

First Order Ordinary Differential Equation



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Classification of Differential Equation

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Ordinary Differentail equation :

A Equation containing ordinary derivative :

NOTE : *in this equation Number of independent variable is **One**.*

For Example :

$$1. \frac{dy}{dx} + x^2 y = \sin x \quad 2. \frac{d^2 y}{dx^2} + 5 \frac{dy}{dx} + x^2 y = \sin^2 x \dots$$

$$3. \frac{d^3 y}{dx^3} + 5 \frac{d^2 y}{d^2 x} + 6 \frac{dy}{dx} + 9 y = e^x.$$

Partial Differential Equation

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Partial Differential equation :

A Equation containing Partial derivative :

NOTE : *in this equation Number of independent variable are two or more.*

For Example :

$$1. \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$2. c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

Order and Degree of Differential Equation

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Order of Differential equation :

A ODE is said to be of n th order if the highest order derivative of unknown function y is n .

For Example :

1. $\frac{dy}{dx} + 3y = 5x$ (First Order).

2. $\left(\frac{d^2y}{dx^2}\right)^3 + 3\frac{dy}{dx} + 7y = e^x$ (2nd order).

Degree of Differential Equation

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Degree of Differentail equation :

A ODE is said to be of n th degree if power of the highest order derivative of unknown function y is n when the power of each term in y and its derivative are free from radical sign.

For Example :

1. $\frac{dy}{dx} + 3y = 5x$ (*First Order, First degree*).

2. $\left(\frac{d^2y}{dx^2}\right)^3 + 3\frac{dy}{dx} + 7y = e^x$ (*2nd order, 3rd degree*).

3. $\frac{dy}{dx} + 3\sqrt{y} = 5x$ (*First Order, 2nd degree*)

4. $\sqrt{\frac{dy}{dx}} + 3y = 5x$ (*First Order, 1st degree*)

Solution of Ordinary Differential Equation

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A solution of ordinary differential equation is a function $y = f(x)$ which satisfy the given differential equation.

Classification of solution :

1. General solution : A solution containing arbitrary constants.

Note : (1) Number of arbitrary constants in *a general solution is equal to the order of differential equation.*

(2) **Geometrically** general solution is the family of curve.

For Example :

$\frac{dy}{dx} = 2x$, its solution is $y = x^2 + c$, it is a family of parabolas symmetric about y – axis.

Classification of solution :

2. Particular solution : A solution *obtained by Assigning a particular values to arbitrary constants in a general solution.*

(2) Geometrically *particular solution is the a curve.*

For Example : $\frac{dy}{dx} = 2x$, $y(0) = 0$, *its solution is*

$y = x^2$, *it is a parabola with vertex at 0, symmetric about y – axis.*

Formation of a Differential Equation

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I. *If The equation of family of curve is known.*

To form the Differential equation of Family of curve given by $f(x, y, c_1, c_2, \dots, c_n) = 0 \dots (01)$

Method : *Since given family contain n arbitrary constants the required DE will be of n th order.*

$\therefore$ Differentiating (01), n – times we get

$$f'(x, y, y', c_1, c_2, \dots, c_n) = 0$$

$$f'(x, y, y', y'', c_1, c_2, \dots, c_n) = 0$$

...

$$f'(x, y, y', y'', \dots y^{(n)}, c_1, c_2, \dots, c_n) = 0$$

and eliminating $c_1, c_2, \dots, c_n$ from above $(n+1)$

equations we get the required differential equation.

Formation of a Differential Equation

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Formation of a differential equation from a Given physical situation : We required the knowledge of Standard laws of physics

Examples :

1. *Falling body.*
2. *Escape velocity*
3. *Simple harmonic motion.*
4. *Population dynamics*
5. *Simple electrical circuits*
6. *mass spring system.*
7. *Newtons Law of colling.*

We will discuss in detail when we study the Applications of ODE.

Illustrative Example

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Example : Obtain the differentail equation of family circles with fixed radius r and centre at (h, k) .

Solution : The family of circle is

$$(x - h)^2 + (y - k)^2 = r^2 \dots (01)$$

sin ce it contains 2 arbitrary constants

$\therefore$ differentaiting twice, we get

$$2(x-h) + 2(y-k)y' = 0$$

$$\text{and } 1 + (y-k)y'' + (y')^2 = 0$$

$$\therefore y - k = -\frac{1 + (y')^2}{y''}$$

$$\text{but } (x-h) = -(y-k)y' = \frac{1 + (y')^2}{y''}$$

substituting in (01), we get

$$\left(1 + (y')^2\right)^3 = r^2 (y'')^2.$$

First Order Differential Equation

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A Differential equation of the form

$$\frac{dy}{dx} = f(x, y)$$

or $M(x, y) dx + N(x, y) dy = 0$

is called first order differential equation.

Methods of solving First Order DE.

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Variable Separable Method

If a given Differential equation can be expressed as

$$\phi(x)dx + \psi(y)dy = 0$$

called as Variable Separable form

its solution is

$$\int \phi(x)dx + \int \psi(y)dy = c$$

Illustrative Examples

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1. Solve $y\sqrt{1-x^2}dy + x\sqrt{1-y^2}dx = 0$

Solution: $\frac{ydy}{\sqrt{1-y^2}} + x\frac{xdx}{\sqrt{1-x^2}} = 0$

$$\therefore \int (1-y^2)^{-1/2} (2y)dy + \int (1-x^2)^{-1/2} (2x)dx = 0$$

$$\text{i.e. } \frac{\sqrt{1-y^2}}{1/2} + \frac{\sqrt{1-x^2}}{1/2} = c$$

$$\sqrt{1-x^2} + \sqrt{1-y^2} = c.$$

Rule of integration used: $\int f^n(x) f'(x) dx = \frac{f^{n+1}(x)}{n+1}.$

Solve :

$$(1) \frac{y}{x} \frac{dy}{dx} = \sqrt{1+x^2+y^2+x^2y^2}$$

$$(2) \frac{dy}{dx} = xe^{y-x^2}, \text{ if } y=0, \text{ when } x=0.$$

$$(3) \frac{dy}{dx} = e^{3x-2y} + x^2 e^{-2y}.$$

Differential Equations solvable By Substitution

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Differential Equation

Substitution

1. $\frac{dy}{dx} = f(ax + by + c)$

$$ax + by + c = v$$

2. $x \frac{dy}{dx} + y = f(xy)$

$$xy = v$$

3. *A de containing terms*

$$x = r \cos \theta, y = r \sin \theta$$

$$xdx + ydy, xdy - ydx, (x^2 + y^2)^n \quad x^2 + y^2 = r^2, \tan^{-1}\left(\frac{y}{x}\right) = \theta$$

$$\therefore xdx + ydy = rdr, xdy - ydx = r^2 d\theta.$$

Exercise

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Solve

$$1. \frac{dy}{dx} = \sin(x + y).$$

$$2. x \frac{dy}{dx} + y = \sec^2(xy).$$

$$3. (x dx + y dy) + \frac{x dy - y dx}{(x^2 + y^2)} = 0.$$

Homogeneous Differential Equation

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A Differential equation of the form

$$\frac{dy}{dx} = f(x, y) \quad \text{or} \quad M(x, y)dx + N(x, y)dy = 0$$

is homogeneous *DE* if $f(x, y) = F\left(\frac{y}{x}\right)$

i.e. $f(x, y)$ is homogeneous function of degree zero.

or M and N are homogeneous functions of same degree.

Method of solving Homogeneous differential Equation

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$$\text{Put } \frac{y}{x} = v \Rightarrow y = vx$$

$$\therefore \frac{dy}{dx} = v + x \frac{dv}{dx}$$

Substitute in given de, it reduces to variable separable form.

Example: Solve $(x^2 - y^2)dx = 2xydy$

$$\frac{dy}{dx} = \frac{(x^2 - y^2)}{2xy} = \frac{1}{2} \left(\frac{x}{y} - \frac{y}{x} \right) = F \left(\frac{y}{x} \right).$$

$\therefore$ Homogeneous d.e.

$$\therefore \text{Put } \frac{y}{x} = v \Rightarrow y = vx \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

Substituting in given d.e

$$v + x \frac{dv}{dx} = \frac{1}{2} \left(\frac{1}{v} - v \right) \Rightarrow x \frac{dv}{dx} = \frac{1}{2} \left(\frac{1}{v} - v \right) - v$$

$$\text{i.e } x \frac{dv}{dx} = \frac{1 - 3v^2}{2v} \rightarrow \text{Separable form}$$

$$\frac{2v}{1-3v^2} dv = \frac{dx}{x}$$

$$\text{integrating} \quad \int \frac{-6v}{1-3v^2} dv = -3 \int \frac{dx}{x} + c$$

$$\therefore \ln(1-3v^2) = -3 \ln x + \ln c$$

$$\text{i.e. } \ln(1-3v^2) = \ln\left(\frac{c}{x^3}\right)$$

$$\therefore 1-3v^2 = \frac{c}{x^3} \text{ now put } v = \frac{y}{x} \Rightarrow 1-3\left(\frac{y}{x}\right)^2 = \frac{c}{x^3}$$

$$\therefore x^2 - 3y^2 = \frac{c}{x}.$$

Exercise

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$$1. (x^2 y - 2xy^2) dx - (x^3 - 3x^2 y) dy = 0$$

$$2. y^2 + x^2 \frac{dy}{dx} = xy \frac{dy}{dx}$$

$$3. x^3 dx - y^3 dy = 3xy (y dx - x dy)$$

$$4. y dx - x dy = \sqrt{x^2 + y^2} dx$$

Differential Equation reducible to Homogeneous de

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A Differential equation of the form

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$$

where

Case 1: If $\frac{a_1}{a_2} = \frac{b_1}{b_2}$, then

put $a_1x + b_1y = v$

Case 2 : If $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, then

$$\text{put } x = X + h, y = Y + k \quad \therefore \frac{dY}{dX} = \frac{dY}{dy} \frac{dy}{dx} \frac{dx}{dX} = \frac{dy}{dx}$$

$\therefore$ Given d.e becomes

$$\frac{dY}{dX} = \frac{a_1 X + b_1 Y + a_1 h + b_1 k + c_1}{a_2 X + b_2 Y + a_2 h + b_2 k + c_2}$$

choose h and k such that

$$a_1 h + b_1 k + c_1 = 0 \text{ and } a_2 h + b_2 k + c_2 = 0$$

$$\therefore \frac{dY}{dX} = \frac{a_1 X + b_1 Y}{a_2 X + b_2 Y} \rightarrow \text{homogeneous d.e}$$

Example : $\frac{dy}{dx} = \frac{x+y+1}{2x+2y+3}$, here $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{1}{2}$

$$\therefore \text{ put } x+y=v \Rightarrow 1 + \frac{dy}{dx} = \frac{dv}{dx}$$

$$\therefore \frac{dv}{dx} - 1 = \frac{v+1}{2v+3} \Rightarrow \frac{dv}{dx} = \frac{v+1}{2v+3} + 1$$

$$\therefore \frac{dv}{dx} = \frac{v+1}{2v+3} + 1 = \frac{3v+4}{2v+3}$$

separating Variables

$$\frac{2v+3}{3v+4} dv = dx \Rightarrow \frac{2}{3} \left(\frac{3v+4-4+\frac{9}{2}}{3v+4} \right) dv = dx$$

$$\therefore \frac{2}{3} dv + \frac{1}{9} \frac{3}{3v+4} dv = dx$$

integrating

$$6v + \ln(3v+4) = 9x + c$$

$$6(x+y) + \ln(3x+3y+4) = 9x + c$$

$$6y - 3x + \ln(3x+3y+4) = c.$$

Example : $\frac{dy}{dx} = \frac{2x - y - 4}{y - 3x + 3}$, here $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

$\therefore$ put $x = X + h$, $y = Y + k$

$$\therefore \frac{dy}{dx} = \frac{dY}{dX} \Rightarrow \frac{dY}{dX} = \frac{2X - Y + 2h - k - 4}{Y - 3X - 3h + k + 3}$$

Choose h, k such that

$$2h - k - 4 = 0 \text{ and } -3h + k + 3 = 0$$

solving simultaneously

$$h = -1, k = -6.$$

$$\therefore \frac{dY}{dX} \Rightarrow \frac{dY}{dX} = \frac{2X - Y}{Y - 3X} = \frac{2 - \frac{Y}{X}}{\frac{Y}{X} - 3}$$

put $Y = vX \Rightarrow$

$$v + X \frac{dv}{dX} = \frac{2 - v}{v - 3} \Rightarrow X \frac{dv}{dX} = \frac{2 - v}{v - 3} - v$$

$$\therefore X \frac{dv}{dX} = \frac{2 - v}{v - 3} - v = \frac{2 + 2v - v^2}{v - 3}$$

separating variables

$$\frac{v-3}{2+2v-v^2} dv = \frac{dX}{X} \quad \therefore \frac{2v-2-4}{2+2v-v^2} dv = \frac{2dX}{X}$$

$$\therefore \left(\frac{2v-2}{2+2v-v^2} - \frac{4}{2+2v-v^2} \right) dv = \frac{2dX}{X}$$

Integrating

$$\int \frac{2v-2}{2+2v-v^2} dv + 4 \int \frac{1}{(v-1)^2 - 3} dv = \int \frac{2dX}{X} + c$$

$$\therefore \ln(2+2v-v^2) + \frac{4}{2\sqrt{3}} \ln \left(\frac{v-1-\sqrt{3}}{v-1+\sqrt{3}} \right) = 2 \ln(X) + c.$$

where $v = \frac{y+6}{x+1}$, $X = x+1$.

Exercise

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$$1. (4x - 6y)dx + (3y - 2x - 2)dy = 0$$

$$2. \frac{dy}{dx} = \frac{x + 2y - 3}{2x + y - 3}.$$

$$3. \frac{dy}{dx} = \frac{x + y - 2}{x - y - 3}.$$

Exact Differential Equation

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A Differential $M(x, y)dx + N(x, y)dy$ is said to be Exact or total differential if there exists a function $u(x, y)$ such that

$$du = M(x, y)dx + N(x, y)dy$$

$$\text{but } du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$$M = \frac{\partial u}{\partial x}, \quad N = \frac{\partial u}{\partial y}$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial N}{\partial x}$$

Condition for Exactness

A Differentail *Equation*

$M(x, y)dx + N(x, y)dy = 0$ is Exact Iff $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Solution of Exact Differentail Equation

If $M(x, y)dx + N(x, y)dy = 0$, then

its solution is

$$\int_{y-\text{constant}} Mdx + \int (\text{Terms in } N \text{ free from } x) dy = c$$

$$\text{or } \int_{x-\text{constant}} Ndy + \int (\text{Terms in } M \text{ free from } y) dx = c$$

Example :

$$\text{Solve } \left(y^2 e^{xy^2} + 4x^3 \right) dx + \left(2xye^{xy^2} - 3y^2 \right) dy = 0$$

Solution : Comparing with $Mdx + Ndy = 0$

$$M = y^2 e^{xy^2} + 4x^3, \quad N = 2xye^{xy^2} - 3y^2$$

$$\frac{\partial M}{\partial y} = 2ye^{xy^2} + y^2 e^{xy^2} = \frac{\partial N}{\partial x}$$

$\therefore$ Given d.e. is exact

$\therefore$ its solution is

$$\int_{y-\text{constant}} M dx + \int (\text{Terms in } N \text{ free from } x) dy = c$$

$$\int_{y-\text{constant}} \left(y^2 e^{xy^2} + 4x^3 \right) dx + \int (-3y^2) dy = c$$

$$e^{xy^2} + x^4 - y^3 = c$$

Differential Equations Reducible to Exact

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Integrating Factor (IF)

If $Mdx + Ndy = 0$ is not exact it may be reduced to exact by multiplying it by some function $R(x, y)$ called as Integrating Factor.

Methods of Finding Integrating Factor

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Integrating Factor by inspection :

Study the Following differentials

1. $ydx + xdy = d(xy)$

2. $\frac{xdy - ydx}{x^2} = d\left(\frac{y}{x}\right)$

3. $\frac{xdy - ydx}{xy} = d\left(\ln\left(\frac{y}{x}\right)\right)$

4. $\frac{xdy - ydx}{x^2 + y^2} = d\left(\tan^{-1}\left(\frac{y}{x}\right)\right)$

5. $\frac{xdy - ydx}{x^2 - y^2} = \frac{1}{2} d\left(\ln\left(\frac{x+y}{x-y}\right)\right).$

Rule 1 : For homogeneous differential Equation

$$IF = \frac{1}{xM + yN}, \quad xM + yN \neq 0$$

Rule 2 : If the differential equation is of the type

$$IF = \frac{1}{xM - yN}, \quad xM - yN \neq 0$$

Rule 3 : (a) if $\frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = f(x),$

then $if = e^{\int f(x) dx}.$

Rule 3 : (b) *if* $\frac{1}{-M} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = f(y),$

then if $= e^{\int f(y) dy}$

Rule 4 : *If the given d.e. can be expressed as*
 $x^{\alpha} y^{\beta} (mydx + nx dy) + x^{\alpha'} y^{\beta'} (m'ydx + n'x dy) = 0$
then Assume if $= x^a y^b.$

Exercise :

$$1. xdy - ydx + 3(x^2 + y^2) = 0$$

$$2. xdx + ydy = \frac{a^2(xdy - ydx)}{xy}$$

$$3. \frac{dy}{dx} = \frac{x^3 + y^3}{x^2 y}$$

$$4. (x^3 y^2 + x)dy + (x^2 y^3 - y)dx = 0$$

$$5. (x^2 y^2 + xy + 1)ydx + (x^2 y^2 - xy + 1)xdy = 0$$

$$6. (x^4 e^x - 2mxy^2)dx + 2mx^2 ydy = 0$$

First Order Linear Differential Equation

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A Differential equation which can be expressed as

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where P and Q are functions of x or constant is called first order linear differential equation.

Method of solving :

Integrating factor = $IF = e^{\int P(x)dx}$.

Solution : $y(IF) = \int Q(x)(IF)dx + C$

where C is constant of integration.

Illustrative Examples

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$$1. \frac{dy}{dx} = -\frac{x + y \cos x}{1 + \sin x}$$

Obsevation : It contains single term of y .

$\therefore$ We can express it as $\frac{dy}{dx} + P(x)y = Q(x)$.

$$\frac{dy}{dx} + \frac{\cos x}{1 + \sin x} y = -\frac{x}{1 + \sin x}$$

$$\therefore p(x) = \frac{\cos x}{1 + \sin x}, \quad Q(x) = -\frac{x}{1 + \sin x}$$

$$\therefore if = e^{\int p(x)dx} = e^{\int \frac{\cos x}{1+\sin x} dx} = e^{\ln(1+\sin x)} = 1 + \sin x$$

$\therefore$ Solution is

$$y(if) = \int Q(x)(if) dx + c$$

$$y(1 + \sin x) = -\int \frac{x}{1 + \sin x} (1 + \sin x) dx + c$$

$$y(1 + \sin x) = -\int x dx + c$$

$$y(1 + \sin x) = -\frac{x^2}{2} + c$$

2. Solve $ye^y dx = (y^3 + 2xe^y)dy$.

Obsevation : It contains single term of x .

$\therefore$ We can express it as $\frac{dx}{dy} + P(y)x = Q(y)$.

$$\frac{dx}{dy} - \frac{2}{y}x = y^2e^{-y}$$

$$\therefore P(y) = -\frac{2}{y}, Q(y) = y^2e^{-y}$$

$$\therefore if = e^{\int P(y)dy} = e^{-\int \frac{2}{y}} = e^{-2\ln(y)} = \frac{1}{y^2}.$$

∴ Solution is

$$x(if) = \int Q(y)(if) dy + c$$

$$x\left(\frac{1}{y^2}\right) = \int y^2 e^{-y} \frac{1}{y^2} dy + c$$

$$\therefore x\left(\frac{1}{y^2}\right) = \int e^{-y} dy + c = -e^{-y} + c$$

Differential Equations Reducible to Linear D.E.

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Bernaullies Diferential Equation :

A differential equation of the type

$$\frac{dy}{dx} + p(x)y = Q(x)y^n \text{ is called}$$

Bernaullies Diferential Equation

Method to reduce it to linear D.E.

Dividing by y^n

$$y^{-n} \frac{dy}{dx} + p(x)y^{1-n} = Q(x)$$

$$\text{Let } y^{1-n} = v \Rightarrow (1-n) y^{-n} \frac{dy}{dx} = \frac{dv}{dx}$$

$\therefore$ given d.e. reduced to

$$\frac{1}{1-n} \frac{dv}{dx} + p(x)v = Q(x)$$

$$\text{i.e. } \frac{dv}{dx} + (1-n)p(x)v = (1-n)Q(x)$$

This is linear in v .

Example : Solve $\frac{dy}{dx} = \frac{x^2 + y^2 + 1}{2xy}$

Rearranging it

$$2y \frac{dy}{dx} - \frac{1}{x} y^2 = x + \frac{1}{x} \text{ (This is Bernaulies de)}$$

$$\therefore \text{ put } y^2 = v \Rightarrow 2y \frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{dv}{dx} - \frac{1}{x} v = x + \frac{1}{x} \text{ (This is LDE)}$$

where $p(x) = -\frac{1}{x}$, $Q(x) = x + \frac{1}{x}$

$$\therefore if = e^{\int p(x)dx} = e^{-\int \frac{1}{x}dx} = e^{-\ln(x)} = \frac{1}{x}$$

Solution is

$$v(if) = \int Q(x)(if)dx + c$$

$$i.e. \quad v \frac{1}{x} = \int \left(x + \frac{1}{x}\right) \frac{1}{x} dx + c$$

$$v \frac{1}{x} = x - \frac{1}{x} + c \Rightarrow y^2 = x^2 - 1 + cx.$$

Generalized Bernaullies Diferential Equation :

A differential equation of the type

$$f'(y) \frac{dy}{dx} + p(x) f(y) = Q(x) \text{ is called}$$

generalized Bernaullies Diferential Equation

Method to reduce it to linear D.E.

Method of Solving :

$$\text{Put } f(y) = v \Rightarrow f'(y) \frac{dy}{dx} = \frac{dv}{dx}$$

$\therefore$ *given de becomes*

$$\frac{dv}{dx} + p(x)v = Q(x) \text{ (This is LDE)}$$

Solve : $\tan y \frac{dy}{dx} + \tan x = \cos y \cos^2 x$

Dividing by $\cos y$

$$\sec y \tan y \frac{dy}{dx} + \tan x \sec y = \cos^2 x$$

here $f(y) = \sec y = v(\text{say}) \Rightarrow \sec y \tan y \frac{dy}{dx} = \frac{dv}{dx}$

$\therefore$ given de becomes

$$\frac{dv}{dx} + \tan x v = \cos^2 x \quad (\text{This is LDE})$$

$$\frac{dv}{dx} + \tan x v = \cos^2 x \quad (\text{This is LDE})$$

$$\text{if } = e^{\int \tan x dx} = e^{\ln(\sec x)} = \sec x$$

$\therefore$ solution is

$$v \sec x = \int \sec x \cos^2 x dx + c$$

$$\sec y \sec x = \int \cos x dx + c$$

$$\sec y \sec x = \sin x + c$$

References

1. *Higher Engineering Mathematics,*
by Dr.B.S.Grewal, Khanna Publication.
2. *Advanced Engineering Mathemtics,*
Erwin Kreyszig, Wiley India.