

Addis Ababa Science & Technology University
Department of Electrical & Electronics Engineering

Probability and Random Process (EEEG-2114)

Chapter 3: Multiple Random Variables

Multiple Random Variables

Outline

- Introduction
- The Joint Cumulative Distribution Function
- The Joint Probability Density and Mass Functions
- Marginal Statistics
- Independence
- Conditional Distributions
- Correlation and Covariance
- Functions of Two Random Variables

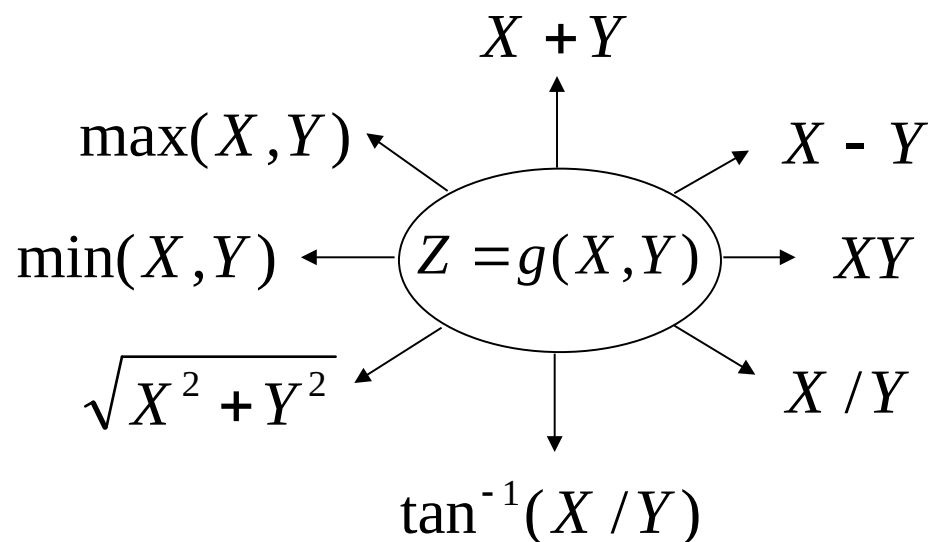
One Function of Two Random Variables

Given two random variables X and Y and a function $g(x,y)$, we form a new random variable Z as

$$Z = g(X, Y).$$

Given the joint pdf $f_{XY}(x, y)$, how does one obtain $f_Z(z)$, the pdf of Z ? Problems of this type are of interest from a practical standpoint. For example, a receiver output signal usually consists of the desired signal buried in noise, and the above formulation in that case reduces to $Z = X + Y$.

It is important to know the statistics of the incoming signal for proper receiver design. In this context, we shall analyze problems of the following type:



The cdf of Z is given by:

$$F_Z(z) = P(Z(\omega) \leq z) = P(g(X, Y) \leq z)$$

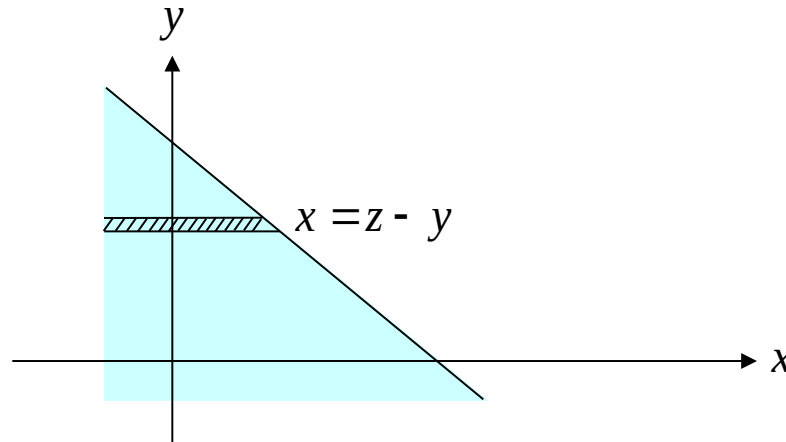
$$= \int \int_{x,y} f_{XY}(x, y) dx dy,$$

Example -1: Let $Z = X + Y$. Find $f_Z(z)$.

Solution:

$$F_Z(z) = P(X + Y \leq z) = \int_{y=-\infty}^{+\infty} \int_{x=-\infty}^{z-y} f_{XY}(x, y) dx dy,$$

Since the required region in the xy plane where $x + y \leq z$ is the shaded area shown below to the left of the line $x + y = z$. Integrating over the horizontal strip along the x -axis first (inner integral) followed by sliding that strip along the y -axis from $-\infty$ to $+\infty$ (outer integral) we cover the entire shaded area.



We can find $f_z(z)$ by differentiating $F_z(z)$ directly. In this context, it is useful to recall the differentiation rule by Leibnitz. Suppose

$$H(z) = \int_{a(z)}^{b(z)} h(x, z) dx.$$

Then,

$$\frac{dH(z)}{dz} = \frac{db(z)}{dz} h(b(z), z) - \frac{da(z)}{dz} h(a(z), z) + \int_{a(z)}^{b(z)} \frac{\partial h(x, z)}{\partial z} dx.$$

Using the above two equations, we get

$$\begin{aligned} f_z(z) &= \int_{-\infty}^{+\infty} \left(\frac{\partial}{\partial z} \int_{-\infty}^{z-y} f_{XY}(x, y) dx \right) dy = \int_{-\infty}^{+\infty} \left(f_{XY}(z-y, y) - 0 + \int_{-\infty}^{z-y} \frac{\partial f_{XY}(x, y)}{\partial z} dx \right) dy \\ &= \int_{-\infty}^{+\infty} f_{XY}(z-y, y) dy. \end{aligned} \quad (i)$$

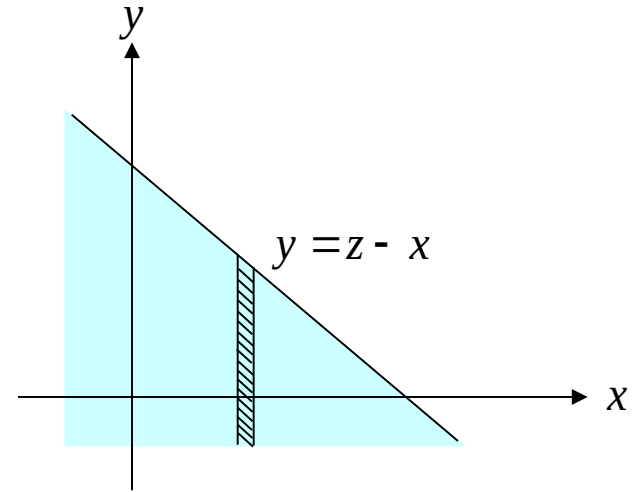
Alternatively, the above integration can be carried out first along the y-axis followed by the x-axis as below.

In the second case

$$F_Z(z) = \int_{x=-\infty}^{+\infty} \int_{y=-\infty}^{z-x} f_{XY}(x, y) dx dy,$$

and differentiating the above equation gives

$$\begin{aligned} f_Z(z) &= \frac{dF_Z(z)}{dz} = \int_{x=-\infty}^{+\infty} \left(\frac{\partial}{\partial z} \int_{y=-\infty}^{z-x} f_{XY}(x, y) dy \right) dx \\ &= \int_{x=-\infty}^{+\infty} f_{XY}(x, z-x) dx. \end{aligned} \quad (ii)$$



If X and Y are independent, then

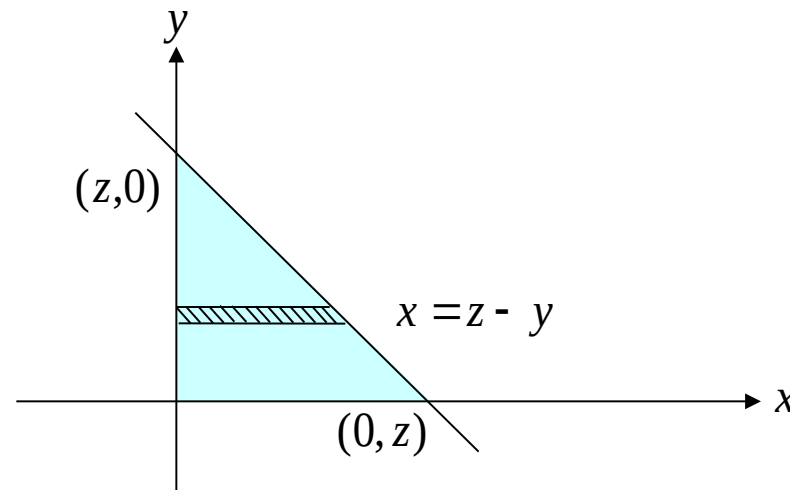
$$f_{XY}(x, y) = f_X(x) f_Y(y) \quad (iii)$$

and inserting equation (iii) into equations (i) and (ii), we get

$$f_Z(z) = \int_{y=-\infty}^{+\infty} f_X(z-y) f_Y(y) dy = \int_{x=-\infty}^{+\infty} f_X(x) f_Y(z-x) dx. \quad (iv)$$

The above integral is the standard convolution of the functions $f_X(z)$ and $f_Y(z)$ expressed two different ways. We thus reach the following conclusion: If two random variables are independent, then the density of their sum equals the convolution of their density functions.

As a special case, suppose that $f_X(x) = 0$ for $x < 0$ and $f_Y(y) = 0$ for $y < 0$, then we can use the following figure to determine the pdf of Z .



In the above case,

$$F_Z(z) = \int_{y=0}^z \int_{x=0}^{z-y} f_{XY}(x, y) dx dy$$

or

$$f_Z(z) = \int_{y=0}^z \left(\frac{\partial}{\partial z} \int_{x=0}^{z-y} f_{XY}(x, y) dx \right) dy = \begin{cases} \int_0^z f_{XY}(z-y, y) dy, & z > 0, \\ 0, & z \leq 0. \end{cases}$$

On the other hand, by considering vertical strips first in above figure, we get

$$F_Z(z) = \int_{x=0}^z \int_{y=0}^{z-x} f_{XY}(x, y) dy dx$$

or

$$f_Z(z) = \int_{x=0}^z f_{XY}(x, z-x) dx = \begin{cases} \int_{y=0}^z f_X(x) f_Y(z-x) dx, & z > 0, \\ 0, & z \leq 0, \end{cases}$$

if X and Y are independent random variables.

Example-2: Suppose X and Y are independent exponential r.vs with common parameter λ , and let $Z = X + Y$.

Determine

Solution: We have $f_Z(z)$.

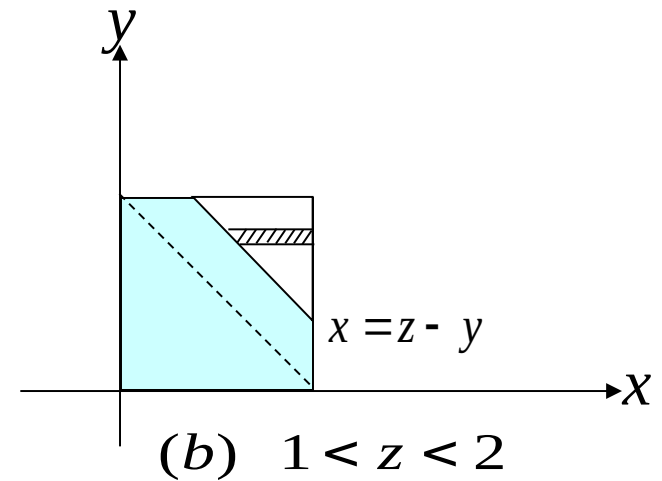
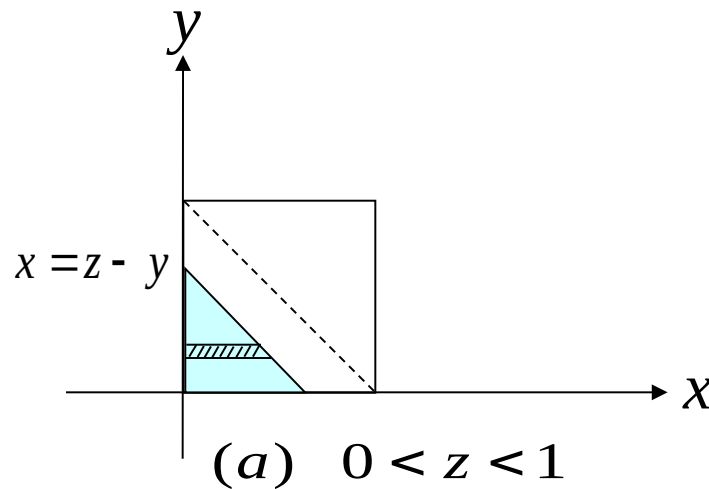
and we can make use of (13) to obtain the pdf of $Z = X + Y$.

$$f_Z(z) = \int_0^z \lambda^2 e^{-\lambda x} e^{-\lambda(z-x)} dx = \lambda^2 e^{-\lambda z} \int_0^z dx = z \lambda^2 e^{-\lambda z} U(z).$$

As the next example shows, care should be taken in using the convolution formula for r.vs with finite range.

Example-3: X and Y are independent uniform r.vs in the common interval $(0,1)$. Determine $f_Z(z)$, where $Z = X + Y$.

Solution: Clearly, $Z = X + Y \Rightarrow 0 < Z < 2$ here, and as following figure shows there are two cases of z for which the shaded areas are quite different in shape and they should be considered separately



For $0 \leq z < 1$,

$$F_Z(z) = \int_{y=0}^z \int_{x=0}^{z-y} 1 \, dx dy = \int_{y=0}^z (z - y) dy = \frac{z^2}{2}, \quad 0 \leq z < 1.$$

For $1 \leq z < 2$, notice that it is easy to deal with the unshaded region. In that case

$$F_Z(z) = 1 - P(Z > z) = 1 - \int_{y=z-1}^1 \int_{x=z-y}^1 1 \, dx dy$$

$$= 1 - \int_{y=z-1}^1 (1 - z + y) dy = 1 - \frac{(2 - z)^2}{2}, \quad 1 \leq z < 2.$$

Finally, differentiating the cdf, we obtain

$$f_z(z) = \frac{dF_z(z)}{dz} = \begin{cases} z & 0 \leq z < 1, \\ 2 - z, & 1 \leq z < 2. \end{cases}$$

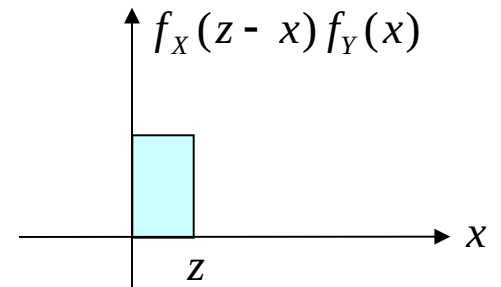
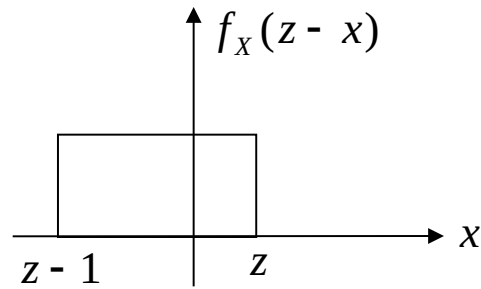
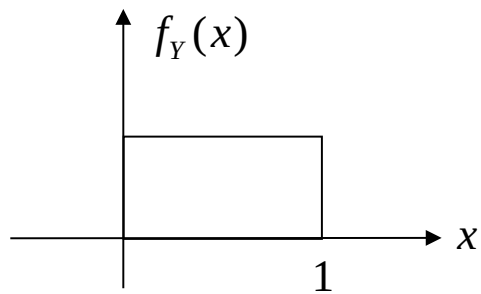
By direct convolution of $f_x(x)$ and $f_y(y)$, we obtain the same result as above. In fact, for $0 \leq z < 1$ [Figure (a)]

$$f_z(z) = \int f_x(z - x) f_y(x) dx = \int_0^z 1 \, dx = z.$$

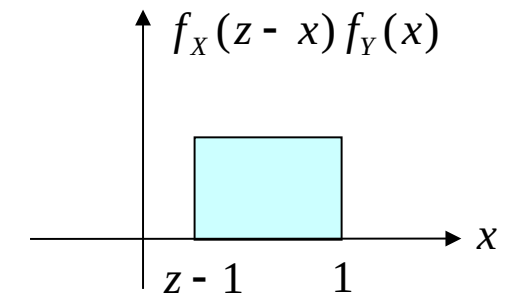
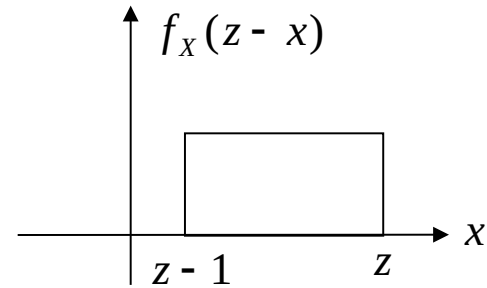
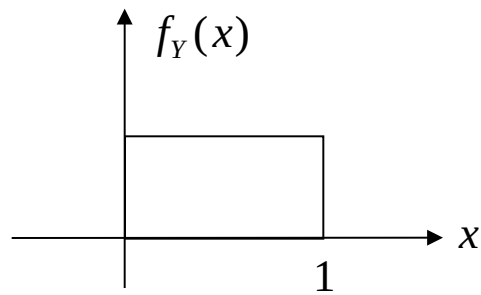
and for $1 \leq z < 2$ [Figure (b)]

$$f_z(z) = \int_{z-1}^1 1 \, dx = 2 - z.$$

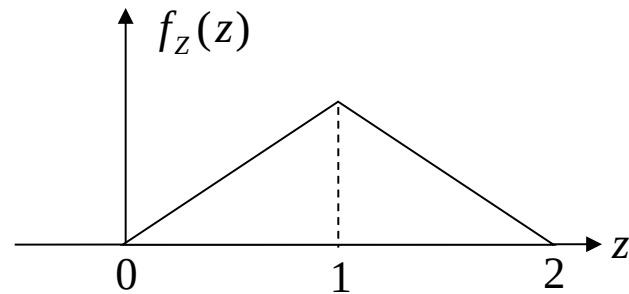
Figure (c) shows $f_z(z)$ which agrees with the convolution of two rectangular waveforms as well.



(a) $0 \leq z < 1$



(b) $1 \leq z < 2$



(c)

Example-4: Let $Z = X - Y$. Determine its pdf $f_Z(z)$.

Solution:

$$F_Z(z) = P(X - Y \leq z) = \int_{y=-\infty}^{+\infty} \int_{x=-\infty}^{z+y} f_{XY}(x, y) dx dy$$

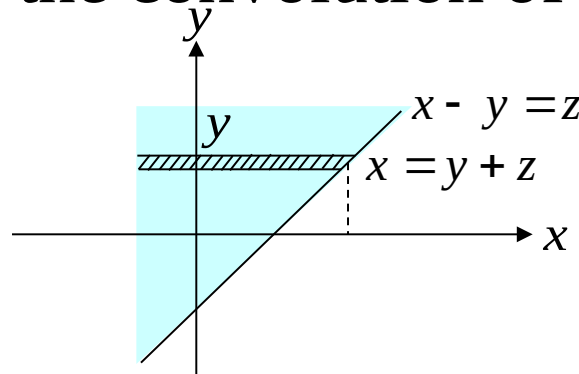
and hence

$$f_Z(z) = \frac{dF_Z(z)}{dz} = \int_{y=-\infty}^{+\infty} \left(\frac{\partial}{\partial z} \int_{x=-\infty}^{z+y} f_{XY}(x, y) dx \right) dy = \int_{-\infty}^{+\infty} f_{XY}(y + z, y) dy.$$

If X and Y are independent, then the above formula reduces to

$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(z + y) f_Y(y) dy = f_X(-z) \otimes f_Y(y),$$

which represents the convolution of $f_X(-z)$ with $f_Y(z)$.



As a special case, suppose

$$f_X(x) = 0, \quad x < 0, \quad \text{and} \quad f_Y(y) = 0, \quad y < 0.$$

In this case, Z can be negative as well as positive, and that gives rise to two situations that should be analyzed separately, since the region of integration for $z \geq 0$ and $z < 0$ are quite different. For $z \geq 0$, from Figure (a)

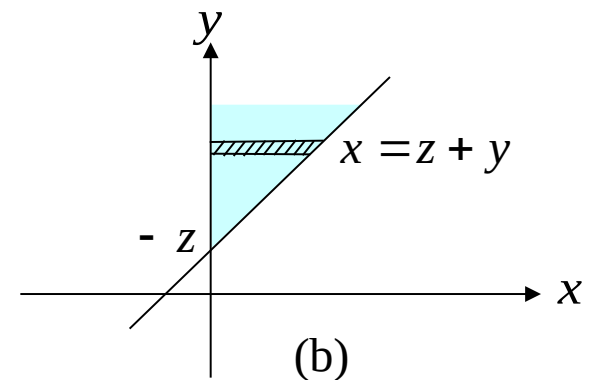
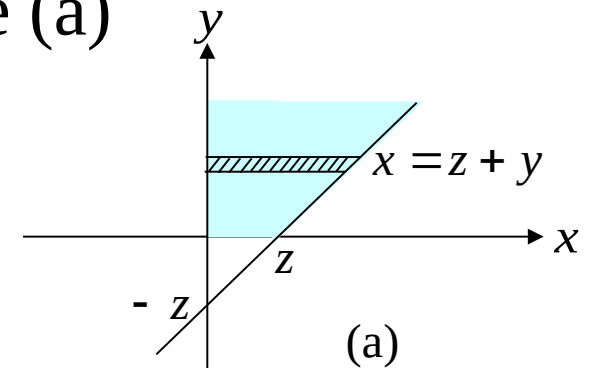
$$F_Z(z) = \int_{y=0}^{+\infty} \int_{x=0}^{z+y} f_{XY}(x, y) dx dy$$

and for $z < 0$, from Figure (b)

$$F_Z(z) = \int_{y=-z}^{+\infty} \int_{x=0}^{z+y} f_{XY}(x, y) dx dy$$

After differentiation, this gives

$$f_Z(z) = \begin{cases} \int_0^{+\infty} f_{XY}(z+y, y) dy, & z \geq 0, \\ \int_{-z}^{+\infty} f_{XY}(z+y, y) dy, & z < 0. \end{cases}$$

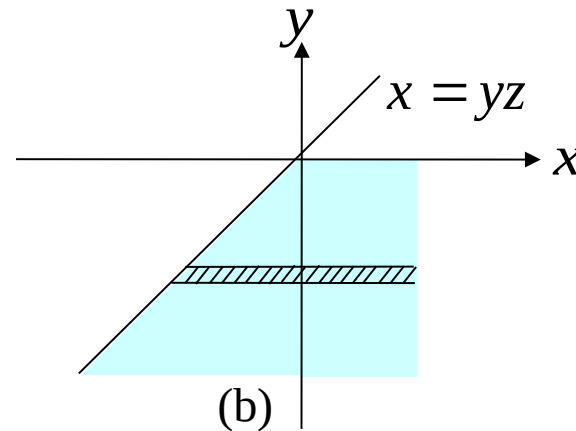
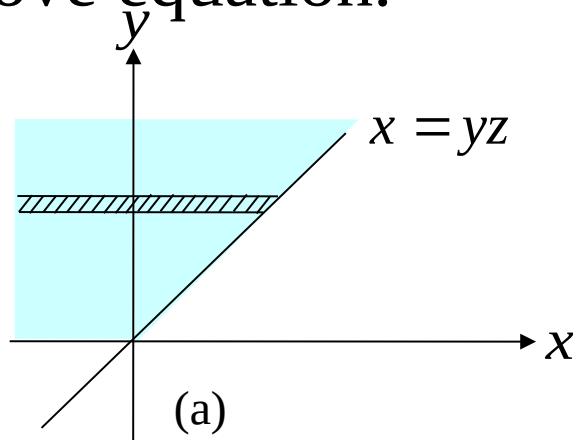


Example-5: Given $Z = X / Y$, obtain its density function.

Solution: We have
 $F_Z(z) = P(X / Y \leq z)$.
 The inequality $X / Y \leq z$ can be rewritten as $X \leq Yz$ if $Y > 0$,
 if $X \geq Yz$ if $Y < 0$.

$$\begin{aligned} P(X / Y \leq z) &= P(X / Y \leq z, Y > 0) + P(X / Y \leq z, Y < 0) \\ &= P(X \leq Yz, Y > 0) + P(X \geq Yz, Y < 0). \end{aligned}$$

Figure (a) shows the area corresponding to the first term, and Figure (b) shows that corresponding to the second term in the above equation.



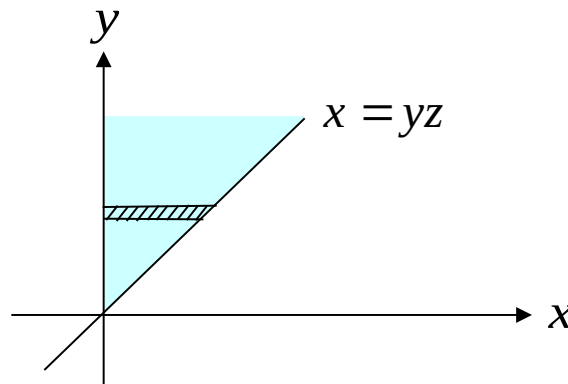
Integrating over these two regions, we get

$$F_Z(z) = \int_{y=0}^{+\infty} \int_{x=-\infty}^{yz} f_{XY}(x, y) dx dy + \int_{y=-\infty}^0 \int_{x=yz}^{+\infty} f_{XY}(x, y) dx dy.$$

Differentiation with respect to z gives

$$\begin{aligned} f_Z(z) &= \int_0^{+\infty} y f_{XY}(yz, y) dy + \int_{-\infty}^0 (-y) f_{XY}(yz, y) dy \\ &= \int_{-\infty}^{+\infty} |y| f_{XY}(yz, y) dy, \quad -\infty < z < +\infty. \end{aligned}$$

Note that if X and Y are nonnegative random variables, then the area of integration reduces to that shown below



Exercise

The joint pdf of two random variables X and Y is given by:

$$f_{XY}(x, y) = \begin{cases} x + y, & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the pdf of Z if :

a. $Z = X + Y$

c. $Z = XY$

b. $Z = \frac{X}{Y}$

d. $Z = X - Y$

Assignment-III

1. Suppose that two continuous random variables X and Y have joint pdf given by:

$$f_{XY}(x, y) = \begin{cases} k(2x + y), & 2 < x < 6, 0 < y < 5 \\ 0, & \text{otherwise} \end{cases}$$

where k is a constant.

- a. Find the constant k .
- b. Find the joint cdf of X and Y .
- c. Find the marginal cdf and pdf of X and Y .
- d. Are X and Y independent?
- e. Find the conditional pdf of X and Y .
- e. Evaluate the following probabilities.

i. $P(3 < X < 4, Y > 2)$

iii. $P(X + Y > 4)$

ii. $P(X > 3)$

iv. $P(0 < Y < 2)$

Assignment-III Cont'd.....

2. Let the joint pmf of two discrete random variables X and Y is given by:

$$P_{XY}(x_i, y_j) = \begin{cases} k(x_i + y_j), & x_i = 1, 2, 3; y = 1, 2 \\ 0, & \text{otherwise} \end{cases}$$

where k is a constant.

- Find the value of k .
- Find the marginal pmf of X and Y .
- Are X and Y independent?

Assignment-III Cont'd.....

3. Suppose that two continuous random variables X and Y are independent and uniform in the interval $(0, 4)$.

Find the pdf of Z if :

a. $Z = X + Y$

b. $Z = Y - X$

c. $Z = X - Y$

d. $Z = XY$

e. $Z = \frac{Y}{X}$