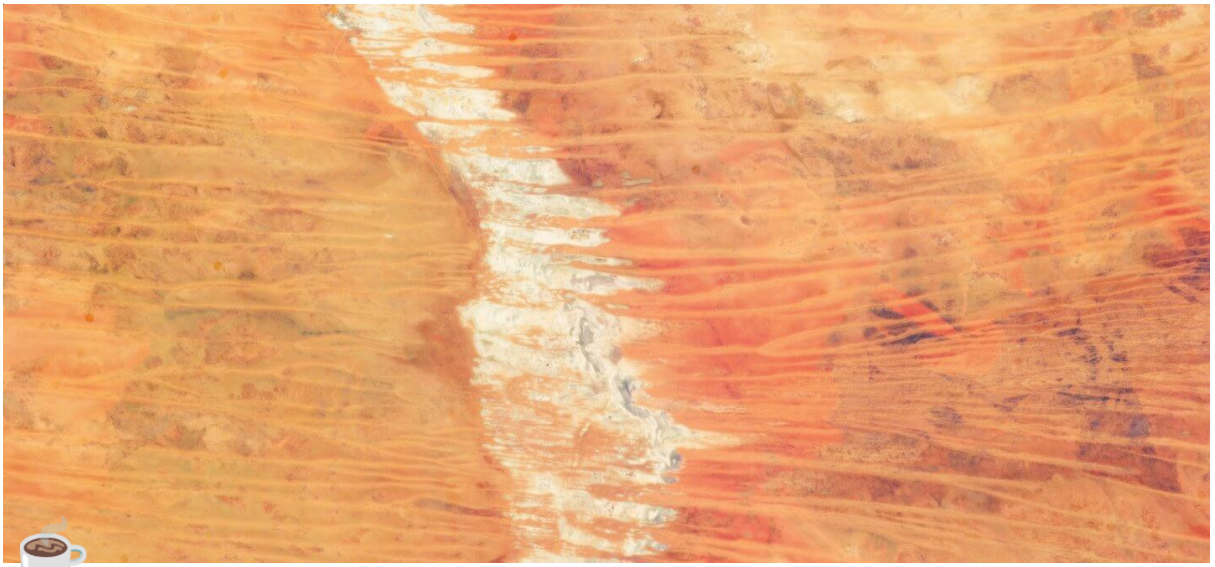


Mock Final Exam — Mechanical Vibrations

Mechanical Vibrations — Comprehensive Mock Final Exam

Format: Part I — Conceptual (short answers) · Part II — Numerical (worked problems)
Coverage: the full syllabus — derivation of EOMs, natural frequency, damping & logarithmic decrement, SDOF / 2DOF / MDOF response, base excitation, vibration measurement, and isolation & control.

Instructions

- Attempt the questions in order within each part. Show all working clearly; marks are awarded for method as well as the final answer.
- Non-programmable calculators are permitted. Assume `g = 9.81 m/s²` where needed.
- Every question has a full worked solution hidden in a toggle. For a realistic timed run, attempt a whole section first, then open the toggles to mark yourself.
- This is a comprehensive revision paper. See the timing note at the end for how to turn it into a 3-hour simulated sitting.

▼ Formula Cheat Sheet (open only when studying — keep closed for a timed run)

Single Degree of Freedom (SDOF)

Natural frequency: $\omega_n = \sqrt{k/m}$

Damping ratio: $\zeta = \dfrac{c}{2m\omega_n} = \dfrac{c}{c_c}$, where $c_c = 2m\omega_n$

Damped natural frequency: $\omega_d = \omega_n\sqrt{1-\zeta^2}$

Logarithmic decrement: $\delta = \dfrac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \dfrac{1}{n}\ln\dfrac{x_0}{x_n}$

Underdamped free response:

$$x(t) = e^{-\zeta\omega_n t}\left[x_0\cos\omega_d t + \frac{v_0+\zeta\omega_n x_0}{\omega_d}\sin\omega_d t\right]$$

SDOF Harmonic / Forced Vibration

For $m\ddot{x} + kx = F_0 \sin \omega t$ with $f_0 = F_0/m$, total (undamped) response:

$$x(t) = x_0 \cos \omega_n t + \left[\frac{v_0}{\omega_n} - \frac{\omega f_0}{\omega_n(\omega_n^2 - \omega^2)}\right] \sin \omega_n t + \frac{f_0}{\omega_n^2 - \omega^2} \sin \omega t$$

Magnification factor: $M = \dfrac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$, with $r = \omega/\omega_n$

Vibration Control & Measurement

Transmissibility: $T_r = \dfrac{\sqrt{1 + (2\zeta r)^2}}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$ undamped:

$T_r = \left|\dfrac{1}{1 - r^2}\right|$

Rotating unbalance force: $F = m_u e\, \omega^2$

Effective isolation requires $r > \sqrt{2}$.

2DOF & MDOF Free Vibration

Matrix form: $[M]\ddot{X} + [K]X = 0$. Assume $X = \bar{X} \cos \omega t$ → eigenvalue problem $\left([K] - \omega^2[M]\right) \bar{X} = 0$.

Frequency equation: $\det\left([K] - \omega^2[M]\right) = 0$ → polynomial in ω^2.

Mode shapes: substitute each ω_i^2 back to obtain the amplitude ratio. The same procedure applies to MDOF systems with $n \times n$ matrices, giving n natural frequencies and n mode shapes.

Part I — Conceptual & Short-Answer Questions

🧠 **Conceptual & short-answer** — fifteen short, high-value questions covering definitions, comparisons, matching, and short derivations. Suggested ~2 marks each (≈30 marks total).

Q1. Explain the difference between a **discrete (lumped-parameter)** system and a **continuous (distributed)** system, and give one example of each.

▼ ✅ Solution

- **Discrete system:** mass, stiffness and damping are modelled as a finite number of separate (lumped) elements, giving a finite number of degrees of freedom and **ordinary** differential equations. *Example:* a spring–mass–damper, or a car modelled as masses on springs.
- **Continuous system:** mass and elasticity are distributed continuously, giving **infinite** degrees of freedom and **partial** differential equations. *Example:* the transverse vibration of a beam, string, or shaft.

Q2. State **three** differences between Coulomb (dry-friction) damping and viscous damping.

▼ ✅ Solution

1. **Effect on frequency:** Viscous damping lowers the natural frequency ($\omega_d = \omega_n\sqrt{1-\zeta^2}$); Coulomb damping does **not** change the frequency of oscillation.
2. **Decay shape:** Viscous-damped amplitude decays **exponentially**; Coulomb-damped amplitude decays **linearly** (a constant amount, $4F/k$ per cycle).
3. **Cessation of motion:** A viscously damped system theoretically oscillates forever (amplitude → 0 asymptotically); a Coulomb-damped system **stops** in a finite time when the spring force can no longer overcome static friction.

(Also acceptable: damping force $\propto$ velocity for viscous vs. constant magnitude μN opposing motion for Coulomb.)

Q3. Define the following terms: (a) damping ratio, (b) logarithmic decrement, (c) critical damping, (d) loss coefficient.

▼ ✅ Solution

(a) Damping ratio ζ: ratio of actual damping to critical damping, $\zeta = c/c_c$.

(b) Logarithmic decrement δ: natural log of the ratio of two successive amplitudes of free decay, $\delta = \ln(x_i/x_{i+1}) = 2\pi\zeta/\sqrt{1-\zeta^2}$.

(c) Critical damping c_c: the smallest amount of damping for which the system returns to equilibrium **without oscillating** ($\zeta = 1$), $c_c = 2m\omega_n$.

(d) Loss coefficient η: energy dissipated per radian per unit strain energy, $\eta = \dfrac{\Delta W}{2\pi W} \approx 2\zeta$ for light damping.

Q4. Give the symbol, formula, and physical meaning of each: (a) damped natural frequency, (b) magnification factor, (c) displacement transmissibility, (d) critical damping ratio.

▼ ✅ Solution

(a) Damped natural frequency $\omega_d = \omega_n\sqrt{1-\zeta^2}$ — the actual oscillation frequency of an underdamped system in free decay.

(b) Magnification factor $M = \dfrac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$ — ratio of dynamic to static amplitude under harmonic forcing.

(c) Displacement transmissibility $T_d = \dfrac{\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$ — ratio of mass displacement to base displacement in base excitation.

(d) Critical damping ratio $\zeta = \dfrac{c}{c_c}$, with $c_c = 2m\omega_n$ — the fraction of critical damping; $\zeta = 1$ is the boundary between oscillatory and non-oscillatory return.

Q5. Define **resonance**. At what frequency does it occur, why is it dangerous, and how does damping affect the response at resonance?

▼ ✅ Solution

Resonance occurs when the excitation frequency equals the system's natural frequency ($\omega = \omega_n$, i.e. the frequency ratio $r = 1$). At this condition the magnification factor is a maximum and, for an **undamped** system, the amplitude theoretically grows without bound.

- **Why dangerous:** the very large amplitudes produce high dynamic stresses that can cause fatigue or sudden structural failure.
- **Effect of damping:** damping limits the peak to a finite value $M_{max} \approx \dfrac{1}{2\zeta}$ (for light damping) and shifts the true peak slightly below ω_n to $\omega = \omega_n\sqrt{1-2\zeta^2}$. More damping gives a lower, broader resonance peak.

Q6. Derive the expression for the logarithmic decrement δ from the underdamped free-vibration response.

▼ ✅ Solution

The underdamped response is $x(t) = Xe^{-\zeta\omega_n t}\cos(\omega_d t - \phi)$. Two peaks one period $T_d = 2\pi/\omega_d$ apart have the ratio

$$\frac{x(t)}{x(t+T_d)} = \frac{e^{-\zeta\omega_n t}}{e^{-\zeta\omega_n (t+T_d)}} = e^{\zeta\omega_n T_d}$$

Taking the natural log:

$$\delta = \ln \frac{x_1}{x_2} = \zeta \omega_n T_d = \zeta \omega_n \cdot \frac{2\pi}{\omega_d} = \frac{2\pi \zeta \omega_n}{\omega_n \sqrt{1-\zeta^2}} = \frac{2\pi \zeta}{\sqrt{1-\zeta^2}}$$

Q7. Why is vibration measurement important in engineering practice? Give two reasons.

▼ ✅ Solution

- **Condition monitoring / predictive maintenance:** detecting changes in vibration signature reveals faults (unbalance, misalignment, bearing wear, looseness) before failure.
- **Design verification:** confirms natural frequencies and damping so resonance with operating speeds is avoided.
- **Other valid points:** validating theoretical models, ensuring human comfort / ride quality, meeting safety and regulatory limits, and balancing rotating machinery.

Q8. Name and briefly explain the working principle of **three** types of vibration exciter (shaker).

▼ ✅ Solution

1. **Electrodynamic (electromagnetic) exciter:** a current-carrying coil in a magnetic field produces a force proportional to the current; capable of high frequencies and precise control.
2. **Mechanical (rotating-unbalance) exciter:** counter-rotating unbalanced masses generate a harmonic force $F = m_u e\, \omega^2$; simple and good for large forces at lower frequencies.
3. **Hydraulic (electrohydraulic) exciter:** a servo-valve controls hydraulic fluid to a piston, delivering very large forces and displacements at low frequencies.

Q9. List, in order, the main steps of an experimental **modal analysis** procedure.

▼ ✅ Solution

1. **Preparation / setup** — mount the structure, attach transducers, plan measurement points.
2. **Excitation** — apply a known input (impact hammer or shaker).
3. **Data acquisition** — measure input force and output response; compute Frequency Response Functions (FRFs).
4. **Modal parameter identification** — extract natural frequencies, damping ratios, and mode shapes from the FRFs.
5. **Validation / refinement** — check the model (e.g. mode-shape animation, correlation) and refine as needed.

Q10. What is the purpose of vibration isolation? Explain force transmissibility and why the condition $r > \sqrt{2}$ is required for effective isolation.

▼ ✅ Solution

Purpose: to reduce the dynamic force (or motion) transmitted between a machine and its supporting structure, protecting the foundation (force isolation) or the equipment (motion isolation).

Force transmissibility T_r is the ratio of the force transmitted to the foundation to the applied harmonic force. For an undamped mount $T_r = \left| \dfrac{1}{1-r^2} \right|$.

Why $r > \sqrt{2}$: isolation (i.e. $T_r < 1$) requires $|1 - r^2| > 1$, which holds only when $r^2 > 2$, i.e. $r > \sqrt{2}$. Below this the mount **amplifies** the force; the operating speed must therefore lie well above the mounted natural frequency.

Q11. (a) What is an accelerometer, and what is its main advantage over a vibrometer (seismometer)? (b) Why is critical damping important?

▼ ✅ Solution

(a) An accelerometer is a seismic instrument with a **high** natural frequency ($\omega_n \gg \omega$, so $r \ll 1$) whose output is proportional to the **acceleration** of the base. Advantage: it is small, lightweight, has a wide usable

frequency range, and the measured signal can be integrated to obtain velocity and displacement — unlike a vibrometer, which needs a very low natural frequency and is therefore bulky.

(b) Critical damping ($\zeta = 1$) gives the **fastest possible return to equilibrium without oscillation/overshoot**. It is the design target for instruments and devices (e.g. meters, door closers, gun recoil) that must settle quickly without vibrating.

Q12. *(Multiple choice and matching.)*

(a) The characteristic (frequency) equation of an n-DOF system is **a polynomial of degree** n in $\lambda = \omega^2$.

(b) Transmissibility is the ratio of the **force transmitted to the foundation** to the **exciting force**.

(c) Match each quantity (I–IV) with its expression.

▼ ✅ Solution

(a) From $\det([K] - \omega^2[M]) = 0$ the frequency equation is an n-th degree polynomial in $\lambda = \omega^2$ → **polynomial of degree** n.

(b) Force transmitted to exciting force (for base excitation, output-to-input displacement).

(c) Matching:

- **I. Magnification factor (undamped)** → $\left|\dfrac{1}{1-r^2}\right|$
- **II. Magnification factor (damped)** → $\dfrac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$
- **III. Damped natural frequency** → $\omega_n\sqrt{1-\zeta^2}$
- **IV. Displacement transmissibility** → $\sqrt{\dfrac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}}$

Q13. List and briefly discuss the factors that affect the success of a dynamic vibration absorber.

▼ ✅ Solution

- **Tuning:** the absorber works only when $\sqrt{k_2/m_2}$ equals the excitation frequency; the main-mass amplitude is zero only at perfect tuning.
- **Mass ratio $\mu = m_2/m_1$:** a larger absorber mass widens the gap between the two new natural frequencies, giving a broader useful band and less sensitivity to speed drift.
- **Absorber damping:** an undamped absorber gives exact cancellation at one frequency but creates two new resonances; damping tames those peaks but spoils perfect cancellation (a trade-off).
- **Steady operating speed:** the classical absorber is effective only near one frequency, so it suits machines that run at a constant speed.
- **Placement of new natural frequencies:** the two new natural frequencies must be kept clear of the operating-speed range.

Q14. Explain **coordinate coupling**, distinguish static (elastic) from dynamic (inertia) coupling, and state what principal coordinates achieve.

▼ ✅ Solution

- **Coupling** means the equations of motion are not independent — motion in one coordinate excites the other.
- **Static (elastic) coupling:** off-diagonal terms appear in the **stiffness** matrix $[K]$.
- **Dynamic (inertia) coupling:** off-diagonal terms appear in the **mass** matrix $[M]$.
- Coupling depends on the chosen coordinates; **principal (modal) coordinates** diagonalize both $[M]$ and $[K]$, decoupling the system into independent SDOF equations.

Q15. Define the **whirling / critical speed** of a rotating shaft. At what speed does it occur and why does it matter?

▼ ✅ Solution

Whirling is the lateral (bending) vibration of a spinning shaft caused by residual unbalance, in which the bowed shaft rotates about the bearing axis. It becomes violent at the **critical speed**, where the spin speed equals the shaft's lateral natural frequency:

$$\omega_{cr} = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{\delta_{st}}}$$

There the deflection and bearing loads are theoretically unbounded, so shafts are run well below — or accelerated quickly through — ω_{cr}.

Part II — Numerical & Analytical Problems

🧮 **Numerical & analytical problems** — grouped by topic following the exam syllabus. Marks shown against each problem are for self-assessment.

A. Single-Degree-of-Freedom (SDOF) Free Vibration

Problem A1 — Equation of Motion & Classification ***(8 marks)***

The free vibration of a single-degree-of-freedom system is governed by

$$\ddot{x} + 20\dot{x} + 900x = 0$$

(a) Find the natural frequency ω_n. **(2)**

(b) Find the damping ratio ζ and classify the system. **(3)**

(c) Find the damped natural frequency ω_d. **(1.5)**

(d) Find the logarithmic decrement δ. **(1.5)**

▼ ✅ Solution

Compare with the standard form $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$.

(a) $\omega_n^2 = 900 \Rightarrow \omega_n = 30 \text{ rad/s}$.

(b) $2\zeta\omega_n = 20 \Rightarrow \zeta = \dfrac{20}{2(30)} = 0.333$. Since $0 < \zeta < 1$, the system is **underdamped**.

(c) $\omega_d = \omega_n\sqrt{1-\zeta^2} = 30\sqrt{1-0.111} = 30(0.943) = 28.3 \text{ rad/s}$.

(d) $\delta = \dfrac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \dfrac{2\pi(0.333)}{0.943} = 2.22$.

Problem A2 — Natural Frequency from Static Deflection ***(4 marks)***

A spring–mass system has a static deflection of $\delta_{st} = 5 \text{ mm}$ under gravity. Find its natural frequency in rad/s and Hz.

▼ ✅ Solution

At static equilibrium $k\delta_{st} = mg$, so $k/m = g/\delta_{st}$ and

$$\omega_n = \sqrt{\frac{g}{\delta_{st}}} = \sqrt{\frac{9.81}{0.005}} = \sqrt{1962} = 44.3 \text{ rad/s}$$

$$f_n = \frac{\omega_n}{2\pi} = \frac{44.3}{2\pi} = 7.05 \text{ Hz}$$

Problem A3 — Stiffness & Young's Modulus ***(7 marks)***

A steel rod of length $L = 0.5\ \text{m}$ and cross-sectional area $A = 2 \times 10^{-4}\ \text{m}^2$ acts as an **axial spring** supporting a mass $m = 25\ \text{kg}$. The measured natural frequency of axial vibration is $f_n = 200\ \text{Hz}$.

(a) Determine the equivalent stiffness k. **(4)**

(b) Hence determine Young's modulus E of the rod material. **(3)**

▼ ✅ Solution

(a) Convert to angular frequency: $\omega_n = 2\pi f_n = 2\pi(200) = 1256.6\ \text{rad/s}$.

From $\omega_n = \sqrt{k/m}$:

$$k = m\,\omega_n^2 = 25 \times (1256.6)^2 = 3.95 \times 10^7\ \text{N/m}$$

(b) An axial bar has stiffness $k = \dfrac{AE}{L}$, so

$$E = \frac{kL}{A} = \frac{(3.95 \times 10^7)(0.5)}{2 \times 10^{-4}} = 9.87 \times 10^{10}\ \text{Pa} \approx 98.7\ \text{GPa}$$

Problem A4 — Damping Ratio from Overshoot ***(4 marks)***

A shock absorber is to be designed so that the **overshoot** of free vibration is limited to 15% of the initial displacement (i.e. successive amplitude ratio $x_1/x_2 = 1/0.15$). Find the required damping ratio ζ.

▼ ✅ Solution

Logarithmic decrement: $\delta = \ln\dfrac{x_1}{x_2} = \ln\dfrac{1}{0.15} = \ln(6.667) = 1.897$.

From $\delta = \dfrac{2\pi\zeta}{\sqrt{1-\zeta^2}}$, solve for ζ:

$$\zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}} = \frac{1.897}{\sqrt{39.48 + 3.60}} = \frac{1.897}{6.564} = 0.289$$

So $\zeta \approx 0.29$ (underdamped).

Problem A5—Underdamped Response from Initial Conditions ***(6 marks)***

A viscously damped system has $m = 2\ \text{kg}$, $c = 8\ \text{N}\cdot\text{s/m}$, $k = 200\ \text{N/m}$, with $x(0) = 0.02\ \text{m}$ and $\dot{x}(0) = 0$. Find ω_n, ζ, ω_d, and write the response $x(t)$.

▼ ✅ Solution

$\omega_n = \sqrt{k/m} = \sqrt{200/2} = 10\ \text{rad/s}$.

$\zeta = \dfrac{c}{2m\omega_n} = \dfrac{8}{2(2)(10)} = 0.2$ (underdamped).

$\omega_d = 10\sqrt{1 - 0.2^2} = 10(0.980) = 9.80\ \text{rad/s}$.

Using $x(t) = e^{-\zeta\omega_n t}\left[x_0\cos\omega_d t + \dfrac{v_0 + \zeta\omega_n x_0}{\omega_d}\sin\omega_d t\right]$ with $\zeta\omega_n = 2$:

coefficient $= \dfrac{0 + 2(0.02)}{9.80} = 0.00408$.

$$x(t) = e^{-2t}\left[0.02\cos 9.80t + 0.00408\sin 9.80t\right]\ \text{m}$$

Problem A6 — Equivalent Stiffness, Damping Ratio & Damped Frequency ***(5 marks)***

A mass $m = 10\ \text{kg}$ is supported by **two parallel springs** $k_1 = 2000\ \text{N/m}$ and $k_2 = 3000\ \text{N/m}$ and a damper $c = 120\ \text{N}\cdot\text{s/m}$. Find the equivalent stiffness, damping ratio, and damped natural frequency.

▼ ✅ Solution

Parallel springs: $k_{eq} = k_1 + k_2 = 5000\ \text{N/m}$.

$\omega_n = \sqrt{k_{eq}/m} = \sqrt{5000/10} = \sqrt{500} = 22.36\ \text{rad/s}.$

$\zeta = \dfrac{c}{2m\omega_n} = \dfrac{120}{2(10)(22.36)} = 0.268.$

$\omega_d = 22.36\sqrt{1 - 0.268^2} = 22.36(0.963) = 21.5\ \text{rad/s}.$

Problem A7 — Natural Frequency by the Energy Method ***(5 marks)***

A uniform solid cylinder of mass $m = 2\ \text{kg}$ and radius r rolls without slipping on a horizontal surface, and its centre is attached to a spring $k = 300\ \text{N/m}$. Using the energy method, find the natural frequency of small oscillations.

▼ ✅ Solution

For rolling without slipping $\dot{x} = r\dot{\theta}$, and the cylinder's moment of inertia is $I = \frac{1}{2}mr^2$.

Energies: $T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2 = \frac{3}{4}m\dot{x}^2$ and $V = \frac{1}{2}kx^2$, so the effective mass is $\frac{3}{2}m$.

$$\omega_n = \sqrt{\frac{k}{\frac{3}{2}m}} = \sqrt{\frac{2k}{3m}} = \sqrt{\frac{2(300)}{3(2)}} = \sqrt{100} = 10\ \text{rad/s}$$

Problem A8 — Logarithmic Decrement over Several Cycles ***(4 marks)***

In a free-vibration test the amplitude decays from $25\ \text{mm}$ to $5\ \text{mm}$ over **4 complete cycles**. Determine the logarithmic decrement and the damping ratio.

▼ ✅ Solution

Over $n = 4$ cycles: $\delta = \dfrac{1}{n}\ln\dfrac{x_0}{x_n} = \dfrac{1}{4}\ln\dfrac{25}{5} = \dfrac{1}{4}\ln 5 = \dfrac{1.609}{4} = 0.402.$

$$\zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}} = \frac{0.402}{\sqrt{39.48 + 0.162}} = \frac{0.402}{6.296} = 0.0639$$

So $\zeta \approx 0.064$ (lightly damped).

Problem A9 — Overdamped System Classification ***(5 marks)***

A system has $m = 4\ \text{kg}$, $k = 1600\ \text{N/m}$ and $c = 200\ \text{N}\cdot\text{s/m}$. Classify the damping and write the general form of the free response.

▼ ✅ Solution

$\omega_n = \sqrt{k/m} = \sqrt{1600/4} = 20\ \text{rad/s}$; critical damping $c_c = 2m\omega_n = 160\ \text{N}\cdot\text{s/m}$.

$\zeta = c/c_c = 200/160 = 1.25 > 1$ → **overdamped** (non-oscillatory).

The roots are $s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -25 \pm 20\sqrt{0.5625} = -25 \pm 15$, i.e. $s_1 = -10,\ s_2 = -40\ \text{s}^{-1}$.

$$x(t) = Ae^{-10t} + Be^{-40t}$$

with A, B from the initial conditions; the motion decays without oscillating.

B. SDOF Forced Vibration & Base Excitation

⚠️ **Likely exam question — Rotating Unbalance.** In the mid-exam you were given the **unbalance mass** m and **eccentricity** e — expect this type again on the final. Key formulas:

- Harmonic driving force (grows with ω^2): $F_0 = me\,\omega^2$.
- Machine steady-state amplitude (total mass M): $X = \dfrac{me}{M} \cdot \dfrac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$, with $r = \omega/\omega_n$.
- High-speed limit: $X \to \dfrac{me}{M}$ as $r \to \infty$.
- Force transmitted to the foundation: $F_T = T_r\, F_0$.

→ Worked examples: **Problem B8 (worked in the mid-exam style)**, plus **B6, B7, E2, E5**.

Problem B1 — Total Harmonic Response ***(9 marks)***

An **undamped** mass–spring system has $m = 1\ \text{kg}$ and $k = 25\ \text{N/m}$. It is driven by the harmonic force

$$F(t) = 10 \sin 10t \ \ (\text{N})$$

Find the **total response** $x(t)$ for each of the following sets of initial conditions:

(a) $x(0) = 0,\ \dot{x}(0) = 0$. **(4.5)**

(b) $x(0) = 0.05\ \text{m},\ \dot{x}(0) = 0$. **(4.5)**

▼ ✅ Solution

Set-up. $\omega_n = \sqrt{k/m} = \sqrt{25/1} = 5\ \text{rad/s}$. Forcing: $F_0 = 10\ \text{N},\ \omega = 10\ \text{rad/s},\ f_0 = F_0/m = 10$.

Particular-solution amplitude: $\dfrac{f_0}{\omega_n^2 - \omega^2} = \dfrac{10}{25 - 100} = \dfrac{10}{-75} = -0.1333$.

General solution: $x(t) = x_0 \cos \omega_n t + B \sin \omega_n t - 0.1333 \sin \omega t$, where

$$B = \frac{v_0}{\omega_n} - \frac{\omega f_0}{\omega_n(\omega_n^2 - \omega^2)}$$

(a) $x_0 = 0,\ v_0 = 0$:

$$B = 0 - \frac{(10)(10)}{5(-75)} = -\frac{100}{-375} = +0.2667.$$

$$x(t) = 0.267 \sin 5t - 0.133 \sin 10t \ \ (\text{m})$$

(b) $x_0 = 0.05,\ v_0 = 0$: the B term is unchanged ($B = 0.2667$), and the cosine term now appears:

$$x(t) = 0.05 \cos 5t + 0.267 \sin 5t - 0.133 \sin 10t \ \ (\text{m})$$

(Quick check (a): $x(0) = 0$ ✓, $\dot{x}(0) = 5(0.267) - 10(0.133) = 1.33 - 1.33 = 0$ ✓.)

Problem B2 — Magnification Factor & Phase Angle ***(4 marks)***

A machine is driven by a harmonic force with frequency ratio $r = \omega/\omega_n = 0.5$ and damping ratio $\zeta = 0.1$. Find the magnification factor M and the phase angle ϕ.

▼ ✅ Solution

$(1 - r^2) = 1 - 0.25 = 0.75$, $(2\zeta r) = 2(0.1)(0.5) = 0.1$.

$$M = \frac{1}{\sqrt{(0.75)^2 + (0.1)^2}} = \frac{1}{\sqrt{0.5625 + 0.01}} = \frac{1}{0.7566} = 1.32$$

$$\phi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = \tan^{-1}\left(\frac{0.1}{0.75}\right) = 7.6^\circ$$

Problem B3 — Base-Excitation Displacement Transmissibility ***(6 marks)***

A system is subjected to harmonic **base excitation** $y(t) = Y\sin\omega t$. Derive the displacement transmissibility $T_d = X/Y$.

▼ ✅ Solution

The equation of motion (relative to ground) is $m\ddot{x} + c(\dot{x}-\dot{y}) + k(x-y) = 0$, i.e.

$$m\ddot{x} + c\dot{x} + kx = ky + c\dot{y} = kY\sin\omega t + c\omega Y\cos\omega t$$

The right side is a harmonic force of amplitude $Y\sqrt{k^2+(c\omega)^2}$. Dividing the steady-state amplitude by Y and using $r = \omega/\omega_n$, $c = 2m\zeta\omega_n$ gives

$$T_d = \frac{X}{Y} = \sqrt{\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}}$$

The phase lag is $\phi = \tan^{-1}\left(\dfrac{2\zeta r^3}{1+(4\zeta^2-1)r^2}\right)$.

Problem B4 — Damped Forced Steady-State Amplitude ***(6 marks)***

A damped system has $m = 10\ \text{kg}$, $k = 4000\ \text{N/m}$, $c = 80\ \text{N·s/m}$ and is driven by $F_0 = 100\ \text{N}$ at $\omega = 15\ \text{rad/s}$. Find the steady-state amplitude X and phase angle ϕ.

▼ ✅ Solution

$\omega_n = \sqrt{4000/10} = 20\ \text{rad/s}$, $r = 15/20 = 0.75$, $\zeta = \dfrac{c}{2m\omega_n} = \dfrac{80}{2(10)(20)} = 0.2$.

Static deflection $X_{st} = F_0/k = 100/4000 = 0.025\ \text{m}$.

$(1-r^2) = 0.4375$, $(2\zeta r) = 0.3$.

$$X = \frac{X_{st}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} = \frac{0.025}{\sqrt{0.1914+0.09}} = \frac{0.025}{0.5305} = 0.0471\ \text{m}$$

$$\phi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = \tan^{-1}\left(\frac{0.3}{0.4375}\right) = 34.4^\circ$$

So $X \approx 47.1\ \text{mm}$, lagging the force by 34.4°.

Problem B5 — Vehicle on a Wavy Road (Base Excitation) ***(6 marks)***

A vehicle of mass $m = 1200\ \text{kg}$ on a suspension of stiffness $k = 4\times10^5\ \text{N/m}$ and damping ratio $\zeta = 0.3$ travels at $v = 20\ \text{m/s}$ over a road whose surface is sinusoidal with wavelength $5\ \text{m}$ and amplitude $Y = 0.02\ \text{m}$. Find the amplitude of the vehicle's vertical motion.

▼ ✅ Solution

$\omega_n = \sqrt{k/m} = \sqrt{4\times10^5/1200} = 18.26\ \text{rad/s}$.

Forcing frequency from the road: $\omega = \dfrac{2\pi v}{\lambda} = \dfrac{2\pi(20)}{5} = 25.13\ \text{rad/s}$, so $r = 25.13/18.26 = 1.376$.

$(1-r^2)^2 = 0.799$, $(2\zeta r)^2 = (0.826)^2 = 0.682$.

$$T_d = \sqrt{\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}} = \sqrt{\frac{1.682}{1.481}} = 1.066$$

$$X = T_d\, Y = 1.066(0.02) = 0.0213\ \text{m} \approx 21.3\ \text{mm}.$$

Problem B6 — Forced Amplitude due to Rotating Unbalance ***(6 marks)***

A machine of total mass $M = 80\ \text{kg}$ has a rotating unbalance $m_u e = 0.2\ \text{kg}\cdot\text{m}$. It is mounted on a spring $k = 3\times10^5\ \text{N/m}$ with damping ratio $\zeta = 0.1$ and runs at $\omega = 120\ \text{rad/s}$. Find the steady-state amplitude of the machine.

▼ ✅ Solution

$\omega_n = \sqrt{3\times10^5/80} = 61.24\ \text{rad/s}$, $r = 120/61.24 = 1.96$.

For rotating unbalance, $X = \dfrac{m_u e}{M}\cdot\dfrac{r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$.

$r^2 = 3.84$, $(1-r^2)^2 = 8.07$, $(2\zeta r)^2 = 0.154$, so the denominator $= \sqrt{8.22} = 2.87$.

$$X = \frac{0.2}{80}\cdot\frac{3.84}{2.87} = 0.0025(1.34) = 3.35\times10^{-3}\ \text{m} \approx 3.35\ \text{mm}$$

Problem B7 — Engine Isolation over a Speed Range ***(8 marks)***

An internal-combustion engine of mass $M = 200\ \text{kg}$ has a rotating unbalance $m_u e = 1\ \text{kg}\cdot\text{m}$ and operates between 800 and $2000\ \text{rpm}$. (a) Find the unbalance force transmitted to the floor at the two extreme speeds if the engine is bolted **directly** to a rigid floor. (b) The engine is instead placed on (lightly damped, treated as undamped) isolators. Find the isolator stiffness k so the transmitted force never exceeds $6000\ \text{N}$ across the operating range.

▼ ✅ Solution

(a) $\omega = 2\pi N/60$: at $800\ \text{rpm}$, $\omega = 83.8\ \text{rad/s}$; at $2000\ \text{rpm}$, $\omega = 209.4\ \text{rad/s}$. The unbalance force is $F_0 = m_u e\,\omega^2$:

$$F_{800} = 1(83.8)^2 = 7.0\times10^3\ \text{N}, \qquad F_{2000} = 1(209.4)^2 = 4.39\times10^4\ \text{N}$$

(b) With undamped mounts the transmitted force is $F_T = \dfrac{m_u e\,\omega^2}{r^2-1} = \dfrac{m_u e\,\omega^2\,\omega_n^2}{\omega^2-\omega_n^2}$. Since every operating speed is **above** resonance, F_T is largest at the **lowest** speed (800 rpm). Setting $F_T = 6000\ \text{N}$ there ($\omega^2 = 7018$):

$$\frac{7018\,\omega_n^2}{7018-\omega_n^2} = 6000 \;\Rightarrow\; \omega_n^2 = 3235 \;\Rightarrow\; \omega_n = 56.9\ \text{rad/s}$$

$$k = M\omega_n^2 = 200(3235) = 6.47\times10^5\ \text{N/m}$$

(Check: $r_{800} = 1.47 > \sqrt{2}$ ✓, and at 2000 rpm $F_T \approx 3.5\times10^3\ \text{N} < 6000$ ✓.)

Problem B8 — Rotating Unbalance from Given Mass & Eccentricity (mid-exam style) ***(7 marks)***

A small machine of total mass $M = 60\ \text{kg}$ is mounted on springs of total stiffness $k = 1.2\times10^5\ \text{N/m}$ with damping ratio $\zeta = 0.08$. Its rotor carries an **unbalance mass** $m = 0.5\ \text{kg}$ at an **eccentricity** $e = 50\ \text{mm}$ and spins at $1200\ \text{rpm}$.

(a) Find the rotating-unbalance force amplitude F_0. **(2)**

(b) Find the frequency ratio r and the steady-state amplitude X of the machine. **(3)**

(c) Find the force transmitted to the foundation. **(2)**

▼ ✅ Solution

Set-up. $\omega = \dfrac{2\pi(1200)}{60} = 125.7\ \text{rad/s}$, $\omega_n = \sqrt{1.2\times10^5/60} = \sqrt{2000} = 44.7\ \text{rad/s}$, so $r = 125.7/44.7 = 2.81$.

(a) The unbalance force amplitude grows with ω^2:

$$F_0 = me\,\omega^2 = 0.5(0.05)(125.7)^2 = 0.025(15800) = 395\ \text{N}$$

(b) $r^2 = 7.90$, $(1-r^2)^2 = (-6.90)^2 = 47.6$, $(2\zeta r)^2 = (0.450)^2 = 0.202$, denominator $= \sqrt{47.8} = 6.91$.

$$X = \frac{me}{M}\cdot\frac{r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} = \frac{0.025}{60}\cdot\frac{7.90}{6.91} = (4.17\times10^{-4})(1.143) = 4.77\times10^{-4}\ \text{m} \approx 0.48\ \text{mm}$$

(High-speed check: as $r\to\infty$, $X\to me/M = 0.42$ mm — consistent.)

(c) $T_r = \sqrt{\dfrac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}} = \sqrt{\dfrac{1.202}{47.8}} = 0.159$, so the transmitted force is

$$F_T = T_r F_0 = 0.159(395) = 62.8\ \text{N}$$

C. Two-Degree-of-Freedom (2DOF) Systems

⭐ The 2DOF and MDOF problems are the highest-weight items on the paper. Budget your time so you can complete them fully.

Problem C1 — Natural Frequencies & Mode Shapes ***(10 marks)***

A rigid machine body of mass $m = 100\ \text{kg}$ and centroidal mass moment of inertia $J_0 = 20\ \text{kg}\cdot\text{m}^2$ rests on two springs. Spring $k_1 = 60{,}000\ \text{N/m}$ is at distance $l_1 = 0.5\ \text{m}$ to the **left** of the centre of gravity (G); spring $k_2 = 60{,}000\ \text{N/m}$ is at distance $l_2 = 0.8\ \text{m}$ to the **right** of G. Use the vertical displacement x of G and the rotation θ as coordinates.

(a) Derive the equations of motion in matrix form. **(3)**

(b) Form the frequency equation and find the two natural frequencies. **(4)**

(c) Determine the two mode shapes (amplitude ratios). **(3)**

▼ ✅ Solution

(a) Equations of motion. Summing vertical forces and moments about G:

$$m\ddot{x} + (k_1+k_2)x + (-k_1l_1 + k_2l_2)\theta = 0$$

$$J_0\ddot{\theta} + (-k_1l_1 + k_2l_2)x + (k_1l_1^2 + k_2l_2^2)\theta = 0$$

Evaluate coefficients:

- $k_1 + k_2 = 120{,}000$
- $-k_1l_1 + k_2l_2 = -60000(0.5) + 60000(0.8) = -30000 + 48000 = 18{,}000$
- $k_1l_1^2 + k_2l_2^2 = 60000(0.25) + 60000(0.64) = 15000 + 38400 = 53{,}400$

Matrix form $[M]\ddot{X} + [K]X = 0$:

$$\begin{bmatrix} 100 & 0 \\ 0 & 20 \end{bmatrix}\begin{Bmatrix}\ddot{x}\\ \ddot{\theta}\end{Bmatrix} + \begin{bmatrix} 120000 & 18000 \\ 18000 & 53400 \end{bmatrix}\begin{Bmatrix} x \\ \theta\end{Bmatrix} = \begin{Bmatrix}0\\0\end{Bmatrix}$$

(b) Frequency equation. Assume $X = \bar{X}\cos\omega t$ and set $\det([K]-\omega^2[M]) = 0$. Let $\lambda = \omega^2$:

$$(120000 - 100\lambda)(53400 - 20\lambda) - (18000)^2 = 0$$

Expanding gives $2000\lambda^2 - 7.74\times10^6\lambda + 6.084\times10^9 = 0$, i.e.

$$\lambda^2 - 3870\lambda + 3{,}042{,}000 = 0$$

$$\lambda = \frac{3870 \pm \sqrt{3870^2 - 4(3{,}042{,}000)}}{2} = \frac{3870 \pm 1676}{2}$$

$\lambda_1 = 1097, \quad \lambda_2 = 2773.$

Therefore:

$$\omega_1 = \sqrt{1097} = 33.1 \text{ rad/s}, \qquad \omega_2 = \sqrt{2773} = 52.7 \text{ rad/s}$$

(c) Mode shapes. From the first row, $\dfrac{x}{\theta} = \dfrac{-18000}{120000 - 100\lambda}$.

- Mode 1 ($\lambda_1 = 1097$): $\dfrac{x}{\theta} = \dfrac{-18000}{120000 - 109700} = \dfrac{-18000}{10300} = -1.75 \text{ m/rad}$ (translation and rotation **out of phase**).
- Mode 2 ($\lambda_2 = 2773$): $\dfrac{x}{\theta} = \dfrac{-18000}{120000 - 277300} = \dfrac{-18000}{-157300} = +0.114 \text{ m/rad}$ (**in phase**).

Problem C2 — Full Modal Analysis (A–H) ***(12 marks)***

Consider the fixed–fixed 2DOF system of Figure 1 with $m_1 = 1$ kg, $m_2 = 4$ kg, and three springs $k_1 = k_3 = 10$ N/m, $k_2 = 2$ N/m (the centre spring k_2 connects the two masses).

A. Write the equations of motion in matrix form.

B. Form the eigenvalue problem.

C. Find the eigenvalues. **D.** Find the natural frequencies.

E. Find the mode shapes. **F.** Mass-normalize and verify orthonormality.

G. Write the decoupled modal equations.

▼ ✅ Solution

A. Equations of motion.

$$\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0$$

With numbers, $[K] = \begin{bmatrix} 12 & -2 \\ -2 & 12 \end{bmatrix}$.

B. Eigenvalue problem. Assume $x = X\cos\omega t$, let $\lambda = \omega^2$:

$$\det([K] - \lambda[M]) = \det \begin{bmatrix} 12 - \lambda & -2 \\ -2 & 12 - 4\lambda \end{bmatrix} = 0$$

C. Eigenvalues.

$$(12 - \lambda)(12 - 4\lambda) - 4 = 4\lambda^2 - 60\lambda + 140 = 0 \;\Rightarrow\; \lambda^2 - 15\lambda + 35 = 0$$

$$\lambda = \frac{15 \pm \sqrt{225 - 140}}{2} = \frac{15 \pm 9.22}{2} \;\Rightarrow\; \lambda_1 = 2.89,\ \lambda_2 = 12.11$$

D. Natural frequencies. $\omega_1 = \sqrt{2.89} = 1.70$ rad/s, $\omega_2 = \sqrt{12.11} = 3.48$ rad/s.

E. Mode shapes. From row 1, $\dfrac{X_2}{X_1} = \dfrac{12 - \lambda}{2}$:

- Mode 1 ($\lambda_1 = 2.89$): $X_2/X_1 = 4.56 \rightarrow \{1,\ 4.56\}^T$ (in phase).

- Mode 2 ($\lambda_2 = 12.11$): $X_2/X_1 = -0.055 \rightarrow \{1,\ -0.055\}^T$ (out of phase).

F. Mass normalization. $\mu_i = \phi_i^T [M] \phi_i$:

$\mu_1 = 1^2(1) + 4.56^2(4) = 84.2 \Rightarrow \sqrt{\mu_1} = 9.18.$

$\mu_2 = 1^2(1) + (-0.055)^2(4) = 1.012 \Rightarrow \sqrt{\mu_2} = 1.006.$

Normalized modal (modal) matrix:

$$[P] = \begin{bmatrix} 0.109 & 0.994 \\ 0.497 & -0.0547 \end{bmatrix}$$

Then $[P]^T[M][P] = [I]$ and $[P]^T[K][P] = \begin{bmatrix} 2.89 & 0 \\ 0 & 12.11 \end{bmatrix}$ (orthonormality confirmed).

G. Decoupled modal equations. With principal coordinates q_1, q_2:

$$\ddot{q}_1 + 2.89\, q_1 = 0, \qquad \ddot{q}_2 + 12.11\, q_2 = 0$$

(H. Response: solve each decoupled equation for the given initial conditions in modal coordinates, $q_i(t) = q_i(0)\cos\omega_i t + \dfrac{\dot{q}_i(0)}{\omega_i}\sin\omega_i t$, then transform back with $x = [P]q$.)

Problem C3 — 2DOF Chain with a Free End ***(8 marks)***

Two masses $m_1 = 2\ \text{kg}$ and $m_2 = 1\ \text{kg}$ are connected in series by two springs: $k_1 = 100\ \text{N/m}$ joins a fixed wall to m_1, and $k_2 = 100\ \text{N/m}$ joins m_1 to m_2 (the right end is free). Find the natural frequencies and mode shapes.

▼ ✅ Solution

$$[M] = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \qquad [K] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} = \begin{bmatrix} 200 & -100 \\ -100 & 100 \end{bmatrix}$$

With $\lambda = \omega^2$, $\det([K] - \lambda[M]) = (200 - 2\lambda)(100 - \lambda) - 100^2 = 0$:

$$2\lambda^2 - 400\lambda + 10000 = 0 \ \Rightarrow\ \lambda^2 - 200\lambda + 5000 = 0$$

$\lambda = \dfrac{200 \pm \sqrt{40000 - 20000}}{2} = \dfrac{200 \pm 141.4}{2}$, so $\lambda_1 = 29.3,\ \lambda_2 = 170.7.$

$$\omega_1 = \sqrt{29.3} = 5.41\ \text{rad/s}, \qquad \omega_2 = \sqrt{170.7} = 13.07\ \text{rad/s}$$

Mode shapes from row 1, $\dfrac{X_2}{X_1} = \dfrac{200 - 2\lambda}{100}$:

- Mode 1: $X_2/X_1 = +1.41 \rightarrow \{1,\ 1.41\}$ (in phase).
- Mode 2: $X_2/X_1 = -1.41 \rightarrow \{1,\ -1.41\}$ (out of phase).

Problem C4 — Coordinate Coupling of a Rigid Bar ***(8 marks)***

A uniform rigid bar of mass $m = 40\ \text{kg}$ and length $L = 2\ \text{m}$ rests on two springs, $k_1 = 8000\ \text{N/m}$ at the left end and $k_2 = 12000\ \text{N/m}$ at the right end. Using the vertical displacement x of the mass centre G (at the middle) and the rotation θ: (a) write the equations of motion and identify the coupling; (b) find where to place the coordinate origin so the system decouples; (c) find the natural frequencies.

▼ ✅ Solution

$J_G = \frac{1}{12}mL^2 = \frac{1}{12}(40)(2^2) = 13.33\ \text{kg}\cdot\text{m}^2$; $l_1 = l_2 = 1\ \text{m}$.

(a) $[M] = \begin{bmatrix} 40 & 0 \\ 0 & 13.33 \end{bmatrix}$, $[K] = \begin{bmatrix} k_1 + k_2 & -k_1 l_1 + k_2 l_2 \\ -k_1 l_1 + k_2 l_2 & k_1 l_1^2 + k_2 l_2^2 \end{bmatrix} = \begin{bmatrix} 20000 & 4000 \\ 4000 & 20000 \end{bmatrix}$.

The off-diagonal stiffness term $-k_1 l_1 + k_2 l_2 = 4000 \neq 0$, so the system has **static (elastic) coupling** (the mass matrix is diagonal).

(b) Decoupling requires a new origin at distance d from G (toward the stiffer spring) with $k_1(l_1 + d) = k_2(l_2 - d)$: $8000(1 + d) = 12000(1 - d) \Rightarrow 20000d = 4000 \Rightarrow d = 0.2\ \text{m}$.

(c) With $\lambda = \omega^2$, $(20000 - 40\lambda)(20000 - 13.33\lambda) - 4000^2 = 0 \Rightarrow \lambda^2 - 2000\lambda + 720{,}180 = 0$:

$$\lambda = \frac{2000 \pm \sqrt{2000^2 - 4(720180)}}{2} = \frac{2000 \pm 1059}{2} \Rightarrow \lambda_1 = 471,\ \lambda_2 = 1530$$

$\omega_1 = 21.7\ \text{rad/s},\ \omega_2 = 39.1\ \text{rad/s}$.

Problem C5 — Torsional 2DOF System (Lagrangian) ***(8 marks)***

Two disks with mass moments of inertia $J_1 = 1\ \text{kg}\cdot\text{m}^2$ and $J_2 = 2\ \text{kg}\cdot\text{m}^2$ are on a shaft. The segment from the fixed wall to disk 1 has torsional stiffness $k_{t1} = 2000\ \text{N}\cdot\text{m/rad}$ and the segment between the disks has $k_{t2} = 2000\ \text{N}\cdot\text{m/rad}$. Using Lagrange's equations with coordinates θ_1, θ_2, find the natural frequencies and mode shapes.

▼ ✅ Solution

Lagrangian. $T = \frac{1}{2}J_1\dot{\theta}_1^2 + \frac{1}{2}J_2\dot{\theta}_2^2$, $V = \frac{1}{2}k_{t1}\theta_1^2 + \frac{1}{2}k_{t2}(\theta_2 - \theta_1)^2$. Applying $\frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}_i} + \frac{\partial V}{\partial \theta_i} = 0$:

$$[J] = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \qquad [K_t] = \begin{bmatrix} k_{t1} + k_{t2} & -k_{t2} \\ -k_{t2} & k_{t2} \end{bmatrix} = \begin{bmatrix} 4000 & -2000 \\ -2000 & 2000 \end{bmatrix}$$

With $\lambda = \omega^2$, $(4000 - \lambda)(2000 - 2\lambda) - 2000^2 = 0 \Rightarrow \lambda^2 - 5000\lambda + 2{,}000{,}000 = 0$:

$$\lambda = \frac{5000 \pm \sqrt{25 \times 10^6 - 8 \times 10^6}}{2} = \frac{5000 \pm 4123}{2} \Rightarrow \lambda_1 = 438,\ \lambda_2 = 4562$$

$\omega_1 = 20.9\ \text{rad/s},\ \omega_2 = 67.5\ \text{rad/s}$.

Mode shapes from row 1, $\frac{\theta_2}{\theta_1} = \frac{4000 - \lambda}{2000}$: Mode 1 → $\{1,\ 1.78\}$ (in phase); Mode 2 → $\{1,\ -0.28\}$ (out of phase).

D. Multi-Degree-of-Freedom (MDOF) Systems

Problem D1 — 3-DOF Fixed–Fixed Chain ***(10 marks)***

Three equal masses $m = 1\ \text{kg}$ are connected in a line by four equal springs $k = 100\ \text{N/m}$, with both ends fixed (a fixed–fixed chain). Using coordinates x_1, x_2, x_3:

(a) Write the equations of motion in matrix form.

(b) Set up the frequency equation and find the three natural frequencies.

(c) Determine the three mode shapes and describe them physically.

▼ ✅ Solution

(a) Equations of motion. With both ends fixed, each mass is restrained by the springs on either side:

$$[M] = m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad [K] = k\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

so that $[M]\ddot{X} + [K]X = 0$.

(b) Frequency equation. Let $\lambda = \frac{m\omega^2}{k}$ and set $\det([K] - \omega^2[M]) = 0$:

$$\det\begin{bmatrix}2-\lambda & -1 & 0\\ -1 & 2-\lambda & -1\\ 0 & -1 & 2-\lambda\end{bmatrix} = (2-\lambda)\left[(2-\lambda)^2-2\right] = 0$$

Hence $\lambda = 2$ or $(2-\lambda) = \pm\sqrt{2}$, giving

$$\lambda_1 = 2-\sqrt{2} = 0.586, \qquad \lambda_2 = 2, \qquad \lambda_3 = 2+\sqrt{2} = 3.414$$

With $\omega = \sqrt{\lambda k/m}$ and $k/m = 100$:

$$\omega_1 = 7.65\ \text{rad/s}, \qquad \omega_2 = 14.14\ \text{rad/s}, \qquad \omega_3 = 18.48\ \text{rad/s}$$

(c) Mode shapes. Substituting each λ back gives the amplitude ratios $X_1 : X_2 : X_3$:

- **Mode 1** ($\lambda_1 = 0.586$): $\{1,\ \sqrt{2},\ 1\} = \{1,\ 1.41,\ 1\}$ — all three masses move **in phase** (fundamental mode).
- **Mode 2** ($\lambda_2 = 2$): $\{1,\ 0,\ -1\}$ — the centre mass stays **stationary** (a node) while the outer masses move in opposition.
- **Mode 3** ($\lambda_3 = 3.414$): $\{1,\ -\sqrt{2},\ 1\} = \{1,\ -1.41,\ 1\}$ — adjacent masses move **out of phase** (highest mode).

Problem D2 — 3-DOF Fixed–Free Chain ***(10 marks)***

Three equal masses $m = 1\ \text{kg}$ are connected in a line by three equal springs $k = 100\ \text{N/m}$; the left end is fixed to a wall and the right end is free. Find the three natural frequencies and mode shapes.

▼ ✅ Solution

$$[M] = m\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}, \qquad [K] = k\begin{bmatrix}2&-1&0\\-1&2&-1\\0&-1&1\end{bmatrix}$$

Let $\mu = \dfrac{m\omega^2}{k}$. The frequency determinant $\det([K]-\omega^2[M]) = 0$ expands to

$$\mu^3 - 5\mu^2 + 6\mu - 1 = 0 \ \Rightarrow\ \mu_1 = 0.198,\ \mu_2 = 1.555,\ \mu_3 = 3.247$$

With $k/m = 100$, $\omega = \sqrt{100\,\mu}$:

$$\omega_1 = 4.45\ \text{rad/s}, \qquad \omega_2 = 12.47\ \text{rad/s}, \qquad \omega_3 = 18.02\ \text{rad/s}$$

Mode shapes (set $X_1 = 1$; the free end carries the largest amplitude in the fundamental):

- Mode 1 (μ_1): $\{1,\ 1.80,\ 2.25\}$ — all masses in phase.
- Mode 2 (μ_2): $\{1,\ 0.45,\ -0.80\}$ — one internal node.
- Mode 3 (μ_3): $\{1,\ -1.25,\ 0.55\}$ — two nodes (highest mode).

Problem D3 — 3-DOF Free–Free Chain (Rigid-Body Mode) ***(8 marks)***

Three equal masses $m = 1\ \text{kg}$ are connected in a line by two equal springs $k = 100\ \text{N/m}$, with **both ends free**. Find the natural frequencies and mode shapes, and identify the rigid-body mode.

▼ ✅ Solution

$$[K] = k\begin{bmatrix}1&-1&0\\-1&2&-1\\0&-1&1\end{bmatrix}$$

With $\mu = \dfrac{m\omega^2}{k}$, the determinant gives $\mu(\mu-1)(\mu-3)=0$, so $\mu = 0,\ 1,\ 3$. With $k/m = 100$:

$$\omega_1 = 0, \qquad \omega_2 = \sqrt{100} = 10 \text{ rad/s}, \qquad \omega_3 = \sqrt{300} = 17.32 \text{ rad/s}$$

Mode shapes:

- $\omega_1 = 0$: $\{1,\ 1,\ 1\}$ — **rigid-body mode** (all masses translate together, springs unstretched).
- $\omega_2 = 10$: $\{1,\ 0,\ -1\}$ — centre mass stationary, ends out of phase.
- $\omega_3 = 17.32$: $\{1,\ -2,\ 1\}$ — ends in phase, centre opposed.

E. Vibration Isolation & Control

Problem E1 — Isolator Stiffness for a Target Transmissibility ***(3 marks)***

A machine of mass $m = 120 \text{ kg}$ runs at a constant speed of $\omega = 150 \text{ rad/s}$. It is to be mounted on **undamped** isolators so that the force transmissibility is reduced to $T_r = 0.1$. Determine the required isolator stiffness k.

▼ ✅ Solution

For an undamped isolator, $T_r = \dfrac{1}{|1-r^2|}$. With $T_r = 0.1$:

$$|1-r^2| = 10$$

For isolation $r > \sqrt{2}$, so $1 - r^2 < 0$: $r^2 - 1 = 10 \Rightarrow r^2 = 11 \Rightarrow r = 3.32$.

Since $r = \omega/\omega_n$: $\omega_n = \dfrac{150}{3.32} = 45.2 \text{ rad/s}$.

$$k = m\,\omega_n^2 = 120 \times (45.2)^2 = 2.45 \times 10^5 \text{ N/m}$$

Problem E2 — Rotating Unbalance & Isolation ***(6 marks)***

An electric motor of mass $M = 50 \text{ kg}$ carries an unbalanced mass $m = 0.5 \text{ kg}$ at radius $e = 0.1 \text{ m}$ and runs at 1500 rpm. (a) Find the rotating-unbalance force amplitude. (b) Design undamped isolators so that only 10% of this force reaches the floor ($T_r = 0.1$): find the required total stiffness k and the transmitted force.

▼ ✅ Solution

(a) $\omega = \dfrac{2\pi(1500)}{60} = 157.1 \text{ rad/s}$.

$$F_0 = me\,\omega^2 = 0.5(0.1)(157.1)^2 = 1234 \text{ N}$$

(b) Undamped: $T_r = \dfrac{1}{|1-r^2|} = 0.1 \Rightarrow r^2 = 1 + 10 = 11 \Rightarrow r = 3.32$.

$\omega_n = \omega/r = 157.1/3.32 = 47.3 \text{ rad/s}$.

$$k = M\omega_n^2 = 50(47.3)^2 = 1.12 \times 10^5 \text{ N/m}$$

Transmitted force $= T_r F_0 = 0.1(1234) = 123 \text{ N}$.

Problem E3 — Undamped Vibration Absorber Design ***(8 marks)***

A machine of mass $m_1 = 25 \text{ kg}$ on a spring $k_1 = 250{,}000 \text{ N/m}$ runs at $\omega = 100 \text{ rad/s}$ — exactly at its natural frequency (resonance) — due to a rotating unbalance. An **undamped absorber** of mass $m_2 = 5 \text{ kg}$ is added. (a) Find the absorber stiffness k_2 that drives the main-mass vibration to zero. (b) Find the two natural frequencies of the combined system.

▼ ✅ Solution

(a) Zero main-mass amplitude requires the absorber tuned to the excitation: $\omega^2 = k_2/m_2$, so

$$k_2 = m_2\,\omega^2 = 5(100)^2 = 50{,}000\ \text{N/m}$$

(b) Combined 2DOF system, $[M] = \begin{bmatrix} 25 & 0 \\ 0 & 5 \end{bmatrix}$, $[K] = \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} = \begin{bmatrix} 300000 & -50000 \\ -50000 & 50000 \end{bmatrix}$. With $\lambda = \omega^2$:

$$(300000 - 25\lambda)(50000 - 5\lambda) - (50000)^2 = 0 \;\Rightarrow\; \lambda^2 - 22000\lambda + 10^8 = 0$$

$$\lambda = \frac{22000 \pm \sqrt{22000^2 - 4(10^8)}}{2} = \frac{22000 \pm 9165}{2} \Rightarrow \lambda_1 = 6418,\ \lambda_2 = 15583$$

$$\omega_1 = 80.1\ \text{rad/s}, \qquad \omega_2 = 124.8\ \text{rad/s}$$

The operating speed (100 rad/s) now lies safely **between** the two new natural frequencies.

Problem E4 — Damped Force Transmissibility ***(5 marks)***

A machine of mass $m = 500\ \text{kg}$ is mounted on isolators of total stiffness $k = 2 \times 10^6\ \text{N/m}$ and damping ratio $\zeta = 0.1$. It runs at $\omega = 120\ \text{rad/s}$ producing a harmonic force of amplitude $F_0 = 2000\ \text{N}$. Find the transmissibility and the force transmitted to the foundation.

▼ ✅ Solution

$\omega_n = \sqrt{2 \times 10^6/500} = 63.25\ \text{rad/s}$, $r = 120/63.25 = 1.897$.

$(1-r^2)^2 = (-2.599)^2 = 6.76$, $(2\zeta r)^2 = (0.379)^2 = 0.144$.

$$T_r = \sqrt{\frac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2}} = \sqrt{\frac{1.144}{6.90}} = 0.407$$

Transmitted force $= T_r F_0 = 0.407(2000) = 814\ \text{N}$ (about a 59% reduction).

Problem E5 — Machine on a Cantilever Beam with Absorber ***(10 marks)***

A $35\ \text{kg}$ machine is attached to the free end of a cantilever beam of length $L = 2\ \text{m}$, $E = 210 \times 10^9\ \text{Pa}$ and $I = 1.3 \times 10^{-7}\ \text{m}^4$. It runs at $180\ \text{rpm}$ with a rotating unbalance $m_u e = 0.3\ \text{kg}\cdot\text{m}$. (a) Find the stiffness of an undamped absorber of mass $5\ \text{kg}$ that eliminates steady-state vibration at 180 rpm. (b) Find the two natural frequencies of the combined system. (c) Outline how to find the speed range for which the machine amplitude stays below 8 mm.

▼ ✅ Solution

Beam (cantilever, end load) stiffness: $k_1 = \dfrac{3EI}{L^3} = \dfrac{3(210\times10^9)(1.3\times10^{-7})}{2^3} = 10{,}238\ \text{N/m}$.

Operating frequency: $\omega = \dfrac{2\pi(180)}{60} = 18.85\ \text{rad/s}$.

(a) Tune the absorber so $\omega^2 = k_2/m_2$: $k_2 = m_2\omega^2 = 5(18.85)^2 = 1776\ \text{N/m}$.

(b) Combined system $[M] = \begin{bmatrix} 35 & 0 \\ 0 & 5 \end{bmatrix}$, $[K] = \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} = \begin{bmatrix} 12014 & -1776 \\ -1776 & 1776 \end{bmatrix}$. With $\lambda = \omega^2$: $\lambda^2 - 698.4\lambda + 103{,}900 = 0$, giving $\lambda_1 = 214.8,\ \lambda_2 = 483.6$:

$$\omega_1 = 14.66\ \text{rad/s}, \qquad \omega_2 = 21.99\ \text{rad/s}$$

(c) The main-mass amplitude is $X_1 = \dfrac{(k_2 - m_2\omega^2)\,F_0}{(k_1 + k_2 - m_1\omega^2)(k_2 - m_2\omega^2) - k_2^2}$ with $F_0 = m_u e\,\omega^2$. Setting $|X_1| = 8\ \text{mm}$ and solving for ω gives the two bounding speeds; the acceptable band lies **around the tuning**

speed (180 rpm), between the two new resonances ω_1 and ω_2, where $|X_1| < 8$ mm.

Problem E6 — Operating-Speed Range for an Amplitude Limit *(5 marks)*

An undamped machine has $m = 50$ kg, $k = 2 \times 10^5$ N/m and is driven by a constant-amplitude force $F_0 = 500$ N. For what range of operating frequencies will the steady-state amplitude stay below 2 mm?

▼ ✅ Solution

$\omega_n = \sqrt{2 \times 10^5 / 50} = 63.25$ rad/s; static deflection $X_{st} = F_0/k = 500/(2 \times 10^5) = 2.5$ mm.

Undamped amplitude $X = \dfrac{X_{st}}{|1 - r^2|} < 2$ mm requires $|1 - r^2| > 1.25$.

Since $1 - r^2 > 1.25$ is impossible, we need $r^2 - 1 > 1.25 \Rightarrow r^2 > 2.25 \Rightarrow r > 1.5$.

$$\omega > 1.5\,\omega_n = 1.5(63.25) = 94.9 \text{ rad/s} \quad (\approx 906 \text{ rpm})$$

So the machine must run **above 94.9 rad/s** (well past resonance).

Problem E7 — Isolation Efficiency & Static Deflection *(5 marks)*

A 300 kg machine runs at 1200 rpm and is to be isolated on undamped mounts so that 90% of the disturbing force is isolated ($T_r = 0.1$). Find the required mount stiffness and the static deflection of the mounts.

▼ ✅ Solution

$\omega = \dfrac{2\pi(1200)}{60} = 125.7$ rad/s. Undamped: $T_r = \dfrac{1}{r^2 - 1} = 0.1 \Rightarrow r^2 = 11 \Rightarrow r = 3.317$.

$\omega_n = \omega / r = 125.7 / 3.317 = 37.9$ rad/s.

$$k = m\omega_n^2 = 300(37.9)^2 = 4.30 \times 10^5 \text{ N/m}$$

Static deflection $\delta_{st} = \dfrac{mg}{k} = \dfrac{300(9.81)}{4.30 \times 10^5} = 6.84 \times 10^{-3}$ m $= 6.84$ mm. Isolation efficiency $= 1 - T_r = 90\%$.

F. Vibration Measurement

Problem F1 — Accelerometer Measurement Error *(3 marks)*

An accelerometer has a natural frequency $f_n = 500$ Hz and damping ratio $\zeta = 0.6$. It is used to measure a vibration whose frequency is $f = 150$ Hz. Calculate the magnification factor M and hence the percentage measurement error.

▼ ✅ Solution

Frequency ratio: $r = f / f_n = 150/500 = 0.3$.

$$M = \frac{1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$

$(1 - r^2)^2 = (1 - 0.09)^2 = (0.91)^2 = 0.8281$.

$(2\zeta r)^2 = (2 \times 0.6 \times 0.3)^2 = (0.36)^2 = 0.1296$.

Sum $= 0.9577$, $\sqrt{0.9577} = 0.9786$.

$$M = \frac{1}{0.9786} = 1.022$$

Percentage error $= (M - 1) \times 100\% = +2.2\%$. The instrument **over-reads** by about 2.2%.

Problem F2 — Vibrometer (Seismometer) Measurement ***(5 marks)***

A vibrometer (seismic displacement pickup) has a natural frequency $\omega_n = 4\ \text{rad/s}$ and damping ratio $\zeta = 0.6$. It is used to measure a vibration of frequency $\omega = 30\ \text{rad/s}$. Find the ratio of the recorded to the true displacement and the percentage error.

▼ ✅ Solution

A vibrometer reads the relative (seismic-mass) displacement Z, where $\dfrac{Z}{Y} = \dfrac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$ with $r = \omega/\omega_n = 30/4 = 7.5$.

$r^2 = 56.25$, $(1-r^2)^2 = 3052.6$, $(2\zeta r)^2 = 81$, denominator $= \sqrt{3133.6} = 55.98$.

$$\frac{Z}{Y} = \frac{56.25}{55.98} = 1.005$$

Error $\approx +0.5\%$ — the large frequency ratio ($r \gg 1$) lets the vibrometer track true displacement closely.

Problem F3 — Accelerometer Flat-Response Check ***(5 marks)***

An accelerometer has damping ratio $\zeta = 0.7$ and natural frequency $f_n = 600\ \text{Hz}$. It measures a vibration at $f = 180\ \text{Hz}$. Find the amplitude-distortion factor and the percentage measurement error.

▼ ✅ Solution

$r = f/f_n = 180/600 = 0.3$. The accelerometer factor is $M = \dfrac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$.

$(1-r^2)^2 = 0.8281$, $(2\zeta r)^2 = (0.42)^2 = 0.1764$, sum $= 1.0045$.

$$M = \frac{1}{\sqrt{1.0045}} = 0.9978$$

Error $= -0.22\%$. With $\zeta \approx 0.7$ the response stays flat to within a fraction of a percent — which is why this damping value is chosen for accelerometers.

Problem F4 — Damping from the Half-Power Bandwidth ***(5 marks)***

A frequency-response test of a lightly damped structure shows a resonance peak at $f_n = 40\ \text{Hz}$, with half-power (−3 dB) points at $f_1 = 38.5\ \text{Hz}$ and $f_2 = 41.5\ \text{Hz}$. Determine the damping ratio and the quality factor Q.

▼ ✅ Solution

Half-power bandwidth method: $\zeta = \dfrac{f_2 - f_1}{2f_n} = \dfrac{41.5 - 38.5}{2(40)} = \dfrac{3}{80} = 0.0375$.

$$Q = \frac{1}{2\zeta} = \frac{f_n}{f_2 - f_1} = \frac{40}{3} = 13.3$$

So the structure is lightly damped ($\zeta \approx 3.75\%$).

📊 Marks Summary

Section	Topic	Problems	Marks
Part I	Conceptual — fundamentals, damping, measurement, isolation, resonance, coupling & MDOF	Q1–Q15	30
II-A	SDOF free vibration	A1–A9	48
II-B	SDOF forced & base excitation	B1–B8	52
II-C	2DOF systems (incl. coupling & torsional/Lagrangian)	C1–C5	46

Section	Topic	Problems	Marks
II-D	MDOF systems	D1–D3	28
II-E	Vibration isolation & control (incl. absorbers)	E1–E7	42
II-F	Vibration measurement	F1–F4	18
Total	**51 problems** (15 conceptual + 36 numerical)		**264**

⏱️ **How to use this as a timed mock.** This is a comprehensive 51-problem revision bank — far more than a single 3-hour sitting. For a realistic exam simulation modelled on the **June 2024 final** (now updated for 2026), attempt **all of Part I** plus **one problem from each Part II topic A–F** (≈ 50 marks) within 3 hours. Spend the most time on the high-weight 2DOF / MDOF problems (C1–C2, D1–D2), and treat the short isolation and measurement items (E1, F1) as quick wins. The remaining problems are extra practice across the full syllabus.

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