$ be two functions. We define the following four functions:

1. $(f+g)(x)=f(x)+g(x)$ The sum of the two functions
2. $(f-g)(x)=f(x)-g(x)$ The difference of the two functions
3. $(f\cdot g)(x)=f(x)g(x)$ The product of the two functions
4. $\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}$ The quotient of the two functions (provided $g(x)\neq 0$)

Since an $x-$value must be an inout into both f and g, the domain of $(f+g)(x)$ is the set of all x common to the domain of f and g. This is usually written as $Dom(f+g)=Dom(f)\bigcap Dom(g)$. Similar statements hold for the domains of the difference and product of two functions. In the case of the quotient, we must impose the additional restriction that all elements in the domain of g for which $g(x)=0$ are excluded.

Example :

1. Let $f(x)=3x^2+2$ and $g(x)=5x-4$. Find each of the following and its domain

a) $(f+g)(x)$ b) $(f-g)(x)$ c) $(f.g)(x)$ d) $\left(\frac{f}{g}\right)(x)$

Solution:

a) $(f+g)(x)=f(x)+g(x)=(3x^2+2)+(5x-4)=3x^2+5x-2$

b) $(f-g)(x)=f(x)-g(x)=(3x^2+2)-(5x-4)=3x^2-5x+6$

c) $(f\cdot g)(x)=(3x^2+2)(5x-4)=15x^3-12x^2+10x-8$

d) $\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}=\frac{3x^2+2}{5x-4}$

We have

$$Dom(f+g)=Dom(f-g)=Dom(fg)=Dom(f)\bigcap Dom(g)=\Re\bigcap\Re=\Re$$

$$Dom\left(\frac{f}{g}\right)=\left[Dom(f)\bigcap Dom(g)\right]\setminus\{x:g(x)=0\}=\Re\setminus\left\{\frac{5}{4}\right\}$$

2. Let $f(x)=\sqrt[4]{x+1}$ and $g(x)=\sqrt{9-x^2}$, with respective domains $[-1,\infty)$ and $[-3,3]$. Find formulas for $f+g, f-g, f\cdot g, \frac{f}{g}$ and f^3 and give their domains.

Solution:

Formula	Domain
$(f+g)(x)=f(x)+g(x)=\sqrt[4]{x+1}+\sqrt{9-x^2}$	$[-1,3]$
$(f-g)(x)=f(x)-g(x)=\sqrt[4]{x+1}-\sqrt{9-x^2}$	$[-1,3]$
$(f\cdot g)(x)=f(x)\cdot g(x)=\sqrt[4]{x+1}\cdot\sqrt{9-x^2}$	$[-1,3]$
$\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}=\frac{\sqrt[4]{x+1}}{\sqrt{9-x^2}}$	$[-1,3)$
$f^3(x)=(f(x))^3=\left(\sqrt[4]{x+1}\right)^3=(x+1)^{\frac{3}{4}}$	$[-1,\infty)$

There is yet another way of producing a new function from two given functions.

Definition : (Composition of functions)

Given two functions $f(x)$ and $g(x)$, the composition of the two functions is denoted by $f\circ g$ and is defined by:

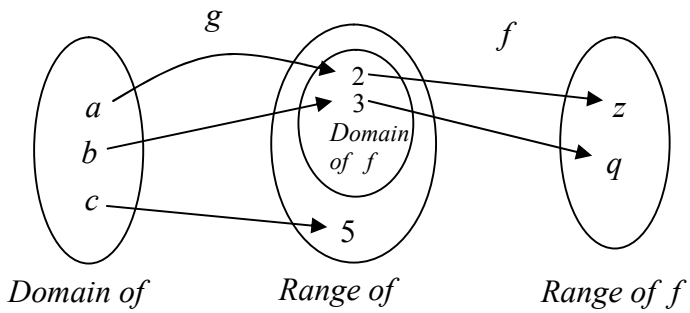
$$(f\circ g)(x)=f[g(x)].$$

$(f\circ g)(x)$ is read as "f composed with g of x". The domain of $f\circ g$ consists of those $x's$ in the domain of g whose range values are in the domain of f, i.e. those $x's$ for which $g(x)$ is in the domain of f.

Example :

1. Suppose $f=\{(2,z),(3,q)\}$ and $g=\{(a,2),(b,3),(c,5)\}$. The function $(f\circ g)(x)=f(g(x))$ is found by taking elements in the domain of g and evaluating as follows: $(f\circ g)(a)=f(g(a))=f(2)=z,\quad (f\circ g)(b)=f(g(b))=f(3)=q$

If we attempt to find $f(g(c))$ we get $f(5)$, but 5 is not in the domain of $f(x)$ and so we cannot find $(f\circ g)(c)$. Hence, $f\circ g=\{(a,z),(b,q)\}$. The figure below illustrates this situation.



2. Given $f(x)=5x^2-3x+2$ and $g(x)=4x+3$, find

a) $(f\circ g)(-2)$ b) $(g\circ f)(2)$ c) $(f\circ g)(x)$ d) $(g\circ f)(x)$

Solution:

a) $(f\circ g)(-2)=f(g(-2))$ …… First evaluate $g(-2)=4(-2)+3=-5$

$=f(-5)$

$=5(-5)^2-3(-5)+2=142$

b) $(g\circ f)(2)=g(f(2))$ …….First evaluate $f(2)=5(2)^2-3(2)+2=16$

$=g(16)$

$=4(16)+3=67$

c) $(f\circ g)(x)=f(g(x))$ ……. But $g(x)=4x+3$

$=f(4x+3)$

$=5(4x+3)^2-3(4x+3)+2$

$=80x^2+108x+38$

d) $(g\circ f)(x)=g(f(x))$ ……. But $f(x)=5x^2-3x+2$

$=g(5x^2-3x+2)$

$=4(5x^2-3x+2)+3$

$=20x^2-12x+11$

3. Given $f(x)=\dfrac{x}{x+1}$ and $g(x)=\dfrac{2}{x-1}$, find

a) $(f\circ g)(x)$ and its domain b) $(g\circ f)(x)$ and its domain

Solution: a) $(f\circ g)(x)=f\left(\dfrac{2}{x-1}\right)=\dfrac{\dfrac{2}{x-1}}{\dfrac{2}{x-1}+1}=\dfrac{2}{x+1}$. Thus, $Dom(f\circ g)=\{x:x\neq\pm1\}$.

b) $(g\circ f)(x)=g(f(x))=\dfrac{2}{\dfrac{x}{x+1}-1}=-2x-2$. Since x must first be an input into $f(x)$ and so must be in the domain of f, we see that $Dom(g\circ f)=\{x:x\neq-1\}$.

4. Let $f(x)=\dfrac{6x}{x^2-9}$ and $g(x)=\sqrt{3x}$. Find $(f\circ g)(12)$ and $(g\circ f)(x)$ and its domain.

Solution: We have $(f\circ g)(12)=f(g(12))=f(\sqrt{36})=f(6)=\frac{36}{27}=\frac{4}{3}$.

$$(f\circ g)(x)=f(g(x))=f(\sqrt{3x})=\frac{6\sqrt{3x}}{(\sqrt{3x})^2-9}=\frac{6\sqrt{3x}}{3x-9}=\frac{2\sqrt{3x}}{x-3}.$$

The domain of $f\circ g$ is $[0,3)\cup(3,\infty)$.

We now explore the meaning of equality of two functions. Let $f: A \to B$ and $g: A \to B$ be two functions. Then, f and g are subsets of $A \times B$. Suppose $f = g$. Let x be any element of A. Then, $(x, f(x)) \in f = g$ and thus $(x, f(x)) \in g$. Since g is a function and $(x, f(x))$, $(x, g(x)) \in g$, we must have $f(x) = g(x)$. Conversely, assume that $g(x) = f(x)$ for all $x \in A$. Let $(x, y) \in f$. Then, $y = f(x) = g(x)$. Thus, $(x, y) \in g$, which implies that $f \subseteq g$. Similarly, we can show that $g \subseteq f$. It now follows that $f = g$. Thus two functions $f: A \to B$ and $g: A \to B$ are equal if and only if $f(x) = g(x)$ for all $x \in A$. In general we have the following definition.

Definition : (Equality of functions)

Two functions are said to be equal if and only if the following two conditions hold:

i) The functions have the same domain;

ii) Their functional values are equal at each element of the domain.

Example :

1. Let $f: Z \to Z^+ \cup \{0\}$ and $g: Z \to Z^+ \cup \{0\}$ be defined by $f = \{(n, n^2) : n \in Z\}$ and $g = \{(n, |n|^2) : n \in Z\}$. Now, for all $n \in Z$, $f(n) = n^2 = |n|^2 = g(n)$. Thus, $f = g$.
2. Let $f(x) = \dfrac{x^2 - 25}{x - 5}, x \in \Re \setminus \{5\}$, and $g(x) = x + 5, x \in \Re$. The function f and g are not equal because $Dom(f) \neq Dom(g)$.

Exercise

1. For $f(x)=x^2+x$ and $g(x)=\dfrac{2}{x+3}$, find each value:
   a) $(f-g)(2)$ c) $g^2(3)$ e) $(g\circ f)(1)$
   b) $\left(\dfrac{f}{g}\right)(1)$ d) $(f\circ g)(1)$ f) $(g\circ g)(3)$
2. If $f(x)=x^3+2$ and $g(x)=\dfrac{2}{x-1}$, find a formula for each of the following and state its domain.
   a) $(f+g)(x)$ c) $\left(\dfrac{g}{f}\right)(x)$
   b) $(f\circ g)(x)$ d) $(g\circ f)(x)$
3. Let $f(x)=x^2$ and $g(x)=\sqrt{x}$.
   a) Find $(f\circ g)(x)$ and its domain.
   b) Find $(g\circ f)(x)$ and its domain
   c) Are $(f\circ g)(x)$ and $(g\circ f)(x)$ the same functions? Explain.
4. Let $f(x)=5x-3$. Find $g(x)$ so that $(f\circ g)(x)=2x+7$.
5. Let $f(x)=2x+1$. Find $g(x)$ so that $(f\circ g)(x)=3x-1$.
6. If f is a real function defined by $f(x)=\dfrac{x-1}{x+1}$. Show that $f(2x)=\dfrac{3f(x)+1}{f(x)+3}$.
7. Find two functions f and g so that the given function $h(x)=(f\circ g)(x)$, where
   a) $h(x)=(x+3)^3$
   b) $h(x)=\dfrac{1}{x}+6$
   c) $h(x)=\sqrt{5x-3}$
   d) d) $h(x)=\dfrac{1}{x+6}$
8. Let $f(x)=4x-3, g(x)=\dfrac{1}{x}$ and $h(x)=x^2-x$. Find
   a) $f(5x+7)$
   b) $5f(x)+7$
   c) $f(g(h(3)))$
   d) $f(1)\cdot g(2)\cdot h(3)$
   e) $f(x+a)$
   f) $f(x)+a$

INVERSE OF FUNCTIONS

After completing this section, the student should be able to:

- define one to oneness and ontoness of a function
- check invertibility of a function
- find the inverse of an invertible function

Since a function is a relation , the inverse of a function f is denoted by f^{-1} and is defined by:

$$f^{-1} = \{(y,x) : (x,y) \in f\}$$

For instance, if $f = \{(2,4),(3,6),(1,7)\}$, then $f^{-1} = \{(4,2),(6,3),(7,1)\}$. Note that the inverse of a function is not always a function. To see this consider the function $f = \{(2,4),(3,6),(5,4)\}$. Then, $f^{-1} = \{(4,2),(6,3),(4,5)\}$, which is not a function.

As we have seen above not all functions have an inverse, so it is important to determine whether or not a function has an inverse before we try to find the inverse. If the function does not have an inverse, then we need to realize that it does not have an inverse so that we do not waste our time trying to find something that does not exist.

A one to one function is special because only one to one functions have inverse. If a function is one to one, to find the inverse we will follow the steps below:

1. Interchange x and y in the equation $y = f(x)$
2. Solving the resulting equation for y, we will obtaining the inverse function.

Note that the domain of the inverse function is the range of the original function and the range of the inverse function is the domain of the original function.

Example

1. Given $y = f(x) = x^3$. Find f^{-1} and its domain.

Solution: We begin by interchanging x and y, and we solve for y.

$y = x^3$	Interchange x and y
$x = y^3$	Take the cube root of both sides
$\sqrt[3]{x} = y$	This is the inverse of the function

Thus, $f^{-1}(x) = \sqrt[3]{x}$. The domain of f^{-1} is the set of all real numbers.

2. Let $y = f(x) = \dfrac{x}{x+2}$. Find $f^{-1}(x)$.

Solution: Again we begin by interchanging x and y, and then we solve for y.

$$y = \frac{x}{x+2} \qquad \text{Interchange } x \text{ and } y$$

$$x = \frac{y}{y+2} \qquad \text{Solving for } y$$

$$x(y+2) = y \;\Leftrightarrow\; xy + 2x = y \;\Leftrightarrow\; 2x = y(1-x) \;\Leftrightarrow\; y = \frac{2x}{1-x}$$

Thus, $f^{-1}(x) = \frac{2x}{1-x}$.

Remark: Even though, in general, we use an exponent of -1 to indicate a reciprocal, inverse function notation is an exception to this rule. Please be aware that $f^{-1}(x)$ is not the reciprocal of f. That is,

$$f^{-1}(x) \neq \frac{1}{f(x)}$$

If we want to write the reciprocal of the function $f(x)$ by using a negative exponent, we must write

$$\frac{1}{f(x)} = [f(x)]^{-1}.$$

Exercise

1. Find $f^{-1}(x)$ if

a) $f(x) = 7x - 6$

b) $f(x) = \frac{2x-9}{4}$

c) $f(x) = 1 - \frac{3}{x}$

d) $f(x) = \frac{4-x}{3x}$

e) $f(x) = \frac{5x+3}{1-2x}$

f) $f(x) = \sqrt[3]{x+1}$

g) $f(x) == -(x+2)^2 - 1$

h) $f(x) = \frac{2x}{1+x}$

CHAPTER TWO

RATIONAL FUNCTIONS AND THEIR GRAPHS

A rational function is a function of the form $f(x)=\frac{n(x)}{d(x)}$ where both $n(x)$ and $d(x)$ are polynomials and $d(x)\neq 0$.

Example: The functions $f(x)=\frac{3}{x+5}$, $f(x)=\frac{x-1}{x^2-4}$ and $f(x)=\frac{x^5+2x^3-x+1}{x\ +5x}$ are examples of rational function.

Note that the domain of the rational function $f(x)=\frac{n(x)}{d(x)}$ is $\{x : d(x)\neq 0\}$

Example : Find the domain and zeros of the function $f(x)=\frac{3x-5}{x^2-x-12}$.

Solution: The values of x for which $x^2-x-12=0$ are excluded from the domain of f. Since $x^2-x-12=(x-4)(x+3)$, we have $Dom(f)=\{x : x\neq -3,4\}$. To find the zeros of $f(x)$, we solve the equation $\frac{n(x)}{d(x)}=0 \Leftrightarrow n(x)=0 \ \& \ q(x)\neq 0$

Therefore, to find the zeros of $f(x)$, we solve $3x-5=0$, giving $x=\frac{5}{3}$. Since $\frac{5}{3}$ does not make the denominator zero, it is the only zero of $f(x)$.

The following terms and notations are useful in our next discussion.

Given a number *a*,

- x approaches a from the right means x takes any value near and near to a but $x > a$. This is denoted by: $x\to a^+$ (read: 'x approaches a from the right').

  For instance, $x\to 1^+$ means x can be 1.001, 1.0001, 1.00001, 1.000001, etc.

- x approaches a from the left means x takes any value near and near to a but $x < a$.

  This is denoted by: $x\to a^-$ (read: 'x approaches a from the left').

  For instance, $x\to 1^-$ means x can be 0.99, 0.999, 0.9999, 0.9999, etc.

- $x\to\infty$ (read: 'x approaches or tends to *infinity*') means the value of x gets indefinitely larger and larger in magnitude (keep increasing without bound). For instance, x can be 10^6, 10^{10}, 10^{12}, etc.

- $x\to -\infty$ (read: 'x approaches or tends to negative *infinity*') means the value of x is negative and gets indefinitely larger and larger negative in magnitude (keep decreasing without bound). For instance, x can be -10^6, -10^{10}, -10^{12}, etc.

The same meanings apply also for the values of a function f if we wrote $f(x)\to\infty$ or $f(x)\to-\infty$. The following figure illustrates these notion and notations.

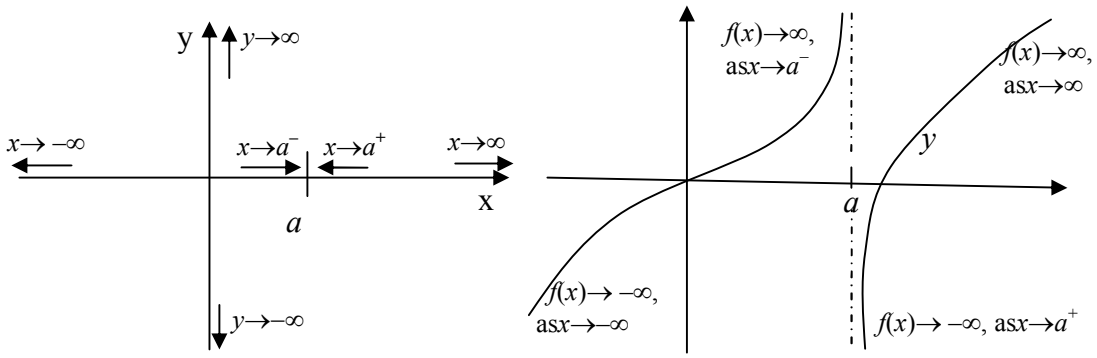


Fig. 2.1. Graphical illustration of the idea of $x\to a^+$, $f(x)\to\infty$, etc.

We may also write $f(x)\to b$ (read: '$f(x)$ approaches b') to mean the function values, $f(x)$, becomes arbitrarily closer and closer to b (i.e., approximately b) but not exactly equal to b. For instance, if $f(x)=\frac{1}{x}$, then $f(x)\to 0$ as $x\to\infty$; i.e., $\frac{1}{x}$ is approximately 0 when x is arbitrarily large.

The following steps are usually used to sketch (or draw) the graph of a rational function $f(x)$.

1. Identify the domain and simplify it.
2. Find the intercepts of the graph whenever possible. Recall the following:
   - y–intercept is the point on y-axis where the graph of $y=f(x)$ intersects with the y-axis. At this point $x=0$. Thus, $y=f(0)$, or $(0, f(0))$ is the y-intercept if $0\in\text{Dom}(f)$.
   - x–intercept is the point on x-axis where the graph of $y=f(x)$ intersects with the x-axis. At this point $y=0$. Thus, $x=a$ or $(a, 0)$ is x-intercept if $f(a)=0$.
3. Determine the asymptotes of the graph. Here, remember the following.

- Vertical Asymptote: The vertical line $x=a$ is called a vertical asymptote(VA) of $f(x)$ if
  i) $a\notin\text{dom}(f)$, i.e., f is not defined at $x=a$; and
  ii) $f(x)\to\infty$ or $f(x)\to-\infty$ when $x\to a^+$ or $x\to a^-$. In this case, the graph of f is almost vertically rising upward (if $f(x)\to\infty$) or sinking downward (if $f(x)\to-\infty$) along with the vertical line $x=a$ when x approaches a either from the right or from the left.

Example : Consider $f(x)=\frac{1}{(x-a)^n}$, where $a\geq 0$ and n is a positive integer.

Obviously $a\notin\text{Dom}(f)$. Next, we investigate the trend of the values of $f(x)$ near a. To do this, we consider two cases, when n is even or odd:

Suppose n is even: In this case $(x-a)^n>0$ for all $x\in\Re\setminus\{a\}$; and since $(x-a)^n\to 0$ as $x\to a^+$ or $x\to a^-$. Hence, $f(x)=\frac{1}{(x-a)^n}\to\infty$ as $x\to a^+$ or $x\to a^-$. Therefore, $x=a$ is a VA of $f(x)$.

Moreover, $y= 1/a^n$ or $(0, 1/a^n)$ is its y-intercept since $f(0)=1/a^n$. However, it has no x-intercept since $f(x) >0$ for all x in its domain (See, Fig. 2.2 (A)).

Suppose n is odd: In this case $(x-a)^n > 0$ for all $x>a$ and $1/(x-a)^n \to \infty$ when $x \to a^+$ as in the above case. Thus, $x=a$ is its VA. However, $1/(x-a)^n \to -\infty$ when $x \to a^-$ since $(x-a)^n < 0$ for $x<a$. Moreover, $y= -1/a^n$ or $(0, -1/a^n)$ is its y-intercept since $f(0) = -1/a^n$. However, it has no x-intercept also in this case. (See, Fig. 2.2 (B)).

Note that in both cases, $f(x) = \dfrac{1}{(x-a)^n} \to 0$ as $x \to \infty$ or $x \to -\infty$.

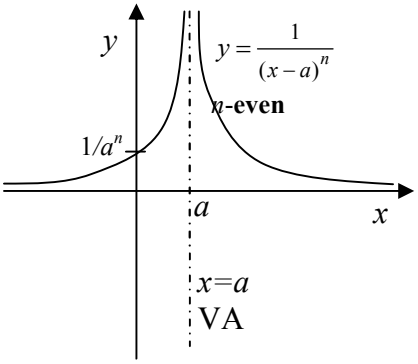


Fig. 2.2 (A)

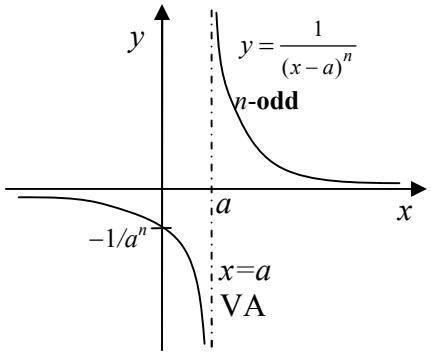


Fig. 2.2 (B)

Remark: Let $f(x) = \dfrac{n(x)}{d(x)}$ be a rational function. Then,

1. if $d(a) = 0$ and $n(a) \neq 0$, then $x=a$ is a VA of f .
2. if $d(a) = 0 = n(a)$, then $x=a$ may or may **not** be a VA of f. In this case, simplify $f(x)$ and look for VA of the simplest form of f.

- Horizontal Asymptote: A horizontal line $y=b$ is called horizontal asymptote (HA) of $f(x)$ if the value of the function becomes closer and closer to b (i.e., $f(x) \to b$) as $x \to \infty$ or as $x \to -\infty$.
  In this case, the graph of f becomes almost a horizontal line along with (or near) the line $y=b$ as $x \to \infty$ and as $x \to -\infty$. For instance, from the above example, the HA of $f(x) = \dfrac{1}{(x-a)^n}$ is $y=0$ (the x-axis) , for any positive integer n (See, Fig. 2.2).

Remark: A rational function $f(x) = \dfrac{n(x)}{d(x)}$ has a HA only when $degree(n(x)) \leq degree(d(x))$.

In this case, (i) If $degree(n(x)) < degree(d(x))$, then $y = 0$ (the x-axis) is the HA of f.

(ii) If $degree(n(x)) = degree(d(x)) = n$, i.e., $f(x) = \dfrac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0}$,

then $y = \dfrac{a_n}{b_n}$ is the HA of f.

- Oblique Asymptote: The oblique line $y=ax+b$, $a\neq 0$, is called an oblique asymptote (OA) of f if the value of the function, $f(x)$, becomes closer and closer to $ax+b$ (i.e., $f(x)$ becomes approximately $ax+b$) as either $x\to\infty$ or $x\to -\infty$. In this case, the graph of f becomes almost a straight line along with (or near) the oblique line $y=ax+b$ as $x\to\infty$ and as $x\to -\infty$.

Note: A rational function $f(x)=\frac{n(x)}{d(x)}$ has an OA only when $degree(n(x)) = degree(d(x)) + 1$. In this case, using long division, if the quotient of $n(x) \div d(x)$ is $ax+b$, then $y= ax+b$ is the OA of f.

Example : Sketch the graphs of (a) $f(x)=\frac{x+2}{x-1}$ (b) $g(x)=\frac{x^2+3x+2}{x^2-1}$

Solution: (a) Since $x-1=0$ at $x=1$, $dom(f) = \Re\backslash\{1\}$.

- Intercepts: y-intercept: $x=0 \Rightarrow y=f(0) = -2$. Hence, $(0, -2)$ is y-intercept.
  x-intercept: $y=0 \Rightarrow x+2=0 \Rightarrow x= -2$. Hence, $(-2, 0)$ is x-intercept.
- Asymptotes:
  - VA: Since $x-1=0$ at $x=1$ and $x+2\neq 0$ at $x=1$, $x=1$ is VA of f. In fact, if $x\to 1^+$, then $x+2 \approx 3$ but the denominator $x-1$ is almost 0 (but positive).
    Consequently, $f(x)\to\infty$ as $x\to 1^+$.
    Moreover, $f(x)\to -\infty$ as $x\to 1^-$ (since , if $x\to 1^-$ then $x-1$ is almost 0 but negative) .

    (So, the graph of f rises up to $+\infty$ at the right side of $x=1$, and sink down to $-\infty$ at the left side of $x=1$)
  - HA: Note that if you divide $x+2$ by $x-1$, the quotient is 1 and remainder is 3. Thus, $f(x)=\frac{x+2}{x-1}=1+\frac{3}{x-1}$. Thus, if $x\to\infty$ (or $x\to -\infty$), then $\frac{3}{x-1}\to 0$ so that $f(x)\to 1$.
    Hence, $y=1$ is the HA of f.

Using these information, you can sketch the graph of f as displayed below in Fig. 2.3 (A).

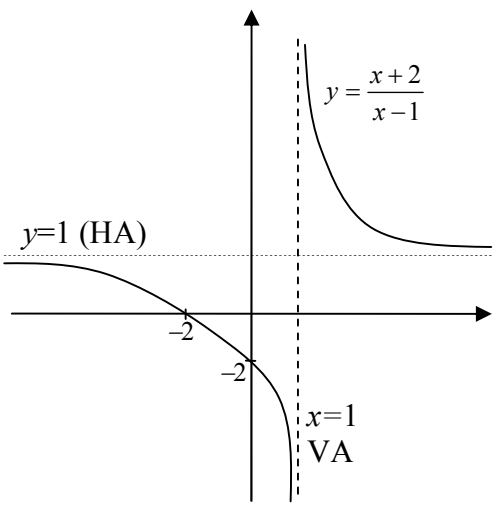
(b) Both the denominator and numerator are 0 at $x=1$. So, first factorize and simplify them:
$x^2+3x+2=(x+2)(x+1)$ and $x^2-1 = (x-1)(x+1)$. Therefore,

$$g(x)=\frac{x^2+3x+2}{x^2-1}=\frac{(x+2)(\cancel{x+1})}{(x-1)(\cancel{x+1})}, \qquad x\neq -1$$

$$=\frac{x+2}{x-1}. \qquad (\text{So, } dom(g) = \Re\backslash\{1,-1\})$$

This implies that only $x=1$ is VA.

Hence, the graph of $g(x)=\frac{x+2}{x-1}$, $x\neq -1$, is exactly the same as that of $f(x)=\frac{x+2}{x-1}$ except that $g(x)$ is not defined at $x=-1$. Therefore, the graph of g and its VA are the same as that of f except that there should be a 'hole' at the point corresponding to $x=-1$ on the graph of g as shown on Fig. 2.3(B) below.



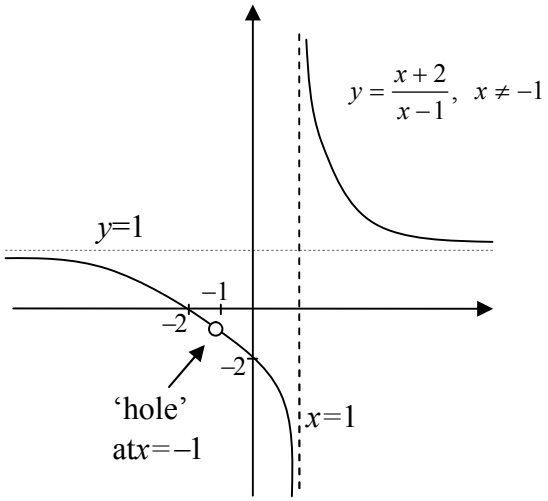


Fig 2.3 (A) $f(x)=\frac{x+2}{x-1}$ (B) $g(x)=\frac{x^2+3x+2}{x^2-1}=\frac{x+2}{x-1},\ x\neq -1$

Exercise

1. Find the domain and the real zeros of the given function.

   a) $f(x)=\frac{3}{x^2-25}$ b) $g(x)=\frac{x-3}{x^2 4x-12}$ c) $f(x)=\frac{(x-3)^2}{x^3-3x^2+2x}$ d) $f(x)=\frac{x^2-16}{x^2+4}$

2. Sketch the graph of

   a) $f(x)=\frac{1-x}{x-3}$ b) $f(x)=\frac{x^2+1}{x}$ c) $f(x)=\frac{1}{x}+2$ d) $f(x)=\frac{x^2}{x^2-4}$

3. Determine the behavior of $f(x)=\frac{x^3-8x-3}{x-3}$ when x is near 3.

4. The graph of any rational function in which the degree of the numerator is exactly one more than the degree of the denominator will have an oblique (or slant) asymptote.

   a) Use long division to show that

   $$y=f(x)=\frac{x^2-x+6}{x-2}=x+1+\frac{8}{x-2}$$

   b) Show that this means that the line $y=x+1$ is a slant asymptote for the graph and sketch the graph of $y=f(x)$.

WORK SHEET ONE (SELF EXAM)

RELATED WITH RELATION AND FUNCTION

Choose the correct answer among the given alternatives

1. Let D = {0, 1, 2, 4} and $R = \{(x, y): x, y \in D \text{ and } y < x\}$. from the following ordered pairs which one belongs to R?

   A. (3, 1) B. (2, 0) C. (4, 4) D. (0, 4)

2. Let $f(x) = x^2 + 4x + 1$.Thus the graph of f is obtained by shifting the graph of

   A. $g(x) = (x + 2)^2$ by 3 units downward B. $g(x) = (x + 2)^2$ by 3 units upward

   C. $g(x) = (x + 2)^2$ by 1 unit downward D. $g(x) = (x + 2)^2$ by 1 unit upward

3. Let $R = \{(x, y): x, y \in \mathbb{R} \text{ and } y < 2x - x^2\}$ be a relation. What is the range of R?

   A. $\{y \in \mathbb{R} : y < 1\}$ B. $\{y \in \mathbb{R} : y > 1\}$ C. $\{y \in \mathbb{R} : y < 2\}$ D. $\{y \in \mathbb{R} : y > 2\}$

4. let f be a function defined by $f(x) = \sqrt{x - 1}$, then what is the domain of f?

   A. {1, 2, 4, 5} B.{2, 3, 5, 6} C. $\{x \in \mathbb{R} : x \geq 1\}$ D.$\{x \in \mathbb{R} : 1 \leq x \leq 10\}$

5. let $A = \{-1, 0, 1\}$. A relation R on a set A is defined by

   $R = \{(x, y) | \ y < 2x, where \ \ x, y \in A\}$.Which one of the following sets is equal toR?

   A. $\{(0, -1), (1, -1), (1, 0), (1, 1)\}$ B.$\{(0, -1)\}$

   C. $\{(0, -1), (1, -1)\}$ D.$\{(0, -1), (-1, 0), (0, 1)\}$

6.Given the functions $f = \{(1, 0), (-1, 2), (3, 4), (2, 1)\}$ and

   $g = \{(0, 0), (2, 3), (3, -1), (1, -1)\}$. Then which one of the following is correct?

   A. $2f(2) = 2g(1)$

   B. $(f + g)(-1) = 1$

   C. $(f \cdot g)(2) = -1$

   D. $(f - g)(3) = 5$

7. If f is a function which is given by $f(x) = x^2 + (1 - x)^2 - 13$, then what is the minimum value of this function?

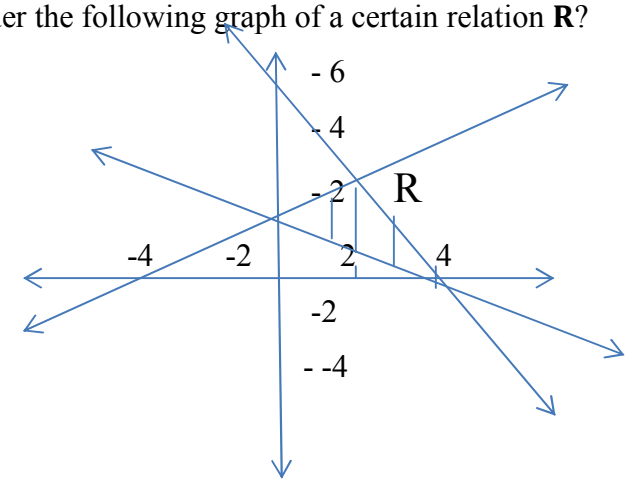
   A. $\frac{1}{2}$

   B. -12

   C. $-\frac{25}{2}$

   D. $-\frac{21}{2}$

8.Consider the following graph of a certain relation **R**?



From the graph the domain of this relation is?

A. $\{x \in \mathbb{R} : 0 \leq x \leq 4\}$ B. $\{x \in \mathbb{R} : 2 \leq x \leq 4\}$

C. $\{x \in \mathbb{R} : 0 \leq x \leq 3\}$ D. $\{x \in \mathbb{R} : -4 \leq x \leq 4\}$

9. If a function f is given by $f(x) = \sqrt{9 - x^2}$, where $x \in$ [- 3, 3].then the range of f is?

A. [- 9, 9] B. [- 3, 3] C.[0, 3] D.$\mathbb{R}$

10. $f(x) = x^2 + 6x + 9$, then which one of the following statement is true about the graph of ?

A. The range of f is $\{y \mid y \geq 0\}$.

B. The graph of f is open down ward.

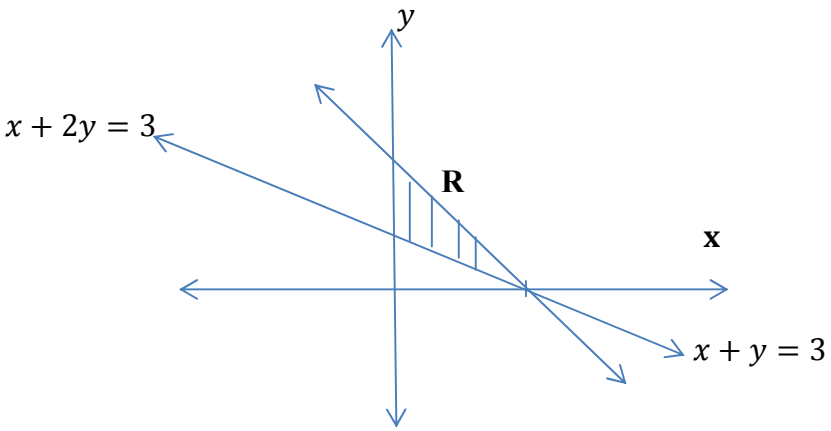
C. The minimum value of this function is 9.

D. The graph of f is symmetrical with respect to the y-axis.

11.If set A= $\{\frac{1}{4}, \frac{1}{2}, 2, 4\}$,then which one of the following is a function from A onto set A?

A. $\{(x,y): y \leq x\}$ B. $\{(x,y): y = \frac{1}{x}\}$ C. $\{(x,y): y = x^2\}$ D. $\{(x,y): y = 2x\}$

12.The shaded region on the graph shown below is the solution of a relation **R.**



Which one of the following system of linear inequalities represents the relation **R**?

A. $\begin{cases} x+y \leq 3 \\ x+2y \geq 3 \\ x \geq 0 \end{cases}$ B. $\begin{cases} x+y \leq 3 \\ x+2y \geq 3 \\ y \geq 0 \end{cases}$ C. $\begin{cases} x+y \leq 3 \\ x+2y \geq 3 \end{cases}$ D. $\begin{cases} x+y \geq 3 \\ x+2y \leq 3 \end{cases}$

WORK OUT (show your steps clearly and neately)

1. Which of the following sets of ordered pairs are functions?
   a. $(1,2),(2,3),(3,4),(4,5)$
   b. $(4,3),(2,2),(-3,4),(-3,-3)$
   c. $(1,2),(2,3),(1,3),(4,5)$
   d. $(1,-1),(1,-6),(4,2),(2,-3)$
2. Let $R = \{(x,\ y): y\ is\ shorter\ than\ x\}$.
   a. Does(x,x) belong to the relation? Why?
   b. Is true that if (x,y) belongs to R, then (y,x) belongs to R?
   c. If (x,y) and (y,z) belong to R, then is it true that (x,z) belongs to R?
3. Find the domain and range of each of the following relations:
   a. R is the relation which contains the set of order pairs $(x,y): y = -3x$.
   b. R is the relation which contains the set of order pairs $(x,y): x$ contains $-2,-1,0,1,3,5$ and $y = 2-x$
   c. R is the relation which contains the set of order pairs $(x,y): y = \sqrt{1-x^2}$.
   d. The set of ordered pairs (x,y), where y is a sister of x.
   e. The set of ordered pairs (x,y), where a pupil in $y's$ class is x.
4. Sketch the graph of each of the following relations and find the domain and range.
   a. R is the relation which contains the set of order pairs $(x,y): y = x^3$.
   b. R is the relation which contains the set of order pairs $(x,y): y \leq -x-1$.
   c. R is the relation which contains the set of order pairs $(x,y): y > 3x-1$.
   d. R is the relation which contains the set of order pairs $(x,y): y \geq 2x-3$.
   e. R is the relation which contains the set of order pairs $(x,y): y \geq 2x$ and $y < x+1$.

f. R is the relation which contains the set of order pairs (x, y): $y < -x + 2$ and $y > x - 3$.

5. For the following graph (Figure 1.44), specify the relation and write down the domain and range:

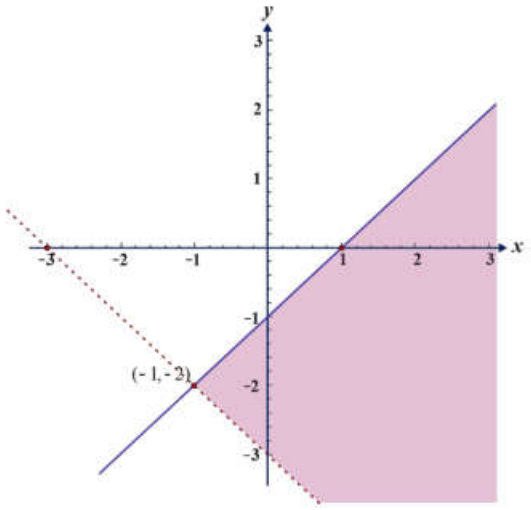


Figure 1.44

6. Let $f(x) = 2x^2 - 5x - 3$ and $g(x) = -2x^2 + x + 7$

a. Find:

i) $f + g$ **ii)** $f - g$

iii) $(f + g)(-1)$ **iv)** $(2f - g)(3)$

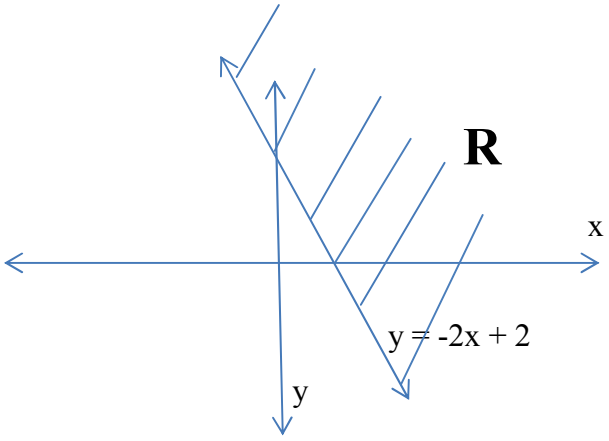
b. Determine the domain of $f - g$.

7. Let $f(x) = 2x^2 - 1$ and $g(x) = x - 3$, then

a. Evaluate: **i)** fg **ii)** $\frac{f}{g}$ **iii)** $(fg)(2)$ **iv)** $\left(\frac{f}{g}\right)(5)$

b. Find the domain of $\frac{f}{g}$.

8.This is a graph of the relation R = {(x, y)| y ≥-2x + 2}. Determine the range value when the domain value is –1.



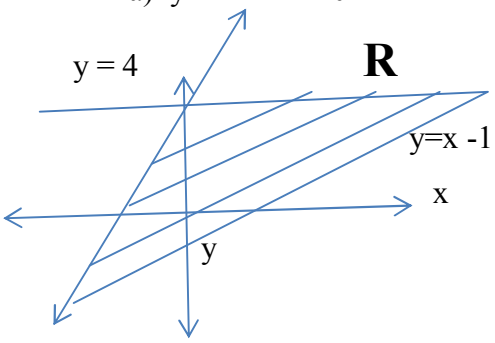
9. Sketch the graph and determine the domain and range of the relation given by

a) R= {(x, y)| y ≥x – 3.y ≥ -3x + 3 and y<6}

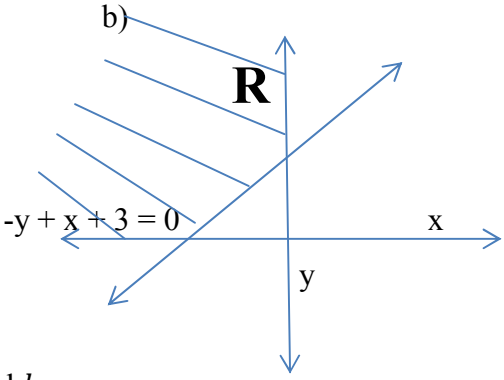
b) R= {(x, y)| y >-3x – 3 and y < x – 2}

10. Determine the relation from the following graphs and determine the domain and range?

a) $y - 2x - 4 = 0$



b)



11. In each of the following cases, find $a$and b.

(i) $(2a + b,\ a - b) = (8, 3)$ (ii) $\left(\frac{a}{4}, a - 2b\right) = (0, 6 + b)$

12. Given A = {1, 2, 3, 4, 5}, **R** = {(x, y): x ∈ A, y ∈ A}. Find the ordered pairs which satisfy the conditions given below:

(i) $x + y = 5$ (ii) $x + y < 5$ (iii) $x + y > 8$

13. Given R = {(x, y): x, y ∈ W, $x^2 + y^2 = 25$}. Find the domain and Range of R.

14. If R_1= {(x, y) | y = 2x + 7,where x ∈ R & $-5 \leq x \leq 5$}is a relation. Then find the domain and Range of R_1.

15. If R_2 = {(x, y) | x and y are integers and $x^2 + y^2 = 64$} is a relation.Then find R_2.

16. If R_3 = {(x, |x|) | x is a real number} is a relation. Then find domain and range of R_3.

17.Let R_1= {(x, y) | $y \geq -2x + 3$}, R_2 = {(x, y) | $y \leq \frac{-2}{3}x + 2$}. then Sketch the graph of $R_1 \cap R_2$

18. Is the given relation a function? Give reasons for your answer.

(i) h = {(4, 6), (3, 9), (– 11, 6), (3, 11)}

(ii) f = {(x, x) | x is a real number}

(iii) g = $\{(a, \frac{1}{a})$ |a is a positive integer }

(iv) s = $\{(n, n^2)$ | n is a positive integer}

(v) t = {(x, 3) | x is a real number}

19. If f and g are real functions defined by $f(x) = x^2 + 7$ and $g(x) = 3x + 5$, find each of the following

(a) $f(3) + g(-5)$

(b) $f\left(\frac{1}{2}\right) \times g(14)$

(c) $f(-2) + g(-1)$

20. Let f and g be real functions defined by $f(x) = 2x + 1$ and $g(x) = 4x - 7$.

(a) For what real numbers x, $f(x) = g(x)$?

(b) For what real numbers x, $f(x) < g(x)$?

21. If f and g are two real valued functions defined as $f(x) = 2x + 1$, $g(x) = x^2 + 1$, then find.

(i) f+ g

(ii) f − g

(iii) f.g

(iv) $\frac{f}{g}$

(v) For what real numbers x, $f(x) > g(x)$?

22. Express the following functions as set of ordered pairs and determine their range.

$f: X \to R$, $f(x) = x^3 + 1$, where $X = \{-1, 0, 3, 9, 7\}$

23. Find the values of x for which the functions $f(x) = 3x^2 - 1$ and $g(x) = 3 + x$ are equal

24. Is $g = \{(1, 1), (2, 3), (3, 5), (4, 7)\}$ a function? Justify If this is described by the relation, $g(x) = \alpha x + \beta$, then what values should be assigned to α and β?

25.Let R be a relation on the set $A=\{6,5,4,3,2,1\}$ defined by $R = \{(a, b): a + b \leq 9\}$

i) List the elements of R

ii) determine domain and range of R

WORK SHEET TWO

RELATED WITH MODULUS, FLOOR AND SIGNUM FUNCTIONS

1. Draw the graph of the power function given below & find its Domain and Range
   A. $f(x) = x^{\frac{8}{3}}$
   B. $f(x) = x^{-\frac{6}{3}}$
   C. $f(x) = x^{\frac{5}{4}}$
   D. $f(x) = x^{-\frac{1}{7}}$
   E. $f(x) = x^{-\frac{5}{2}}$
   F. $f(x) = x^{-\frac{1}{2}}$
   G. $f(x) = x^{-\frac{1}{5}}$
   H. $f(x) = x^{\frac{1}{4}}$
   I. $f(x) = x^{\frac{1}{7}}$
2. Draw the graph of the Absolute function given below & find its Domain and Range
   A. $f(x) = |x|$
   B. $f(x) = |x+1|$
   C. $f(x) = |x-2|$
   D. $f(x) = |x| - 3$
   E. $f(x) = |x| + 4$
   F. $f(x) = 2|x-4|$
   G. $f(x) = 3|x+5|$
   H. $f(x) = |x-2| + 1$
   I. $f(x) = 3 - |x|$
3. Find the solution set of the following Modulus Functions
   A. $|x+5| = 14$
   B. $|x-4| + 10 = 4$
   C. $2|4x-2| = 4$
   D. $|x+2| - 9 = -9$
   E. $|6x+2| - 8 = 4$
   F. $\left|\frac{1}{2}x - 10\right| - 50 = -10$
   G. $|x-20| = 0$

4. Draw the graph of the Signum function given below & find its Domain and Range
   A. $f(x) = \text{sgn}(2x)$
   B. $f(x) = 2\,\text{sgn}(3x)$
   C. $f(x) = \text{sgn}(x) + 2$
   D. $f(x) = 2 - \text{sgn}(x)$
5. Evaluate the following greatest integer (Floor)
   A. $\lfloor 10.8 \rfloor$
   B. $\lfloor -3.8 \rfloor$
   C. $\lfloor 24 \rfloor$
   D. $\lfloor -4.2 \rfloor$
   E. $\lfloor 11.2 \rfloor$
   F. $\lfloor \pi \rfloor$
   G. $\lfloor 0 \rfloor$
   H. $\left\lfloor \frac{7}{8} \right\rfloor$
   I. $\left\lfloor \frac{3}{2} \right\rfloor$
6. Draw the graph of the greatest integer (Floor) function given below & find its Domain and Range
   A. $f(x) = \lfloor 3x \rfloor$
   B. $f(x) = \lfloor x + 2 \rfloor$
   C. $f(x) = \lfloor x - 4 \rfloor$
   D. $f(x) = \lfloor 5x - 3 \rfloor$
   E. $f(x) = \lfloor 4x + 4 \rfloor$
   F. $f(x) = 3\lfloor x \rfloor$
7. Solve the following equations
   A. $\lfloor x + 2 \rfloor = 6$
   B. $\lfloor x - 2 \rfloor = 8$
   C. $\lfloor 2x + 4 \rfloor = 10$
   D. $\lfloor 1 + \lfloor x \rfloor \rfloor = 5$
   E. $\lfloor 3 + \lfloor 2x \rfloor \rfloor = 9$
8. Let $f(x) = \lfloor x - 12 \rfloor$ then Evaluate the following at each of the following number
   A. -2
   B. -4
   C. 14
   D. 12
   E. $\frac{3}{4}$
   F. $-\frac{5}{4}$
   G. 0

WORK SHEET THREE

RELATED WITH RATONAL EXPRESSIONS AND RATIONAL FUNCTION

1. Find the domain of and simplify each of the following rational expressions
   A. $\frac{x+2}{x^2-4}$
   B. $\frac{x+3}{x^3-9x}$
   C. $\frac{4x^2-36}{x^2-6x+9}$
   D. $\frac{x^2+1}{x^3+2x^2+x}$
2. Write the following rational expressions as sum of partial sums
   A. $\frac{3x+6}{x^2-4}$
   B. $\frac{x^2+x+2}{x^2+6x+8}$
   C. $\frac{x+2}{x^2-4x+4}$
   D. $\frac{x^3+2x^2+2}{x^2-x-6}$
3. Solve each of the following rational equations
   A. $\frac{3x}{x+4}-\frac{1}{x+2}+\frac{2}{(x+2)(x+4)}=0$
   B. $2-x=\frac{4}{x+3}$
   C. $\frac{6}{x^2}=\frac{x-5}{x}$
4. Solve each of the following rational inequalities
   A. $\frac{4}{x+2}\le\frac{5}{x-1}$
   B. $\frac{2x}{x-4}\le\frac{x-4}{x-4}$
   C. $\frac{x^2-6}{3x^2}\ge\frac{5}{3x}$
   D. $\frac{x+2}{x-1}\le 0$
   E. $\frac{x+2}{x-1}>0$
   F. $\frac{x+2}{x-1}<0$

5. Sketch the graph of the following functions and Find its doamin

A. $f(x)=\dfrac{1}{x+3}$

B. $g(x)=\dfrac{1}{x^2-9}$

C. $h(x)=\dfrac{4x}{9x+3}$

D. $k(x)=\dfrac{2x-1}{-5x-2x^2+3}$

E. $m(x)=\dfrac{x^2+1}{x+3}$

F. $n(x)=\dfrac{1}{x^2+6x+9}$

G. $o(x)=\dfrac{x^3+6x^2+5x}{x^2-4x+4}$

H. $p(x)=\dfrac{1}{x-3}$

I. $q(x)=\dfrac{x^3+1}{x^2+4x+4}$

J. $r(x)=\dfrac{x+1}{x^2+5x+6}$

K. $s(x)=\dfrac{x^2}{x^2+3x+2}$

6. Let the ratio of girls in a certain in a certain school is 4:5. If the total number of students in the school is 1260, Find the number of boys and the number of girls in the school.

GOOD PRACTICE

‘TO LEARN MATHEMATICS DO MATHEMATICS WITH YOUR GROUPS AGAIN AND AGAIN’