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Engineering
Probability and Random Process
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Chapter 4: Random Processes

Random Processes

Outline

- Introduction
- Definition of a Random Process
- Characterization of Random Processes
- Mean, Correlation, and Covariance Functions
- Classification of Random Processes
- Power Spectral Densities of Random Processes
- Response of Linear Systems to Random Inputs

Introduction

- The theory of random processes was first developed in connection with the study of fluctuations and noise in physical systems.
- A random process is the mathematical model of an empirical process whose development is governed by probability laws.
- Random processes provides useful models for the studies of such diverse fields as statistical physics, communication and control, time series analysis, population growth, and management sciences.

Definition of a Random Process

- A random process is a family of random variables $\{X(t), t \in T\}$ defined on a given probability space, indexed by the parameter t , where t varies over an index set T .
- In a random process $\{X(t), t \in T\}$, the index set T is called the parameter set of the random process.
- The values assumed by $X(t)$ are called **states**, and the set of all possible values forms the state space E of the random process.
- If the index set T of a random process is discrete, then the process is called a **discrete-time random process**.

Definition of a Random Process

Cont'd.....

- A discrete-time random process is also called a **random sequence** and is denoted by $\{X_n, n = 1, 2, 3, \dots\}$.
- If T is continuous, then we have a continuous-time random process.
- In fact, a random process $\{X(t), t \in T\}$ is a function of two arguments $\{X(t, \omega), t \in T, \omega \in \Omega\}$.
- For a fixed time $t=t_k$, $X(t_k, \omega) = X_k(\omega)$ is a random variable denoted by $X(t_k)$, as ω varies over the sample space Ω .

Definition of a Random Process

Cont'd.....

- On the other hand, for a fixed sample point $\omega_i \in \Omega$, $X(t, \omega_i) = X_i(t)$ is a single function of time $\mathbf{t}$, called a sample function or a realization of the process.
- The totality of all sample functions is called an ***ensemble***.
- Of course if both $\boldsymbol{\omega}$ and $\mathbf{t}$ are fixed, $X(t_k, \omega_i)$ is simply a real number.
- In the following discussion, we use the notation $\mathbf{X}(\mathbf{t})$ to represent $\mathbf{X}(\mathbf{t}, \boldsymbol{\omega})$.

Characterization of Random Processes

- If $\mathbf{X}(t)$ is a random process, then for fixed $t=t_1$, $\mathbf{X}_1=\mathbf{X}(t_1)$ represents a random variable.
- Its distribution function is given by:

$$F_x(x_1, t_1) = P\{X(t) \leq x_1\}$$

- Notice that $F_x(\mathbf{x}, t)$ depends on t , since for a different t , we obtain a different random variable.
- The first-order probability density function of the process $\mathbf{X}(t)$ is defined as:

$$f_x(x_1, t_1) = \frac{dF_x(x_1, t_1)}{dx_1}$$

Characterization of Random Processes

Cont'd.....

- For $t = t_1$ and $t = t_2$, $\mathbf{X}(t)$ represents two different random variables $\mathbf{X}_1 = \mathbf{X}(t_1)$ and $\mathbf{X}_2 = \mathbf{X}(t_2)$ respectively.
- Their joint distribution is given by:
- The second-order probability density function of the random process $\mathbf{X}(t)$ is:

$$f_x(x_1, x_2, t_1, t_2) = P\{X(t_1) \leq x_1, X(t_2) \leq x_2\}$$

- Similarly $f_x(x_1, x_2, t_1, t_2)$ represents the n^{th} order density function of the process $\mathbf{X}(t)$.

$$f_x(x_1, x_2, t_1, t_2) = \frac{\partial^2 F_x(x_1, x_2, t_1, t_2)}{\partial x_1 \partial x_2}$$

$$f_x(x_1, x_2, \dots, x_n, t_1, t_2, \dots, t_n)$$

Mean, Correlation, and Covariance Functions

- As in the case of random variables, random processes are often described by using statistical averages.
- The mean of $\mathbf{X}(t)$ is defined by:

where $\mathbf{X}(t)$ is treated as a random variable for a fixed value of t .
$$\mu_x(t) = E[X(t)]$$

- In general, $\mu_x(t)$ is a function of time, and it is often called the ensemble average of $\mathbf{X}(t)$.

Mean, Correlation, and Covariance Functions

- A measure of dependence among the random variables of $\mathbf{X}(t)$ is provided by its autocorrelation function, defined by:

$$R_{xx}(t_1, t_2) = E[X(t_1)X(t_2)]$$

- Note that:

- $R_{xx}(t_1, t_2) = R_{xx}(t_2, t_1)$ and $R_{xx}(t, t) = E[X^2(t)]$
The autocovariance function of $\mathbf{X}(t)$ is defined by:

$$\begin{aligned} C_{xx}(t_1, t_2) &= \text{Cov}[X(t_1), X(t_2)] = E\{[X(t_1) - \mu_X(t_1)][X(t_2) - \mu_X(t_2)]\} \\ &= R_{xx}(t_1, t_2) - \mu_X(t_1)\mu_X(t_2) \end{aligned}$$

Mean, Correlation, and Covariance Functions

- It is clear that if the mean of $\mathbf{X}(t)$ is zero, then:

$$C_{XX}(t_1, t_2) = R_{XX}(t_1, t_2)$$

- Note that the variance of $\mathbf{X}(t)$ is given by:

- If $\mathbf{X}(t)$ is a complex random process, then its autocorrelation function $\mathbf{R}_{XX}(t_1, t_2)$ and autocovariance function $\mathbf{C}_{XX}(t_1, t_2)$ are defined, respectively, by:

$$R_{XX}(t_1, t_2) = E[X(t_1)X^*(t_2)] \text{ and}$$

$$C_{XX}(t_1, t_2) = E\left\{[X(t_1) - \mu_X(t_1)][X(t_2) - \mu_X(t_2)]^*\right\}$$

Classification of Random Processes

i. Stationary Processes

- A random process $\{X(t), t \in T\}$ is said to be stationary or strict-sense stationary (SSS) if, for all n and for every set of time instants $(t_i \in T, i = 1, 2, \dots, n)$,
- Hence, the distribution of a stationary process will be unaffected by a shift in the time origin, and $\mathbf{X}(t)$ and $\mathbf{X}(t+\tau)$ will have the same distributions for any τ .
- Nonstationary processes are characterized by distributions depending on the points $t_1, t_2, \dots, t_n$.

Classification of Random Processes

Cont'd.....

ii. Wide-Sense Stationary Processes

- A random process $\mathbf{X}(t)$ is wide-sense stationary (WSS) if:

1. $E[X(t)] = \mu_X$ (constant)

2. $R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)] = R_{XX}(|t_2 - t_1|)$

- Note that a strict-sense stationary process is also a WSS process, but, in general, the converse is not true.

Classification of Random Processes

Cont'd.....

- Other types of random processes include:
 - ✓ Independent Processes
 - ✓ Markov Processes
 - ✓ Normal Processes
 - ✓ Ergodic Processes
 - ✓ Poisson Processes

Power Spectral Densities of Random Processes

- The autocorrelation function of a continuous-time random process $\mathbf{X}(t)$ is defined as:

$$R_{XX}(\tau) = E[X(t)X(t+\tau)]$$

- **Properties of $R_{XX}(\tau)$:**

1. $R_{XX}(-\tau) = R_{XX}(\tau)$
2. $|R_{XX}(\tau)| \leq R_{XX}(0)$
3. $R_{XX}(0) = E[X^2(t)] \geq 0$

Power Spectral Densities of Random Processes.....

- In case of a discrete-time random process $\mathbf{X(n)}$, the autocorrelation function of $\mathbf{X(n)}$ is defined by:

$$R_{XX}(k) = E[X(n)X(n+k)]$$

- **Properties of $R_X(k)$:**

1. $R_{XX}(-k) = R_{XX}(k)$
2. $|R_{XX}(k)| \leq R_{XX}(0)$
3. $R_{XX}(0) = E[X^2(n)] \geq 0$

Power Spectral Densities of Random Processes.....

- Two processes $\mathbf{X}(t)$ and $\mathbf{Y}(t)$ are called (mutually) orthogonal if:

$$R_{XY}(\tau) = 0, \quad \text{for all } \tau$$

- Similarly, the cross-correlation function of two discrete-time jointly **WSS** random processes $\mathbf{X}(n)$ and $\mathbf{Y}(n)$ is defined by:
- $$R_{XY}(k) = E[X(n)Y(n+k)]$$
- The various properties of $R_{XY}(k)$ similar to those of $R_{XY}(\tau)$ can be obtained by replacing τ by k in the above equations.

Power Spectral Densities of Random Processes.....

- The power spectral density (or power spectrum) $S_{XX}(\omega)$ of a continuous-time random process $X(t)$ is defined as the Fourier transform of $R_{XX}(\tau)$, i.e. ,

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$$

- Thus, taking the inverse Fourier transform of $S_X(\omega)$, we obtain:

- The above equations are known as the Wiener-Khinchin relations.

Power Spectral Densities of Random Processes.....

- **Properties of $S_{XX}(\omega)$:**

1. $S_{XX}(\omega)$ is real and $S_{XX}(\omega) \geq 0$

2. $S_{XX}(-\omega) = S_{XX}(\omega)$

3. $E[X^2(t)] = R_{XX}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) d\omega$

- Similarly, the power spectral density $S_{XX}(\Omega)$ of a discrete-time random process $\mathbf{X}(\mathbf{n})$ is defined as the Fourier transform of $\mathbf{R}_{XX}(\mathbf{k})$:

$$S_{XX}(\Omega) = \sum_{k=-\infty}^{\infty} R_{XX}(k) e^{-j\Omega k}$$

Power Spectral Densities of Random Processes.....

- Thus, taking the inverse Fourier transform of

$S_{XX}(\Omega)$, we obtain:

$$R_{XX}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} S_{XX}(\Omega) e^{j\Omega k} d\Omega$$

- **Properties of $S_{XX}(\Omega)$**

1. $S_{XX}(\Omega)$ is real and $S_{XX}(\Omega) \geq 0$

2. $S_{XX}(-\Omega) = S_{XX}(\Omega)$

3. $E[X^2(n)] = R_{XX}(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} S_{XX}(\Omega) d\Omega$

Power Spectral Densities of Random Processes.....

- The cross power spectral density (or cross power spectrum) $S_{XY}(\omega)$ of two continuous-time random processes $X(t)$ and $Y(t)$ is defined as the Fourier transform of $R_{XY}(\tau)$:

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-j\omega\tau} d\tau$$

- Thus, taking the inverse Fourier transform of $S_{XY}(\omega)$, we get:

$$R_{XY}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XY}(\omega) e^{j\omega\tau} d\omega$$

Power Spectral Densities of Random Processes.....

- **Properties of $S_{XY}(\omega)$:**
- Unlike $S_{XX}(\omega)$, which is a real-valued function of ω , $S_{XY}(\omega)$, in general, is a complex-valued function.

1. $S_{XY}(\omega) = S_{YX}^*(-\omega)$

- Similarly, the cross power spectral density $S_{XY}(\Omega)$ of two discrete-time random processes $X(n)$ and $Y(n)$ is defined:

2. $S_{XY}(\Omega) = \sum_{k=-\infty}^{\infty} R_{XY}(k) e^{-j\Omega k}$

Power Spectral Densities of Random Processes.....

- Taking the inverse Fourier transform of $\mathbf{S}_{XY}(\Omega)$, we get:

$$R_{XY}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} S_{XY}(\Omega) e^{j\Omega k} d\Omega$$

- **Properties of $\mathbf{S}_{XY}(\omega)$:**
- Unlike $\mathbf{S}_{XX}(\Omega)$, which is a real-valued function of Ω , $\mathbf{S}_{XY}(\Omega)$, in general, is a complex-valued function.

$$1. S_{XY}(\Omega + 2\pi) = S_{XY}(\Omega)$$

$$2. S_{XY}(\Omega) = S_{YX}(-\Omega)$$

$$3. S_{XY}(-\Omega) = S_{XY}^*(\Omega)$$

Example on Random Processes

Example:

Consider a random process $X(t)$ defined by

$$X(t) = A \cos(\omega_0 t + \phi)$$

where ω_0 and A are constants and ϕ is a uniform random variable over the interval $(0, 2\pi)$

- Find the mean $\mu_X(t)$.
- Find the autocorrelation function $R_{XX}(t_1, t_2)$.
- Find the autocovariance function $C_{XX}(t_1, t_2)$.
- Determine whether $X(t)$ is WSS random process or not
- Find the power spectral density of $X(t)$.

Example on Random Processes

Cont'd.....

Solution:

$$a. \mu_X(t) = E[X(t)] = E[A \cos(\omega_0 t + \phi)] = A E[\cos(\omega_0 t + \phi)]$$

$$\text{But } \cos(\omega_0 t + \phi) = \cos(\omega_0 t) \cos \phi - \sin(\omega_0 t) \sin \phi$$

$$\begin{aligned} \Rightarrow \mu_X(t) &= E[X(t)] = A E[\cos(\omega_0 t) \cos \phi - \sin(\omega_0 t) \sin \phi] \\ &= A \cos(\omega_0 t) E[\cos \phi] - A \sin(\omega_0 t) E[\sin \phi] \end{aligned}$$

$$E[\cos \phi] = \frac{1}{2\pi} \int_0^{2\pi} \cos \phi d\phi = 0$$

$$\text{Similarly } E[\sin \phi] = \frac{1}{2\pi} \int_0^{2\pi} \sin \phi d\phi = 0$$

$$\therefore \mu_X(t) = E[X(t)] = 0$$

Example on Random Processes

Cont'd.....

Solution:

$$\begin{aligned} b. R_{XX}(t_1, t_2) &= E[X(t_1)X(t_2)] \\ &= E[A \cos(\omega_0 t_1 + \phi) A \cos(\omega_0 t_2 + \phi)] \\ &= A^2 E[\cos(\omega_0 t_1 + \phi) \cos(\omega_0 t_2 + \phi)] \\ &= \frac{A^2}{2} E[\cos \omega_0(t_2 - t_1) + \cos(\omega_0(t_1 + t_2) + 2\phi)] \end{aligned}$$

But, $E[\cos \omega_0(t_2 - t_1)] = \cos \omega_0(t_2 - t_1)$ and

$$E[\cos(\omega_0(t_1 + t_2) + 2\phi)] = 0$$

$$\therefore R_{XX}(t_1, t_2) = \frac{A^2}{2} \cos \omega_0(t_2 - t_1)$$

Example on Random Processes

Cont'd.....

Solution:

$$c. C_{XX}(t_1, t_2) = R_{XX}(t_1, t_2) - \mu_X(t_1)\mu_X(t_2)$$

$$= \frac{A^2}{2} \cos \omega_0(t_2 - t_1) - 0$$

$$\therefore C_{XX}(t_1, t_2) = \frac{A^2}{2} \cos \omega_0(t_2 - t_1)$$

d. Since the mean is constant and the autocorrelation function depends on time difference only, $X(t)$ is a WSS random process.

Example on Random Processes

Cont'd.....

Solution:

e. Since $X(t)$ is a WSS random process, the autocorrelation function can be simply written as:

$$R_{XX}(\tau) = \frac{A^2}{2} \cos(\omega_0 \tau)$$

The power spectral density of $X(t)$ is given by:

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$$

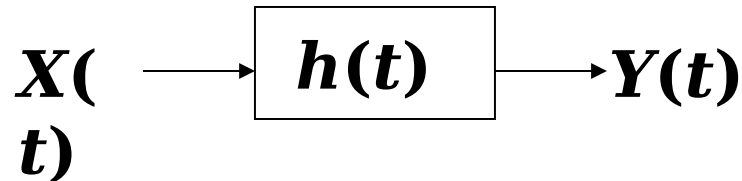
But from Fourier transform pair table we have

$$FT\{\cos(\omega_0 t)\} = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

$$\therefore S_{XX}(\omega) = \frac{A^2\pi}{2} \delta(\omega - \omega_0) + \frac{A^2\pi}{2} \delta(\omega + \omega_0)$$

Response of Linear Systems to Random Inputs

- If a WSS random process $\mathbf{X}(t)$ with autocorrelation function $\mathbf{R}_{XX}(\tau)$ is applied to a linear system with impulse response $\mathbf{h}(t)$, then the cross correlation function $\mathbf{R}_{XY}(\tau)$ and the output autocorrelation function $\mathbf{R}_{YY}(\tau)$ are given as follows.



$$R_{XY}(\tau) = R_{XX}(\tau) * h^*(-\tau)$$

And

$$\begin{aligned} R_{YY}(\tau) &= R_{XY}(\tau) * h(\tau) \\ &= R_{XX}(\tau) * h^*(-\tau) * h(\tau) \end{aligned}$$

Response of Linear Systems to Random Inputs.....

- Using properties of Fourier transform, we get:

$$f(t) \xleftrightarrow{FT} F(\omega) \quad \text{and} \quad g(t) \xleftrightarrow{FT} G(\omega)$$

- Then using the above property, the cross and output power spectral densities can be evaluated as:

$$S_{XY}(\omega) = FT\{R_{XX}(\tau) * h^*(-\tau)\} = S_{XX}(\omega)H^*(\omega)$$

And

$$\begin{aligned} S_{YY}(\omega) &= FT\{R_{YY}(\tau)\} = FT\{R_{XY}(\tau) * h(\tau)\} \\ &= S_{XY}(\omega)H(\omega) = S_{XX}(\omega)|H(\omega)|^2 \end{aligned}$$

Example on Response of Linear Systems

Example:

Consider a WSS random process $\mathbf{X}(t)$ with autocorrelation

function given by:

$$R_{XX}(\tau) = e^{-a|\tau|}, \text{ where } a \text{ is a real positive constant}$$

Let the random process $\mathbf{X}(t)$ be applied to the input of an LTI system with impulse response given by:

$$h(t) = e^{-bt}u(t), \text{ where } b \text{ is a real positive constant}$$

Find the autocorrelation function of the output $\mathbf{Y}(t)$ of the system.

Example on Response of Linear Systems

Cont'd.....

Solution:

The frequency response $H(\omega)$ of the system is:

$$H(\omega) = FT\{h(t)\} = \frac{1}{j\omega + b}$$

The power spectral density of $X(t)$ is:

$$S_{XX}(\omega) = FT\{R_{XX}(\tau)\} = \frac{2a}{\omega^2 + a^2}$$

Then the power spectral density of $Y(t)$ is given by:

$$S_{YY}(\omega) = S_{XX}(\omega)|H(\omega)|^2 = \left(\frac{1}{\omega^2 + b^2}\right)\left(\frac{2a}{\omega^2 + a^2}\right)$$

Example on Response of Linear Systems

Cont'd.....

Solution:

$$\Rightarrow S_{YY}(\omega) = \frac{a}{(a^2 - b^2)b} \left(\frac{2b}{\omega^2 + b^2} \right) - \frac{b}{(a^2 - b^2)b} \left(\frac{2a}{\omega^2 + b^2} \right)$$

Taking the inverse Fourier transform of both sides of the above equation we obtain

$$R_{YY}(\tau) = \frac{1}{(a^2 - b^2)b} (ae^{-b|\tau|} - be^{-a|\tau|})$$

Exercise on Random Processes

1. Consider a random process $X(t)$ defined by

$$X(t) = A \cos(\omega_0 t + \phi)$$

where ω_0 and ϕ are constants and A is a uniform random variable over the interval $(0, 2)$

- Find the mean $\mu_X(t)$.
- Find the autocorrelation function $R_{XX}(t_1, t_2)$.
- Find the autocovariance function $C_{XX}(t_1, t_2)$.
- Determine whether $X(t)$ is WSS random process or not
- Find the power spectral density of $X(t)$.

Exercise on Random Processes

2. Consider a random process $\mathbf{X}(t)$ defined by:

$$X(t) = A \sin(\omega_0 t + \phi)$$

where A and ϕ are independent random variables which are

uniformly distributed over the interval $[0, 1]$ and $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

respectively and ω_0 is a constant.

a. Find the mean $\mu_X(t)$.

b. Find the autocorrelation function $R_{XX}(t_1, t_2)$.

c. Find the autocovariance function $C_{XX}(t_1, t_2)$.

d. Determine whether $\mathbf{X}(t)$ is WSS random process or not

e. Find the power spectral density of $X(t)$.

Exercise on Random Processes

Cont'd.....

3. Two random processes $X(t)$ and $Y(t)$ are given by:

$$X(t) = A \cos(\omega_0 t + \phi) \text{ and } Y(t) = A \sin(\omega_0 t + \phi)$$

where A and ω_0 are constants and ϕ is a uniform random variable over the interval $(0, 2\pi)$.

a. Find the cross correlation function of $X(t)$ and $Y(t)$.

b. Verify that $R_{XY}(-\tau) = R_{XY}(\tau)$

Exercise on Random Processes

Cont'd.....

4. Consider a discrete-time WSS random process $X(n)$ with autocorrelation function given by:

$$R_{XX}(k) = 2e^{-|k|}$$

Find the power spectral density of $X(n)$.