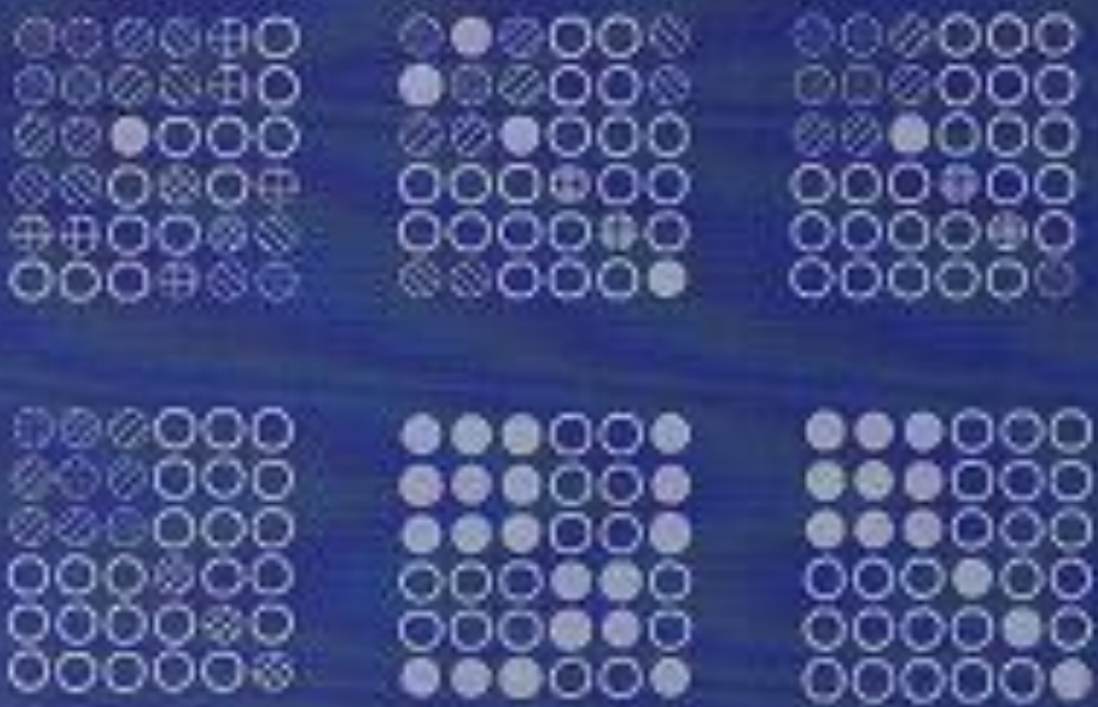
APPLIED ELASTICITY

Matrix and Tensor Analysis of Elastic Continua

2nd Edition

John D. Renton



APPLIED ELASTICITY, 2nd Edition
Matrix and Tensor Analysis of Elastic Continua

Talking of education, "People have now a-days" (said he) "got a strange opinion that every thing should be taught by lectures. Now, I cannot see that lectures can do so much good as reading the books from which the lectures are taken. I know nothing that can be best taught by lectures, expect where experiments are to be shewn. You may teach chymestry by lectures. — You might teach making of shoes by lectures!"

James Boswell: *Life of Samuel Johnson, 1766* [1709-1784]

ABOUT THE AUTHOR

In 1947 John D Renton was admitted to a Reserved Place (entitling him to free tuition) at King Edward's School in Edgbaston, Birmingham which was then a Grammar School. After six years there, followed by two doing National Service in the RAF, he became an undergraduate in Civil Engineering at Birmingham University, and obtained First Class Honours in 1958. He then became a research student of Dr A H Chilver (now Lord Chilver) working on the stability of space frames at Fitzwilliam House, Cambridge. Part of the research involved writing the first computer program for analysing three-dimensional structures, which was used by the consultants Ove Arup in their design project for the roof of the Sydney Opera House. He won a Research Fellowship at St John's College Cambridge in 1961, from where he moved to Oxford University to take up a teaching post at the Department of Engineering Science in 1963. This was followed by a Tutorial Fellowship to St Catherine's College in 1966.

Two main strands of research have been the behaviour of regular structures (such as trusses and plates) and the stability of continua. The former led to a general beam theory, equally applicable to continuous beams and trusses (see *Elastic Beams and Frames*, 2nd Ed. 2002 Horwood Publishing). The stability of continua, being the only way to establish the correct governing equations in terms of tensor calculus, gave rise to the present book (see Chapter 4). Both books contain much of the work in the author's published papers.

The author's other interests include judo (he was in both the Birmingham and Cambridge University teams) and photography (he does his own chemical colour printing). He thought it possible that special photographic techniques might rescue the Pre-Raphaelite murals illustrating King Arthur and the Knights of the Round Table which were in the Oxford Union from total obscurity, and with the aid of the head photographer at the Physics laboratory, Cyril Band, this was done. A booklet *The Oxford Union Murals* the author wrote on them in 1976 is now in its fourth edition, the murals having been fully restored and illuminated for the many visitors who now come to see them.

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Preface

This book was resulted from a need to solve certain three-dimensional problems in an organised manner. for those familiar with matrix algebra, much can be done without the explicit use of tensors. Chapter 1 was written with such readers in mind, so that only the briefest summary of matrix methods is given. Those unfamiliar with these techniques might well start at Chapter 3.

Elastic analysis using Cartesian tensor notation follows quite naturally from matrix notation and so forms the subject of Chapter 2. The concepts of thermoelasticity are examined more readily at this level. For most purposes, the slight differences between adiabatic and isothermal behaviour can be ignored, but temperature changes under adiabatic conditions can now be used to give thermal maps of stress fields. Elsewhere in the book, the material is understood to be hyperelastic. That is, the state of stress is given by the rate of potential with strain, as in Green's formula (2.52).

Matrix algebra is inadequate for analysing problems related to curvilinear coordinates. They are best solved using the curvilinear tensors discussed in Chapter 3. Care has been taken to give a physical and geometrical grounding to the quantities used. For example, all too often the Christoffel symbol is defined by a formula.

Large deformation theory is left until the last chapter. Here, the reference state is expressed in terms of the undeformed geometry. This is because the undeformed configuration of the body is known and the deformed state is sought. A consequence of this approach is that lower case letters refer to the undeformed geometry and upper case letters to the deformed geometry, which is the opposite of the notation often used elsewhere. Also, attention has tended to focus on the exact analysis of mathematically defined materials. However, these only approximate to the behaviour of real materials. Examples of such exact analyses are given in section 4.5. An alternative approach is to start from a description based on the known properties of real materials, accepting that this is likely to be incomplete. For many purposes such a description may well suffice and in section 4.6 it is used to derive a small deflection theory of the stability of elastic continua.

Other topics which do not fit readily into the main flow of the book will be found in

1

Matrix methods

This chapter starts with a brief résumé of the properties of matrices, a topic which will already be familiar to most readers. Those requiring a fuller exposition may consult Barnett (1979), Bell (1975) or Graham (1980) for example. Some readers will also have encountered the matrix representation of a cross product, and possibly that of differential vector operators. An introduction to subscript notation is also outlined; this notation becomes essential in later chapters.

By using matrix methods, a number of equations can be encapsulated in a single matrix equation and readily transformed from one Cartesian coordinate system to another. These useful properties are also intrinsic in the tensor notation used in later chapters. For some purposes, anisotropic behaviour can be examined more easily in terms of matrix notation than in tensor notation†. This is because the symmetry of the components of stress and strain can be used implicitly in writing the equations, as in section 1.8.

1.1 SUMMARY OF MATRIX PROPERTIES

An $m \times n$ matrix $\mathbf{A}$ is an array of m rows and n columns of elements a_{ij}, where the subscripts i and j indicate that the element is in the ith row and jth column of the matrix. For example, if $\mathbf{A}$ is a 2×3 matrix, it is given by

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{23} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \tag{1.1}$$

A matrix with only one row is called a **row vector**, and a matrix with only one column is

†Even some mathematicians who should know better seem unaware of the distinction. Tensors are like vectors and their elements have transformation properties which do not necessarily apply to matrices.

called a **column vector.** Two matrices **A** and **B** can be added and subtracted if m and n is the same both of them. Then if the elements of **B** are b_{ij}, the **sum** of the matrices **A** and **B** is the array of elements $a_{ij}+b_{ij}$ and the **difference** is the array of elements $a_{ij}-b_{ij}$. For 2×3 matrices, these take the forms

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{bmatrix} = \mathbf{B} + \mathbf{A} \tag{1.2}$$

$$\mathbf{A} - \mathbf{B} = \begin{bmatrix} a_{11}-b_{11} & a_{12}-b_{12} & a_{13}-b_{13} \\ a_{21}-b_{21} & a_{22}-b_{22} & a_{23}-b_{23} \end{bmatrix} = -(\mathbf{B} - \mathbf{A}) \tag{1.3}$$

A matrix can be multiplied by a **scalar**, λ say, (a simple quantity possessing magnitude only). This has the effect of multiplying each element by λ. Thus

$$\lambda\mathbf{A} = \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \lambda a_{13} \\ \lambda a_{21} & \lambda a_{22} & \lambda a_{23} \end{bmatrix} \tag{1.4}$$

A **matrix product CD** of two matrices **C** and **D** can be formed if the number of columns of **C** is the same as the number of rows of **D**. If **C** is an $m\times n$ matrix and **D** is an $n\times p$ matrix, then the product is an $m\times p$ matrix. A typical element of the ith row and jth column of the product is given by the sum of the products $c_{ir}d_{rj}$ for all values of r from 1 to n. For example

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}\begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \end{bmatrix} = \begin{bmatrix} c_{11}d_{11}+c_{12}d_{21} & c_{11}d_{12}+c_{12}d_{22} & c_{11}d_{13}+c_{12}d_{23} \\ c_{21}d_{11}+c_{22}d_{21} & c_{21}d_{12}+c_{22}d_{22} & c_{21}d_{13}+c_{22}d_{23} \end{bmatrix} \tag{1.5}$$

The rows of the **transpose** of a matrix are formed from the columns of the original matrix, and conversely the rows of the original matrix become the columns of the transpose. The transpose of a matrix will be denoted by the superscript $^{\mathrm{T}}$. Thus the transpose of the matrix **A** in (1.1) is

$$\mathbf{A}^{\mathrm{T}} = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix} \tag{1.6}$$

The transpose of a product of matrices is equal to the product of their transposes in the reverse order. For example

$$(\mathbf{ABC})^{\mathrm{T}} = \mathbf{C}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}} \tag{1.7}$$

A **square matrix** has an equal number of rows and columns. Such a matrix, **M** say, with elements m_{ij}, is said to be **symmetric** if it is equal to its transpose ($m_{ij} = m_{ji}$) and **skew-symmetric** if it is equal to minus its transpose ($m_{ij} = -m_{ji}$). The elements m_{jj} of a square matrix lie on the diagonal across it from top left to bottom right. This is known as the **leading diagonal**. The sum of the terms on the leading diagonal is known as the **trace** of the matrix. The trace of matrix **M** is denoted by tr(**M**). A **unit matrix**, usually denoted by **I**, is one which has unit elements on its leading diagonal and all the other elements are zero. Multiplying any matrix **A** by a (compatible) unit matrix leaves **A** unchanged.

The solution of an equation of the form

$$\mathbf{M}\mathbf{v} = \lambda\mathbf{v} \tag{1.8}$$

for a given $n \times n$ matrix **M** is known as an **eigenvalue problem**, where the column vector **v** and the scalar λ are to be found. There may be n linearly-independent column vectors, $\mathbf{v}_p$, known as **eigenvectors**, which satisfy the equation for particular corresponding values λ_p of the scalar, known as **eigenvalues**. If two eigenvalues, λ_p and λ_q, are distinct and **M** is a symmetric matrix, the orthogonality conditions

$$\mathbf{v}_p^{\mathrm{T}}\mathbf{M}\mathbf{v}_q = \mathbf{v}_p^{\mathrm{T}}\mathbf{v}_q = 0 \qquad (p \neq q) \tag{1.9}$$

are satisfied. The general form of $\mathbf{v}^{\mathrm{T}}\mathbf{M}\,\mathbf{v}$, where **v** is a column vector, is known as a **quadratic form**. It is said to be **positive definite** if it is positive (or zero) for any real column vector **v**. This is true if, and only if, all the principal minors of **M** are positive. A **principal minor** of **M** is the determinant of the first k rows and columns of **M** where k is an integer between 1 and n.

The **determinant**†, det(**M**), of an $n \times n$ square matrix **M** is given by

$$\det(\mathbf{M}) = \sum_{i=1}^{n}\sum_{j=1}^{n}\cdots\sum_{k=1}^{n} \epsilon_{ij\ldots k} m_{1i} m_{2j}\cdots m_{nk} \tag{1.10}$$

where $\epsilon_{ij\ldots k}$ is +1 if $ij\ldots k$ is an even permutation of 1 2 . . . n, -1 if $ij\ldots k$ is an odd permutation of 1 2 . . . n, and is zero in all other cases. A **permutation** of 1 2 . . . n is **even** if it produced by an even number of interchanges of adjacent numbers and is **odd** if it is produced by an odd number of interchanges.

Each term in the determinant contains just one element from any given row (or column). This means that the determinant can be expressed as a sum of the elements, m_{rs} in row r, each multiplied by a coefficient, M_{rs}, formed from a set of products of elements of the other rows, that is,

$$\det(\mathbf{M}) = \sum_{s=1}^{n} m_{rs} M_{rs} \tag{1.11}$$

The coefficients M_{rs} are known as **cofactors**. As each term in (1.10) also contains just one

†Alternative definitions will suffice for the purposes of this chapter.

element from any given column, a similar expression can be written in terms of a linear combination of column elements.

A square matrix **M** has a **reciprocal** $\mathbf{M}^{-1}$ provided that its determinant is non-zero. The reciprocal satisfies the equation

$$\mathbf{M}\,\mathbf{M}^{-1} = \mathbf{M}^{-1}\mathbf{M} = \mathbf{I} \tag{1.12}$$

The matrix $\mathbf{M}^{-1}$ is given by

$$\mathbf{M}^{-1} = \mathrm{adj}(\mathbf{M})/\det(\mathbf{M}) \tag{1.13}$$

where adj(**M**) is the **adjugate** of **M** and is given by replacing each element m_{ij} of **M** with its cofactor M_{ij}, and taking the transpose of the resulting matrix.

The equation

$$\mathbf{M}\mathbf{v} = \mathbf{0} \tag{1.14}$$

only has a solution for non-zero **v** if the determinant of **M** is zero. The determinant of a product of square matrices is equal to the product of their determinants.

1.1.1 Subscript notation

A matrix, **A**, can be represented by a typical element, a_{ij}, where the first subscript gives the row and the second subscript the column of the element. In this notation, addition and subtraction of matrices, and multiplication by a scalar, are denoted by typical elements of the resulting matrices, $a_{ij}+b_{ij}$, $a_{ij}-b_{ij}$, and λa_{ij}. A single repetition of a (Latin) subscript within a term is taken to imply summation over all values of that subscript. This is known as the **Einstein summation convention**. Thus the trace of a square matrix **M** is given simply by m_{jj}. It could equally well have been written as m_{kk}. The particular letter used for the repeated subscript has no significance as it does not take a specific value but implies summation over all the possible values of that subscript. For this reason, it is known as a **dummy subscript**. A matrix product, such as that given by (1.5), is expressed in this notation as $c_{ik}\,d_{kj}$. A unit matrix is expressed by δ_{ij} which is known as the **Kronecker delta**. It is equal to one if its subscripts are the same and equal to zero if they are not. Certain results can readily be established using this notation. Suppose that u_i, v_j and w_j are typical elements of column vectors **u**, **v** and **w**, the subscripts indicating the row of the element. Then

$$(\mathbf{u}\mathbf{v}^{\mathrm{T}})\mathbf{w} \sim (u_i v_j)w_j = v_j w_j u_i \sim (\mathbf{v}^{\mathrm{T}}\mathbf{w})\mathbf{u}\ ,\ (\mathbf{w}^{\mathrm{T}}\mathbf{v})\mathbf{u} \text{ or } (\mathbf{u}\mathbf{w}^{\mathrm{T}})\mathbf{v}$$

so that

$$(\mathbf{u}\mathbf{v}^{\mathrm{T}})\mathbf{w} = (\mathbf{v}^{\mathrm{T}}\mathbf{w})\mathbf{u} = (\mathbf{w}^{\mathrm{T}}\mathbf{v})\mathbf{u} = (\mathbf{u}\mathbf{w}^{\mathrm{T}})\mathbf{v} \tag{1.15}$$

where the symbol ~ indicates correspondence between the usual matrix notation and the subscript notation for a typical element. Such relationships are particularly useful when the column vectors in turn represent vectors.

1.2 VECTOR REPRESENTATION

A vector **v** may be represented in a right-handed system of Cartesian coordinates (x, y, z)

by

$$\boldsymbol{v} = v_x\boldsymbol{i} + v_y\boldsymbol{j} + v_z\boldsymbol{k} \tag{1.16}$$

where $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ are unit vectors in the x, y and z directions respectively, as shown in Fig. 1.1, and v_x, v_y and v_z are the components of $\boldsymbol{v}$ in these directions. A vector $\boldsymbol{v}$ can be expressed in matrix form by the column vector

$$\mathbf{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \tag{1.17}$$

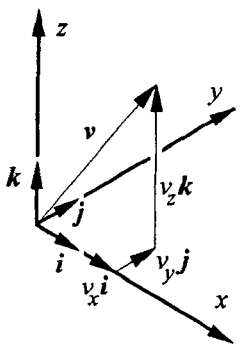


Figure 1.1 Vector components.

the position of the term in the column vector determining whether it should be multiplied by $\boldsymbol{i}, \boldsymbol{j}$ or $\boldsymbol{k}$. As above, a vector will be denoted in bold italic and its matrix equivalent in bold sanserif.

1.2.1 Scalar products

The scalar product of two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ can be written in vector form as

$$\boldsymbol{v}\cdot\boldsymbol{w} = v_x w_x + v_y w_y + v_z w_z = |\boldsymbol{v}||\boldsymbol{w}|\cos\theta \quad (= \boldsymbol{w}\cdot\boldsymbol{v}) \tag{1.18}$$

where $|\boldsymbol{v}|$ and $|\boldsymbol{w}|$ are the magnitudes of $\boldsymbol{v}$ and $\boldsymbol{w}$, and θ is the angle between the vectors. This is given in matrix form by

$$\boldsymbol{v}\cdot\boldsymbol{w} \equiv \mathbf{v}^{\mathrm{T}}\mathbf{w} = [v_x\ v_y\ v_z]\begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix} \quad (= \mathbf{w}^{\mathrm{T}}\mathbf{v}) \tag{1.19}$$

1.2.2 Vector products

The vector product $\boldsymbol{v} \times \boldsymbol{w}$ is given by

$$\boldsymbol{u} = \boldsymbol{v}\times\boldsymbol{w} = (v_y w_z - v_z w_y)\boldsymbol{i} + (v_z w_x - v_x w_z)\boldsymbol{j} + (v_x w_y - v_y w_x)\boldsymbol{k} = -\boldsymbol{w}\times\boldsymbol{v} \tag{1.20}$$

where

$$|\boldsymbol{u}| = |\boldsymbol{v}||\boldsymbol{w}|\sin\theta \tag{1.21}$$

and $\boldsymbol{u}$ is normal to the plane containing $\boldsymbol{v}$ and $\boldsymbol{w}$, and is such that the sense of the rotation θ from $\boldsymbol{v}$ to $\boldsymbol{w}$ is clockwise as viewed in the direction of $\boldsymbol{u}$. In matrix form this can be represented by

$$\mathbf{u} = [\mathbf{v}]\mathbf{w} = -[\mathbf{w}]\mathbf{v} \tag{1.22}$$

where $[\mathbf{v}]$ is the skew-symmetric matrix

$$[\mathbf{v}] = \begin{bmatrix} 0 & -v_z & v_y \\ v_z & 0 & -v_x \\ -v_y & v_x & 0 \end{bmatrix} \tag{1.23}$$

and [**w**] is similarly defined.

A result of particular importance is that the vector product of a vector with itself is a zero vector. In matrix form, this is given by

$$[\mathbf{v}]\mathbf{v} = \begin{bmatrix} 0 & -v_z & v_y \\ v_z & 0 & -v_x \\ -v_y & v_x & 0 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{1.24}$$

A useful property of the matrix representation of a cross product is

$$[\mathbf{v}][\mathbf{v}] = \mathbf{v}\mathbf{v}^{\mathrm{T}} - (\mathbf{v}^{\mathrm{T}}\mathbf{v})\mathbf{I} \tag{1.25}$$

where **I** is the unit matrix given by

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.26}$$

Example 1.1

Suppose that a force $\boldsymbol{F}$ acts on a particle at a point P given by the position vector $\boldsymbol{r}$ in Fig. 1.2. If the particle moves to a point P′, adjacent to P, given by the relative position vector d$\boldsymbol{r}$, the work done, dW, is

$$\mathrm{d}W = \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r} = \mathbf{F}^{\mathrm{T}}\mathrm{d}\mathbf{r} \tag{1.27}$$

where **F** and d**r** are the column vector representations of $\boldsymbol{F}$ and d$\boldsymbol{r}$. The work done is expressed in terms of its magnitude alone, so that it is unrelated to any coordinate system. All such quantities are known as scalars, and all scalar products are scalars. Again, the square of the distance OP is a scalar and is given by

$$|\boldsymbol{r}|^2 = \boldsymbol{r}\cdot\boldsymbol{r} = \mathbf{r}^{\mathrm{T}}\mathbf{r} \tag{1.28}$$

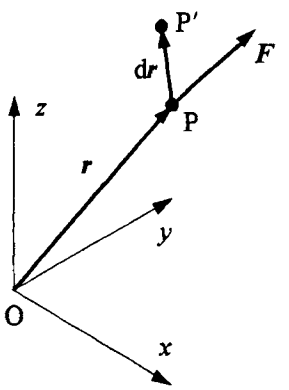


Figure 1.2 Vector mechanics.

Vector products are always vectors. for example, the moment $\boldsymbol{M}$ produced about O by $\boldsymbol{F}$ can be written as

$$M = r \times F \tag{1.29}$$

where

$$M = M_x i + M_y j + M_z k \tag{1.30}$$

M_x, M_y and M_z being the clockwise moments about the x, y and z axes respectively. This can be written in matrix form as

$$\mathbf{M} = [\mathbf{r}]\mathbf{F} \tag{1.31}$$

1.3 COORDINATE TRANSFORMATION

The elements in $\mathbf{v}$, the column representation of the vector v, are related to the coordinate system used, (x, y, z). If $\bar{\mathbf{v}}$ is the representation of v with respect to another Cartesian coordinate system, ($\bar{x}, \bar{y}, \bar{z}$), as shown in Fig. 1.3, there is a linear relationship between $\mathbf{v}$ and $\bar{\mathbf{v}}$ given by

$$\bar{\mathbf{v}} = \mathbf{T}\mathbf{v} \tag{1.32}$$

where $\mathbf{T}$ is known as a **transformation matrix**. For example, if the new coordinate system is generated by rotation the old one through an angle θ about its z axis, as shown in Fig. 1.4, then (1.32) takes the form

$$\begin{bmatrix} \bar{v}_x \\ \bar{v}_y \\ \bar{v}_z \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \tag{1.33}$$

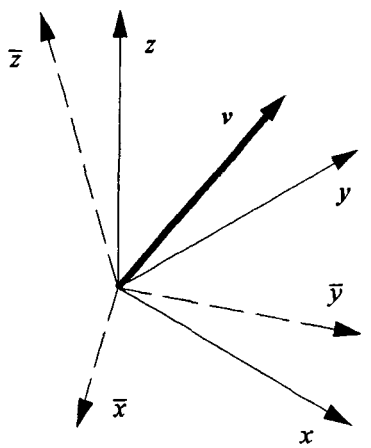


Figure 1.3 Two Cartesian coordinate systems.

1.3.1 Orthogonality of the transformation matrix

Because the scalar product is a scalar quantity and so is the same in all coordinate systems,

$$v \cdot w = \mathbf{v}^T\mathbf{w} = \bar{\mathbf{v}}^T\bar{\mathbf{w}} = \mathbf{v}^T\mathbf{T}^T\mathbf{T}\mathbf{w} \tag{1.34}$$

from (1.19) and (1.32), so that

$$\mathbf{v}^T(\mathbf{T}^T\mathbf{T} - \mathbf{I})\mathbf{w} = 0 \tag{1.35}$$

where $\mathbf{I}$ is the unit matrix given by (1.26). For this to be true for any v and w,

$$\mathbf{T}^T\mathbf{T} = \mathbf{I} \tag{1.36}$$

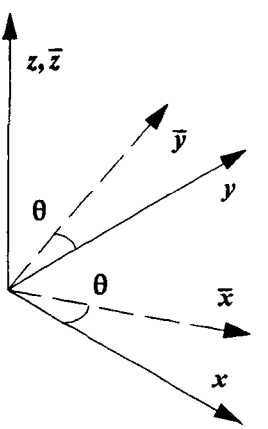


Figure 1.4 Rotated coordinates.

so that the inverse of a transformation matrix is equal to its transpose which means that it is **orthogonal**. The inverse of (1.32) can now be written as

$$\mathbf{v} = \mathbf{T}^T\bar{\mathbf{v}} \tag{1.37}$$

For the case given by (1.33), the inverse relationship is given by replacing θ by $-\theta$ in the matrix. It can be seen that this the same as taking the transpose.

1.3.2 Transformation of a matrix

Two column representations of vectors, **v** and **w**, may be related by a matrix **A**, where

$$\mathbf{v} = \mathbf{A}\mathbf{w} \tag{1.38}$$

as in (1.22) and (1.31) for example. In a new coordinate system, this becomes

$$\bar{\mathbf{v}} = \bar{\mathbf{A}}\bar{\mathbf{w}} \tag{1.39}$$

Then from the above and (1.32) and (1.37),

$$\bar{\mathbf{A}}\bar{\mathbf{w}} = \bar{\mathbf{v}} = \mathbf{T}\mathbf{v} = \mathbf{T}\mathbf{A}\mathbf{w} = (\mathbf{T}\mathbf{A}\mathbf{T}^{\mathrm{T}})\bar{\mathbf{w}} \tag{1.40}$$

Because this is true for all possible $\bar{\mathbf{v}}$ and $\bar{\mathbf{w}}$,

$$\bar{\mathbf{A}} = \mathbf{T}\mathbf{A}\mathbf{T}^{\mathrm{T}} \tag{1.41}$$

which is the transformation law for any matrix relating the column representations of two vectors and the inverse transformation is

$$\mathbf{A} = \mathbf{T}^{\mathrm{T}}\bar{\mathbf{A}}\mathbf{T} \tag{1.42}$$

Thus, for example, the transformation of [**r**] in (1.31) is

$$[\bar{\mathbf{r}}] = \mathbf{T}[\mathbf{r}]\mathbf{T}^{\mathrm{T}} \tag{1.43}$$

If **A** is a symmetric matrix, then so is $\bar{\mathbf{A}}$, for

$$\bar{\mathbf{A}}^{\mathrm{T}} = (\mathbf{T}\mathbf{A}\mathbf{T}^{\mathrm{T}})^{\mathrm{T}} = \mathbf{T}\mathbf{A}^{\mathrm{T}}\mathbf{T}^{\mathrm{T}} = \mathbf{T}\mathbf{A}\mathbf{T}^{\mathrm{T}} = \bar{\mathbf{A}} \tag{1.44}$$

and likewise, a skew-symmetric matrix, such as [**r**], remains skew symmetric.

1.4 DIFFERENTIAL OPERATORS

The operator ∇ is given by

$$\nabla = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z} \tag{1.45}$$

and can be used to express gradient, divergence and curl in vector form. The column vector representation of this is given by

$$\vec{\nabla} \text{ or } \overleftarrow{\nabla} = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \tag{1.46}$$

The arrows point to the term to be differentiated. Thus differentiation of the succeeding term is indicated if the arrow points to the right and differentiation of the preceding term is indicated if the arrow points to the left.

1.4.1 Gradient, divergence and curl

If scalars or vectors at all points in a zone of space can be expressed by scalar or vector functions of position, then a **scalar** or **vector field** is said to exist. Potential energy in a gravitational field is an example of a scalar field, and the (vector) displacements of points in a body is an example of a vector field.

The **gradient** of a scalar function ϕ is denoted by

$$\nabla\phi = \boldsymbol{i}\frac{\partial\phi}{\partial x} + \boldsymbol{j}\frac{\partial\phi}{\partial y} + \boldsymbol{k}\frac{\partial\phi}{\partial z} \tag{1.47}$$

and is represented in column vector form by $\vec{\nabla}\phi$.

The **divergence** of a vector function $\boldsymbol{v}$ is given by the scalar product

$$\nabla\cdot\boldsymbol{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \tag{1.48}$$

and also by the matrix product $\vec{\nabla}^{\mathrm{T}}\mathbf{v}$.

The **curl** of a vector $\boldsymbol{v}$ is given by

$$\nabla\times\boldsymbol{v} = \boldsymbol{i}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \boldsymbol{j}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \boldsymbol{k}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) \tag{1.49}$$

and is represented in column vector form by $[\vec{\nabla}]\mathbf{v}$.

1.4.2 Derived results

The following results can be found from the above definitions and the expressions for vector operations in section 1.2.

$$\text{Div curl } \boldsymbol{v} \quad = \nabla\cdot(\nabla\times\boldsymbol{v}) = \vec{\nabla}^{\mathrm{T}}[\vec{\nabla}]\mathbf{v} = (-[\vec{\nabla}]\vec{\nabla})^{\mathrm{T}}\mathbf{v} = 0$$

$$\text{Curl grad } \phi \quad = \nabla\times(\nabla\phi) \sim [\vec{\nabla}]\vec{\nabla}\phi = \mathbf{0}$$

$$\text{Curl curl } \boldsymbol{v} \quad = \nabla\times(\nabla\times\boldsymbol{v}) \sim [\vec{\nabla}][\vec{\nabla}]\mathbf{v} = (\vec{\nabla}\vec{\nabla}^{\mathrm{T}} - \vec{\nabla}^{\mathrm{T}}\vec{\nabla})\mathbf{I}\mathbf{v}$$

$$= \vec{\nabla}(\vec{\nabla}^{\mathrm{T}}\mathbf{v}) - (\vec{\nabla}^{\mathrm{T}}\vec{\nabla})\mathbf{v} \sim \nabla(\nabla\cdot\boldsymbol{v}) - (\nabla\cdot\nabla)\boldsymbol{v}$$

see (1.25). As before, the symbol ~ is used to show correspondence.

1.5 THE STRAIN MATRIX

Let P and P′ be adjacent points in the solid body shown in Fig. 1.5. Their relative position is given by

$$\mathrm{d}\boldsymbol{r} = \boldsymbol{i}\mathrm{d}x + \boldsymbol{j}\mathrm{d}y + \boldsymbol{k}\mathrm{d}z \tag{1.50}$$

After some deflection of the body, the displacements of P and P′ are given by $\boldsymbol{u}$ and $\boldsymbol{u}+\mathrm{d}\boldsymbol{u}$ respectively, where the increment in the displacement is given by

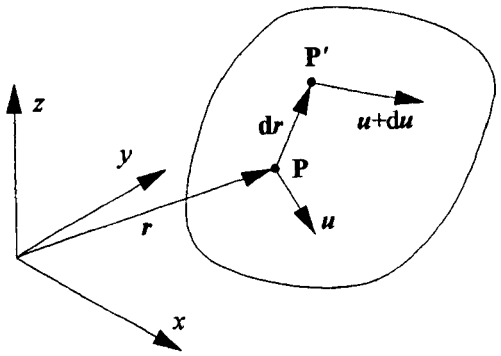


Figure 1.5 Displacement field within a body.

$$\mathbf{d}\boldsymbol{u} = i\,\mathrm{d}u_x + j\,\mathrm{d}u_y + k\,\mathrm{d}u_z \tag{1.51}$$

and

$$\begin{aligned}
\mathrm{d}u_x &= \frac{\partial u_x}{\partial x}\mathrm{d}x + \frac{\partial u_x}{\partial y}\mathrm{d}y + \frac{\partial u_x}{\partial z}\mathrm{d}z \\
\mathrm{d}u_y &= \frac{\partial u_y}{\partial x}\mathrm{d}x + \frac{\partial u_y}{\partial y}\mathrm{d}y + \frac{\partial u_y}{\partial z}\mathrm{d}z \\
\mathrm{d}u_z &= \frac{\partial u_z}{\partial x}\mathrm{d}x + \frac{\partial u_z}{\partial y}\mathrm{d}y + \frac{\partial u_z}{\partial z}\mathrm{d}z
\end{aligned} \tag{1.52}$$

or

$$\mathbf{du} = \mathbf{u}\overleftarrow{\nabla}^{\mathrm{T}}\mathbf{dr} = \boldsymbol{\varepsilon}\,\mathbf{dr} \tag{1.53}$$

where

$$\mathbf{u} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} \;;\; \mathbf{du} = \begin{bmatrix} \mathrm{d}u_x \\ \mathrm{d}u_y \\ \mathrm{d}u_z \end{bmatrix} \;;\; \boldsymbol{\varepsilon} = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} \\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial z} \end{bmatrix} \;;\; \mathbf{dr} = \begin{bmatrix} \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{bmatrix}. \tag{1.54}$$

The general form of $\boldsymbol{\varepsilon}$ is known as the **Jacobian matrix**.

The column vectors **du** and **dr** represent the vectors d$\boldsymbol{u}$ and d$\boldsymbol{r}$, and so with respect to a new coordinate system ($\bar{x}, \bar{y}, \bar{z}$),

$$\mathbf{d\bar{u}} = \mathbf{T}\,\mathbf{du} \;;\; \mathbf{dr} = \mathbf{T}^{\mathrm{T}}\mathbf{d\bar{r}} \tag{1.55}$$

in accordance with (1.32) and (1.37). In the new coordinate system, the relationship equivalent to (1.53) is

$$\mathbf{d\bar{u}} = \bar{\boldsymbol{\varepsilon}}\,\mathbf{d\bar{r}} = \mathbf{T}(\boldsymbol{\varepsilon}\mathbf{T}^{\mathrm{T}}\mathbf{d\bar{r}}) = (\mathbf{T}\boldsymbol{\varepsilon}\mathbf{T}^{\mathrm{T}})\mathbf{d\bar{r}}$$

so that

$$\bar{\boldsymbol{\varepsilon}} = \mathbf{T}\boldsymbol{\varepsilon}\mathbf{T}^{\mathrm{T}} \tag{1.56}$$

in accordance with (1.41). Taking the transpose of this,

$$\bar{\boldsymbol{\varepsilon}}^{\mathrm{T}} = \mathbf{T}\boldsymbol{\varepsilon}^{\mathrm{T}}\mathbf{T}^{\mathrm{T}} \tag{1.57}$$

The **strain matrix** $\mathbf{e}$ and the **rotation matrix** $\boldsymbol{\omega}$ are defined as

$$\mathbf{e} = \tfrac{1}{2}(\boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}^{\mathrm{T}}) \quad ; \quad \boldsymbol{\omega} = \tfrac{1}{2}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{T}}) \tag{1.58}$$

for small deflections. In the new coordinate system,

$$\bar{\mathbf{e}} = \tfrac{1}{2}(\bar{\boldsymbol{\varepsilon}} + \bar{\boldsymbol{\varepsilon}}^{\mathrm{T}}) = \tfrac{1}{2}\mathbf{T}(\boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}^{\mathrm{T}})\mathbf{T}^{\mathrm{T}} = \mathbf{T}\mathbf{e}\mathbf{T}^{\mathrm{T}} \tag{1.59}$$

and $\boldsymbol{\omega}$ obeys the same transformation law. The matrix $\mathbf{e}$ is given by

$$\mathbf{e} = \begin{bmatrix} \varepsilon_{xx} & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{zx} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_{yy} & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{yz} & \varepsilon_{zz} \end{bmatrix} = \tfrac{1}{2}(\mathbf{u}\overleftarrow{\nabla}^{\mathrm{T}} + \vec{\nabla}\mathbf{u}^{\mathrm{T}}) \tag{1.60}$$

where the elements are given by the usual expressions for small strains,

$$\begin{gathered} \varepsilon_{xx} = \frac{\partial u_x}{\partial x} \; ; \; \varepsilon_{yy} = \frac{\partial u_y}{\partial y} \; ; \; \varepsilon_{zz} = \frac{\partial u_z}{\partial z} \; ; \\ \gamma_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \; ; \; \gamma_{zx} = \frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z} \; ; \; \gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \; . \end{gathered} \tag{1.61}$$

The terms ε_{xx}, ε_{yy} and ε_{zz} are known as **normal strains** and γ_{yz}, γ_{zx} and γ_{xy} are called **shear strains.**

1.5.1 Mohr's circle for strain

For a change of coordinates of the type given by (1.33), the new strains in the xy plane are given by

$$\begin{bmatrix} \bar{\varepsilon}_{xx} & \frac{1}{2}\bar{\gamma}_{xy} \\ \frac{1}{2}\bar{\gamma}_{xy} & \bar{\varepsilon}_{yy} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} & \frac{1}{2}\gamma_{xy} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_{yy} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{1.62}$$

which, after some manipulation, can be written as

$$\begin{aligned} \bar{\varepsilon}_{xx} &= \tfrac{1}{2}(\varepsilon_{xx} + \varepsilon_{yy}) + \tfrac{1}{2}(\varepsilon_{xx} - \varepsilon_{yy})\cos 2\theta + \tfrac{1}{2}\gamma_{xy}\sin 2\theta \\ \bar{\varepsilon}_{yy} &= \tfrac{1}{2}(\varepsilon_{xx} + \varepsilon_{yy}) - \tfrac{1}{2}(\varepsilon_{xx} - \varepsilon_{yy})\cos 2\theta - \tfrac{1}{2}\gamma_{xy}\sin 2\theta \\ \tfrac{1}{2}\bar{\gamma}_{xy} &= -\tfrac{1}{2}(\varepsilon_{xx} - \varepsilon_{yy})\sin 2\theta + \tfrac{1}{2}\gamma_{xy}\cos 2\theta \end{aligned} \tag{1.63}$$

This means that for the particular angle θ given by

$$\tan 2\theta = \frac{\gamma_{xy}}{(\varepsilon_{xx} - \varepsilon_{yy})} \tag{1.64}$$

the shear strain $\bar{\gamma}_{xy}$ is zero. The inverse relationship is given by replacing θ by $-\theta$ throughout. In the particular case where $\bar{\gamma}_{xy}$ is zero, the inverse relationship is given by

$$\begin{aligned} \varepsilon_{xx} &= \tfrac{1}{2}(e_{\mathrm{I}} + e_{\mathrm{II}}) + \tfrac{1}{2}(e_{\mathrm{I}} - e_{\mathrm{II}})\cos 2\theta \\ \varepsilon_{yy} &= \tfrac{1}{2}(e_{\mathrm{I}} + e_{\mathrm{II}}) - \tfrac{1}{2}(e_{\mathrm{I}} - e_{\mathrm{II}})\cos 2\theta \\ \tfrac{1}{2}\gamma_{xy} &= \tfrac{1}{2}(e_{\mathrm{I}} - e_{\mathrm{II}})\sin 2\theta \end{aligned} \tag{1.65}$$

where these particular values of $\bar{\varepsilon}_{xx}$ and $\bar{\varepsilon}_{yy}$ are denoted by e_{I} and e_{II} respectively. They are known as the **principal strains** in the xy plane.

This relationship means that ε_{xx}, ε_{yy} and $\frac{1}{2}\gamma_{xy}$ can be represented by the ordinates of two points on a circle, as shown in Fig. 1.6. This is known as **Mohr's circle**. In going from the e_{I} ordinate to the ε_{xx} ordinate, the angle 2θ is anticlockwise. This is in the same sense as the rotation θ from the direction of the e_{I} strain to the ε_{xx} strain, as viewed along the z axis.

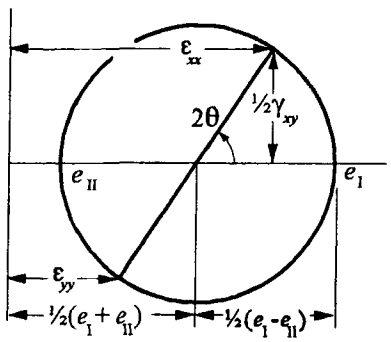


Figure 1.6 Mohr's circle for strain.

These results are independent of the material properties of the body. In some texts, Mohr's circle for strain is derived from Mohr's circle for stress, assuming isotropic, linear-elastic behaviour, but it can be seen from the above derivation that no such assumptions are necessary. Similar Mohr's circles can be drawn for any other second-order symmetric matrix obeying the transformation law given by (1.41).

1.5.2 Principal strains in three dimensions

Suppose that, for some orientation of the three-dimensional coordinate system, there are no shear strains so that the strain matrix takes the form

$$\mathbf{e}_{\mathrm{p}} = \begin{bmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{bmatrix} \tag{1.66}$$

The coordinates in this particular orientation are known as the **principal strain coordinates** and the strains e_1, e_2 and e_3 are known as the **principal strains**. (The principal strains in the xy plane, e_{I} and e_{II}, are not necessarily a subset of the principal strains.) The matrix of principal strains is found by some coordinate transformation of the matrix given by (1.60). Then from (1.41),

$$\begin{aligned} \mathbf{e}_{\mathrm{p}} &= \mathbf{T}\mathbf{e}\mathbf{T}^{\mathrm{T}} \\ \text{or} \qquad \mathbf{T}^{\mathrm{T}}\mathbf{e}_{\mathrm{p}} &= \mathbf{e}\mathbf{T}^{\mathrm{T}} \end{aligned} \tag{1.67}$$

Suppose that the columns of $\mathbf{T}^{\mathrm{T}}$ are given by the column vectors $\mathbf{t}_1$, $\mathbf{t}_2$, $\mathbf{t}_3$. That is

$$\mathbf{T}^{\mathrm{T}} = [\mathbf{t}_1 \ \mathbf{t}_2 \ \mathbf{t}_3] \tag{1.68}$$

Then $$[\mathbf{t}_1\ \mathbf{t}_2\ \mathbf{t}_3]\mathbf{e}_p = \mathbf{e}[\mathbf{t}_1\ \mathbf{t}_2\ \mathbf{t}_3]$$

On comparing the columns on each side of this equation,

$$\mathbf{t}_1 e_1 = \mathbf{e}\mathbf{t}_1\ ;\ \mathbf{t}_2 e_2 = \mathbf{e}\mathbf{t}_2\ ;\ \mathbf{t}_3 e_3 = \mathbf{e}\mathbf{t}_3\ ,$$

$$\text{or} \qquad \mathbf{e}\mathbf{t}_\alpha = e_\alpha \mathbf{t}_\alpha \qquad (\alpha = 1 \text{ to } 3). \tag{1.69}$$

which is the form of the eigenvalue problem given by (1.8). The principal strains e_α are thus the eigenvalues of the matrix $\mathbf{e}$ and the set of $\mathbf{t}_\alpha$ is formed from the eigenvectors of $\mathbf{e}$. Because the inverse of a transformation matrix is its transpose,

$$\mathbf{I} = \mathbf{T}\mathbf{T}^T = \begin{bmatrix} \mathbf{t}_1^T \\ \mathbf{t}_2^T \\ \mathbf{t}_3^T \end{bmatrix} [\mathbf{t}_1\ \mathbf{t}_2\ \mathbf{t}_3]$$

or in terms of the Kronecker delta,

$$\mathbf{t}_\alpha^T \mathbf{t}_\beta = \delta_{\alpha\beta} \qquad (\alpha, \beta = 1 \text{ to } 3) \tag{1.70}$$

so that the eigenvectors $\mathbf{t}_\alpha$ are not only orthogonal but are **normalised eigenvectors**, that is, the product of the transpose of such an eigenvector with itself is unity.

1.5.3 Strain invariants

Equation (1.69) can be rewritten as

$$(\mathbf{e} - \mathbf{I}e_\alpha)\mathbf{t}_\alpha = 0$$

so that for non-zero $\mathbf{t}_\alpha$, the determinant of $(\mathbf{e} - \mathbf{I}e_\alpha)$ must be zero, that is,

$$\begin{vmatrix} \varepsilon_{xx} - e_\alpha & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{zx} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_{yy} - e_\alpha & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{yz} & \varepsilon_{zz} - e_\alpha \end{vmatrix} = 0$$

so that

$$e_\alpha^3 - e_\alpha^2 K_1 + e_\alpha K_2 - K_3 = 0 \tag{1.71}$$

where K_1, K_2 and K_3 are given by

$$\begin{aligned} K_1 &= \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} \\ K_2 &= \varepsilon_{xx}\varepsilon_{yy} + \varepsilon_{yy}\varepsilon_{zz} + \varepsilon_{zz}\varepsilon_{xx} - \tfrac{1}{4}(\gamma_{yz}^2 + \gamma_{zx}^2 + \gamma_{xy}^2) \\ K_3 &= \varepsilon_{xx}\varepsilon_{yy}\varepsilon_{zz} + \tfrac{1}{4}(\gamma_{yz}\gamma_{zx}\gamma_{xy} - \varepsilon_{xx}\gamma_{yz}^2 - \varepsilon_{yy}\gamma_{zx}^2 - \varepsilon_{zz}\gamma_{xy}^2) \end{aligned} \tag{1.72}$$

The solution of (1.71) gives the three principal strains e_1, e_2 and e_3 associated with the state of strain at a particular locality in the body. These same strains must result from (1.71) regardless of the coordinate system used to define the strain matrix given by (1.60). For this to be true, K_1, K_2 and K_3 must take the same values for any choice of coordinate

orientation at the locality, and so they are known as the **strain invariants** of small deflection theory. It can be seen that K_1 is the trace of this matrix. It is also the local **volumetric strain** dV/V where V is the volume of an elementary cuboid $dx \times dy \times dz$ and dV is the increment in this volume after strain (terms of the order of the square of a strain and higher being ignored, cf. section 1.11.3). The third invariant, K_3, is the determinant of the strain matrix. It can be seen from (1.71) that if K_3 is zero, at least one of the principal strains must be zero, so that the local deformation is one of plane strain. As K_1, K_2 and K_3 are invariant, they can be evaluated in any coordinate system. Choosing in particular the principal strain coordinates,

$$\begin{aligned} K_1 &= e_1 + e_2 + e_3 \\ K_2 &= e_1e_2 + e_2e_3 + e_3e_1 \\ K_3 &= e_1e_2e_3 \end{aligned} \tag{1.73}$$

1.5.4 Compatibility conditions

Because the set of six strains given by (1.61) are derived from only three independent displacements, the strains are interrelated. Thus for example

$$\frac{\partial^2\gamma_{xy}}{\partial x\partial y} = \frac{\partial^3 u_x}{\partial x\partial y^2} + \frac{\partial^3 u_y}{\partial x^2\partial y} = \frac{\partial^2\varepsilon_{xx}}{\partial y^2} + \frac{\partial^2\varepsilon_{yy}}{\partial x^2} \tag{1.74}$$

which is the only compatibility condition for plane strain in the xy plane. In general, there are two similar conditions

$$\begin{aligned} \frac{\partial^2\gamma_{yz}}{\partial y\partial z} &= \frac{\partial^2\varepsilon_{yy}}{\partial z^2} + \frac{\partial^2\varepsilon_{zz}}{\partial y^2} \\ \frac{\partial^2\gamma_{zx}}{\partial z\partial x} &= \frac{\partial^2\varepsilon_{zz}}{\partial x^2} + \frac{\partial^2\varepsilon_{xx}}{\partial z^2} \end{aligned} \tag{1.75}$$

and three other conditions

$$\begin{aligned} \frac{1}{2}\frac{\partial}{\partial x}\left(-\frac{\partial\gamma_{yz}}{\partial x} + \frac{\partial\gamma_{zx}}{\partial y} + \frac{\partial\gamma_{xy}}{\partial z}\right) &= \frac{\partial^3 u_x}{\partial x\partial y\partial z} = \frac{\partial^2\varepsilon_{xx}}{\partial y\partial z} \\ \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial\gamma_{yz}}{\partial x} - \frac{\partial\gamma_{zx}}{\partial y} + \frac{\partial\gamma_{xy}}{\partial z}\right) &= \frac{\partial^2\varepsilon_{yy}}{\partial z\partial x} \\ \frac{1}{2}\frac{\partial}{\partial z}\left(\frac{\partial\gamma_{yz}}{\partial x} + \frac{\partial\gamma_{zx}}{\partial y} - \frac{\partial\gamma_{xy}}{\partial z}\right) &= \frac{\partial^2\varepsilon_{zz}}{\partial x\partial y} \end{aligned} \tag{1.76}$$

1.5.5 Bianchi's identities

The above six conditions have been shown by Love (1944, section 17) to be sufficient but they are only weakly independent. In terms of the strain matrix, **e**, they are given by

$$\mathbf{c} = [\vec{\nabla}]\mathbf{e}[\vec{\nabla}] = \mathbf{0} \tag{1.77}$$

where (1.74) and (1.75) are given by equating the on-diagonal terms of the product, **c**, to

zero. Equating the off-diagonal terms of **c** (which is symmetrical) to zero gives (1.76).It can be seen that (1.77) can be obtained directly from the second definition of **e** in (1.60), using (1.24), and noting that $[\vec{\nabla}]$ is equal to $-[\vec{\nabla}]^T$. It also follows from (1.24) and the above definition of **c** that

$$\mathbf{c}\vec{\nabla} = [\vec{\nabla}]\mathbf{e}[\vec{\nabla}]\vec{\nabla} \equiv \mathbf{0} \tag{1.78}$$

These three equations are known as **Bianchi's identities**. They illustrate the fact that there is some relationship between the compatibility conditions. Washizu† has shown that it only necessary for the on-diagonal terms of **c** to be zero within a simply-connected body and the off-diagonal terms to be zero on the boundary, or vice-versa, for all the conditions to be satisfied throughout the body. (A simply-connected body is one in which any closed curve within the body can be continuously contracted to a point; a ring is not simply-connected.)

1.6 THE STRESS MATRIX

The cuboid with side lengths dx, dy and dz shown in Fig. 1.7 defines an elementary volume of a solid body. It is convenient to define its faces as follows. The pair of faces perpendicular to the x axis will be called x faces. The face further in the x direction is the positive x face and the other is the negative x face. The y and z faces can be defined in a similar fashion. The faces shown in Fig. 1.7 are all positive faces. A **surface traction** is defined as a force per unit area of the surface of the body on which it acts. A **stress** is also a force per unit area, acting on any surface on or through the body (real or imaginary), and is associated with a coordinate system. The above faces of the cuboid form elementary parts of the x, y and z **coordinate surfaces**. The sense of the stresses shown acting on the positive faces is taken to be positive in the positive x, y and z directions. Each internal positive face is in contact with a negative face of an adjacent cuboid. As the stresses must be equal and opposite, the stresses on the negative faces are taken to be in the negative senses of x, y and z. The stresses acting normal to the faces are denoted by σ and are called **normal stresses**. Those acting in the planes of the faces are denoted by τ and are called **shear stresses**. The subscripts used refer firstly to the face on which the stress acts and secondly to the direction in which it acts. Subscripts alone are sufficient to distinguish between normal and shear stresses, and in later chapters the single symbol σ will be used for both.

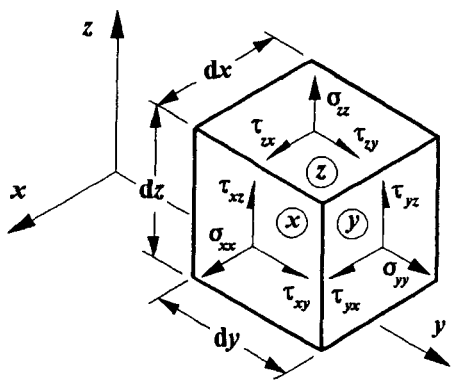


Figure 1.7 Stresses on an elementary cuboid.

†Washizu, K. (1958) A note on the conditions of compatibility. *Journal of Mathematics and Physics* **36** 306-312

1.6.1 Moment equilibrium of a cuboid

The normal stresses produce no moments about the centroid of the cuboid shown in Fig. 1.7. The shear stresses on each pair of faces form couples. About an axis in the x direction, the τ_{yz} stresses on the pair of y faces produce a positive couple and the τ_{zy} stresses on the pair of z faces produce a negative couple. For equilibrium about this axis,

$$(\tau_{yz}\,dx\,dz)\,dy - (\tau_{zy}\,dx\,dy)\,dz = 0$$

or

$$\tau_{yz} = \tau_{zy} \tag{1.79}$$

and similarly, on taking moments about the other two axes,

$$\tau_{zx} = \tau_{xz} \;;\; \tau_{xy} = \tau_{yx} \tag{1.80}$$

Any variation in shear stress between pairs of faces produces smaller-order terms which can be neglected in the above equations. If there had been distributed moments on the faces of the cuboid, their effects would have to be taken into account in these equations. Such **couple stresses** are discussed in Appendix A6.

1.6.2 Equilibrium of surface tractions

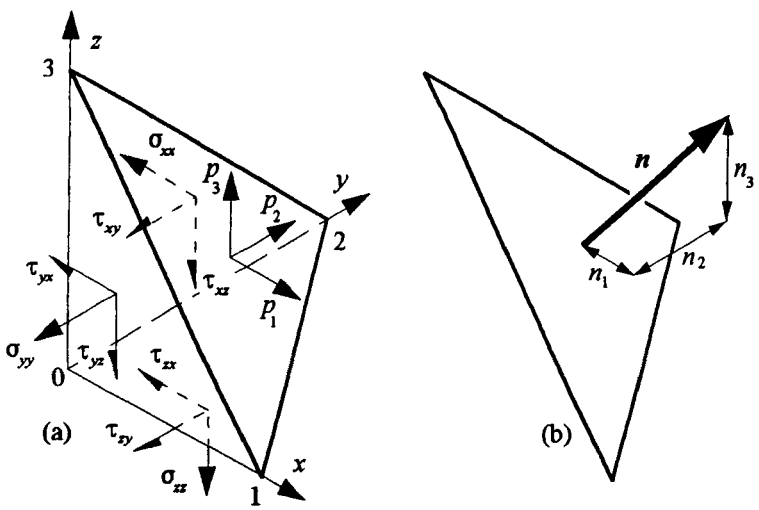


Figure 1.8 Equilibrium of stresses and surface tractions.

Fig. 1.8a shows a tetrahedral element with negative x, y and z faces, 023, 031 and 012 respectively. The fourth face, 123, is at some arbitrary orientation to the others. On this face, surface tractions p_1, p_2 and p_3 act in the x, y and z directions respectively. Fig.1.8b shows an outward unit vector $\boldsymbol{n}$ acting normal to this face. The projections of this unit vector have magnitudes n_1, n_2 and n_3 in the x, y and z directions respectively.

For the equilibrium of this element in the x direction,

$$p_1 \times \text{area}(123) = \sigma_{xx} \times \text{area}(023) + \tau_{yx} \times \text{area}(031) + \tau_{zx} \times \text{area}(012) \tag{1.81}$$

(Any terms involving body forces or mass accelerations of the element are related to its volume, which is of a smaller order of magnitude.) The areas on the right-hand side of the equation are projections of the area 123 on the x, y and z surfaces. The angles between the area 123 and these surfaces are the same as the angles between the unit normal vector $\boldsymbol{n}$ and the x, y and z directions. This means that the ratios of the projected areas to the area 123 are given by n_1, n_2 and n_3 respectively. Thus (1.81) can be written as

$$p_1 = \sigma_{xx}n_1 + \tau_{yx}n_2 + \tau_{zx}n_3 \tag{1.82}$$

Two similar equations can be written for equilibrium in the y and z directions,

$$p_2 = \tau_{xy}n_1 + \sigma_{yy}n_2 + \tau_{zy}n_3 \tag{1.83}$$

$$p_3 = \tau_{xz}n_1 + \tau_{yz}n_2 + \sigma_{zz}n_3 \tag{1.84}$$

In matrix form, (1.82) to (1.84) become

$$\mathbf{p} = \boldsymbol{\sigma}\mathbf{n} \tag{1.85}$$

where

$$\mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \; ; \; \boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} \; ; \; \mathbf{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \tag{1.86}$$

It follows from (1.79) and (1.80) that $\boldsymbol{\sigma}$ is a symmetric matrix.

1.6.3 Coordinate transformation

Both $\mathbf{p}$ and $\mathbf{n}$ are column vectors of resolved parts, $\mathbf{p}$ of a force per unit area vector $\boldsymbol{p}$ and $\mathbf{n}$ of a unit normal vector $\boldsymbol{n}$. This means that they are column vector representations which transform according to the laws given in section 1.3. Then in a new coordinate system, (1.85) takes the form

$$\bar{\mathbf{p}} = \bar{\boldsymbol{\sigma}}\bar{\mathbf{n}} = \mathbf{T}\mathbf{p} = \mathbf{T}\boldsymbol{\sigma}\mathbf{n} = \mathbf{T}\boldsymbol{\sigma}\mathbf{T}^{\mathrm{T}}\bar{\mathbf{n}}$$

so that

$$\bar{\boldsymbol{\sigma}} = \mathbf{T}\boldsymbol{\sigma}\mathbf{T}^{\mathrm{T}} \tag{1.87}$$

which is the same rule as that given by (1.41) and (1.59).

1.6.4 Mohr's circle for stress

Because $\boldsymbol{\sigma}$ is symmetric, the transformation rules used to establish Mohr's circle for strain again apply. By comparing terms in the two matrices given by (1.60) and (1.86), the equivalent of (1.65) can be deduced. This is

$$\begin{aligned} \sigma_{xx} &= \tfrac{1}{2}(\sigma_{\mathrm{I}} + \sigma_{\mathrm{II}}) + \tfrac{1}{2}(\sigma_{\mathrm{I}} - \sigma_{\mathrm{II}})\cos 2\theta \\ \sigma_{yy} &= \tfrac{1}{2}(\sigma_{\mathrm{I}} + \sigma_{\mathrm{II}}) - \tfrac{1}{2}(\sigma_{\mathrm{I}} - \sigma_{\mathrm{II}})\cos 2\theta \\ \tau_{xy} &= \tfrac{1}{2}(\sigma_{\mathrm{I}} - \sigma_{\mathrm{II}})\sin 2\theta \end{aligned} \tag{1.88}$$

where σ_{I} and σ_{II} are known as the **principal stresses** in the xy plane. The x and y axes are

in directions at an anticlockwise angle θ to the directions of the σ_I and σ_{II} stresses respectively, as viewed along the z axis.

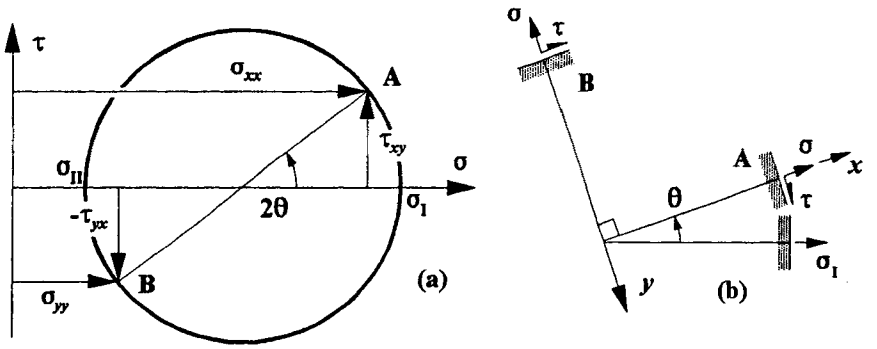


Figure 1.9 Mohr's circle for stress.

The equivalent of Fig. 1.6 is shown in Fig. 1.9a. In Fig. 1.9b, A is a positive x face and B is a negative y face, in the sense defined earlier. As shown, the coordinates of points on the circle are (σ,τ) where σ is the local outwards normal stress at an angle θ and τ is the clockwise shear stress. On face A, σ and τ are in the same sense as σ_{xx} and τ_{xy}. On face B, σ and τ are in the same sense as σ_{yy} and $-\tau_{yx}$.

1.6.5 Principal stresses

Suppose that the surface traction $\boldsymbol{p}$ acting on the surface 123 shown in Fig. 1.8 is given by $\sigma\boldsymbol{n}$, so that it is normal to the surface and of magnitude σ. In this case, (1.85) becomes

$$\boldsymbol{\sigma n} = \sigma\mathbf{n} \tag{1.89}$$

which is the form of the eigenvalue problem given by (1.8). The three eigenvalues found from this equation are known as the three **principal stresses** σ_1, σ_2 and σ_3. The normals to the surfaces on which they act are given by the corresponding eigenvectors. Because the eigenvectors are orthogonal, the three principal stresses act on mutually perpendicular planes. The normals to these planes are known as the **principal stress coordinates**.

1.6.6 Stress invariants

As in section 1.5.3, the eigenvalue problem given here by (1.89) can be recast in the form

$$(\boldsymbol{\sigma} - \mathbf{I}\sigma)\mathbf{n} = \mathbf{0}$$

so that for non-zero $\mathbf{n}$, the determinant of $(\boldsymbol{\sigma} - \mathbf{I}\sigma)$must be zero, giving

$$\sigma^3 - \sigma^2 I_1 + \sigma I_2 - I_3 = 0 \tag{1.90}$$

where I_1, I_2 and I_3 are the **stress invariants** of small deflection theory and are given by

$$\begin{aligned} I_1 &= \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = \sigma_1 + \sigma_2 + \sigma_3 \\ I_2 &= \sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} - \tau_{yz}^2 - \tau_{zx}^2 - \tau_{xy}^2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 \\ I_3 &= \sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\tau_{yz}\tau_{zx}\tau_{xy} - \sigma_{xx}\tau_{yz}^2 - \sigma_{yy}\tau_{zx}^2 - \sigma_{zz}\tau_{xy}^2 = \sigma_1\sigma_2\sigma_3 \end{aligned} \tag{1.91}$$

Just as previously for strain, these invariants have the same values in all coordinate systems. The above expressions for the invariants in terms of the principal stresses then follow as particular cases of the general expressions. As will be seen later, the principal coordinates for stress and for strain are the same for isotropic materials although this is not necessarily true for other materials.

The roots of (1.90) are more readily found by recasting the problem in terms of **stress deviations**. The normal stress deviations are defined as the variations of the normal stresses from the average normal stress, $I_1/3$, and so are given by

$$s_{xx} = \sigma_{xx} - I_1/3 \; ; \; s_{yy} = \sigma_{yy} - I_1/3 \; ; \; s_{zz} = \sigma_{zz} - I_1/3 \; . \tag{1.92}$$

The shear stress deviations, s_{xy}, s_{yz} and s_{zx}, are taken to be identical to the ordinary shear stresses, τ_{xy}, τ_{yz} and τ_{zx}. If the principal stress deviation is defined as s $(= \sigma - I_1/3)$, then in terms of stress deviations, (1.90) takes the form

$$s^3 - J_2 s - J_3 = 0 \tag{1.93}$$

where J_2 and J_3 are known as the **stress deviation invariants** and are given by

$$J_2 = -s_{xx}s_{yy} - s_{yy}s_{zz} - s_{zz}s_{xx} + s_{yz}^2 + s_{zx}^2 + s_{xy}^2 = I_1^2/3 - I_2 \tag{1.94}$$

$$J_3 = s_{xx}s_{yy}s_{zz} + 2s_{yz}s_{zx}s_{xy} - s_{xx}s_{yz}^2 - s_{yy}s_{zx}^2 - s_{zz}s_{xy}^2 = 2I_1^3/27 - I_1I_2/3 + I_3 \tag{1.95}$$

In some texts, J_1 is defined as the coefficient of s^2 in (1.93), i.e. zero. Equation (1.93) is the canonical form of the cubic equation to which Cardan's solution can be applied. The particular solution in which all the roots of the equation are real is called the irreducible case. This solution gives the principal stress as

$$\sigma_k = 2\sqrt{J_2/3}\cos[(\alpha + 2k\pi)/3] + I_1/3 \qquad (k = 1 \text{ to } 3) \tag{1.96}$$

where

$$\cos\alpha = J_3/[2\sqrt{(J_2/3)^3}] \tag{1.97}$$

Not only can J_2 and J_3 be expressed in terms of I_1, I_2 and I_3 but so can any other possible stress invariant.

1.6.7 Failure criteria

An **isotropic material** is one which appears to have the same properties regardless of the orientation of the coordinate system relative to which they are measured. This means that any failure criterion for an isotropic material must be invariant. If it expressed as a function of stress, it must then be a function of I_1, I_2 and I_3, or alternatively as a function of I_1, J_2 and J_3. This places a useful constraint not only on the possible functions for failure criteria, but also on the potential functions of stress used to describe the behaviour of non-linear isotropic materials.

The best known failure criterion for ductile isotropic materials is due to **von Mises**. According to this criterion, failure of the material is determined from a limiting value of J_2. This limiting value can be found from a simple uniaxial tension test. If the failure stress in this test is σ_f, then from (1.91) to (1.94), the limiting value of J_2 is $\sigma_f^2/3$. The possible stress states up to failure are governed by the condition

$$J_2 \le \sigma_f^2/3 \tag{1.98}$$

A criterion proposed by Drucker and Prager† for a perfectly plastic soils takes the form

$$\left|\alpha I_1 + \sqrt{J_2}\right| \le \left|(\alpha + 1/\sqrt{3})\sigma_f\right| \tag{1.99}$$

where α is positive. This is a generalisation of failure related to Coulomb friction where the failure shear stress is some multiple of the normal stress on the same plane.

1.6.8 Equations of motion of a cuboid

In the equations derived earlier, stress variations, body forces and mass accelerations gave rise to terms of negligible magnitude. On considering the equations of motion of the cuboid shown in Fig. 1.7, these effects must be taken into account.

Fig. 1.10 shows those stresses and the component of the body force per unit mass, X, which produce forces in the x direction. (The virtue of using a body force per unit mass rather than a body force per unit volume becomes apparent in Chapter 4, where reference is made to the same set of material points regardless of the large deformations of the cuboid.) Taking the density of the cuboid to be ρ, the relationship between the net force and the acceleration $\ddot{u}_x$ in the x direction is given by

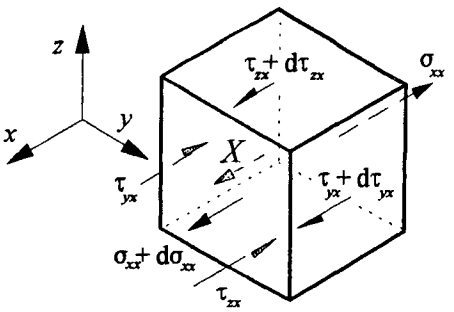


Figure 1.10 Equilibrium in the presence of a body force.

$$d\sigma_{xx}dy\,dz + d\tau_{yx}dz\,dx + d\tau_{zx}dx\,dy + \rho X dx\,dy\,dz = \rho\ddot{u}_x dx\,dy\,dz$$

but

$$d\sigma_{xx} = \frac{\partial\sigma_{xx}}{\partial x}dx \quad , \quad d\tau_{yx} = \frac{\partial\tau_{yx}}{\partial y}dy \quad , \quad d\tau_{zx} = \frac{\partial\tau_{zx}}{\partial z}dz \ ,$$

so that on removing a common factor of dx dy dz,

$$\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} + \frac{\partial\tau_{zx}}{\partial z} + \rho X = \rho\ddot{u}_x \tag{1.100a}$$

and similarly

$$\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\tau_{zy}}{\partial z} + \rho Y = \rho\ddot{u}_y \tag{1.100b}$$

$$\frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\sigma_{zz}}{\partial z} + \rho Z = \rho\ddot{u}_z \tag{1.100c}$$

where Y, Z, $\ddot{u}_y$ and $\ddot{u}_z$ are the body forces per unit mass and accelerations in the y and z directions. These three equations can be written in the form

$$\boldsymbol{\sigma}\tilde{\nabla} + \rho\mathbf{f} = \rho\ddot{\mathbf{u}} \tag{1.101}$$

†Drucker, D.C. and Prager, W. (1952) Soil mechanics and plastic analysis or limit design. *Quarterly of Applied Mathematics*, **10** 157-165

where

$$\mathbf{f} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} , \quad \ddot{\mathbf{u}} = \begin{bmatrix} \ddot{u}_x \\ \ddot{u}_y \\ \ddot{u}_z \end{bmatrix} \tag{1.102}$$

1.7 ISOTROPIC ELASTICITY

The equations relating strain to stress in a linearly-elastic isotropic material are

$$\begin{aligned} \varepsilon_{xx} &= [(1+\nu)\sigma_{xx} - \nu(\sigma_{xx}+\sigma_{yy}+\sigma_{zz})]/E \\ \varepsilon_{yy} &= [(1+\nu)\sigma_{yy} - \nu(\sigma_{xx}+\sigma_{yy}+\sigma_{zz})]/E \\ \varepsilon_{zz} &= [(1+\nu)\sigma_{zz} - \nu(\sigma_{xx}+\sigma_{yy}+\sigma_{zz})]/E \end{aligned} \tag{1.103}$$

and

$$\tau_{yz} = \tau_{zy} = G\gamma_{yz} , \ \tau_{zx} = \tau_{xz} = G\gamma_{zx} , \ \tau_{xy} = \tau_{yx} = G\gamma_{xy} \tag{1.104}$$

where **Young's modulus,** E, **Poisson's ratio**, ν, and the **shear modulus**, G, are related by

$$E = 2G(1+\nu) \tag{1.105}$$

These relationships are the most general possible for an isotropic material, as will be seen in section 1.8.3. From the above and the definitions of the strain and stress matrices, (1.60) and (1.86), it follows that

$$\mathbf{e} = \frac{1}{2G}\boldsymbol{\sigma} - \frac{\nu}{2G(1+\nu)}\mathbf{I}\,\mathrm{tr}(\boldsymbol{\sigma}) \tag{1.106}$$

or, using the subscript notation of section 1.1.1,

$$e_{ij} = \frac{1}{2G}\sigma_{ij} - \frac{\nu}{2G(1+\nu)}\delta_{ij}\sigma_{kk} \tag{1.106a}$$

Using the summation convention, the trace of $\mathbf{e}$ is

$$e_{jj} = \frac{1}{2G}\sigma_{jj} - \frac{3\nu}{2G(1+\nu)}\sigma_{kk}$$

or

$$\mathrm{tr}(\mathbf{e}) = \left(\frac{1}{2G} - \frac{3\nu}{2G(1+\nu)}\right)\mathrm{tr}(\boldsymbol{\sigma})$$

which, on substituting in (1.106), gives

$$\boldsymbol{\sigma} = 2G\left(\mathbf{e} + \frac{\nu}{1-2\nu}\mathbf{I}\,\mathrm{tr}(\mathbf{e})\right) \tag{1.107}$$

From either (1.106) or (1.107), it follows that if either $\boldsymbol{\sigma}$ or $\mathbf{e}$ is a diagonal matrix, they both are. Thus for an isotropic, linearly-elastic material, the principal directions, defining the principal axes and planes, are common to both stress and strain.

1.7.1 Equations of motion in terms of displacements

Substituting the expression for $\boldsymbol{\sigma}$ given by (1.107) into (1.101) and expressing $\mathbf{e}$ in terms of $\mathbf{u}$ using (1.60),

$$G\left(\mathbf{u}\vec{\nabla}^{\mathrm{T}} + \vec{\nabla}\mathbf{u}^{\mathrm{T}} + \frac{2\nu}{1-2\nu}\mathbf{I}\vec{\nabla}^{\mathrm{T}}\mathbf{u}\right)\vec{\nabla} + \rho\mathbf{f} = \rho\ddot{\mathbf{u}}$$

Making use of (1.15), this becomes

$$G(\vec{\nabla}^T\vec{\nabla})\mathbf{u} + \frac{G}{1-2\nu}(\vec{\nabla}\vec{\nabla}^T)\mathbf{u} + \rho\mathbf{f} = \rho\ddot{\mathbf{u}} \tag{1.108}$$

In the absence of body forces and accelerations, this equation takes the form

$$\{(1-2\nu)\vec{\nabla}^T\vec{\nabla} + \vec{\nabla}\vec{\nabla}^T\}\mathbf{u} = \mathbf{0} \tag{1.109}$$

One form of solution of this equation, in terms of harmonic functions, is discussed in Appendix A3.

1.7.2 The Galerkin vector

Now

$$\vec{\nabla}^T\vec{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \nabla^2 \tag{1.110}$$

where ∇^2 is known as the **Laplacian operator**. Because it is a scalar,

$$\nabla^2\vec{\nabla}\vec{\nabla}^T = \vec{\nabla}\nabla^2\vec{\nabla}^T = \vec{\nabla}\vec{\nabla}^T\nabla^2 \tag{1.111}$$

Then if a column vector $\mathbf{g}$ is related to $\mathbf{u}$ by

$$2G\mathbf{u} = \{2(1-\nu)\vec{\nabla}^T\vec{\nabla} - \vec{\nabla}\vec{\nabla}^T\}\mathbf{g} \tag{1.112}$$

it follows from (1.109) to (1.112) that

$$\begin{aligned}\{(1-2\nu)\vec{\nabla}^T\vec{\nabla} + \vec{\nabla}\vec{\nabla}^T\}\{2(1-\nu)\vec{\nabla}^T\vec{\nabla} - \vec{\nabla}\vec{\nabla}^T\}\mathbf{g} \\ = 2(1-\nu)(1-2\nu)\nabla^2\nabla^2\mathbf{g} = \mathbf{0}\end{aligned} \tag{1.113}$$

where $\mathbf{g}$ is known as the **Galerkin vector**. Writing it as

$$\mathbf{g} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} \tag{1.114}$$

(1.113) can be expressed as the three conditions

$$\nabla^2\nabla^2 g_i = 0 \quad (i = 1 \text{ to } 3) \tag{1.113a}$$

Any solution of these **biharmonic** equations can be used to generate possible displacement fields for isotropic materials which are not subject to body forces and accelerations. Expressed in full, (1.112) becomes

$$2G\begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} 2(1-\nu)\nabla^2 - \dfrac{\partial^2}{\partial x^2} & -\dfrac{\partial^2}{\partial x\partial y} & -\dfrac{\partial^2}{\partial x\partial z} \\ -\dfrac{\partial^2}{\partial x\partial y} & 2(1-\nu)\nabla^2 - \dfrac{\partial^2}{\partial y^2} & -\dfrac{\partial^2}{\partial y\partial z} \\ -\dfrac{\partial^2}{\partial x\partial z} & -\dfrac{\partial^2}{\partial y\partial z} & 2(1-\nu)\nabla^2 - \dfrac{\partial^2}{\partial z^2} \end{bmatrix}\begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} \tag{1.112a}$$

The stresses can be expressed in terms of g_i using (1.60) and (1.107). For

example, in terms of g_3 alone,

$$\sigma_{xx} = \frac{\partial}{\partial z}\left(\nu\nabla^2 - \frac{\partial^2}{\partial x^2}\right)g_3 \ , \ \sigma_{yy} = \frac{\partial}{\partial z}\left(\nu\nabla^2 - \frac{\partial^2}{\partial y^2}\right)g_3 \ ,$$

$$\sigma_{zz} = \frac{\partial}{\partial z}\left((2-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right)g_3 ,$$

$$\tau_{yz} = \frac{\partial}{\partial y}\left((1-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right)g_3 \ , \ \tau_{zx} = \frac{\partial}{\partial x}\left((1-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right)g_3 \ , \tag{1.115}$$

$$\tau_{xy} = -\frac{\partial^3}{\partial x \partial y \partial z}g_3 \ .$$

The stresses in terms of g_1 and g_2 can be found by one and two cyclic permutations respectively of x, y and z throughout (1.115).

1.7.3 Love's function

The results can be expressed in terms of the cylindrical polar coordinates (r, θ, z) used in section A.1.1 by use of the relationships

$$\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \sin\theta\frac{1}{r}\frac{\partial}{\partial \theta} \ , \ \frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \cos\theta\frac{1}{r}\frac{\partial}{\partial \theta} \ .$$

Rotating the Cartesian axes so that θ is zero,

$$\frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial r^2} \ , \ \frac{\partial^2}{\partial x \partial y} = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial \theta}\right) \ , \ \frac{\partial^2}{\partial y^2} = \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \ .$$

The directions of increasing x, y and z are now aligned with the directions of increasing r, θ and z respectively. Then from (1.115),

$$\begin{aligned}
\sigma_{rr} &= \frac{\partial}{\partial z}\left(\nu\nabla^2 - \frac{\partial^2}{\partial r^2}\right)g_3 \\
\sigma_{\theta\theta} &= \frac{\partial}{\partial z}\left(\nu\nabla^2 - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)g_3 \\
\sigma_{zz} &= \frac{\partial}{\partial z}\left((2-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right)g_3 \\
\tau_{\theta z} &= \frac{1}{r}\frac{\partial}{\partial \theta}\left((1-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right)g_3 \\
\tau_{zr} &= \frac{\partial}{\partial r}\left((1-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right)g_3 \\
\tau_{r\theta} &= -\frac{\partial^3}{\partial r \partial \theta \partial z}\left(\frac{1}{r}g_3\right)
\end{aligned} \tag{1.116}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \tag{1.117}$$

For rotationally symmetric problems, g_3 is not a function of θ and it is then identical to Love's χ function, see Love (1944) section 188.

1.7.4 The Airy stress function

Plane strain in the xy plane is given by taking g_1 and g_2 as functions of x and y only and taking g_3 as zero. Then u_z is zero and (1.112a) reduces to

$$2G\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 2(1-\nu)\nabla^2 - \dfrac{\partial^2}{\partial x^2} & -\dfrac{\partial^2}{\partial x \partial y} \\ -\dfrac{\partial^2}{\partial x \partial y} & 2(1-\nu)\nabla^2 - \dfrac{\partial^2}{\partial y^2} \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \tag{1.112b}$$

Weber† has shown that if the solution is expressed in terms of g_2 alone, where

$$\frac{\partial g_2}{\partial y} = \phi - 2(1-\nu)\lambda\,, \quad \phi = \psi + x\frac{\partial \lambda}{\partial x} \quad \text{and} \quad \nabla^2\psi = \nabla^2\lambda = 0\,, \tag{1.118}$$

then

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2}\,, \quad \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2}\,, \quad \sigma_{zz} = \nu\nabla^2\phi \quad \text{and} \quad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} \tag{1.119}$$

where

$$\nabla^2\nabla^2\phi = 0 \tag{1.120}$$

and $\phi(x,y)$ is known as the **Airy stress function**. The same function ϕ can also be derived from a similar relationship in terms of g_1. It gives all possible states of stress under conditions of plane strain. It is often used to analyse general conditions of plane stress. Then σ_z is taken as zero instead of being given by (1.119). The equilibrium conditions in the absence of body forces and mass accelerations given by (1.100) are still satisfied, as is the compatibility condition given by (1.74). However, the compatibility conditions given by (1.75) and (1.76) imply that

$$\frac{\partial^2}{\partial x^2}(\nabla^2\phi) = \frac{\partial^2}{\partial y^2}(\nabla^2\phi) = \frac{\partial^2}{\partial x \partial y}(\nabla^2\phi) = 0 \tag{1.121}$$

are additional constraints on the state of plane stress, unless Poisson's ratio is zero. A discussion of the conditions of plane stress will be found in Sokolnikoff (1956) section 67, where they are justified for the mean values of the stresses across the thickness of a plate. Lists of two- and three-dimensional biharmonic functions, suitable for use as χ functions, Airy stress functions, and Galerkin vectors will be found in Appendix A2.

†Weber, C. (1925) Achsensymmetrische Deformation von Umdrehungskörpern. *Zeitschrift für angewandte Mathematik und Mechanik*, **10**, 466-468

1.8 LINEAR ANISOTROPIC BEHAVIOUR

The most general linear relationship between stress and strain cannot be expressed in terms of the stress and strain matrices, $\boldsymbol{\sigma}$ and $\mathbf{e}$. Such a relationship would require a four-dimensional array of stiffnesses or compliances. This can be represented in tensor notation, as will be seen in the next chapter. If the stresses and strains are represented by column vectors, then a relationship can be written in matrix form as

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} & s_{15} & s_{16} \\ s_{21} & s_{22} & s_{23} & s_{24} & s_{25} & s_{26} \\ s_{31} & s_{32} & s_{33} & s_{34} & s_{35} & s_{36} \\ s_{41} & s_{42} & s_{43} & s_{44} & s_{45} & s_{46} \\ s_{51} & s_{52} & s_{53} & s_{54} & s_{55} & s_{56} \\ s_{61} & s_{62} & s_{63} & s_{64} & s_{65} & s_{66} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{bmatrix} \tag{1.122}$$

or

$$\boldsymbol{\gamma} = \mathbf{S}\,\boldsymbol{\tau} \tag{1.123}$$

where $\boldsymbol{\gamma}$ and $\boldsymbol{\tau}$ are the above column vectors of the strains and stresses and $\mathbf{S}$ is the matrix of **compliances** s_{ij}. In analysing anisotropic problems, this is more commonly used than the inverse relationship

$$\boldsymbol{\tau} = \mathbf{C}\,\boldsymbol{\gamma} \tag{1.124}$$

where $\mathbf{C}$ is the matrix of **stiffnesses** c_{ij}.

1.8.1 Work and energy

During a small change in the deformation of an elastic body, suppose that the positive x face of the elementary cuboid shown in Fig. 1.7 moves through a small displacement d_{xx} relative to the negative x face, as shown in Fig. 1.11. Then

$$d_{xx} = \delta\left(\frac{\partial u_{xx}}{\partial x}\right) = \delta\varepsilon_{xx}\,dx$$

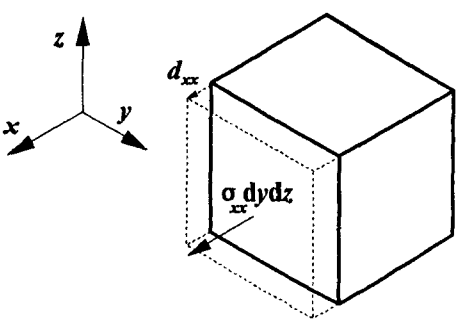


Figure 1.11 Work done in extending a cuboid.

(Here, an incremental change in a quantity during deformation is indicated by the symbol δ preceding it.) The only net work done on the cuboid is by the stress σ_{xx}, and is given by

$$\delta W_{xx} = \sigma_{xx}\,dy\,dz\,d_{xx} = \sigma_{xx}\,\delta\varepsilon_{xx}\,dx\,dy\,dz \tag{1.125}$$

If the positive y face moves through a small displacement d_{yz} relative to the negative y face, as shown in Fig. 1.12, then

$$d_{yz} = \delta\left(\frac{\partial u_z}{\partial y}\right) dy$$

In this case, the only net work done is by the shear stress τ_{yz} and is given by

$$\delta W_{yz} = \tau_{yz} dx dz d_{yz}$$
$$= \tau_{yz}\delta\left(\frac{\partial u_z}{\partial y}\right) dx dy dz$$

Similarly, the work done during a small displacement in the y direction of the positive z face relative to the negative z face is

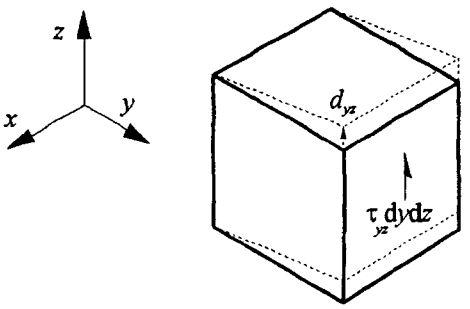


Figure 1.12 Work done in shearing a cuboid.

$$\delta W_{zy} = \tau_{zy}\delta\left(\frac{\partial u_y}{\partial z}\right) dx dy dz$$

Because τ_{yz} and τ_{zy} are equal, the sum of these two increments of work is

$$\delta W_{yz} + \delta W_{zy} = \tau_{yz}\delta\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right) dx dy dz = \tau_{yz}\delta\gamma_{yz}\, dx dy dz \qquad (1.126)$$

The total work done by all such increments is

$$\delta W = (\sigma_{xx}\delta\varepsilon_{xx} + \sigma_{yy}\delta\varepsilon_{yy} + \sigma_{zz}\delta\varepsilon_{zz} + \tau_{yz}\delta\gamma_{yz} + \tau_{zx}\delta\gamma_{zx} + \tau_{xy}\delta\gamma_{xy})\, dx dy dz$$
$$= \boldsymbol{\tau}^{\mathrm{T}}(\delta\boldsymbol{\gamma})\, dV \qquad (1.127)$$

where dV is the volume of the cuboid, $dxdydz$.

If the stress σ_{xx} and the strain ε_{xx} are increased in proportion to one another up to their final values, then the work done per unit volume is shown by the shaded area in Fig. 1.13 and is equal to $\frac{1}{2}\sigma_{xx}\varepsilon_{xx}$. For a linearly elastic body, all the stresses and strains can be increased in proportion to one another and then the total work done per unit volume is

$$W_V = \tfrac{1}{2}\boldsymbol{\tau}^{\mathrm{T}}\boldsymbol{\gamma} \quad (= \tfrac{1}{2}\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{\tau}) \qquad (1.128)$$

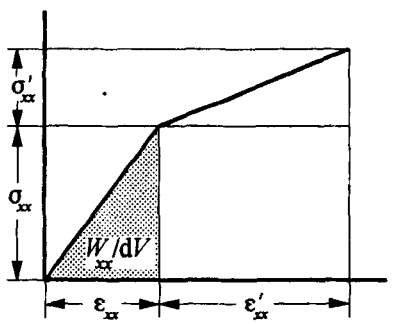


Figure 1.13 Work and energy per unit volume.

Ignoring thermal effects†, the energy stored in an elastic body is equal to the work done on it, and is entirely a function of the final state of stress or strain. Thus (1.128) gives the work done regardless of how the loading was applied to reach the final state. If additional stresses $\boldsymbol{\tau}'$ induce additional strains $\boldsymbol{\gamma}'$, then the total work done per unit volume is

†See the further discussion in section 2.6.

$$W_V^t = \tfrac{1}{2}\sigma_{xx}\varepsilon_{xx} + \sigma_{xx}\varepsilon'_{xx} + \tfrac{1}{2}\sigma'_{xx}\varepsilon'_{xx} + \ldots$$
$$= \tfrac{1}{2}\boldsymbol{\tau}^{\mathrm{T}}\boldsymbol{\gamma} + \boldsymbol{\tau}^{\mathrm{T}}\boldsymbol{\gamma}' + \tfrac{1}{2}\boldsymbol{\tau}'^{\mathrm{T}}\boldsymbol{\gamma}' \qquad (1.129)$$

Because the response is linear, the stress increments $\boldsymbol{\tau}'$ will induce the strain increments $\boldsymbol{\gamma}'$ if applied in the absence of the initial stress and strain, so that the work done per unit volume by applying the stresses and strains in the reverse order is

$$W_V^t = \tfrac{1}{2}\boldsymbol{\tau}'^{\mathrm{T}}\boldsymbol{\gamma}' + \boldsymbol{\tau}'^{\mathrm{T}}\boldsymbol{\gamma} + \tfrac{1}{2}\boldsymbol{\tau}^{\mathrm{T}}\boldsymbol{\gamma} \qquad (1.130)$$

Because the body is elastic, the work done is the same as that given by (1.129), so that

$$\boldsymbol{\tau}^{\mathrm{T}}\boldsymbol{\gamma}' = \boldsymbol{\tau}'^{\mathrm{T}}\boldsymbol{\gamma} \qquad (\text{or} \quad \boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{\tau}' = \boldsymbol{\gamma}'^{\mathrm{T}}\boldsymbol{\tau}) \qquad (1.131)$$

which is a form of **Betti's theorem** for stresses and strains. From (1.123), $\boldsymbol{\gamma}' = \mathbf{S}\boldsymbol{\tau}'$, so that from (1.131),

$$\boldsymbol{\tau}^{\mathrm{T}}\mathbf{S}\boldsymbol{\tau}' = \boldsymbol{\tau}'^{\mathrm{T}}\mathbf{S}\boldsymbol{\tau} = \boldsymbol{\tau}'^{\mathrm{T}}\mathbf{S}^{\mathrm{T}}\boldsymbol{\tau}$$

(the third expression arising from taking the transpose of the first). Then

$$\boldsymbol{\tau}'^{\mathrm{T}}(\mathbf{S} - \mathbf{S}^{\mathrm{T}})\boldsymbol{\tau} = 0$$

As this must be true for all $\boldsymbol{\tau}$ and $\boldsymbol{\tau}'$, it follows that $\mathbf{S}$ is symmetrical, that is,

$$\mathbf{S} = \mathbf{S}^{\mathrm{T}} \qquad (1.132)$$

or s_{ij} is equal to s_{ji} for all i and j ($i, j = 1$ to 6) leaving no more than 21 independent compliances in (1.122). Similarly, it can be shown that the stiffness matrix $\mathbf{C}$ is symmetric.

Novozhilov (1961) argues that there are no more than 18 independent elastic constants, if a change orientation of the material is not to be confused with a change of material. He argues that 'principal coordinates' for the material should be defined so that proper comparisons can be made between materials. He does this by stating that these coordinates must be such that uniform dilatation ($\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz}$) without shear is induced when only normal stresses only are applied. This imposes three conditions on the stiffness coefficients c_{ij} which can be satisfied by choosing the right orientation of the coordinate system. A similar alternative is to say that the orientation is such that only dilatation is induced, ($\gamma_{yz} = \gamma_{zx} = \gamma_{xy} = 0$), when only **hydrostatic loading**, (such that the normal stresses are all equal and the shear stresses are all zero), is applied. From (1.122), this implies that

$$s_{j1} + s_{j2} + s_{j3} = 0 \qquad (j = 4 \text{ to } 6) \qquad (1.133)$$

These three equations can be used to determine the 'principal' axes of the material.

Using (1.123) and (1.124), (1.128) can be expressed by the quadratic forms

$$W_V = \tfrac{1}{2}\boldsymbol{\tau}^{\mathrm{T}}\mathbf{S}\boldsymbol{\tau} = \tfrac{1}{2}\boldsymbol{\gamma}^{\mathrm{T}}\mathbf{C}\boldsymbol{\gamma} \qquad (1.134)$$

These must be positive, or else it would be possible to make the material do work in moving from its unloaded state. This means that these expressions are positive-definite quadratic forms, so that the principal minors of $\mathbf{S}$ and $\mathbf{C}$ are all positive.

1.8.2 Coordinate transformation

Suppose that a strain vector $\bar{\boldsymbol{\gamma}}$ in some new coordinate system is related to $\boldsymbol{\gamma}$ by

$$\bar{\boldsymbol{\gamma}} = \mathbf{T}\boldsymbol{\gamma} \tag{1.135}$$

where $\mathbf{T}$ is a transformation matrix, although not of the kind used earlier. The work done per unit volume is the same in all coordinate systems, so that the work expressed by (1.131) can also be written as

$$\boldsymbol{\gamma}'^{\mathrm{T}}\boldsymbol{\tau} = \bar{\boldsymbol{\gamma}}'^{\mathrm{T}}\bar{\boldsymbol{\tau}} = (\mathbf{T}\boldsymbol{\gamma}')^{\mathrm{T}}\bar{\boldsymbol{\tau}} = \boldsymbol{\gamma}'^{\mathrm{T}}\mathbf{T}^{\mathrm{T}}\bar{\boldsymbol{\tau}}$$

Because this holds for arbitrary $\boldsymbol{\gamma}'^{\mathrm{T}}$, it follows that

$$\boldsymbol{\tau} = \mathbf{T}^{\mathrm{T}}\bar{\boldsymbol{\tau}} \tag{1.136}$$

Then from (1.123), (1.135) and (1.136),

$$\bar{\boldsymbol{\gamma}} = \mathbf{T}\boldsymbol{\gamma} = \mathbf{T}\mathbf{S}\boldsymbol{\tau} = (\mathbf{T}\mathbf{S}\mathbf{T}^{\mathrm{T}})\bar{\boldsymbol{\tau}}$$

so that the compliance matrix in the new coordinate system is given by

$$\bar{\mathbf{S}} = \mathbf{T}\mathbf{S}\mathbf{T}^{\mathrm{T}} \tag{1.137}$$

and similarly the stiffness matrix $\bar{\mathbf{C}}$ in the new coordinate system is found from

$$\bar{\boldsymbol{\tau}} = (\mathbf{T}^{\mathrm{T}})^{-1}\boldsymbol{\tau} = (\mathbf{T}^{\mathrm{T}})^{-1}\mathbf{C}\boldsymbol{\gamma} = (\mathbf{T}^{\mathrm{T}})^{-1}\mathbf{C}\mathbf{T}^{-1}\bar{\boldsymbol{\gamma}}$$

so that

$$\bar{\mathbf{C}} = (\mathbf{T}^{\mathrm{T}})^{-1}\mathbf{C}\mathbf{T}^{-1} \tag{1.138}$$

Note that the reciprocal of $\mathbf{T}$ is not necessarily equal to its transpose. This is because the column vectors $\boldsymbol{\gamma}$ and $\boldsymbol{\tau}$ do not represent vectors in space.

The form of $\mathbf{T}$ for a clockwise rotation about the z axis, as shown in Fig. 1.4, can be deduced from (1.33), (1.56) and (1.135). It is given by

$$\mathbf{T} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 0 & 0 & 0 & \frac{1}{2}\sin 2\theta \\ \sin^2\theta & \cos^2\theta & 0 & 0 & 0 & -\frac{1}{2}\sin 2\theta \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 & \sin\theta & \cos\theta & 0 \\ -\sin 2\theta & \sin 2\theta & 0 & 0 & 0 & \cos 2\theta \end{bmatrix} \tag{1.139}$$

and its inverse, $\mathbf{T}^{-1}$, is given by replacing θ by $-\theta$ throughout. For inversion of the z axis,

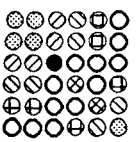
$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \mathbf{T}^{-1} \tag{1.140}$$

From (1.139), the transformation matrix for rotation through 180° about the z axis is the same as that for inversion of the z axis, as given by (1.140). This means that material symmetry under either one of these operations is the same as symmetry under the other operation.

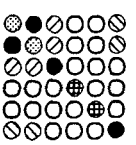
1.8.3 Material symmetry

The material may have a degree of symmetry, so that its properties are the same for different orientations of the coordinates. This means that $\mathbf{S}$ and $\bar{\mathbf{S}}$ (or $\mathbf{C}$ and $\bar{\mathbf{C}}$) are identical under certain transformations. This imposes certain conditions on the elements which can be found by comparing these matrices before and after transformation according to (1.137) and (1.138). Here are the conditions for some materials which are invariant under certain coordinate rotations or axis inversions:

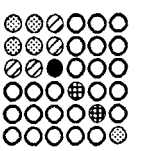
(i) **Ternary symmetry (Trigonal system)**
Invariance under rotations of 120° about the z axis.
(7 independent constants). $s_{11} = s_{22} = s_{12} + \frac{1}{2} s_{66}$;
$s_{13} = s_{23}$; $s_{14} = -s_{24} = \frac{1}{2} s_{56}$; $s_{15} = -s_{25} = -\frac{1}{2} s_{46}$; **(1.141)**
$s_{44} = s_{55}$; $s_{16} = s_{26} = s_{34} = s_{35} = s_{36} = s_{45} = 0$.



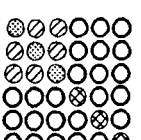
(ii) **Quadric symmetry (Tetragonal system)**
Invariance under rotations of 90° about the z axis.
(7 independent constants).
$s_{11} = s_{22}$; $s_{13} = s_{23}$; $s_{16} = -s_{26}$; $s_{44} = s_{55}$; **(1.142)**
$s_{14} = s_{15} = s_{24} = s_{25} = s_{34} = s_{35} = s_{36} = s_{45} = s_{46} = s_{56} = 0$.



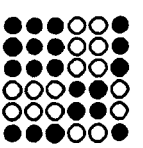
(iii) **Plane isotropy (and Hexagonal system)**
Invariance under rotations of 60° (and all other angles) about the z axis. (5 independent constants).
$s_{11} = s_{22} = s_{12} + \frac{1}{2} s_{66}$; $s_{13} = s_{23}$; $s_{44} = s_{55}$; **(1.143)**
$s_{14} = s_{15} = s_{16} = s_{24} = s_{25} = s_{26} = s_{34} = s_{35} = s_{36} = s_{45} = s_{46} = s_{56} = 0$.



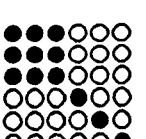
(iv) **Cubic system**
Invariance under rotations of 90° about the x,y and z axes. (3 independent constants).
$s_{11} = s_{22} = s_{33}$; $s_{12} = s_{13} = s_{23}$; $s_{44} = s_{55} = s_{66}$; **(1.144)**
$s_{14} = s_{15} = s_{16} = s_{24} = s_{25} = s_{26} = s_{34} = s_{35} = s_{36} = s_{45} = s_{46} = s_{56} = 0$.



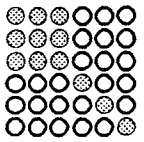
(v) **Monoclinic system**
Invariance under inversion of the z axis.
(13 independent constants).
This is the same as invariance under rotations of 180° about the z axis.
$s_{14} = s_{15} = s_{24} = s_{25} = s_{34} = s_{35} = s_{46} = s_{56} = 0$. **(1.145)**



(vi) **Orthotropy (Orthorhombic system)**
Invariance under the inversion of any two axes. Invariance under inversion of the third axis is then a necessary consequence. (9 independent constants).
$s_{14} = s_{15} = s_{16} = s_{24} = s_{25} = s_{26} = s_{34} = s_{35} = s_{36} = s_{45} = s_{46} = s_{56} = 0$.
(1.146)



(vii) Full isotropy



Invariance under any rotation about any axis (or inversion of any axis).

$$s_{11} = s_{22} = s_{33} = s_{12} + \tfrac{1}{2}s_{44} \; ;$$
$$s_{12} = s_{13} = s_{23} \; ; \; s_{44} = s_{55} = s_{66} ; \qquad \textbf{(1.147)}$$
$$s_{14} = s_{15} = s_{16} = s_{24} = s_{25} = s_{26} = s_{34} = s_{35} = s_{36} = s_{45} = s_{46} = s_{56} = 0 \, .$$

In common usage, s_{11} is defined as $1/E$, s_{12} as $-\nu/E$ and s_{44} as G. The first of the above relationships then implies (1.105).

The arrays of dots indicate how these elements appear in the **S** matrix. Black dots are independent elements, white dots are zero elements and dots with the same patterns indicate equal or related elements. The same relationships hold between the stiffnesses c_{ij} apart from the following exceptions:

(i) Ternary symmetry (Trigonal system)

$$c_{11} = c_{22} = c_{12} + 2c_{66} \; ;$$
$$c_{13} = c_{23} \; ; \; c_{14} = -c_{24} = c_{56} \; ; \; c_{15} = -c_{25} = -c_{46} \; ; \qquad \textbf{(1.141a)}$$
$$c_{44} = c_{55} \; ; \; c_{16} = c_{26} = c_{34} = c_{35} = c_{36} = c_{45} = 0 \, .$$

(iii) Plane isotropy (and Hexagonal system)

$$c_{11} = c_{22} = c_{12} + 2c_{66} \; ; \; c_{13} = c_{23} \; ; \; c_{44} = c_{55} \; ; \qquad \textbf{(1.143a)}$$
$$c_{14} = c_{15} = c_{16} = c_{24} = c_{25} = c_{26} = c_{34} = c_{35} = c_{36} = c_{45} = c_{46} = c_{56} = 0 \, .$$

(vii) Full isotropy

$$c_{11} = c_{22} = c_{33} = c_{12} + 2c_{44} = 2G(1-\nu)/(1-2\nu) \; ; \; c_{12} = c_{13} = c_{23} = 2G\nu/(1-2\nu) \; ;$$
$$c_{44} = c_{55} = c_{66} = G \; ; \qquad \textbf{(1.147a)}$$
$$c_{14} = c_{15} = c_{16} = c_{24} = c_{25} = c_{26} = c_{34} = c_{35} = c_{36} = c_{45} = c_{46} = c_{56} = 0 \, .$$

Further conditions on these stiffnesses and compliances are imposed by the positive definiteness of the quadratic forms in (1.134). For example, in the case of plane isotropy, the principal minors of the stiffness matrix **C** are positive provided that

$$c_{11} \, , \; c_{33} \, , \; c_{44} \, , \; c_{66} > 0 \quad , \quad c_{33}(c_{12} + c_{66}) > c_{13}^2 \, . \qquad \textbf{(1.148)}$$

Such conditions are also referred to as the Born conditions for stability. Applying these conditions to the isotropic form of the compliance matrix, (1.147) gives

$$1/E > 0 \quad , \quad 1/G > 0 \quad , \quad 1 - \nu^2 > 0 \quad , \quad 1 - 3\nu^2 - 2\nu^3 > 0 \, . \qquad \textbf{(1.149)}$$

The last two conditions imply that Poisson's ratio lies between -1 and ½. Normally, Poisson's ratio lies between 0.1 (concrete) and 0.49 (rubber). However, negative values of Poisson's ratio have been found†.

Values of elastic compliances and stiffnesses for anisotropic materials are given by Hearmon (1961) and Brandes (1983). Ledbetter and Kim‡ list the properties of 41 cubic elements and 21 hexagonal elements. Some of the stiffnesses are given in Table 1.1. With the exception of beechwood, these results are for single crystals. The behaviour of media composed of randomly oriented crystals, known as crystalline aggregates, will be discussed next.

†Lakes, R.S. (1987) Negative Poisson's ratio materials. *Science* **238** 551

‡See Chapter 7 of Levy and Furr (2001), Tables 7.1 and 7.2.

Table 1.1 Stiffnesses of some anisotropic materials (in gigapascals)
1 GPa = 10^9 N/m^2 ≈ 145,038 p.s.i.

Orthorhombic	c_{11}	c_{22}	c_{33}	c_{44}	c_{55}	c_{66}	c_{23}	c_{13}	c_{12}
Beechwood	1.7	15.8	3.38	1.56	0.44	1.03	2.22	1.35	1.5
Polyethylene (20°C)	5.55	9.03	257	2.88	0.83	2.51	5.95	2.73	1.67
α-Uranium	215	199	267	124	73.4	74.3	108	21.8	46.5
Hexagonal	$c_{11} = c_{22}$		c_{33}	$c_{44} = c_{55}$		c_{66}	$c_{23} = c_{13}$		c_{12}
Ice (-16°C)	13.6		14.6	3.2		3.5	5.2		6.7
Zinc	165		61.8	39.6		63	50.0		31.1
Cubic	$c_{11} = c_{22} = c_{33}$			$c_{44} = c_{55} = c_{66}$			$c_{23} = c_{13} = c_{12}$		
Aluminium	106.75			28.34			60.41		
Copper	169.68			74.493			122.55		
Diamond	1076			575.8			125		
Lead	49.66			14.98			42.31		
Nickel	248.1			124.2			154.9		

1.8.4 Crystalline aggregates

Large aggregates of randomly oriented crystals have isotropic macroscopic properties. Voigt proposed a model based on a uniform strain in all crystals, in effect producing an average of the stiffnesses c_{ij}. Reuss assumed equal stresses in all the crystals, giving properties based on averaging the compliances s_{ij}. Hill† showed that the results obtained by the methods of Voigt and Reuss should form upper and lower bounds to the true values. However, the results of Odajima and Maeda‡ indicate that the method of Reuss gives much more accurate results for strongly anisotropic crystals. The formulae for Young's modulus and the shear modulus are given by

$$E_R = 5/(3A + 2B + C) \tag{1.150}$$

$$G_R = 5/(4A - 4B + 3C) \tag{1.151}$$

where

$$\begin{aligned} 3A &= s_{11} + s_{22} + s_{33} \\ 3B &= s_{23} + s_{13} + s_{12} \\ 3C &= s_{44} + s_{55} + s_{66} \end{aligned} \tag{1.152}$$

Ledbetter considers seven methods of averaging, which all use (1.188) for the overall bulk

†Hill, R. (1952) *Proceedings of the Physical Society A* **65** 349

‡Odajima, A. (1966) *T. Journal of Polymer Science C* **15** 55

modulus[†]. He concludes that the Hershey-Kröner-Eshelby method gives the most accurate results for stainless steel. For cubic materials, the overall shear modulus, G, is a root of

$$8G^3 + (5c_{11} + 4c_{12})G^2 - c_{44}(7c_{11} - 4c_{12})G - c_{44}(c_{11} - c_{12})(c_{11} + 2c_{12}) = 0$$

1.9 ENGINEERING THEORY OF BEAMS

The fundamental assumption of the engineering theory of flexure is usually taken to be that plane sections remain plane. This assumption can be shown to be the particular case for homogeneous, prismatic, isotropic beams of a more general theory for linearly-elastic beams (and trusses), see Chapter 8 of Renton (2002) for example. This relies on the application of **Saint-Venant's principle**.

This principle considers the effects of replacing one system of loads applied to a small part of the surface of an elastic body with another system of loads, with the same resultants, acting on the same part. It states that the difference in the stresses and strains induced by the two systems is only significant at small distances from the part (comparable with the linear dimensions of the part). This difference will be induced by a loading on the part which is statically equivalent to zero loading, that is, it has zero resultant forces and moments. Let the part to which this difference in loads is applied have linear dimensions of a small order a. Then its area will be no greater than the order of a^2. Let the magnitude of the surface tractions on the part from the difference loading be of order σ. Then the surface forces will be of order σa^2 (or less). The strains at the surface will be of order σ/E. The maximum relative displacements of the part, other than those due to rigid-body motion, are then of order $\sigma a/E$. The difference loading, having no resultant, does no work during rigid-body motion. The work it does is then given by the surface forces moving through the above relative displacements and so is of order $\sigma^2 a^3/E$. The strain energy density in the region of the part is of order σ^2/E. Then the work done can be absorbed by the strain energy of this density in a volume of order a^3 in the region of the part. Stresses of order σ are then unlikely to extend beyond a linear dimension of order a from this part.

This means that the effects of the loads applied at any locality on a beam will decay towards a **characteristic response,** which is dependent only on the resultants of these loads, beyond the immediate vicinity of that locality. This characteristic response is the response at infinite distances from the locality. If the beam is prismatic, then the response to resultant axial, flexural or torsional loading is uniform. (The characteristic response to shear loading varies linearly owing to the linear variation of the bending moment.) The principle may only apply weakly in some cases where the beam is strongly anisotropic[‡] and then the engineering theory of beams is inadequate, because the beam's behaviour cannot adequately be deduced from the resultant loads on it.

Consider the case where only a resultant bending moment, torque and axial force is applied. The characteristic response is then one of constant stress and strain along the beam. Taking the z axis to lie along the beam, it follows from (1.61) that the constant state of strain along the beam implies that

[†]See Chapter 17 of Levy and Furr (2001).

[‡]For a rectangular strip, Horgan (1982, *Journal of Composite Materials* **16** 411) shows that the characteristic decay length for a transversely isotropic rectangular strip of breadth b is of the order of $b\sqrt{(s_{44}/s_{33})}$.

$$\frac{\partial^2 u_x}{\partial x \partial z} = \frac{\partial^2 u_y}{\partial y \partial z} = \frac{\partial^2 u_z}{\partial z^2} = 0 ,$$
$$\frac{\partial^2 u_x}{\partial y \partial z} + \frac{\partial^2 u_y}{\partial x \partial z} = \frac{\partial^2 u_y}{\partial z^2} + \frac{\partial^2 u_z}{\partial y \partial z} = \frac{\partial^2 u_z}{\partial x \partial z} + \frac{\partial^2 u_x}{\partial z^2} = 0 . \tag{1.153}$$

The most general form of solution of these equations, excluding rigid-body rotations about the x and y axes, is

$$\begin{aligned} u_x &= -yz\vartheta_z + \tfrac{1}{2}\vartheta_y z^2 + U(x,y) \\ u_y &= xz\vartheta_z - \tfrac{1}{2}\vartheta_x z^2 + V(x,y) \\ u_z &= z(\varepsilon_z - \vartheta_y x + \vartheta_x y) + W(x,y) \end{aligned} \tag{1.154}$$

where ϑ_x, ϑ_y and ϑ_z are the (overall) rates of rotation about the x, y and z axes and ε_z is the (overall) tensile axial strain.

Because the stresses are also constant along the beam, the conditions of internal equilibrium given by (1.100a) to (1.100c) in the absence of body forces and accelerations become

$$\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} &= 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} &= 0 \\ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} &= 0 \end{aligned} \tag{1.155}$$

and from (1.82) to (1.84), the conditions on the (unloaded) sides of the beam are

$$\begin{aligned} \sigma_{xx} n_1 + \tau_{xy} n_2 &= 0 \\ \tau_{xy} n_1 + \sigma_{yy} n_2 &= 0 \\ \tau_{xz} n_1 + \tau_{yz} n_2 &= 0 \end{aligned} \tag{1.156}$$

There are four independent solutions of (1.154) which satisfy (1.155), (1.156) and the stress-strain relationships. These are most readily derived in terms of compliances, using (1.122) and (1.61). Lekhnitskii (1981) expresses these in terms of the **principal axes** x and y of the cross-section. These axes are such that

$$\int_A x \, \mathrm{d}A = \int_A y \, \mathrm{d}A = \int_A xy \, \mathrm{d}A = 0 \tag{1.157}$$

where the integration is over the area A of the cross-section. For a simple resultant axial force P acting along the **centroidal axis**, the line with coordinates $(0, 0, z)$,

$$\begin{aligned} u_x &= \frac{P}{A}\left(s_{13} x + \tfrac{1}{2} s_{36} y\right) \\ u_y &= \frac{P}{A}\left(\tfrac{1}{2} s_{36} x + s_{23} y\right) \\ u_z &= \frac{P}{A}\left(s_{33} z + s_{35} x + s_{34} y\right) \end{aligned} \tag{1.158}$$

Comparing this with (1.154), it can be seen that there is an overall strain ε_z equal to Ps_{33}/A.

A moment M_x producing only an axial stress σ_{zz} equal to $M_x y/I_{xx}$, where I_{xx} is the second moment of area of the cross-section about the x axis, produces deflections of the form

$$\begin{aligned} u_x &= \frac{M_x}{2I_{xx}}\left(s_{35}yz + 2s_{13}xy + s_{36}y^2\right) \\ u_y &= \frac{M_x}{2I_{xx}}\left(-s_{35}xz - s_{33}z^2 - s_{13}x^2 + s_{23}y^2\right) \\ u_z &= \frac{M_x}{2I_{xx}}\left(2s_{33}yz + s_{35}xy + s_{34}y^2\right) \end{aligned} \qquad \textbf{(1.159)}$$

Again, on making a comparison of these deflections with the forms given by (1.154), flexure and torsion are seen to be coupled, because

$$\vartheta_z = -\frac{M_x}{2I_{xx}}s_{35} \quad , \quad \vartheta_x = \frac{M_x}{I_{xx}}s_{33} \qquad \textbf{(1.160)}$$

Notice also that the last term in the expression for u_z shows that plane sections do not in general remain plane during flexure. A similar result can be found for the effects of a moment about the y axis, giving

$$\vartheta_z = -\frac{M_y}{2I_{yy}}s_{34} \quad , \quad \vartheta_y = \frac{M_y}{I_{yy}}s_{33} \qquad \textbf{(1.161)}$$

The fourth solution relates to the torsion of a beam. In general, this is very much more complex than the Saint-Venant solution for isotropic beams, except for hexagonal materials, as described by (1.143). Then the Saint-Venant solution can be applied in a modified form, taking c_{44} as the shear modulus, G. Lekhnitskii (ibid. Chapter 6) describes methods for analysing the torsion of anisotropic beams and there are some particular solutions, such as those found by Saint-Venant[†] for the torsion of an orthotropic rod of elliptic cross-section. This problem also reduces to that for an isotropic rod after suitable changes of variable. Betti's reciprocal theorem can be invoked to show that if a bending moment produces torsion, then a torque T will produce flexure. From (1.160) and (1.161), this is given by

$$\vartheta_x = -\frac{T}{2I_{xx}}s_{35} \quad , \quad \vartheta_y = -\frac{T}{2I_{yy}}s_{34} \qquad \textbf{(1.162)}$$

In the engineering theory of beams, these characteristic responses for uniform extension, flexure and torsion are used as approximations to the problems of variable resultant moments and axial loading. Unless these resultants vary extremely rapidly along the beam, it will adjust to give almost exactly the local characteristic response to the resultants. The resulting rates of rotation and displacement can then be integrated along the beam to give accurate estimates of the deflections of the beam.

The analysis of beams with curvilinear anisotropy is discussed in Appendix A1.

[†]Saint-Venant, A.J.C.B. (1856) Mémoire sur la torsion des prismes. *Mémoires présentés par divers savants à l'académie des sciences. Sciences math. et phys.* **14** 233-560

1.10 ENGINEERING THEORY OF PLATES

As in the previous section, this theory of plates is based on the examination of a uniform state of stress and strain. This time the state is uniform in the x and y directions, which lie in the middle surface of the plate of thickness h as shown in Fig. 1.14. This time, it is not possible to justify its use on the strict application of Saint-Venant's principle. The resultant flexural moments, M_1 and M_2, and torsional moments, H_1 and H_2 per unit length are given by

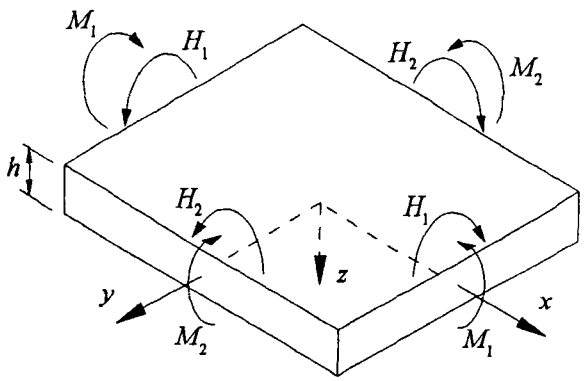


Figure 1.14 A plate element in pure flexure and twist.

$$M_1 = \int_{-h/2}^{h/2} \sigma_{xx} z\,dz, \quad M_2 = \int_{-h/2}^{h/2} \sigma_{yy} z\,dz, \quad H_1 = \int_{-h/2}^{h/2} \tau_{xy} z\,dz = \int_{-h/2}^{h/2} \tau_{yx} z\,dz = H_2 \ (= H).$$

As before, the body forces and accelerations will be taken as zero, so that the internal equilibrium conditions given by (1.100a) to (1.100c) reduce to

$$\frac{\partial \tau_{xz}}{\partial z} = \frac{\partial \tau_{yz}}{\partial z} = \frac{\partial \sigma_{zz}}{\partial z} = 0 \tag{1.163}$$

as the stresses vary only in the z direction. The conditions on the free surfaces $z=\pm h/2$ are

$$\tau_{xz} = \tau_{yz} = \sigma_{zz} = 0 \tag{1.164}$$

But it follows from (1.163) that these conditions apply throughout the body. Under the action of the moments M_1, M_2 and H†, a uniform strain solution can be found of the form

$$\begin{aligned}
u_x &= \frac{6}{h^3}[M_1(s_{15}z^2 + s_{16}yz + 2s_{11}xz) + M_2(s_{25}z^2 + s_{26}yz + 2s_{12}xz) \\
&\quad + H(s_{56}z^2 + s_{66}yz + 2s_{16}xz)] \\
u_y &= \frac{6}{h^3}[M_1(s_{14}z^2 + s_{16}xz + 2s_{12}yz) + M_2(s_{24}z^2 + s_{26}xz + 2s_{22}yz) \\
&\quad + H(s_{46}z^2 + s_{66}xz + 2s_{26}yz)] \\
u_z &= \frac{6}{h^3}[M_1(s_{13}z^2 - s_{11}x^2 - s_{16}xy - s_{12}y^2) + M_2(s_{23}z^2 - s_{12}x^2 - s_{26}xy - s_{22}y^2) \\
&\quad + H(s_{36}z^2 - s_{16}x^2 - s_{66}xy - s_{26}y^2)]
\end{aligned} \tag{1.165}$$

†The fact that $H_1=H_2=H$ results from the equality of τ_{xy} and τ_{yx}. There are more general problems where this is not true (see Appendix A6). Also, the same methods can be applied to a grid, where for example H_1 may be zero even though it has some of the characteristics of an isotropic plate. See Renton, J.D. , *IASS Symposium*, Nagoya 2001, Paper TP062.

where the stresses are given by

$$\sigma_{xx} = 12M_1 z/h^3 , \quad \sigma_{yy} = 12M_2 z/h^3 , \quad \tau_{xy} = 12Hz/h^3 \tag{1.166}$$

see for example Leknitskii (1981) section 16. In Kirchhoff's plate theory, it is assumed that straight lines perpendicular to the middle surface remain straight during bending. The quadratic terms in z in the first two of equations (1.165) show that this is not always true for anisotropic materials.

The above uniform-state solution will be taken as a suitable approximation for conditions of variable loading. This mode will be characterised by the variable W which is the transverse displacement of the middle surface of the plate. From the third of equations (1.165), this is

$$W = -\frac{6}{h^3}[M_1(s_{11}x^2 + s_{16}xy + s_{12}y^2) + M_2(s_{12}x^2 + s_{26}xy + s_{22}y^2) + H(s_{16}x^2 + s_{66}xy + s_{26}y^2)] \tag{1.167}$$

so that

$$\begin{bmatrix} \dfrac{\partial^2 W}{\partial x^2} \\ \dfrac{\partial^2 W}{\partial y^2} \\ 2\dfrac{\partial^2 W}{\partial x \partial y} \end{bmatrix} = -\frac{12}{h^3}\begin{bmatrix} s_{11} & s_{12} & s_{16} \\ s_{12} & s_{22} & s_{26} \\ s_{16} & s_{26} & s_{66} \end{bmatrix}\begin{bmatrix} M_1 \\ M_2 \\ H \end{bmatrix} \tag{1.168}$$

The inverse of this equation takes the form

$$\begin{bmatrix} M_1 \\ M_2 \\ H \end{bmatrix} = -\begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix}\begin{bmatrix} \dfrac{\partial^2 W}{\partial x^2} \\ \dfrac{\partial^2 W}{\partial y^2} \\ 2\dfrac{\partial^2 W}{\partial x \partial y} \end{bmatrix} \tag{1.169}$$

where, for isotropic materials,

$$D_{11} = D_{22} = D = \frac{Eh^3}{12(1-\nu^2)} , \quad D_{12} = \nu D , \quad D_{16} = D_{26} = 0 , \quad D_{66} = \tfrac{1}{2}(1-\nu)D.$$

The moments shown in Fig. 1.14 will now be considered to vary in the x and y directions. In addition, variable shear forces per unit length S_1 and S_2 and a distributed load q per unit surface area will be taken to act on the plate. Fig. 1.15 shows a δx by δy rectangular element of the plate with these loads acting. Only increments in the moments are shown, because the constant components are self-balancing.

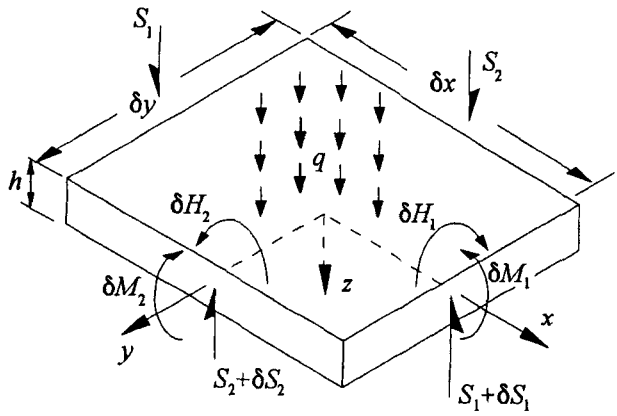


Figure 1.15 Equilibrium of a plate element.

For equilibrium,

$$\begin{aligned} S_1\,\delta y\delta x + \delta M_1\,\delta y + \delta H_2\,\delta x &= 0 \\ S_2\,\delta x\delta y + \delta M_2\,\delta x + \delta H_1\,\delta y &= 0 \\ \delta S_1\,\delta y + \delta S_2\,\delta x - q\,\delta x\delta y &= 0 \end{aligned}$$

As the size of this element tends to zero, these equations become

$$\begin{aligned} S_1 + \frac{\partial M_1}{\partial x} + \frac{\partial H}{\partial y} &= 0 \\ S_2 + \frac{\partial M_2}{\partial y} + \frac{\partial H}{\partial x} &= 0 \\ \frac{\partial S_1}{\partial x} + \frac{\partial S_2}{\partial y} &= q \end{aligned} \qquad \textbf{(1.170)}$$

using $H_1 = H_2 = H$. Then from (1.169) and (1.170), the governing equation for an anisotropic plate is found to be

$$D_{11}\frac{\partial^4 W}{\partial x^4} + 4D_{16}\frac{\partial^4 w}{\partial x^3\partial y} + (2D_{12} + 4D_{66})\frac{\partial^4 W}{\partial x^2\partial y^2} + 4D_{26}\frac{\partial^4 W}{\partial x\partial y^3} + D_{22}\frac{\partial^4 W}{\partial y^4} = q \quad \textbf{(1.171)}$$

For orthotropic plates, D_{16} and D_{26} are zero, making Fourier series solutions of (1.171) possible for distributed loading which is an arbitrary function of x and y. Solutions by this means and others are given by Hearmon (1961) Chapter 7 and Bareš and Massonnet (1968) Chapter 4.

As in the previous section, the theory may only be appropriate for weakly anisotropic materials. Methods of analysing strongly anisotropic materials have been proposed by Rogers†.

† Rogers, T.G. (1977) Deformations of strongly anisotropic materials. *Rheological acta* **16** 123-133

1.11 APPLICATIONS AND WORKED EXAMPLES

1.11.1 Graphical representation of failure criteria for isotropic materials

As mentioned in section 1.6.7, failure criteria for isotropic materials are often expressed in terms of a limiting value of some function of the stress invariants. This limiting state can be represented by a surface in three-dimensional space with Cartesian coordinates $(\sigma_1, \sigma_2, \sigma_3)$.

Consider a vector from the origin O to some point P with coordinates $(\sigma_1, \sigma_2, \sigma_3)$ as shown in Fig. 1.16. Then

$$\underline{OP} = \sigma_1 \boldsymbol{i} + \sigma_2 \boldsymbol{j} + \sigma_3 \boldsymbol{k}$$

If C is a point with coordinates (p, q, r) then

$$\underline{OC} = p\boldsymbol{i} + q\boldsymbol{j} + r\boldsymbol{k}$$

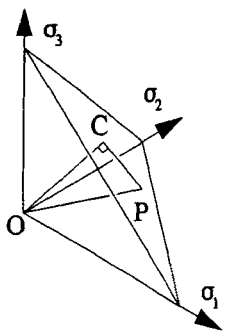


Figure 1.16 Deviatoric plane.

Suppose that $\underline{CP}$ is on the plane perpendicular to $\underline{OC}$. Then

$$\underline{OP}\cdot\underline{OC} = \underline{OC}\cdot\underline{OC} = p^2 + q^2 + r^2$$

All points P on this plane must then satisfy the equation

$$\sigma_1 p + \sigma_2 q + \sigma_3 r = p^2 + q^2 + r^2$$

If OC is the **hydrostatic** or **octahedral axis**, then $p = q = r$ and the equation of the plane perpendicular to this axis, which contains all such points P, is

$$J_1 = I_1 = \sigma_1 + \sigma_2 + \sigma_3 = 3p \qquad \textbf{(1.172)}$$

This is known as the **deviatoric plane**. The vector $\underline{CP}$ is given by

$$\underline{CP} = (\sigma_1 - p)\boldsymbol{i} + (\sigma_2 - p)\boldsymbol{j} + (\sigma_3 - p)\boldsymbol{k} = s_1\boldsymbol{i} + s_2\boldsymbol{j} + s_3\boldsymbol{k} \qquad \textbf{(1.173)}$$

so that it is a vector representation of the principal stress deviations which, from (1.92) are given by

$$s_i = \sigma_i - I_1/3 = \sigma_i - p \qquad (i = 1 \text{ to } 3)$$

The length of the vector $\underline{CP}$ is given by

$$|\underline{CP}|^2 = s_1^2 + s_2^2 + s_3^2 = (s_1 + s_2 + s_3)^2 - 2(s_1s_2 + s_2s_3 + s_3s_1) = 2J_2 \qquad \textbf{(1.174)}$$

using (1.91), (1.92) and (1.94).

It follows immediately from (1.98) that **von Mises' criterion** is represented by a cylindrical failure surface of radius $\sqrt{(2/3)}\sigma_f$ about the hydrostatic axis as shown in Fig. 1.17. From (1.99), the failure surface for the **Drucker-Prager criterion** is given by a right circular cone about the hydrostatic axis, with its apex at

$$\sigma_1 = \sigma_2 = \sigma_3 = (1 + \sqrt{3}\alpha)\sigma_f/(3\sqrt{3}\alpha)$$

This failure surface is shown in Fig.1.18. (The projections are distorted to avoid giving the impression that the hydrostatic axis coincides with the σ_2 axis.)

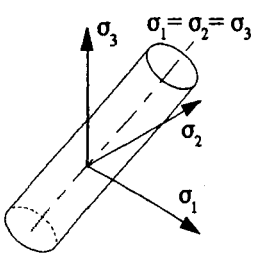


Figure 1.17 von Mises' criterion.

The directions of the principal stresses can have no influence on the failure of an isotropic material. Thus if failure of the material occurs under circumstances represented by a point (a, b, c) in the stress space, it will also occur at the points (a, c, b), (b, c, a), (c, b, a), (c, a, b) and (b, a, c). These all lie on the same deviatoric plane, at a distance $(a+b+c)/\sqrt{3}$ in the positive direction along the hydrostatic axis. This means that any failure surface will exhibit six-fold symmetry about the hydrostatic axis. Drucker [(1967), section 12.5] has discussed another condition governing the nature of the failure surfaces, which is that they are necessarily convex, if the material is stable. (Convexity means that a straight line between any two points in the volume bounded by the surface will also lie within the surface.)

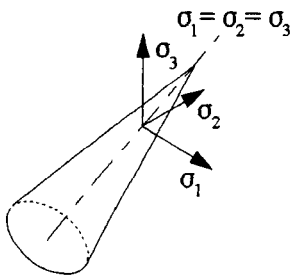


Figure 1.18 Drucker-Prager criterion.

The von Mises' criterion gives a failure surface which is a close upper bound to the experimental data for ductile materials. A close lower bound is given by **Tresca's criterion**. This states that failure occurs when the greatest local shear stress reaches a limiting value. From Mohr's circle for stress, Fig. 1.9a, a maximum shear stress is given by half the difference between the principal stresses. In three dimensions, it then follows that the criterion is given by the failure surface bounded by the six planes given by

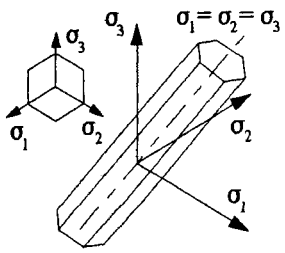


Figure 1.19 Tresca's criterion.

$$\sigma_i - \sigma_j = \sigma_f \quad (i,j = 1,2 \text{ or } 3,\ i \neq j)$$

These six planes intersect to form a regular hexagonal prism about the hydrostatic axis as shown in Fig. 1.19. The distance of this axis to a vertex of the hexagon is $\sqrt{(2/3)}\sigma_f$, so that the prism just fits inside the cylinder given by the von Mises' criterion.

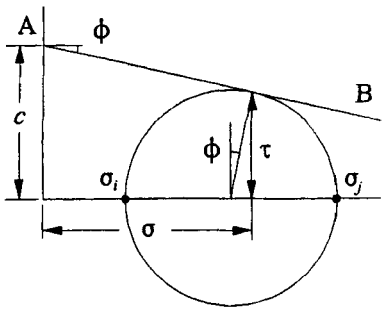


Figure 1.20 Mohr's circle for cohesion.

The Drucker-Prager criterion is suitable for materials which are brittle or have a weak tensile strength, such as concrete or soils. Another commonly used failure criterion is the **Mohr-Coulomb criterion**. This states that failure occurs when the shear stress τ on any plane reaches some multiple of the normal *compression* on it, plus some constant value, c, or

$$\tau = c - \sigma \tan\phi \qquad (1.175)$$

where c is the shear strength of the material under zero normal pressure, known as its **cohesion**. The normal stress σ is taken as tensile-positive here. It acts normal to the plane

on which τ acts. The angle ϕ is known as **the angle of friction**. From (1.175) it follows that failure commences when a Mohr's circle can be drawn which just touches the line AB shown in Fig. 1.20. This is at an angle ϕ to the σ axis and has an intercept on the τ axis at a distance c above the origin. If Mohr's circle does not meet this line, critical conditions have not been reached. Failure occurs as soon as Mohr's circle touches this line, so that (1.175) is satisfied. It will be taken that σ_i and σ_j are two of the three principal stresses in three dimensions. From Fig. 1.20, it can be seen that

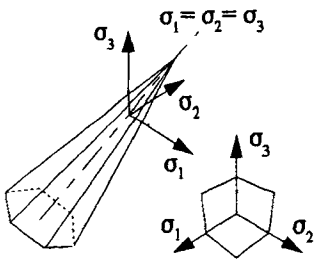


Figure 1.21 Mohr-Coulomb criterion.

$$\sigma = \tfrac{1}{2}(\sigma_i + \sigma_j) + \tau\tan\phi \ , \quad \tau = \tfrac{1}{2}(\sigma_j - \sigma_i)\cos\phi .$$

Substituting these expressions into (1.175) gives the six failure planes

$$\sigma_j(1 + \sin\phi) - \sigma_i(1 - \sin\phi) = 2c\cos\phi \quad (i,j = 1,2 \text{ or } 3,\ i \neq j)$$

This gives the hexagonal pyramidic failure surface with its apex at $\sigma_1 = \sigma_2 = \sigma_3 = c \cot \phi$ as shown in Fig. 1.21.

The irregular hexagonal form of the failure surface, as seen on the deviatoric plane at the bottom right of Fig. 1.21, is closer to most experimental failure surfaces than the circular form given by the Drucker-Prager criterion. Many continuous failure surfaces have been postulated which have a more empirically valid cross-sectional form. One of the simplest is the Lade-Duncan formula for cohesionless soils†,

$$I_1^3 - kI_3 = 0 \qquad \textbf{(1.176)}$$

As shown in Fig. 1.22, the cross-section on the deviatoric plane becomes more triangular as the value of k increases. Failure surfaces are discussed in great detail by Chen and Saleeb (1982).

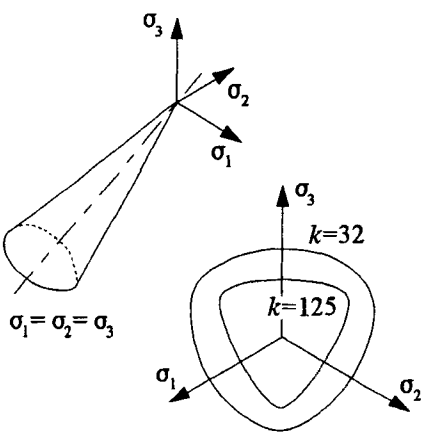


Figure 1.22 Lade-Duncan criterion.

1.11.2 Prandtl's stress function and uniform torsion (cf. sections 1.5, 1.7 & 1.9) Prandtl's stress function can be used to solve problems of Saint-Venant torsion (see section 1.9). It can be derived as a particular case of the Galerkin vector given by

$$g_1 = \frac{-G\alpha}{2(1-\nu)}x^2yz \ , \quad g_2 = \frac{G\alpha}{2(1-\nu)}xy^2z \ , \quad g_3 = g_3(x,y) \qquad \textbf{(1.177)}$$

† Lade, P.V. and Duncan, J.M. (1975) Elastoplastic stress-strain theory for cohesionless soil. *Proc. ASCE*, **101** No. GT10 1037-1053

(see section 1.7.2). Then from (1.112),

$$u_x = -\alpha yz\,, \qquad u_y = \alpha xz\,, \qquad u_z = \frac{(1-\nu)}{G}\nabla^2 g_3 \tag{1.178}$$

so that from (1.60) and (1.107),

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \tau_{xy} = 0$$

$$\tau_{xz} = -G\alpha y + (1-\nu)\nabla^2\frac{\partial g_3}{\partial x} = \frac{\partial \psi}{\partial y}\ \text{(say)} \tag{1.179}$$

$$\tau_{yz} = G\alpha x + (1-\nu)\nabla^2\frac{\partial g_3}{\partial y} = -\frac{\partial \psi}{\partial x}\ \text{(say)} \tag{1.180}$$

giving

$$(1-\nu)\nabla^2\frac{\partial g_3}{\partial x} = G\alpha y + \frac{\partial \psi}{\partial y}$$

and

$$(1-\nu)\nabla^2\frac{\partial g_3}{\partial y} = -G\alpha x - \frac{\partial \psi}{\partial x}$$

These two equations imply that

$$\nabla^2\nabla^2 g_3 = 0$$

for an arbitrary function ψ, but this follows from (1.113a). They also imply that

$$\nabla^2\psi = -2G\alpha \tag{1.181}$$

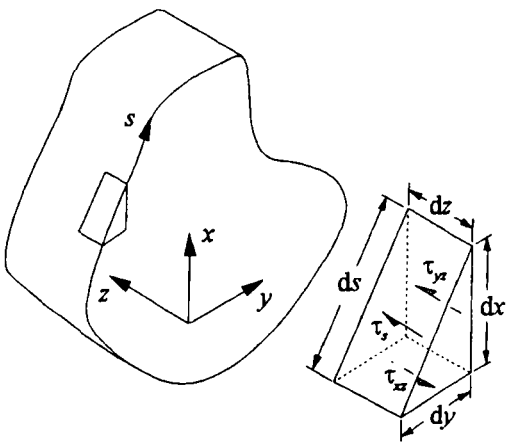


Figure 1.23 Shear equilibrium of a surface element.

where $\psi(x, y)$ is known as **Prandtl's stress function**. Some solutions of (1.181) can be found from linear combinations of the harmonic functions in section A2.1 and simple integrals of these functions.

From (1.154) and (1.178), it can be seen that α may represent the uniform rate of twist ϑ_z of a prismatic bar about its longitudinal z axis. This is the problem to which Prandtl's stress function is normally applied. Fig. 1.23 shows a short length of such a bar and the detail shows a small triangular element of it at its surface. The longitudinal shear surface shear traction, τ_s, is in equilibrium with the internal stresses so that

$$dz(\tau_s ds + \tau_{yz} dx - \tau_{xz} dy) = 0$$

or from (1.179) and (1.180),

$$\tau_s = \frac{\partial \psi}{\partial x}\frac{dx}{ds} + \frac{\partial \psi}{\partial y}\frac{dy}{ds} = \frac{d\psi}{ds} \tag{1.182}$$

If the surface is free from tractions, $d\psi/ds$ is zero so that ψ is constant around the boundary of the cross-section. If the cross-section is simply connected, i.e. bounded by a simple closed curve, ψ can conveniently be taken as zero on this curve. If the section has

cavities, then ψ will have constant (but generally different) values on the boundary of each cavity.

If the surface traction τ_s is zero, its complementary shear stress on the cross-section, which is in a direction normal to the surface boundary, is zero. The resultant shear stress here must then be parallel to the boundary. More generally, the resultant shear stress is parallel to any curve on which ψ is constant. These curves are known as **shear stress trajectories**.

The torque T is given by integrating the torques produced by the shear stresses τ_{zx} and τ_{zy} acting on an elementary area dA (= $dxdy$) over the whole cross-sectional area A as shown in Fig. 1.24, so that

$$T = \int_A (\tau_{zy}x - \tau_{zx}y)\,dA$$
$$= -\int_A \left(x\frac{\partial \psi}{\partial x} + y\frac{\partial \psi}{\partial y} \right) dx\,dy \qquad \textbf{(1.183)}$$

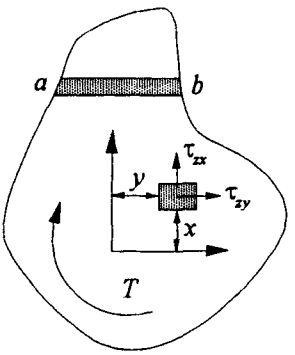


Figure 1.24 Torque on a cross-section.

Integrating along a typical strip ab with a constant x ordinate,

$$\int_a^b y\frac{\partial \psi}{\partial y}\,dy = |\psi y|_a^b - \int_a^b \psi\,dy = -\int_a^b \psi\,dy$$

as ψ can be taken as zero at the boundary points a and b. Integrating over all such strips,

$$-\int\left(\int y\frac{\partial \psi}{\partial y}\,dy \right) dx = \int_A \psi\,dy\,dx = \int_A \psi\,dA$$

A similar result can be obtained for the other term in the integrand of (1.183) by integrating over strips of constant y. Then (1.183) becomes

$$T = 2\int_A \psi\,dA \qquad \textbf{(1.184)}$$

For an **elliptic section**, taking

$$\psi = -G\alpha(b^2x^2 + a^2y^2 - a^2b^2)/(a^2 + b^2)$$

satisfies (1.181) and is zero on an ellipse with semi-axis lengths a and b. Let $X = x/a$, $Y = y/b$. Then the elliptic boundary becomes the circle $X^2 + Y^2 = 1$ and so

$$T = 2\int_A \psi\,dA = -\frac{2G\alpha a^3b^3}{a^2 + b^2}\int_{A'} (X^2 + Y^2 - 1)\,dA'$$

where the second integration is over the area A' of the unit circle. Using polar coordinates,

$$X = r\cos\theta \quad , \quad Y = r\sin\theta \quad , \quad dA' = r\,d\theta\,dr$$

gives

$$T = -\frac{2G\alpha a^3b^3}{a^2 + b^2}\int_0^{2\pi}\left(\int_0^1 (r^2 - 1)r\,dr \right) d\theta = \frac{\pi G\alpha a^3b^3}{a^2 + b^2}$$

so that the **torsional stiffness** of the section is

$$\frac{T}{\alpha} = \frac{G\pi a^3 b^3}{a^2 + b^2} \tag{1.185}$$

1.11.3 Bulk modulus

(cf. section 1.8)

The bulk modulus is given by the hydrostatic loading σ required to produce a unit **volumetric strain**. The volumetric strain is the ratio of the change in volume of a small element of the material to its initial volume. Consider an elementary cuboid with side lengths dx, dy and dz. This has an initial volume V given by $dx \times dy \times dz$. After a small amount of distortion, the volume of the cuboid becomes

$$V + \delta V = (1 + \varepsilon_{xx})dx(1 + \varepsilon_{yy})dy(1 + \varepsilon_{zz})dz \cong (1 + \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz})dxdydz$$

ignoring effects which are of a second order of smallness. Under the loading σ,

$$\delta V/V \cong \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \sigma(s_{11} + s_{12} + s_{13} + s_{21} + s_{22} + s_{23} + s_{31} + s_{32} + s_{33}) \tag{1.186}$$

from (1.122). (This may be simplified by using the symmetry of the compliance matrix **S**.) The **bulk modulus** K is then

$$K = \sigma/(\delta V/V) \tag{1.187}$$

so that

$$K = 1/\sum_{i=1}^{3}\sum_{j=1}^{3} s_{ij} \tag{1.188}$$

from (1.186). Then from (1.147), for an isotropic material

$$K = E/3(1 - 2\nu) \tag{1.189}$$

1.11.4 A tension test

(cf. 1.3, 1.5, 1.8 and 1.9)

A long, prismatic bar of cross-sectional area A is subject to a tensile axial force P as shown in Fig. 1.25. The section is in the form of a right-angled isosceles triangle with equal sides ab and bc. **Strain gauge rosettes** are attached to the lateral surfaces of the bar. A rosette consists of three gauges in different directions enabling the local state of strain on a surface to be found. Each gauge measures the change of length in its own direction in terms of the change in its electrical resistance. Here, pairs of gauges in each rosette are at right angles to one another (1 & 3, 4 & 6 and 7 & 9). The third gauge in each rosette (2, 5 and 8 respectively) is at 45° to these pairs. If the gauges are sufficiently far from the ends of the bar, they can be used to determine up to six elastic constants for an anisotropic material. From (1.158) and (1.61),

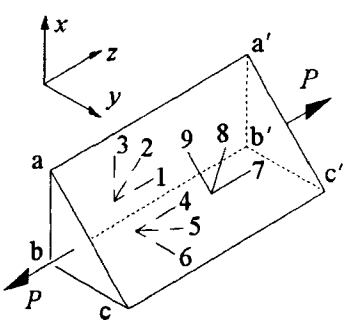


Figure 1.25 Strain gauges on a test specimen.

$$\begin{aligned} \varepsilon_{xx} &= s_{13}P/A\ , \quad \varepsilon_{yy} = s_{23}P/A\ , \quad \varepsilon_{zz} = s_{33}P/A\ , \\ \gamma_{yz} &= s_{34}P/A\ , \quad \gamma_{zx} = s_{35}P/A\ , \quad \gamma_{xy} = s_{36}P/A\ . \end{aligned} \tag{1.190}$$

The rosette 1,2,3 is on the vertical face abb'a' of the bar. The strain e_1, measured by gauge 1, is the strain ε_{zz} and e_3, measured by gauge 3, is ε_{xx}. Then from (1.190), if the force P and the area A are measured, s_{33} and s_{13} can be found. The strain measured by gauge 2 can be found from a coordinate transformation, rotating the x and z axes through 45° clockwise about the y axis. The corresponding transformation matrix is

$$\mathbf{T}_A = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

Using this matrix to transform the strain matrix $\mathbf{e}$ into the matrix $\bar{\mathbf{e}}$ in accordance with (1.59) gives the strain in the new z direction as

$$\bar{\varepsilon}_{zz} = e_2 = \tfrac{1}{2}(\varepsilon_{xx} + \gamma_{xz} + \varepsilon_{zz}) = (s_{13} + s_{35} + s_{33})P/2A$$

As s_{13} and s_{33} have been found already, s_{35} can be found from this equation.

Likewise, from the rosette 4,5,6 on the horizontal face bcc'b', the strains e_4 and e_6 can be used to determine the compliances s_{33} and s_{23}. Rotating the axes y and z through 45° clockwise about the x axis aligns the new y axis with gauge 5. The necessary transformation matrix is

$$\mathbf{T}_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

giving $$e_5 = \bar{\varepsilon}_{yy} = (s_{23} + s_{34} + s_{33})P/2A$$

so that s_{34} can now be found.

On the face acc'a' gauge 7 measures ε_{zz}. If the coordinate system is rotated through 45° anticlockwise about the z axis, the new $\bar{x}$ axis is aligned with gauge 9. The transformation matrix for this case is

$$\mathbf{T}_C = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

cf. (1.33). This gives $$e_9 = \bar{\varepsilon}_{xx} = (s_{13} - s_{36} + s_{23})P/2A$$

so that now s_{36} can be found.

For gauge 8, a second transformation, in addition to the one above, is necessary. The $\bar{x}$ and $\bar{z}$ axes are rotated through 45° clockwise about the $\bar{y}$ axis so that the new $\bar{\bar{z}}$ axis is aligned with gauge 8. The second transformation matrix is exactly the same as $\mathbf{T}_A$. Then the total transformation matrix, $\mathbf{T}_D$, for the coordinate change from the initial state to the final state is

$$\mathbf{T}_D = \mathbf{T}_A\mathbf{T}_C = \begin{bmatrix} 1/2 & -1/2 & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/2 & -1/2 & 1/\sqrt{2} \end{bmatrix}$$

and this gives $e_8 = \bar{\bar{\varepsilon}}_{zz} = [s_{13} - s_{36} + s_{23} + \sqrt{2}(s_{35} - s_{34}) + 2s_{33}]P/4A$

As all six compliances in (1.190) have already been found, e_8 merely provides a check on the other experimental values. The above results could also be obtained by using transformations of the type given by (1.135).

PROBLEMS

Problems are categorised as short (S), medium (M) or long (L). Answers will be found near the end of the book.

(1.1) Show that

$$\mathbf{T} = \frac{1}{\sqrt{6}}\begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} & -\sqrt{3} \\ -2 & 1 & 1 \end{bmatrix}$$

is a valid transformation matrix. (M)

(1.2) Suppose that the axes $\bar{x}$, $\bar{y}$, $\bar{z}$ are generated by a three clockwise rotations of the axes x, y, z. The first is a rotation θ about the x axis, the second is a rotation ϕ about the new y axis and the third is a rotation ψ about the new z axis. (These are known as the **Euler angles**.) Show that the transformation matrix $\mathbf{T}$ in (1.32) can be obtained as the product of three matrices of the kind used in (1.33) and is given by

$$\mathbf{T} = \begin{bmatrix} \cos\theta\cos\phi & \sin\theta\cos\phi & -\sin\phi \\ \cos\theta\sin\phi\sin\psi - \sin\theta\cos\psi & \sin\theta\sin\phi\sin\psi + \cos\theta\cos\psi & \cos\phi\sin\psi \\ \cos\theta\sin\phi\cos\psi + \sin\theta\sin\psi & \sin\theta\sin\phi\cos\psi - \cos\theta\sin\psi & \cos\phi\cos\psi \end{bmatrix}$$

Show that any such product of three transformation matrices satisfies (1.36). (L)

(1.3) A transformation matrix is given by

$$\mathbf{T} = \begin{bmatrix} a & 1/\sqrt{2} & b \\ c & d & 2/\sqrt{5} \\ e & f & 1/\sqrt{5} \end{bmatrix}$$

Show that b must be zero. If this matrix represents a rotation angle between $\pm 90°$, show that, of the possible values of a and d, the positive ones must be chosen. Find the values of the other elements c, e and f, and the angle of rotation. Show that a vector in the direction of the axis of rotation must be a multiple of $\{ 1+\sqrt{2}, 1, \frac{1}{2}(1+\sqrt{5}) \}$. [Use (1.36) and (1.12) to show that $\mathbf{TT}^{\mathrm{T}} = \mathbf{I}$ too. The trace of a transformation matrix is invariant and can be used to find the overall rotation. A vector along the axis of rotation is unchanged by the transformation.] (L)

(1.4) Express the rotation matrix $\boldsymbol{\omega}$ in terms of the differentials of the displacements u_x, u_y and u_z. Show that if the local strains are zero, the motion of P′ relative to P in Fig. 1.5

is given by $[\,\boldsymbol{\theta}\,]\mathbf{dr}$ where $\boldsymbol{\theta}$ is the column vector representation of a small rotation. (S)

(1.5) Show that if the strain matrix is zero at all points within a body, then the most general form of the deflection is given by

$$\mathbf{u} = \{\ a + dy + ez,\ b - dx + fz,\ c - ex - fy\ \}$$

Find the rotation matrix $\boldsymbol{\omega}$. A point P′ in the body has a position $\mathbf{dr} = \{\ dx, dy, dz\ \}$ relative to a point P in the body. Find the motion of the point P′ relative to P in terms of $\boldsymbol{\omega}$ and $\mathbf{dr}$. What are the rotations of the body about the x, y and z axes? (M)

(1.6) A sheet of elastic material is reinforced with parallel, axially-rigid fibres at some angle θ to the x axis. The displacements of the sheet in the xy plane are given by

$$u = (a-1)x + by\ , \qquad v = cx + (d-1)y\ .$$

Show from first principles that the coefficients a to d must satisfy the condition

$$(a^2 + c^2)\cos^2\theta + 2(ab + cd)\sin\theta\cos\theta + (b^2 + d^2)\sin^2\theta = 1\ .$$

A second set of axes, x' and y', are at the same angle θ to the x and y axes respectively. What are the strains with respect to these axes? Explain the apparent difference between the above condition and setting ε'_{xx} to zero. (M)

(1.7) The local principal stresses in a body are 60, 40 and 10 megapascals. Find the stresses with respect to the axes which are related to the principal axes by the transformation matrix $\mathbf{T}$ of problem (1.1). (M)

(1.8) If the local state of stress in a body is given in megapascals by the matrix

$$\boldsymbol{\sigma} = \begin{bmatrix} 20 & -10 & -10 \\ -10 & 0 & 10 \\ -10 & 10 & 0 \end{bmatrix}\ .$$

find the stress deviations and the principal stresses. What are the invariants of the stress matrix? (M)

(1.9) An elementary octahedron is formed by triangular surfaces, the normals to which are each at equal angles to the principal stress coordinates (cf. Fig. 1.8). Show that the normal stress σ_o on these surfaces, known as the **octahedral stress**, is given by the average of the principal stresses. Show also that the resultant shear stress τ_o on these surfaces is given by

$$\tau_o^2 = \tfrac{1}{3}[(s_1 - s_2)^2 + (s_2 - s_3)^2 + (s_3 - s_1)^2]$$

where s_1, s_2 and s_3 are the principal stress deviations. (M)

(1.10) Using Mohr's circle, show that with respect to rotations of the coordinate system about the z axis only, $\sigma_{xx}+\sigma_{yy}$ and $\sigma_{xx}\sigma_{yy} - \tau^2_{xy}$ are invariant. (S)

(1.11) Show that

$$\sigma_{xx}^4 + \sigma_{yy}^4 + \sigma_{zz}^4 + 2(\tau_{yz}^2 + \tau_{zx}^2 + \tau_{xy}^2)^2 + \tau_{xy}^2(\sigma_{xx}^2 + \sigma_{xx}\sigma_{yy} + \sigma_{yy}^2) +$$
$$\tau_{yz}^2(\sigma_{yy}^2 + \sigma_{yy}\sigma_{zz} + \sigma_{zz}^2) + \tau_{zx}^2(\sigma_{zz}^2 + \sigma_{zz}\sigma_{xx} + \sigma_{xx}^2) + 8(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})\tau_{xy}\tau_{yz}\tau_{zx}$$

is invariant by expressing it in terms of I_1, I_2 and I_3. (L)

(1.12) For a linearly-elastic, isotropic body, find expressions for the strain invariants K_1 and K_2 in terms of the stress invariants I_1 and I_2, Young's modulus and Poisson's ratio. The strain energy per unit volume of such a body is given by (1.128). How is this expressed in terms of Young's modulus, the shear modulus and (i) the stress invariants and (ii) the strain invariants? (M)

(1.13) If the body force vector **f** in (1.108) is not zero, show that the components of the Galerkin vector must satisfy the equation

$$(1 - \nu)\nabla^2\nabla^2 g_i = -\rho f_i \qquad (i = 1 \text{ to } 3). \qquad \text{(S)}$$

(1.14) A Rayleigh wave in the earth's crust is described by the displacement field

$$\boldsymbol{u} = \left(Ae^{-\omega mz} - Bcne^{-\omega nz}\right)\sin\left(t - \frac{x}{c}\right)\boldsymbol{i} - \left(Acme^{-\omega mz} - Be^{-\omega nz}\right)\cos\left(t - \frac{x}{c}\right)\boldsymbol{k}$$

From the equations of motion of an isotropic medium, find expressions for the terms m and n. If the earth's surface is taken to be the plane $z = 0$, what conditions constrain the amplitudes A and B and the speed of propagation c? (M)

(1.15) Show that an elastic medium with ternary symmetry as given by (1.141a) is stable if four simple conditions are met. (M)

(1.16) If Novozhilov's condition, given by (1.133), were applied to any of the materials described by (1.141) to (1.147), would the number of independent elastic constants used to describe any of the materials be reduced? (S)

(1.17) Can Novozhilov's condition for the monoclinic system be satisfied by a simple rotation about the z axis? (M)

(1.18) For a plane isotropic material (hexagonal system), find the compliances s_{ij} in terms of the stiffnesses c_{ij}. (M)

(1.19) A prismatic bar has a circular cross-section of radius R. It acts as a cantilever with a transverse force V in the y direction applied to the free end of the bar ($z = 0$), the z axis being along the centroidal axis of the bar. According to the usual engineering theory, the stress distribution induced is

$$\sigma_{xx} = \sigma_{yy} = 0, \quad \sigma_{zz} = -Vyz/I, \quad \tau_{xy} = \tau_{zx} = 0, \quad \tau_{yz} = V(y^2 - R^2)/3I, \quad \text{where } I = \pi R^4/4.$$

By integrating τ_{yz} over the cross-section, show that the resultant of the shear stress is V.

Does this stress distribution satisfy the equations of internal equilibrium? Does it satisfy the condition that the tractions on the sides of the bar ($x^2+y^2=R^2$) are zero? (M)

(1.20) For an orthorhombic (i.e. orthotropic) material, find the elements of the matrix in (1.169) in terms of the stiffnesses c_{ij}. (L)

(1.21) A square beechwood plate is simply supported on its edges, which are of length L. It carries a sinusoidally varying lateral load given by $q = q_0 \cos \pi x/L \cos \pi y/L$. (The origin is at the centre of the plate and the axes are parallel to its edges.) The thickness of the plate is 10mm, L is 1m and q_0 is 1 kilopascal. Using the results of the previous question and taking the lateral deflection W to be of the same form as the loading, find the deflection of the centre of the plate. (S)

(1.22) Show that equations (1.118) imply that $\dfrac{\partial}{\partial x}\left(\nabla^2 g_2\right) = -2\dfrac{\partial^2 \lambda}{\partial x \partial y} + A$, where A is a constant. If A is non-zero, show that if

$$\bar{\lambda} = \lambda + \tfrac{1}{2}Axy, \qquad \frac{\partial g_2}{\partial y} = \bar{\phi} - 2(1-\nu)\bar{\lambda} \qquad \text{and} \qquad \bar{\phi} = \bar{\psi} + x\frac{\partial \bar{\lambda}}{\partial x},$$

then $\nabla^2 \bar{\psi} = \nabla^2 \bar{\lambda} = 0$ and that (1.119) and (1.120) hold for $\bar{\phi}$. (M)

(1.23) Is the bulk modulus given by (1.188) the same in all coordinate systems? Check whether this is so for the materials given by (i) to (vi) of section 1.8.3. (L)

(1.24) Do the Reuss expressions for the moduli given by (1.150) and (1.151) imply the bulk modulus given by (1.188)? (M)

(1.25) Show that for an isotropic elastic body in a state of plane stress, the equations of motion can be written as

$$2\frac{\partial^2 u_x}{\partial x^2} + (1-\nu)\frac{\partial^2 u_x}{\partial y^2} + (1+\nu)\frac{\partial^2 u_y}{\partial x \partial y} = \frac{\rho}{G}(1-\nu)\ddot{u}_x$$

$$2\frac{\partial^2 u_y}{\partial y^2} + (1-\nu)\frac{\partial^2 u_y}{\partial x^2} + (1+\nu)\frac{\partial^2 u_x}{\partial x \partial y} = \frac{\rho}{G}(1-\nu)\ddot{u}_y$$

in the absence of body forces (other than inertial). Taking a solution of the form

$$u_x = \sum_{i=1}^{2} u_i \sin(\lambda x + \alpha)\sinh m_i y \sin(\omega t + \beta)$$

$$u_y = \sum_{i=1}^{2} v_i \cos(\lambda x + \alpha)\cosh m_i y \sin(\omega t + \beta)$$

find the conditions from which the natural frequencies of vibration of a plane strip $2c$ deep can be determined, for a given (flexural) wavelength of $2\pi/\lambda$. (M)

2

Cartesian tensors

One of the advantages of vector equations is that they can be written without reference to any coordinate system. A physical entity such as a force can be represented by a single mathematical entity - a vector. Likewise, the total state of stress (or strain) at a point can be thought of as a single entity and represented by a matrix. Relationships between stress and strain matrices can be written and manipulated in a coordinate-free form, as seen in section 1.7 for example. This does not mean that a force is a vector and a state of stress is a matrix; these are merely representations. Tensor notation is a very useful alternative form of representation. Tensor equations can also be thought of as coordinate-free in that they are not written in relation to specific coordinates but can be interpreted in any system. For this to be possible, all the terms in the equation must transform from one coordinate system to another in the same way. A tensor is defined by the way in which it transforms, so that in a tensor equation each term, which may be a product of tensors, must be the same kind of tensor. Because the transformations are linear, sums and differences of such terms will transform in the same way. Tensor equivalents of the vectors and matrices in the last chapter can be found and other entities, which are beyond the scope of such descriptions, can be expressed by tensors. The general theory is not limited to Cartesian coordinates, although the following chapter will be devoted exclusively to such Cartesian tensors.

2.1 VECTOR AND MATRIX REPRESENTATION

The notation used here is essentially the same as that of the subscripted elements in section 1.1. The numerical value of the subscript is associated with a particular coordinate axis. Thus the coordinates of a point can be represented by x_j, where each ordinate has a particular value of j. In three dimensions,

$$x_1 = x \quad , \quad x_2 = y \quad , \quad x_3 = z \quad . \tag{2.1}$$

It is also useful to represent the unit base vectors in a similar fashion in terms of $\boldsymbol{i}_j$ where

$$\boldsymbol{i}_1 = \boldsymbol{i} \quad , \quad \boldsymbol{i}_2 = \boldsymbol{j} \quad , \quad \boldsymbol{i}_3 = \boldsymbol{k} \tag{2.2}$$

The position vector of a point is then

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} = x_1\boldsymbol{i}_1 + x_2\boldsymbol{i}_2 + x_3\boldsymbol{i}_3 = x_k\boldsymbol{i}_k \tag{2.3}$$

where the repeated Latin index in the last expression automatically indicates summation over all values of k, in the above case from one to three. Because k never refers to an individual value of an index, it is known as a **dummy index**. Where a repeated index is not intended to indicate summation, a Greek letter will be used. Where a term is expressed in relation to a second coordinate system, a bar will be marked over it, as in Chapter 1. Thus the second system itself is denoted by $\bar{x}_m$ and its base vectors by $\boldsymbol{i}_m$ so that $\boldsymbol{r}$ is also given by $\bar{x}_m \boldsymbol{i}_m$.

More generally, a vector $\boldsymbol{v}$ may be expressed as $v_j \boldsymbol{i}_j$ and a vector $\boldsymbol{w}$ as $w_k \boldsymbol{i}_k$. A linear relationship between these two vectors can be written it the form $v_j = a_{jk} w_k$ where it is understood that the equation holds for all values of j, and that there is summation over all values of k. Thus for example in three dimensions it represents the three equations

$$\begin{aligned} v_1 &= a_{11}w_1 + a_{12}w_2 + a_{13}w_3 \\ v_2 &= a_{21}w_1 + a_{22}w_2 + a_{23}w_3 \\ v_3 &= a_{31}w_1 + a_{32}w_2 + a_{33}w_3 \end{aligned}$$

These equations could equally well have been written as $v_p = a_{pq} w_q$. Such changes of index are often necessary when substituting the relationship from one equation into another. It will be assumed that the reader is able to infer such changes of index where they are made. In the above equations, v_j (or v_p) and w_k (or w_q) are first-order tensors and a_{jk} (or a_{pq}) is a second-order tensor. It would appear that this is no different from the subscript notation introduced in section 1.1 for column vectors and matrices. However, any first-order tensor may always be thought of as representing a vector, and so obeys specific transformation rules. As will be seen later, the elements of the matrix σ in (1.86) form a second-order tensor, but the elements of the column vector τ in (1.124) do not form a first-order tensor. Another difference is that there are also third and higher order tensors which have more than two indices. These would correspond to arrays in more than two dimensions and are not covered by normal matrix theory.

2.2 COORDINATE TRANSFORMATION

Two coordinate systems will not necessarily share a common origin, so that in general there will be a linear relationship between them of the form

$$\bar{x}_i = t_{ij}x_j + c_i \tag{2.4}$$

where c_i indicates the shift of origin. Small increments in $\bar{x}_i$ are given by

$$\mathrm{d}\bar{x}_i = \frac{\partial \bar{x}_i}{\partial x_j}\mathrm{d}x_j = t_{ij}\mathrm{d}x_j \tag{2.5}$$

Suppose that these are the increments in the ordinates of a point P′ relative to an adjacent point P, as shown in Fig. 1.2. Then $\mathrm{d}\bar{x}_i$ and $\mathrm{d}x_j$ are the elements of column vector representations, $\mathrm{d}\bar{\mathbf{r}}$ and $\mathrm{d}\mathbf{r}$, of $\mathrm{d}\boldsymbol{r}$ and are related by

$$\mathrm{d}\bar{\mathbf{r}} = \mathbf{T}\,\mathrm{d}\mathbf{r} \tag{2.6}$$

where the elements of $\mathbf{T}$ are t_{ij}.

Equation (1.36) can be written in terms of these elements as $t_{ik}t_{ij} = \delta_{kj}$ **(2.7)**

where δ_{kj} is the Kronecker delta. Hence from (2.5)

$$\delta_{kj}\mathrm{d}x_j = t_{ik}t_{ij}\mathrm{d}x_j = t_{ik}\mathrm{d}\bar{x}_i \qquad \text{or} \qquad \mathrm{d}x_k = \partial\bar{x}_i/\partial x_k\,\mathrm{d}\bar{x}_i \tag{2.8}$$

Because x_k and $\bar{x}_i$ can be any arbitrary Cartesian coordinates, the bar being used merely to distinguish between them, it follows that

$$\mathrm{d}\bar{x}_k = \partial x_i/\partial\bar{x}_k\,\mathrm{d}x_i \tag{2.9}$$

The terms $\mathrm{d}x_i$ form a first-order Cartesian tensor which transforms according to the rules given by (2.5) or (2.9). In more general coordinate systems, these two equations govern the transformations of different kinds of tensor, but in Cartesian coordinate systems, the two equations are equivalent. That is, from (2.5) and (2.9),

$$\frac{\partial\bar{x}_i}{\partial x_j} = t_{ij} = \frac{\partial x_j}{\partial\bar{x}_i} \tag{2.10}$$

but this equation is not generally true for curvilinear coordinate systems. Any set of terms v_j which in a new coordinate system transform to a new set of terms $\bar{v}_i$ according to the rule

$$\bar{v}_i = \frac{\partial\bar{x}_i}{\partial x_j}v_j = \frac{\partial x_j}{\partial\bar{x}_i}v_j \qquad (= t_{ij}v_j) \tag{2.11}$$

are defined as first-order Cartesian tensors. These transformation rules are exactly the same as those for the elements of a column vector representation of a vector discussed in section 1.3. The inverse relationships follow by interchanging the barred and unbarred terms, giving

$$v_i = \frac{\partial x_i}{\partial\bar{x}_j}\bar{v}_j = \frac{\partial\bar{x}_j}{\partial x_i}\bar{v}_j \qquad (= t_{ji}\bar{v}_j) \tag{2.12}$$

2.2.1 Second-order tensors

Suppose that two first-order tensors v_i and w_j are related by an equation of the form

$$v_i = A_{ij}w_j \tag{2.13}$$

(cf. 1.38). In some new coordinate system, the relationship will be

$$\bar{v}_k = \bar{A}_{kl}\bar{w}_l \tag{2.14}$$

Then from (2.11), (2.12) and (2.13),

$$\bar{v}_k = \frac{\partial x_i}{\partial\bar{x}_k}v_i = \frac{\partial x_i}{\partial\bar{x}_k}A_{ij}\frac{\partial x_j}{\partial\bar{x}_l}\bar{w}_l$$

so that on comparing this with (2.14),

$$\bar{A}_{kl} = \frac{\partial x_i}{\partial\bar{x}_k}\frac{\partial x_j}{\partial\bar{x}_l}A_{ij} \qquad (= t_{ki}t_{lj}A_{ij}) \tag{2.15}$$

Thus if the form of (2.13) is to be preserved in a new coordinate system as in (2.14), then

A_{ij} must transform according to the rule given by (2.15). This rule defines a second-order Cartesian tensor and is equivalent to (1.41). The inverse relationship again follows by interchanging barred and unbarred terms, giving

$$A_{kl} = \frac{\partial \bar{x}_i}{\partial x_k}\frac{\partial \bar{x}_j}{\partial x_l}\bar{A}_{ij} \quad (= t_{ik}t_{jl}\bar{A}_{ij})$$

from (2.10).

If an array of elements $B_{ij} = A_{ji}$ becomes the array $\bar{B}_{kl} = \bar{A}_{lk}$ in a second coordinate system, then

$$\bar{B}_{kl} = \bar{A}_{lk} = t_{lj}t_{ki}A_{ji} = t_{ki}t_{lj}B_{ij}$$

so that the array B_{ij} obeys exactly the same transformation rules as A_{ij} and so is a second-order tensor.

2.2.2 Higher-order tensors

Equations (1.56) and (1.87) show that the states of strain and stress can be expressed by matrices which transform in the same way as (1.41). However, no general linear relationship between $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ can be written in matrix form for anisotropic materials, as this would require a four-dimensional matrix. In tensor notation, there is no such limitation. Suppose for example that there is a relationship between a second-order tensor A_{ij} and a first-order tensor v_k of the form

$$A_{ij} = B_{ijk}v_k \tag{2.16}$$

Then in a new coordinate system, there will be a relationship

$$\bar{A}_{pq} = \bar{B}_{pqr}\bar{v}_r \tag{2.17}$$

From (2.12) and (2.15),

$$\bar{A}_{pq} = \frac{\partial x_i}{\partial \bar{x}_p}\frac{\partial x_j}{\partial \bar{x}_q}B_{ijk}\frac{\partial x_k}{\partial \bar{x}_r}\bar{v}_r$$

so that the transformation rule for a third-order tensor is

$$\bar{B}_{pqr} = \frac{\partial x_i}{\partial \bar{x}_p}\frac{\partial x_j}{\partial \bar{x}_q}\frac{\partial x_k}{\partial \bar{x}_r}B_{ijk} \quad (= t_{pi}t_{qj}t_{rk}B_{ijk}) \tag{2.18}$$

This leads to the general transformation rule given in section 2.2.4.

2.2.3 The permutation symbol

The general form of the nth order **permutation symbol** is a subscripted symbol with n indices. If these indices are an even permutation of the integers 1 to n it takes the value +1. If the indices are an odd permutation of these integers, it takes the value -1. In all other cases it is zero. Even and odd permutations are defined in section 1.1. Thus the non-zero elements of the third-order permutation symbol e_{ijk} are

$$e_{123} = e_{231} = e_{312} = +1 \quad , \qquad e_{132} = e_{321} = e_{213} = -1 \tag{2.19}$$

Because the value taken by the symbol is related to the sequence of its indices, it is possible to determine the relative sign of two terms with the same indices. If the number of adjacent index changes necessary to produce one term from another is even, then the signs are the same, otherwise they are opposite. Thus

$$e_{ijk} = e_{jki} = e_{kij} = -e_{ikj} = -e_{kji} = -e_{jik} \tag{2.20}$$

If this symbol obeys the transformation rule for a third-order Cartesian tensor, then

$$\begin{aligned}\bar{e}_{pqr} &= e_{ijk}t_{pi}t_{qj}t_{rk} \\ &= t_{p1}t_{q2}t_{r3} + t_{p2}t_{q3}t_{r1} + t_{p3}t_{q1}t_{r2} - t_{p1}t_{q3}t_{r2} - t_{p3}t_{q2}t_{r1} - t_{p2}t_{q1}t_{r3} \\ &= \begin{vmatrix} t_{p1} & t_{p2} & t_{p3} \\ t_{q1} & t_{q2} & t_{q3} \\ t_{r1} & t_{r2} & t_{r3} \end{vmatrix}\end{aligned}$$

If any two rows of this determinant are the same, then it is zero. Thus p, q and r must be some permutation of 1, 2 and 3. If they are 1, 2 and 3 in that order, the above determinant is det(**T**). Taking the determinant of (1.36),

$$\det(\mathbf{T}^{\mathrm{T}}\mathbf{T}) = \det(\mathbf{T}^{\mathrm{T}})\det(\mathbf{T}) = \det^2(\mathbf{T}) = \det(\mathbf{I}) = 1.$$

The determinant is only negative if the transformation is for an axis inversion. Then taking the transformation to be from one right-handed coordinate system to another, det(**T**), and therefore $\bar{e}_{123}$, is +1. All other non-zero elements of $\bar{e}_{pqr}$ are given by permuting the rows of det(**T**), so that in general

$$\bar{e}_{pqr} = e_{pqr}$$

A matrix interpretation of this is as follows. If

$$v_{ij} = -e_{ijk}v_k \quad (= e_{ikj}v_k \text{ etc. }) \tag{2.21}$$

in one coordinate system, then the elements of v_{ij} and v_k can be arranged in matrix form as respectively

$$[\mathbf{v}] = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \tag{2.22}$$

Thus the permutation symbol can be used to generate the vector product matrix given by (1.23) from the column vector given by (1.17). Because this is formed in the same way regardless of the coordinate system, as long as it is right-handed, the symbol is unchanged by a coordinate transformation. As will be seen in Chapter 3, it is necessary to construct a permutation tensor for use with curvilinear coordinate systems.

2.2.4 The general rule of transformation

From (2.11), (2.15) and (2.18), the general rule for the transformation of a Cartesian tensor of any order can be inferred. This is

$$\begin{aligned}\bar{C}_{p\ldots q} &= \frac{\partial x_i}{\partial \bar{x}_p}\ldots\frac{\partial x_j}{\partial \bar{x}_q}C_{i\ldots j} \\ &= \frac{\partial \bar{x}_p}{\partial x_i}\ldots\frac{\partial \bar{x}_q}{\partial x_j}C_{i\ldots j} \\ &= t_{pi}\ldots t_{qj}C_{i\ldots j}\end{aligned} \tag{2.23}$$

Sets of elements which transform in this way are Cartesian tensors.

2.2.5 Contraction

From (2.7) and (2.18)

$$\bar{B}_{ppr} = t_{pi}t_{pj}t_{rk}B_{ijk} = \delta_{ij}t_{rk}B_{ijk} = t_{ik}B_{iik} \tag{2.24}$$

which is the transformation rule for first-order tensors. Thus summation over a pair of repeated indices produces a tensor of two orders less. This process is known as **contraction**. If contraction is continued until a tensor of order zero is produced (i.e. a scalar), then the contracted forms have the same values in all coordinate systems. Such contracted forms, known as the **invariants** of a tensor, may be produced from tensor products. Thus A_{kk}, $A_{ij}A_{ij}$ and $A_{ij}A_{jk}A_{ki}$ are all invariants of the second-order tensor A_{ij}. Likewise, inner products between different tensors also produce contraction. Thus if A_{ij} and B_{kl} are second-order tensors, the **outer product** $A_{ij}B_{kl}$ forms the elements of a fourth-order tensor of the type C_{ijkl}, but the **inner product** $A_{ij}B_{jl}$ forms the elements of a second-order tensor such as D_{il}.

2.2.6 Scalar and vector products

From the discussion of first-order tensors, it follows that two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ can be represented by $v_j\boldsymbol{i}_j$ and $w_k\boldsymbol{i}_k$ respectively, where v_j and w_k are typical elements of first-order tensors. A scalar product is then given by

$$\boldsymbol{v}\cdot\boldsymbol{w} = (v_j\boldsymbol{i}_j)\cdot(w_k\boldsymbol{i}_k) = (v_jw_k)\boldsymbol{i}_j\cdot\boldsymbol{i}_k = v_jw_k\delta_{jk} = v_jw_j \tag{2.25}$$

giving an inner product of order zero, i.e. a scalar. The vector product $\boldsymbol{u}$ ($= u_i\boldsymbol{i}_i$) of $\boldsymbol{v}$ and $\boldsymbol{w}$ can be deduced from the column vector $\mathbf{u}$ given by (1.22) and the forms given by (2.21) and (2.22). Thus

$$u_i = v_{ij}w_j = -e_{ijk}v_kw_j = e_{ijk}v_jw_k \tag{2.26}$$

The contraction produced by the inner products of the tensors on the right-hand side of this equation confirms that u_i is a first-order tensor.

2.2.7 Isotropic or invariant tensors

A tensor is said to be isotropic or invariant if all its elements are the same in all coordinate systems, that is

$$\bar{c}_{i\ldots j} = c_{i\ldots j}$$

for all coordinate systems $\bar{x}_i$. Jeffreys (1931) points out that properties of such an invariant tensor can be found from the transformation accompanying an arbitrary small rotation $\boldsymbol{\omega}$ of the coordinate system. This will cause an apparent change in a vector $\boldsymbol{v}$ of approximately $-\boldsymbol{\omega}\times\boldsymbol{v}$ as viewed from the coordinate system, or

$$\begin{bmatrix}\bar{v}_1\\ \bar{v}_2\\ \bar{v}_3\end{bmatrix} = \begin{bmatrix}1 & \omega_3 & -\omega_2\\ -\omega_3 & 1 & \omega_1\\ \omega_2 & -\omega_1 & 1\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}$$

The transformation elements t_{ik} found from the above matrix are then

$$t_{ik} = \delta_{ik} - e_{ijk}\omega_j$$

For invariance of a first-order tensor such as v_i,

$$v_i = \bar{v}_i = (\delta_{ik} - e_{ijk}\omega_j)v_k , \qquad \text{or} \qquad e_{ijk}\omega_j v_k = 0$$

for arbitrary ω_j. This is only possible if v_k is zero. For a second-order tensor, A_{ip},

$$A_{ip} = \bar{A}_{ip} = (\delta_{ik} - e_{ijk}\omega_j)(\delta_{pr} - e_{pqr}\omega_q)A_{kr}$$
$$\text{or} \qquad e_{pqr}\omega_q A_{ir} + e_{ijk}\omega_j A_{kp} = 0$$

omitting second-order of smallness terms. For this to be true for arbitrary ω_j,

$$e_{pjr}A_{ir} + e_{ijk}A_{kp} = 0$$
$$\text{giving} \qquad A_{11} = A_{22} = A_{33} \; ; \; A_{ij} = 0 \quad (i \neq j)$$

so that A_{ij} is some scalar multiple of δ_{ij} . The process can be repeated for higher order tensors. An isotropic third-order tensor, B_{ijk} must be a scalar multiple of e_{ijk} and the most general form of an isotropic fourth order tensor is given by

$$d_{ijkl} = a\delta_{ij}\delta_{kl} + b\delta_{ik}\delta_{jl} + c\delta_{il}\delta_{jk} \tag{2.27}$$

where a, b and c are scalars.

2.3 DIFFERENTIATION

If ϕ is a scalar function,

$$\frac{\partial\phi}{\partial\bar{x}_i} = \frac{\partial\phi}{\partial x_1}\frac{\partial x_1}{\partial\bar{x}_i} + \frac{\partial\phi}{\partial x_2}\frac{\partial x_2}{\partial\bar{x}_i} + \frac{\partial\phi}{\partial x_3}\frac{\partial x_3}{\partial\bar{x}_i} = \frac{\partial x_j}{\partial\bar{x}_i}\frac{\partial\phi}{\partial x_j} \tag{2.28}$$

Thus $\partial\phi/\partial x_j$ transforms in accordance with the rule for a first-order tensor given by (2.11). In tensor notation, partial differentiation is indicated by a comma followed by a subscript which is the index of the differentiating ordinate, so that the above equation can be written as

$$\bar{\phi}_{,i} = \frac{\partial x_j}{\partial\bar{x}_i}\phi_{,j} \qquad (= t_{ij}\phi_{,j}) \tag{2.29}$$

As with any first-order tensor, the elements can be thought of as the coefficients of a vector, in this case, grad ϕ (cf. (1.47)).

2.3.1 First-order differentiation

Small increments of a first-order tensor v_j are related to small increments of its representation in a second coordinate system by

$$d\bar{v}_i = \frac{\partial x_j}{\partial\bar{x}_i}dv_j \qquad (= t_{ij}dv_j)$$

Suppose that these increments result from changes in a vector field between adjacent points. Then

$$d\bar{v}_i = \frac{\partial\bar{v}_i}{\partial\bar{x}_k}d\bar{x}_k = \bar{v}_{i,k}d\bar{x}_k$$

and similarly

$$\mathrm{d}v_j = \frac{\partial v_j}{\partial x_l}\mathrm{d}x_l = v_{j,l}\mathrm{d}x_l$$

so that from (2.8),

$$\mathrm{d}\bar{v}_i = \bar{v}_{i,k}\mathrm{d}\bar{x}_k = t_{ij}\mathrm{d}v_j = t_{ij}v_{j,l}\mathrm{d}x_l = t_{ij}v_{j,l}t_{kl}\mathrm{d}\bar{x}_k$$

for arbitrary $\mathrm{d}\bar{x}_k$ so that

$$\bar{v}_{i,k}\ (= t_{ij}t_{kl}v_{j,l}) = \frac{\partial x_j}{\partial \bar{x}_i}\frac{\partial x_l}{\partial \bar{x}_k}v_{j,l} \tag{2.30}$$

It can be seen that this is the same transformation law as that given by (2.15) so that $v_{j,l}$ is a second-order tensor.

Contraction of such a second-order tensor produces a scalar which gives the divergence of a vector, that is,

$$\mathrm{Div}\,\boldsymbol{v} = \nabla\cdot\boldsymbol{v} = v_{k,k} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} \tag{2.31}$$

(cf. (1.48)). The tensor elements of the curl of a vector $\boldsymbol{v}$ can be deduced from (2.26) and takes the form

$$\mathrm{Curl}\,\boldsymbol{v} = \nabla\times\boldsymbol{v} = e_{ijk}v_{k,j}\boldsymbol{i}_i = -e_{kji}v_{k,j}\boldsymbol{i}_i \equiv -e_{ijk}v_{i,j}\boldsymbol{i}_k \quad \text{etc.} \tag{2.32}$$

2.3.2 Higher-order terms

The nature of the partial differentials of higher-order tensors can be determined in the same way. Thus the set of differentials $A_{ij,k}$ of a second-order tensor form a third-order tensor and so on. Because $A_{ij,k}$ is a third-order tensor, the set of its differentials $A_{ij,kl}$ behaves like a fourth-order tensor. In Cartesian space, the order of differentiation, and hence the order of indices after the comma, does not matter. That is to say, $A_{ij,kl}$ and $A_{ij,lk}$ are always the same. It will be seen in the next chapter that there are some intrinsically curved spaces (ones where Cartesian coordinates cannot be used) where the order of differentiation is significant. Examples of such intrinsically curved spaces can be found in the general theory of relativity. At a simpler level, the two-dimensional space generated by the surface of a sphere is intrinsically curved; Cartesian coordinates provide a sufficiently accurate framework only for a small region of the surface.

2.4 REPRESENTATION OF STRAIN

An exact definition of strain in terms of large deformation theory is given in Chapter 4. The definition used in small deflection theory approximates to this, provided that the displacement gradients are small. In effect, this means ignoring terms which are of the order of a displacement gradient (or strain) smaller than the most significant terms in an equation.

The increment $\mathrm{d}\boldsymbol{u}$ in the displacement of a point P′ adjacent to the point P in Fig. 1.5 is given by (1.51) and (1.52) so that

$$\mathrm{d}\boldsymbol{u} = \mathrm{d}u_j\boldsymbol{i}_j = u_{j,k}\mathrm{d}x_k\boldsymbol{i}_j \tag{2.33}$$

The strain tensor e_{jk} is defined by the elements of $\mathbf{e}$ in (1.60) so that

$$e_{jk} = \tfrac{1}{2}(u_{j,k} + u_{k,j}) \tag{2.34}$$

Because $u_{j,k}$ is a second-order tensor, e_{jk} is also, and so the rules for its transformation follow immediately. It is by definition symmetric, that is, e_{jk} and e_{kj} are the same. Similarly, the rotation tensor ω_{jk} is defined by the elements of $\boldsymbol{\omega}$ in (1.58) so that from (1.54),

$$\begin{aligned} \omega_{jk} &= \tfrac{1}{2}(u_{j,k} - u_{k,j}) \\ \text{and so} \qquad d\boldsymbol{u} &= (e_{jk} + \omega_{jk})\,dx_k \boldsymbol{i}_j \end{aligned} \tag{2.35}$$

Scalar invariants of e_{jk} can be formed by contraction, as noted in section 2.2.5, and these are related to the strain invariants of (1.72) by

$$\begin{aligned} K_1 &= e_{ii} \\ K_2 &= \tfrac{1}{2}(e_{ii}e_{jj} - e_{ij}e_{ij}) \\ K_3 &= \tfrac{1}{6}(e_{ii}e_{jj}e_{kk} - 3e_{ij}e_{ij}e_{kk} + 2e_{ij}e_{jk}e_{ki}) \end{aligned} \tag{2.36}$$

Any other invariants can be expressed in terms of these three. For example, $e_{ij}e_{jk}e_{kl}e_{li}$ is also a scalar invariant derived by contraction. However, in three dimensions, two of the subscripts must be equal which means that it can be found in terms of simpler invariants (see problem 2.11).

Differentiating (2.34) twice, $\qquad e_{jk,lm} = \tfrac{1}{2}(u_{j,klm} + u_{k,jlm})$

From this

$$e_{jk,lm} + e_{lm,jk} \{ = \tfrac{1}{2}(u_{j,klm} + u_{k,jlm} + u_{l,mjk} + u_{m,ljk}) \} = e_{jl,km} + e_{km,jl} \tag{2.37}$$

(The order of differentiation and hence the order of the indices after the comma does not matter in Cartesian space.) Equation (2.37) is the equation of compatibility in tensor form. Writing it for every possible combination of the three values each of j, k, l and m gives 81 equations. Of these, the set of equations (1.74) and (1.75) is given six times, the set of equations (1.76) is given twelve times and there are twenty seven identities.

2.5 REPRESENTATION OF STRESS

The elements of the stress matrix $\boldsymbol{\sigma}$ given by (1.86) can be related to the elements of a Cartesian stress tensor σ_{ij} by

$$\begin{bmatrix} \sigma_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} & \sigma_{32} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \tag{2.38}$$

and (1.79) and (1.80) become

$$\sigma_{ij} = \sigma_{ji} \tag{2.39}$$

Unfortunately, the numbered subscripts do not conform to the row and column numbering normally used in matrices, the conventional order of the stress subscripts being defined without regard to their use in matrix arrays. However, because (2.39) is normally true, the order of subscripts is seldom significant. In this notation, (1.85) takes the form

$$\sigma_{ji}n_j = p_i \tag{2.40}$$

Because n_j and p_i are elements of column vector representations, they are first-order

tensors. From (2.13) it then follows that σ_{ij} is a second-order tensor. The components of the body force per unit mass, X,Y,Z and the accelerations $\ddot{u}_x$, $\ddot{u}_y$, $\ddot{u}_z$ in (1.100) can be denoted by the first-order tensors f_i and $\ddot{u}_i$ respectively (i = 1 to 3). Then these equations can be written in tensor form as

$$\sigma_{ji,j} + \rho f_i = \rho \ddot{u}_i \tag{2.41}$$

2.5.1 Invariants

Normally, the stress tensor is a symmetrical second-order tensor like the strain tensor and so the invariants given by (1.91) can be expressed in a form similar to (2.36). The most general form is given by assuming that (2.39) does not hold true and the invariants are then

$$\begin{aligned} I_1 &= \sigma_{ii} \\ I_2 &= \tfrac{1}{2}(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ji}) \\ I_3 &= \tfrac{1}{6}(\sigma_{ii}\sigma_{jj}\sigma_{kk} - 3\sigma_{ij}\sigma_{ji}\sigma_{kk} + 2\sigma_{ij}\sigma_{jk}\sigma_{ki}) \end{aligned} \tag{2.42}$$

The stress deviations can also be expressed in tensor form by

$$s_{ij} = \sigma_{ij} - \tfrac{1}{3}\delta_{ij}\sigma_{kk} \tag{2.43}$$

Because σ_{kk} is a scalar and σ_{ij} and δ_{ij} are second-order tensors, s_{ij} is also a second-order tensor. The stress deviation invariants given by (1.94) and (1.95) are then

$$\begin{aligned} J_2 &= (3\sigma_{ij}\sigma_{ji} - \sigma_{ii}\sigma_{jj}) = \tfrac{1}{2}s_{ij}s_{ji} \\ J_3 &= \tfrac{2}{27}\sigma_{ii}\sigma_{jj}\sigma_{kk} + \tfrac{1}{3}(\sigma_{ij}\sigma_{jk}\sigma_{ki} - \sigma_{ij}\sigma_{ji}\sigma_{kk}) = \tfrac{1}{3}s_{ij}s_{jk}s_{ki} \end{aligned} \tag{2.44}$$

2.6 THERMOELASTIC BEHAVIOUR

The internal energy stored in an elastic body is a function of its state and not its history. For static states, this may consist not only of strain energy but also of thermal energy absorbed as heat. The heat absorbed may be a function of both temperature and strain, thus affecting the stress-strain relationship.

Suppose that an elastic body is in an unstressed and unstrained state at some absolute temperature $T = T_o$. The energy of a small part of the body will be examined. In the above initial state, this part has a volume dV and a density ρ. As the temperature and strain change, the volume and density may change but the mass, ρdV, will remain constant because the part considered consists of the same amount of material at all times. If the work done on the body by the stresses while straining it is W, then from the definitions of σ_{ij} and e_{ij} given by (2.34) and (2.38), (1.127) can be written in the form

$$\begin{aligned} \delta W = \delta U dV = {} & [\sigma_{11}\delta e_{11} + \sigma_{22}\delta e_{22} + \sigma_{33}\delta e_{33} \\ & + \sigma_{23}\delta(e_{23} + e_{32}) + \sigma_{31}\delta(e_{31} + e_{13}) + \sigma_{12}\delta(e_{12} + e_{21})]dV \\ = {} & \sigma_{ij}\delta e_{ij}dV = \sigma_{ji}\delta e_{ij}dV \end{aligned} \tag{2.45}$$

from (2.39), where U is the strain energy stored per unit volume. An increment in the total energy stored per unit volume is given by

$$\delta E = \delta U + \delta Q \tag{2.46}$$

where Q is the heat energy stored per unit volume. The change in Q is a function of the change in temperature and the change in strain, so that

$$\delta Q = \rho C_e \delta T + q_{ij} \delta e_{ij} \tag{2.47}$$

where C_e is the specific heat at constant strain. The change in heat energy is related to the change in entropy δS by

$$\delta Q = T \delta S \tag{2.48}$$

Now δE and δS are perfect differentials†, so that from (2.45) to (2.48),

$$\delta E = \frac{\partial E}{\partial e_{ij}} \delta e_{ij} + \frac{\partial E}{\partial T} \delta T = (\sigma_{ij} + q_{ij}) \delta e_{ij} + \rho C_e \delta T \tag{2.49}$$

$$\delta S = \frac{\partial S}{\partial e_{ij}} \delta e_{ij} + \frac{\partial S}{\partial T} \delta T = \frac{q_{ij}}{T} \delta e_{ij} + \frac{\rho C_e}{T} \delta T \tag{2.50}$$

(Note that e_{ij} is being treated here as if it were independent of e_{ji}. This dependence can be regarded as a condition which will be imposed at the end, following an analysis which is applicable to more general conditions, cf. Appendix A6. Alternatively, this dependence can be imposed from the outset, requiring more cumbersome manipulations but leading to the same final results.) It follows from (2.49) and (2.50) that

$$\frac{\partial^2 E}{\partial e_{ij} \partial T} = \frac{\partial}{\partial T}(\sigma_{ij} + q_{ij}) = \frac{\partial (\rho C_e)}{\partial e_{ij}} \quad \text{and} \quad \frac{\partial^2 S}{\partial e_{ij} \partial T} = \frac{\partial}{\partial T}\left(\frac{q_{ij}}{T} \right) = \frac{\partial}{\partial e_{ij}}\left(\frac{\rho C_e}{T} \right)$$

Then

$$\frac{\partial}{\partial T}(\sigma_{ij} + q_{ij}) = T \frac{\partial}{\partial e_{ij}}\left(\frac{\rho C_e}{T} \right) = T \frac{\partial}{\partial T}\left(\frac{q_{ij}}{T} \right) = \frac{\partial q_{ij}}{\partial T} - \frac{q_{ij}}{T}$$

so that

$$q_{ij} = -T \frac{\partial \sigma_{ij}}{\partial T} \tag{2.51}$$

2.6.1 Isothermal and adiabatic cases

Under **isothermal** conditions, the temperature is constant and it follows from (2.49) and (2.50) that $(\sigma_{ij} + q_{ij})\delta e_{ij}$ and $(q_{ij}/T)\delta e_{ij}$ are perfect differentials. It then follows that at a fixed temperature $\sigma_{ij} \delta e_{ij}$ is a perfect differential, δU, where

$$\frac{\partial U}{\partial e_{ij}} = \sigma_{ij} \left(= \sigma_{ji} = \frac{\partial U}{\partial e_{ji}} \right) \tag{2.52}$$

This is known as **Green's formula**. Then writing U as a power series in strain,

$$U = a + b_{ij} e_{ij} + \frac{1}{2} c_{ijkl} e_{ij} e_{kl} + \frac{1}{3} d_{ijklmn} e_{ij} e_{kl} e_{mn} + \cdots \tag{2.53}$$

The symmetry condition in (2.52) implies that

†See Boley, B.A. and Weiner, J.H. (1960) §1.10. A perfect differential is one which represents a change in a potential which is then uniquely characterised by the differential (apart from a constant of integration).

$$\begin{aligned} c_{ijkl} &= c_{jikl} = c_{ijlk} \cdots \\ d_{ijklmn} &= d_{jiklmn} = d_{ijlkmn} = d_{ijklnm} \cdots \end{aligned} \tag{2.54}$$

and so on. Without loss of generality, the coefficients of $e_{ij}e_{kl}$ and $e_{kl}e_{ij}$ can be taken as the same, as can the coefficients of $e_{ij}e_{kl}e_{mn}$ and $e_{kl}e_{ij}e_{mn}$ and so on, so that

$$\begin{aligned} c_{ijkl} &= c_{klij} \\ d_{ijklmn} &= d_{klijmn} = d_{mnklij} = \cdots \end{aligned} \tag{2.55}$$

Equations (2.54) and (2.55) mean that there are no more than 21 independent constants c_{ijkl} and no more than 56 independent constants d_{ijklmn}. Then from (2.52) to (2.55),

$$\sigma_{ij} = b_{ij} + c_{ijkl}e_{kl} + d_{ijklmn}e_{kl}e_{mn} + \cdots \tag{2.56}$$

If the state of zero strain is also one of zero stress, the terms b_{ij} disappear. Small deflection theory is only a valid approximation provided that the displacement gradients, and hence the strains, are small. This means that the quadratic terms in (2.56) are normally neglected, being very small in comparison with the linear terms. Of course, this is not appropriate if the coefficients c_{ijkl} are the order of a strain smaller than the coefficients d_{ijklmn}. Moreover, in Chapter 4 it will be seen that even when these coefficients are of the same order, both should be retained in the continuum stability equations for small deflections. The approximate nature of the small deflection definition of strain mentioned at the beginning of section 2.4 means that a proper evaluation of the higher-order elastic constants such as d_{ijklmn} has to be made in terms of large deflection theory (see Pollard (1977)). The coefficients c_{ijkl} are related to the elements c_{pq} of the stiffness matrix $\mathbf{C}$ in (1.124). Let x_1, x_2 and x_3 be in the x, y and z directions respectively. Then for sequential values of p from 1 to 6, the pair ij take the values 11, 22, 33, 23, 31 and 12 respectively, and q is related in the same way to the pair kl. Thus c_{25} corresponds to c_{2231} for example.

For the **adiabatic** case, there is no change in the heat energy and so it follows immediately from (2.46) that δU is a total differential. The same arguments can then be applied as before, except that the coefficients b_{ij}, c_{ijkl}, d_{ijklmn} etc. are not usually the same as in the isothermal case.

Thermal expansion under zero stress is normally taken to be proportional to the temperature change. The apparent strains due to such expansion are then also linearly related to the temperature change ΔT by

$$e_{kl}^{a} = \beta_{kl}\Delta T \tag{2.57}$$

Suppose that the stress-strain relationship is adequately given by the linear terms in (2.56). Then because the above strains take place under zero stress,

$$0 = b_{ij} + c_{ijkl}\beta_{kl}\Delta T$$

and so the stress-strain relationship becomes

$$\sigma_{ij} = c_{ijkl}(e_{kl} - \beta_{kl}\Delta T) \tag{2.58}$$

From (2.51) and (2.58),

$$\begin{aligned} q_{ij} &= -Te_{kl}\frac{\partial}{\partial T}(c_{ijkl}) + T\frac{\partial}{\partial T}(c_{ijkl}\beta_{kl}\Delta T) \\ &= -Te_{kl}\frac{\partial}{\partial T}(c_{ijkl}) + Tc_{ijkl}\beta_{kl}\{+T\Delta T\frac{\partial}{\partial T}(c_{ijkl}\beta_{kl})\} \end{aligned} \tag{2.59}$$

where the last term can be ignored provided that ΔT is small or zero. The thermoelastic properties of isotropic materials will be examined in greater detail in section 2.7.3.

2.6.2 Coordinate transformation

Because σ_{ij} and e_{kl} are second-order tensors, it follows from (2.56) that b_{ij}, c_{ijkl} and d_{ijklmn} are second, fourth and sixth-order tensors respectively, and so on. The transformation of the elements c_{ijkl} to a new coordinate system is given by

$$\bar{c}_{pqrs} = t_{pi}t_{qj}t_{rk}t_{sl}c_{ijkl} \tag{2.60}$$

for example. If the new coordinate system is produced by a clockwise rotation θ of the old coordinate system about its x_3 axis, then the transformation terms are given by

$$t_{ij} = \delta_{ij}\cos\theta + \delta_{i3}\delta_{j3}(1-\cos\theta) + (\delta_{i1}\delta_{j2} - \delta_{i2}\delta_{j1})\sin\theta \tag{2.61}$$

For most purposes, the transformation given by (2.60) is more readily performed by the method described in Chapter 1.

2.7 ISOTROPIC MATERIALS

Suppose that the strain energy per unit volume for an isotropic body is expressed by

$$U = c_0 + c_1 f_1(e_{ij}) + c_2 f_2(e_{ij}) + \ldots + c_n f_n(e_{ij})$$

where $c_0, c_1 \ldots c_n$ are independent elastic constants. Because these constants are independent, each function $f_r(e_{ij})$ must be invariant with respect to any coordinate system. This means that U can be expressed in terms of the strain invariants given by (2.36) as

$$U = c_0 + c_1K_1 + c_2K_1^2 + c_3K_2 + c_4K_1^3 + c_5K_1K_2 + c_6K_3 + O(e_{ij}^4)$$

or in terms of the invariants obtained by contraction, (2.53) becomes

$$\begin{aligned} U = a + be_{ii} + \tfrac{1}{2}\lambda e_{ii}e_{jj} + \mu e_{ij}e_{ji} \\ + \tfrac{1}{3}d_1e_{ii}e_{jj}e_{kk} + d_2e_{ii}e_{jk}e_{kj} + \tfrac{1}{3}d_3e_{ij}e_{jk}e_{ki} + O(e_{ij}^4) \end{aligned} \tag{2.62}$$

for isotropic materials, where λ and μ are known as **Lamé's constants**. Then from (2.52),

$$\begin{aligned} \sigma_{ij} = b\delta_{ij} + \lambda\delta_{ij}e_{kk} + 2\mu e_{ij} \\ + \delta_{ij}(d_1e_{kk}e_{ll} + d_2e_{kl}e_{lk}) + 2d_2e_{kk}e_{ij} + d_3e_{jk}e_{ki} + O(e_{ij}^3) \end{aligned} \tag{2.63}$$

Again, if the state of zero strain is necessarily also one of zero stress, b is zero. Then for a linear isotropic material, there are two independent elastic constants. Also, for an isotropic material where the stress is also related to quadratic functions of strain, there are a total of five independent elastic constants. For the normal linear case, (2.63) reduces to

$$\sigma_{ij} = \lambda\delta_{ij}e_{kk} + 2\mu e_{ij} \tag{2.64}$$

The stiffness coefficients c_{ijkl} in (2.56) are given in terms of Lamé's constants by

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{2.65}$$

On comparing this with (2.27), it will be seen that this is not the most general fourth-order tensor. This is because the condition imposed by (2.54) implies that b and c in (2.27) must be equal. The condition of positive definiteness of the strain energy function which gives (1.149) implies that

$$\mu > 0 \quad , \quad 3\lambda + 2\mu > 0 \tag{2.66}$$

Young's modulus and Poisson's ratio are given by

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \quad , \quad \nu = \frac{\lambda}{2(\lambda + \mu)} \tag{2.67}$$

and conversely Lamé's constants can be expressed in terms of the shear modulus and Poisson's ratio by

$$\mu = G \quad , \quad \lambda = \frac{2G\nu}{1 - 2\nu} \tag{2.68}$$

2.7.1 The bulk modulus

If an isotropic material is subject to an isotropic (or 'hydrostatic') stress $p\delta_{ij}$, an isotropic strain $e\delta_{ij}$ must be induced (see section 2.2.7). If an elementary cuboid with side lengths dx_i ($i = 1$ to 3), cf. Fig. 1.7, is subject to such strains, each side length increases to $(1+e)dx_i$ cf. Fig. 1.11. The volume V of the cuboid, initially $dx_1\,dx_2\,dx_3$, increases to

$$V + \delta V = (1 + e)^3 dx_1\,dx_2\,dx_3$$

Then the **volumetric strain** is given by

$$\delta V/V = (1 + e)^3 - 1 \approx 3e \quad (= e_{kk})$$

ignoring quadratic terms and higher terms in strain. For the linear stress-strain relationship given by (2.64),

$$p\delta_{ij} = \sigma_{ij} = (3\lambda + 2\mu)\delta_{ij}e$$

The **bulk modulus** K is defined as the ratio of the isotropic stress to the volumetric strain, or from the above equations,

$$K = p/(\delta V/V) = \lambda + \tfrac{2}{3}\mu \tag{2.69}$$

2.7.2 Incompressibility

A material is said to be incompressible if it undergoes no volume change when stressed. As a rule, the behaviour of the material is then simpler to analyse, and incompressibility is often assumed in the theory of large deformations (see Chapter 4).

For an isotropic material, an isotropic stress can only induce an isotropic strain which necessarily produces a volume change. If the material is incompressible, an isotropic stress can induce no strain. This means that for such a material the state of strain does not uniquely define the state of stress. The equations giving the state of stress in terms of the state of strain must then contain an indeterminate isotropic term. The relationship for an incompressible material is given by (2.63) on allowing e_{kk} to tend to zero as λ tends to infinity. The product λe_{kk} then takes some indeterminate value and the equation becomes

$$\sigma_{ij} = p\delta_{ij} + 2\mu e_{ij} + d_3 e_{jk}e_{ki} \ldots \tag{2.63a}$$

where p is the isotropic stress term. It is determined from the boundary conditions (2.40)

and the equations of motion (2.41) and is not necessarily constant throughout the body. For the incompressible case, (2.67) and (2.68) give

$$E = 3\mu \quad (= 3G) \quad , \quad \nu = \frac{1}{2} \tag{2.67a}$$

and the bulk modulus given by (2.69) becomes infinite.

2.7.3 Thermoelasticity

If the material is isotropic, then the local effect of a temperature change is to produce an apparent isotropic strain. The most general form of the tensor β_{kl} in (2.57) is then

$$\beta_{kl} = \alpha\delta_{kl} \tag{2.70}$$

(see section 2.2.7) where α is known as the **coefficient of linear expansion**. From (2.65) and (2.70), (2.58) now becomes

$$\sigma_{ij} = \lambda\delta_{ij}e_{kk} + 2\mu e_{ij} - \alpha(3\lambda + 2\mu)\delta_{ij}\Delta T \tag{2.71}$$

using the symmetry of e_{ij}. This is known as the **Duhamel-Neumann law**. Then if the strains and ΔT are small, (2.51) gives

$$q_{ij} = -T\frac{\partial\sigma_{ij}}{\partial T} \approx T\frac{\partial}{\partial T}[\alpha(3\lambda + 2\mu)\delta_{ij}\Delta T] \approx T\alpha(3\lambda + 2\mu)\delta_{ij} = 3KT\alpha\delta_{ij} \tag{2.72}$$

using (2.69). For the adiabatic case, δQ is zero and so (2.47) gives

$$\rho C_e\delta T = -q_{ij}\delta e_{ij} = -3KT\alpha\delta e_{ii} \tag{2.73}$$

Then if the change is from the state of zero stress and strain and ΔT is small, δe_{ii} is e_{ii} (or e_{kk}) and (2.71) gives

$$\sigma_{ij} = \left(\lambda + \frac{9K^2\alpha^2T}{\rho C_e}\right)\delta_{ij}e_{kk} + 2\mu e_{ij} \tag{2.74}$$

Thus for the adiabatic case, μ is the same as for the isothermal case, and at a given temperature T, the adiabatic equivalent of λ is given by

$$\lambda_a = \lambda + \frac{9K^2\alpha^2T}{\rho C_e} \tag{2.75}$$

From (2.69), the adiabatic equivalent of the bulk modulus is then

$$K_a = \lambda_a + \frac{2}{3}\mu = K + \frac{9K^2\alpha^2T}{\rho C_e} \tag{2.76}$$

The specific heat normally measured is C_p. This is defined by the relationship between the change of heat absorbed and the change of temperature in the equation

$$\delta Q = \rho C_p\delta T \tag{2.77}$$

under zero stress conditions. Then from (2.47), (2.72) and (2.71),

$$\rho C_p\delta T = \rho C_e\delta T + q_{ij}\delta e_{ij} = \rho C_e\delta T + 3KT\alpha\delta e_{ii} = (\rho C_e + 3KT\alpha.3\alpha)\delta T$$

Then

$$C_p = C_e\left(1 + \frac{9K\alpha^2 T}{\rho C_e}\right) = \frac{K_a}{K}C_e \tag{2.78}$$

from (2.76). Thus the ratio of K_a/K is given by the ratio of C_p/C_e. Gol'denblat (1962), p.199, quotes values of this ratio at 20°C. For iron, it is 1.016, and this is a fairly typical value for engineering metals.

High speed changes of strain, such as might occur when a body vibrates at high frequency, take place under approximately adiabatic conditions. Then (2.73) shows that a temperature change can be related to a change of strain, or using (2.74), (2.75), (2.76) and (2.78),

$$\delta T = -\frac{\alpha T}{\rho C_p}\delta\sigma_{ii} \tag{2.79}$$

Thus a change in the sum of the principal stresses can be related to a change in temperature. This is the basis of stress pattern analysis by thermal emission (SPATE). Stanley and Chan† have measured these changes using an infrared radiometric system with a temperature resolution of 0.002°C. This corresponds to a resolution of 1 MPa (1 N/mm^2) for steel and 0.4 MPa for aluminium.

2.7.4 Beltrami-Michell compatibility conditions

From (2.64),

$$\sigma_{kk} = (2\mu + 3\lambda)e_{kk} \tag{2.80}$$

so that the inverse relationship is given by

$$e_{ij} = \frac{1}{2\mu}\left(\sigma_{ij} - \frac{\lambda}{2\mu + 3\lambda}\delta_{ij}\sigma_{kk}\right) \tag{2.81}$$

Using this equation, the compatibility of strain condition given by (2.37) can be written as

$$\sigma_{ij,kl} + \sigma_{kl,ij} - \frac{\lambda}{2\mu+3\lambda}(\delta_{ij}\sigma_{mm,kl} + \delta_{kl}\sigma_{mm,ij})$$
$$= \sigma_{ik,jl} + \sigma_{jl,ik} - \frac{\lambda}{2\mu+3\lambda}(\delta_{ik}\sigma_{mm,jl} + \delta_{jl}\sigma_{mm,ik})$$

In particular, on setting i equal to j and summing,

$$\sigma_{jj,kl} + \sigma_{kl,jj} - \frac{\lambda}{2\mu+3\lambda}(3\sigma_{mm,kl} + \delta_{kl}\sigma_{mm,jj}) = \sigma_{jk,jl} + \sigma_{jl,jk} - \frac{\lambda}{2\mu+3\lambda}(2\sigma_{mm,kl})$$

or from (2.41), in the absence of accelerations,

$$\sigma_{kl,jj} + \frac{1}{2\mu+3\lambda}[2(\mu+\lambda)\sigma_{mm,kl} - \lambda\delta_{kl}\sigma_{mm,jj}] + \rho(f_{k,l} + f_{l,k}) = 0 \tag{2.82}$$

Putting k equal to l and summing gives

$$\sigma_{jj,ll} = -\frac{2\mu+3\lambda}{2\mu+\lambda}\rho f_{l,l} \tag{2.83}$$

†Stanley, P. and Chan, W.K. (1985) Quantitative stress analysis by means of the thermoelastic effect. *Journal of Strain Analysis* **20** 129-139

and on substituting this result into (2.82),

$$\sigma_{kl,ij} + \frac{2(\mu+\lambda)}{2\mu+3\lambda}\sigma_{jj,kl} + \delta_{kl}\frac{\lambda\rho}{2\mu+\lambda}f_{j,j} + \rho f_{k,l} + \rho f_{l,k} = 0 \qquad (2.84)$$

This gives six independent equations corresponding to the six compatibility conditions. These equations are known as the **Beltrami-Michell compatibility conditions**. Together with the equations of motion, (2.41), (2.84) gives all the conditions concerning the stresses within a body.

2.7.5 Navier's equation

Using (2.34) and (2.64), (2.40) can be written as

$$\mu(u_{i,j} + u_{j,i})n_j + \lambda u_{j,j}n_i = p_i \qquad (2.85)$$

Similarly, (2.41) can be written as

$$\mu u_{i,jj} + (\lambda+\mu)u_{j,ji} + \rho f_i = \rho\ddot{u}_i \qquad (2.86)$$

This is known as **Navier's equation**. If the body forces and accelerations are zero or constant throughout the body, on differentiating (2.86) once,

$$(\lambda + 2\mu)(u_{i,i})_{,jj} = 0$$

Then from (2.34) and (2.36),

$$(u_{i,i})_{,jj} = (e_{ii})_{,jj} = (K_1)_{,jj} = 0 \qquad (2.87)$$

so that the first strain invariant is harmonic. It follows from (2.42) and (2.80) that the first stress invariant, I_1, is also harmonic.

Alternatively, if the body forces and accelerations are zero or constant throughout the body, on differentiating (2.86) twice,

$$\mu u_{i,jjkk} + (\lambda+\mu)(u_{j,j})_{,kki} = 0$$

so that from (2.87),

$$(u_i)_{,jjkk} = 0 \qquad (2.88)$$

and so the displacements are biharmonic. This also follows from (1.112a) which relates these displacements to the biharmonic Galerkin vector.

2.7.6 Conservative force fields

A **conservative force field** is one in which the net work done by the field on a particle which moves on a path through it between two points, A and B say, is independent of the path chosen. In Fig. 2.1, the force field is given by the vector function

$$\boldsymbol{F}(x_1, x_2, x_3) = F_k \boldsymbol{i}_k$$

The work done in moving a particle from A to B is

$$W = \int_A^B \boldsymbol{F}\cdot d\boldsymbol{r} = \int_A^B F_k dx_k$$

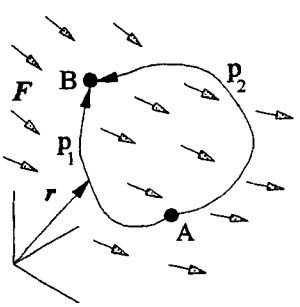


Figure 2.1 Paths in a force field.

along either path p_1 or path p_2 . Suppose that the path is extended to a point B′ adjacent to B but separated from it by a small distance δx_1 in the x_1 direction, as shown in Fig. 2.2. The increment in W is

$$\delta W = \int_B^{B'} \boldsymbol{F}\cdot d\boldsymbol{r} = \int_B^{B'} F_1 dx_1$$

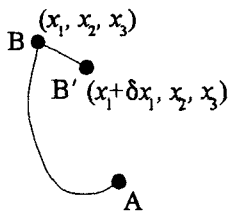


Figure 2.2 Path extension.

where δW is the same along any path between B and B′, but the path along which only x_1 varies has been chosen. In the limit, as δx_1 tends to zero, this gives $W_{,1} = F_1$ and in general,

$$W_{,i} = F_i \tag{2.89}$$

where in this case W is zero at A.

Integrating the work done along any path between two points gives a unique value known as the loss of **potential**, Φ, between the two points. In particular, the net work done along any closed loop beginning and ending at the same point is zero.

2.7.7 Conservative body forces

A conservative body force with components f_i is defined as having a potential ϕ such that

$$-\phi_{,i} = f_i \tag{2.90}$$

Then (2.41) becomes

$$(\sigma_{ji} - \rho\phi\delta_{ji})_{,j} = \rho\ddot{u}_i \tag{2.41a}$$

and the Beltrami-Michell equations take the form

$$\left(\sigma_{kl} - \frac{\lambda\rho\phi}{2\mu+\lambda}\right)_{,ij} + \left(\frac{2(\mu+\lambda)}{2\mu+3\lambda}\sigma_{jj} - 2\rho\phi\right)_{,kl} = 0 \tag{2.84a}$$

where

$$\left(\sigma_{jj} - \frac{2\mu+3\lambda}{2\mu+\lambda}\rho\phi\right)_{,ll} = 0 \tag{2.83a}$$

If v is harmonic, then from (2.83a) σ_{jj} is also harmonic and on differentiating (2.84a) twice,

$$\sigma_{kl,ijmm} = 0 \tag{2.91}$$

that is, σ_{kl} is biharmonic.

2.8 APPLICATIONS AND WORKED EXAMPLES

2.8.1 Generalised plane stress and plane strain

(cf. 2.2.3, 2.4, 2.5 and 2.7)

Suppose that the point P on the surface $x_3 = 0$ is directly below the point P′ on the free surface $x_3 = h$ as shown in Figure 2.3, where h is small. The tractions on the free surface are zero, so that

$$\sigma'_{3j} \ (= \sigma'_{j3}) = 0 \quad (j = 1 \text{ to } 3) \tag{2.92}$$

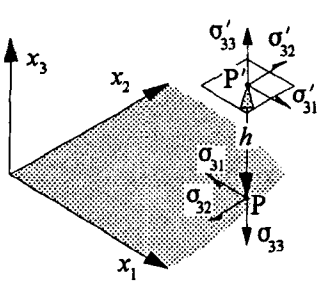


Figure 2.3 Thin plane stresses.

where the prime indicates the state of stress at the point P′. Then using Taylor's expansion,

$$0 = \sigma'_{3j} = \sigma_{3j} + h\sigma_{3j,3} + \tfrac{1}{2}h^2\sigma_{3j,33} + O(h^3) \qquad (j = 1 \text{ to } 3) \tag{2.93}$$

It follows that σ_{3j} is no greater than $O(h)$. On the free surface, σ'_{3j} is identically zero for all x_1 and x_2 so that all the derivatives of σ'_{3j} with respect to x_1 and x_2 are also zero. Using Taylor's expansion as before, it follows that $\sigma_{3j,1}$ ($= \sigma_{j3,1}$) and $\sigma_{3j,2}$ ($= \sigma_{j3,2}$) are no greater than $O(h)$. Then from (2.41), in the absence of body forces,

$$0 = \sigma_{13,1} + \sigma_{23,2} + \sigma_{33,3} = O(h) + \sigma_{33,3} \tag{2.94}$$

From (2.93), it then follows that σ_{33} is $O(h^2)$.

For isotropic materials, from (2.34) and (2.64),

$$\sigma_{ij} = \lambda\delta_{ij}u_{k,k} + \mu(u_{i,j} + u_{j,i}) \tag{2.95}$$

so that from the expression for σ_{33},

$$O(h^2) = \lambda u_{k,k} + 2\mu u_{3,3} = (\lambda + 2\mu)u_{k,k} - 2\mu(u_{1,1} + u_{2,2})$$

Then from (2.95),

$$\sigma_{IJ} = \lambda^*\delta_{IJ}u_{K,K} + \mu(u_{I,J} + u_{J,I}) + O(h^2) \qquad (I,J,K = 1 \text{ to } 2) \tag{2.96}$$

where

$$\lambda^* = 2\lambda\mu/(\lambda + 2\mu) \tag{2.97}$$

and the other stresses are of a small order, as already mentioned. If there are body forces and accelerations, (2.94) does not apply and from (2.93) the error terms in (2.96) are of order h in general.

When the surface $x_3 = 0$ lies midway between two free surfaces, $x_3 = \pm h$, then a second equation like (2.93)

$$0 = \sigma_{3j} - h\sigma_{3j,3} + \tfrac{1}{2}h^2\sigma_{3j,33} + O(h^3) \qquad (j = 1 \text{ to } 3) \tag{2.98}$$

also holds. Taking the sum of this and (2.93) immediately shows that σ_{j3} (and σ_{3j}) are $O(h^2)$. Taking the difference of these two equations shows that $\sigma_{j3,3}$ (and $\sigma_{3j,3}$) are $O(h)$. Two of the equilibrium equations (2.41) then become

$$\sigma_{JI,J} + \rho f_I \approx \rho\ddot{u}_I \qquad (I,J = 1 \text{ to } 2) \tag{2.99}$$

where the error is $O(h)$. The third equation is also satisfied to the same degree if f_3 and $\ddot{u}_3$ are also of order h. If the stresses, body forces and accelerations are replaced by their mean values over the thickness $2h$, it is possible to show that (2.99) is exact, see for example Sokolnikoff (1956) section 67.

Plane strain in the x_1x_2 plane occurs when the out-of-plane strains, e_{i3}, are zero. From (2.64), σ_{13} and σ_{23} are zero so that (2.99) becomes an equality instead of an approximation in this case. From (2.34) and (2.64),

$$\sigma_{IJ} = \lambda\delta_{IJ}u_{K,K} + \mu(u_{I,J} + u_{J,I}) \qquad (I,J,K = 1 \text{ to } 2) \tag{2.100}$$

cf. (2.96). Thus solutions for plane stress and plane strain problems are similar except that the plane stress solutions tend to be approximations and require the use of the modified Lamé constant λ^*.

The stress-strain relationship used for problems of plane elasticity may be deduced from (2.96) or (2.100) and (2.34). It takes a similar form to (2.64), that is

$$\sigma_{IJ} = \lambda^{(*)}\delta_{IJ}e_{KK} + 2\mu e_{IJ} \qquad (I,J,K = 1 \text{ to } 2) \tag{2.101}$$

where the asterisk is needed in the case of plane stress, when the equation may be an approximation as discussed above. Then

$$\sigma_{KK} = 2(\lambda^{(*)} + \mu)e_{KK}$$

giving the inverse of (2.101) as

$$e_{IJ} = \frac{1}{2\mu}(\sigma_{IJ} - \alpha\delta_{IJ}\sigma_{KK}) \tag{2.102}$$

where

$$\alpha = \lambda^{(*)}/(2(\lambda^{(*)} + \mu) \tag{2.103}$$

cf. (2.81). The only compatibility condition interrelating the in-plane strains is given by (1.74). The equivalent of this, derived from (2.37), is

$$e_{11,22} + e_{22,11} = 2e_{12,12}$$

or from (2.102),

$$\sigma_{11,22} + \sigma_{22,11} - \alpha(\sigma_{11,22} + \sigma_{22,11} + \sigma_{11,11} + \sigma_{22,22}) = 2\sigma_{12,12} \tag{2.104}$$

In the absence of body forces and accelerations, the equilibrium equation of the form given by (2.99) when differentiated yields

$$(\sigma_{11,1} + \sigma_{21,2})_{,1} = (\sigma_{12,1} + \sigma_{22,2})_{,2} = 0$$

and on substituting this into (2.104), the compatibility condition becomes

$$(1 - \alpha)\sigma_{II,JJ} = 0 \tag{2.105}$$

cf. (2.83). From (1.119), the stresses are defined in terms of the Airy stress function ϕ[†] by

$$\sigma_{IJ} = e_{IK}e_{JL}\phi_{,KL} \tag{2.106}$$

where e_{IJ} is the two-dimensional permutation symbol, that is,

$$e_{11} = e_{22} = 0 \quad , \quad e_{12} = 1 \quad , \quad e_{21} = -1 \tag{2.107}$$

cf. sections 1.1 and 2.2.3. The form of (2.106) can readily be generalised in terms of curvilinear coordinate systems, as will be seen in Chapter 3.

This definition of stress satisfies the equilibrium equation automatically and satisfies the compatibility condition (2.105) provided that

$$\phi_{,IIJJ} = 0 \tag{2.108}$$

that is, provided that ϕ is a biharmonic function of x_1 and x_2, cf. (1.120). Such biharmonic functions are examined next and in Appendix A2.1.

[†]This is not to be confused with the potential function ϕ used in section 2.7.7.

2.8.2 Complex variable methods (cf. 2.2.1, 2.5 and 2.7)

2.8.2.1 Some properties of functions of complex variables

Let the complex variable z be defined as

$$z \ (= x + iy) = x_1 + ix_2 \tag{2.109}$$

An analytic function

$$f(z) = u(z) + iv(z) \tag{2.110}$$

where $u(z)$ and $v(z)$ are real, is a function which is differentiable in some region. That is,

$$f'(z) = \lim_{\delta z \to 0}\{[f(z + \delta z) - f(z)]/\delta z\} \tag{2.111}$$

is unique, whatever δz is in terms of δx_1 and δx_2, provided that z is within the given region. Thus an analytic function is one which in some sense treats z as if it were a one-dimensional variable, without regard for the separate variations of its components x_1 and x_2. This is similar to the functions used in the theory of travelling waves, $f(x+ct)$ and $g(x-ct)$ where the characteristics, $x+ct$ and $x-ct$, are treated as single variables. Likewise, the function $f(\bar{z})$ is one in which the single variable z is replaced by its complex conjugate, $\bar{z}$. It is also possible to define a function $\bar{f}(z)$ in which the complex operations on z are replaced by their complex conjugates. The function $\bar{f}(\bar{z})$ is derived from $f(z)$ by replacing both the operations and the parameter z by their complex conjugates. It is equal to the complex conjugate of $f(z)$. That is,

$$\overline{f(z)} = \bar{f}(\bar{z}) = u(z) - iv(z) \tag{2.112}$$

In general, any function of x_1 and x_2 can be expressed in terms of operations on z and $\bar{z}$ because

$$x_1 = \frac{1}{2}(z + \bar{z}) \quad , \quad x_2 = \frac{1}{2i}(z - \bar{z}) \tag{2.113}$$

Taking a general function $F(z,\bar{z})$,

$$\begin{aligned} \frac{\partial F}{\partial z} &= \frac{\partial F}{\partial x_1}\frac{\partial x_1}{\partial z} + \frac{\partial F}{\partial x_2}\frac{\partial x_2}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right)F \\ \frac{\partial F}{\partial \bar{z}} &= \frac{\partial F}{\partial x_1}\frac{\partial x_1}{\partial \bar{z}} + \frac{\partial F}{\partial x_2}\frac{\partial x_2}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right)F \end{aligned} \tag{2.114}$$

Therefore

$$4\frac{\partial^2 F}{\partial z \partial \bar{z}} = \left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right)\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right)F = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)F = \nabla^2 F = F_{,II} \tag{2.115}$$

and likewise

$$16\frac{\partial^4 F}{\partial z^2 \partial \bar{z}^2} = \nabla^2(\nabla^2 F) = F_{,IIJJ} \tag{2.116}$$

where, as in section 2.8.1, capital letter subscripts are in the range 1 to 2. Consider the harmonic equation

$$F_{,II} = \nabla^2 F = \frac{\partial^2}{\partial z \partial \bar{z}}F = 0 \tag{2.117}$$

Partial integration with respect to $\bar{z}$ gives

$$\frac{\partial}{\partial z}F = f_1(z)$$

Integrating again with respect to z gives the general solution

$$F(z,\bar{z}) = F_1(z) + f_2(\bar{z}) \tag{2.118}$$

where $\dfrac{\partial F_1}{\partial z} = f_1(z)$ and F_1, f_1 and f_2 are analytic functions.

If $F(z,\bar{z})$ is a real function, then (2.118) can be rewritten in the form

$$F(z,\bar{z}) = \tfrac{1}{2}[f_3(z) + \overline{f_3(z)}] \qquad (f_3 = 2F_1) \tag{2.119}$$

which is the most general real solution of (2.117).

Use of the Airy stress function requires solutions of the biharmonic equation (2.108), where ϕ is real. Then $\nabla^2(\nabla^2\phi)$ is zero where $\nabla^2\phi$ is also real. From (2.116), the solution can be found by integrating (2.119) with respect to z and $\bar{z}$. This gives the most general form of ϕ as

$$\phi = \tfrac{1}{2}[\phi_1(z) + \overline{\phi_1(z)} + \bar{z}\phi_2(z) + z\overline{\phi_2(z)}] \tag{2.120}$$

where ϕ_1 and ϕ_2 are analytic functions. For the purposes of stress analysis, unique solutions are sought. The functions ϕ_1 and ϕ_2 will then be taken as single-valued in the region to which they are applied, that is they are **regular** or **holomorphic** functions. In some other texts, such as Muskhelishvili (1963) and Dugdale and Ruiz (1971), the analytic functions $\psi(z)$ and $\phi(z)$ are used instead of $\phi_1'(z)$ and $\phi_2(z)$ respectively. (A prime is used here to indicate differentiation of an analytic function with respect to z.)

2.8.2.2 Expressions for stress

Differentiating ϕ with respect to x_j gives

$$\begin{aligned}
\phi_{,1} &= \frac{\partial\phi}{\partial z}\frac{\partial z}{\partial x_1} + \frac{\partial\phi}{\partial\bar{z}}\frac{\partial\bar{z}}{\partial x_1} \\
&= \tfrac{1}{2}[\phi_1'(z) + \bar{z}\phi_2'(z) + \overline{\phi_2(z)}] + \tfrac{1}{2}[\bar{\phi}_1'(\bar{z}) + \phi_2(z) + z\bar{\phi}_2'(\bar{z})] \\
\phi_{,2} &= \frac{\partial\phi}{\partial z}\frac{\partial z}{\partial x_2} + \frac{\partial\phi}{\partial\bar{z}}\frac{\partial\bar{z}}{\partial x_2} \\
&= \tfrac{i}{2}[\phi_1'(z) + \bar{z}\phi_2'(z) + \overline{\phi_2(z)}] - \tfrac{i}{2}[\bar{\phi}_1'(\bar{z}) + \phi_2(z) + z\bar{\phi}_2'(\bar{z})]
\end{aligned} \tag{2.121}$$

using (2.109), (2.120) and the first equality in (2.112). Repeating the differentiation in the same manner and using (2.106) or (1.119) to express the stresses in terms of ϕ yields the expressions

$$\begin{aligned}
&\sigma_{11} + \sigma_{22} = 2\phi_2'(z) + 2\bar{\phi}_2'(\bar{z}) = 4\Re\{\phi_2'(z)\} \\
&\sigma_{22} - \sigma_{11} - 2i\sigma_{12} = 2[z\bar{\phi}_2''(\bar{z}) + \bar{\phi}_1''(\bar{z})] \\
\text{or}\quad &\sigma_{22} - \sigma_{11} + 2i\sigma_{12} = 2[\bar{z}\phi_2''(z) + \phi_1''(z)]
\end{aligned} \tag{2.122}$$

where $\Re\{\ \}$ and $\Im\{\ \}$ are used to indicate the real and imaginary parts of an expression. For example, $u(z)$ and $v(z)$ are the real and imaginary parts of $f(z)$ in (2.110).

2.8.2.3 Boundary conditions

Fig. 2.4 shows a plane elastic region R with a boundary contour C. An elementary triangular zone is shown, to an exaggerated scale, on the right of the region. The length of its horizontal face is δx_1, the height of its vertical face is δx_2 and the length of its sloping face, which forms part of the boundary contour, is δs. On this face, surface tractions p_1 and p_2 act horizontally and vertically as shown. The unit normal $\boldsymbol{n}$ to the face has two components,

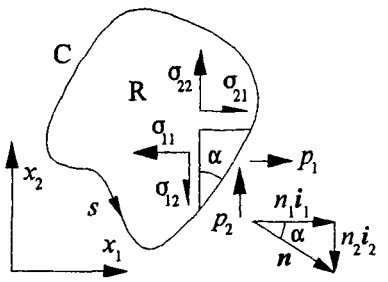


Figure 2.4 Boundary conditions.

$$n_1 = \cos\alpha = \delta x_2/\delta s \quad , \quad n_2 = -\sin\alpha = -\delta x_1/\delta s$$

so that from (2.40),

$$\begin{aligned} \sigma_{11}\delta x_2/\delta s - \sigma_{21}\delta x_1/\delta s &= p_1 \\ \sigma_{12}\delta x_2/\delta s - \sigma_{22}\delta x_1/\delta s &= p_2 \end{aligned} \tag{2.123}$$

or, at the limit $\delta s \to 0$,

$$\sigma_{IJ} e_{IM} \frac{dx_M}{ds} = p_J \tag{2.124}$$

from (2.107). Then from (2.106),

$$p_J = e_{IK} e_{JL} e_{IM} \phi_{,KL} \frac{dx_M}{ds} = e_{JL} \frac{d\phi_{,L}}{ds} \tag{2.125}$$

because

$$e_{IK} e_{IM} = \delta_{KM} \quad \text{and} \quad (\phi_{,L})_{,K} \frac{dx_K}{ds} = \frac{d\phi_{,L}}{ds} .$$

Knowing the surface tractions p_J, it is possible to determine the function $p(s)$ where

$$p(s) = i\int_0^s (p_1 + ip_2)ds$$

integration being along the boundary contour C from some arbitrary origin. Using this expression, together with (2.121) and (2.125),

$$p(s) = \phi_{,1} + i\phi_{,2} + c = \bar{\phi}_1'(\bar{z}) + \phi_2(z) + z\bar{\phi}_2'(\bar{z}) + c \tag{2.126}$$

where c is a constant and z is given by the ordinates on the boundary corresponding to z, that is, $z = z(s)$.

2.8.2.4 The unit circle problem

A unit circle is centred on the origin and has a unit radius. On the boundary of this circle, C, the complex variable z is related to polar coordinates by $z = e^{i\theta}$, $\bar{z} = e^{-i\theta}$.

Suppose that $p(s)$ can be represented by a complex Fourier series of the form

$$p(s) = \sum_{j=-\infty}^{\infty} A_j z^j = \sum_{j=-\infty}^{\infty} A_j e^{j(i\theta)} \tag{2.127}$$

where the complex coefficients A_j are found from the contour integral

$$\oint_C p(s)e^{-ik\theta}d\theta = \sum_{j=-\infty}^{\infty} A_j \int_0^{2\pi} e^{i(j-k)\theta}d\theta = 2\pi A_k \qquad (2.128)$$

all the integrals in the summation being zero unless $j=k$. Because ϕ_1' and ϕ_2 are regular functions, they can be represented by the Taylor's expansions

$$\phi_1' = \sum_{k=0}^{\infty} a_k z^k \quad , \quad \phi_2 = \sum_{k=0}^{\infty} b_k z^k \qquad (2.129)$$

Substituting the expansions given by (2.127) and (2.129) into (2.126), and bearing in mind that the boundary is a unit circle,

$$\sum_{k=0}^{\infty} (\bar{a}_k e^{-ik\theta} + b_k e^{ik\theta}) + \sum_{k=1}^{\infty} k\bar{b}_k e^{-i(k-2)\theta} + c = \sum_{j=-\infty}^{\infty} A_j e^{ij\theta} \qquad (2.130)$$

The form of ϕ is to some extent arbitrary, in that the expressions for the stresses given by (2.122) are not altered by assuming that $b_0 = \Im\{b_1\} = c = 0$.
Then on comparing coefficients of like exponentials,

$$b_k = A_k \quad (k \geq 2) \quad , \quad b_1 + \bar{b}_1 \ (= 2b_1) = A_1 \quad , \quad a_k = \bar{A}_{-k} - (k+2)A_{k+2} \qquad (2.131)$$

From (2.122), (2.128), (2.129) and (2.131) the state of stress within the unit circle can be found for any given surface tractions on the boundary. Note that (2.131) implies that A_1 is always real. This can be deduced from the condition that the surface tractions have no resultant torque.

Example 2.1 A pinch-loaded disc

Fig. 2.5 shows a circular disc with equal and opposite forces P applied to points on the periphery, subtending an angle 2θ at the centre. The coefficients in the complex Fourier series given by (2.127) are found from the following contour integration.

$$2\pi A_k = \oint_C p(s)e^{-ik\theta}d\theta$$
$$= \int_{\alpha}^{2\pi-\alpha} Pe^{-ik\theta}d\theta = -\frac{P}{ik}(e^{ik\alpha} - e^{-ik\alpha})$$

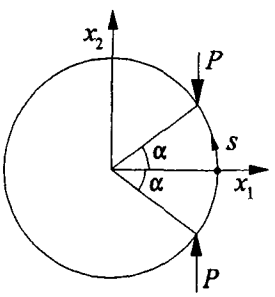


Figure 2.5 A loaded disc.

so that $$A_k = -\frac{P}{\pi k}\sin k\alpha \quad (k \neq 0) \quad \text{and} \quad A_0 = \frac{P}{\pi}(\pi - \alpha).$$

Then from (2.129) and (2.131),

$$\phi_1' = \frac{P}{\pi}\left[\pi - \alpha + \sin 2\alpha + \sum_{k=1}^{\infty}\left(\sin(k+2)\alpha - \frac{1}{k}\sin k\alpha\right)z^k\right]$$

and $$\phi_2 = -\frac{P}{\pi}\left[\frac{1}{2}z\sin\alpha + \sum_{k=2}^{\infty}\left(\frac{1}{k}\sin k\alpha\right)z^k\right]$$

$$\text{so that} \qquad \phi_1'' = \frac{P}{\pi}\sum_{k=1}^{\infty}[k\sin(k+2)\alpha - \sin k\alpha]z^{k-1}$$

$$\text{and} \qquad \phi_2' = -\frac{P}{\pi}[\frac{1}{2}\sin\alpha + \sum_{k=2}^{\infty}(\sin k\alpha)z^{k-1}]$$

Substituting these expressions into (2.122) gives the stress distribution. For example, at the centre of the disc,

$$\sigma_{11} = -\frac{P}{\pi}\sin 3\alpha \quad , \quad \sigma_{22} = \frac{P}{\pi}(\sin 3\alpha - 2\sin\alpha) \quad , \quad \sigma_{12} = 0.$$

2.8.2.5 Strains and displacements

From (2.102) and (2.122),

$$e_{11} + e_{22} = e_{II} = \frac{1-2\alpha}{2\mu}\sigma_{II} = \frac{\kappa}{2\mu}\sigma_{II} = \frac{\kappa}{\mu}[\phi_2'(z) + \overline{\phi_2'(z)}] \tag{2.132}$$

$$\text{where} \quad \kappa = 1 - 2\alpha \begin{cases} = \dfrac{\lambda+2\mu}{3\lambda+2\mu} = \dfrac{1-\nu}{1+\nu} & \text{(plane stress)} \\ = \dfrac{\mu}{\lambda+\mu} = 1-2\nu & \text{(plane strain)} \end{cases}$$

and

$$\begin{aligned} e_{22} - e_{11} - 2ie_{12} &= \frac{1}{2\mu}[\sigma_{22} - \sigma_{11} - 2i\sigma_{12}] = \frac{1}{\mu}[z\overline{\phi_2''(z)} + \overline{\phi_1''(z)}] \\ \text{or} \quad e_{22} - e_{11} + 2ie_{12} &= \frac{1}{\mu}[\bar{z}\phi_2''(z) + \phi_1''(z)] \end{aligned} \tag{2.133}$$

Suppose that the displacements u_1 and u_2 are expressed in terms of a function U as

$$U(z,\bar{z}) = u_1 + iu_2 \tag{2.134}$$

Then from (2.134),

$$e_{II} = u_{1,1} + u_{2,2} = \frac{1}{2}(U+\bar{U})_{,1} - \frac{1}{2}i(U-\bar{U})_{,2} = \frac{\partial U}{\partial z} + \frac{\partial \bar{U}}{\partial \bar{z}} \tag{2.135}$$

using (2.110), (2.112) and (2.114). Likewise,

$$\begin{aligned} e_{22} - e_{11} - 2ie_{12} &= -\tfrac{1}{2}i(U-\bar{U})_{,2} - \tfrac{1}{2}(U+\bar{U})_{,1} - i[\tfrac{1}{2}(U+\bar{U})_{,2} - \tfrac{1}{2}i(U-\bar{U})_{,1}] \\ &= -(U_{,1} + iU_{,2}) = -2\frac{\partial U}{\partial \bar{z}} \\ \text{or} \quad e_{22} - e_{11} + 2ie_{12} &= -2\frac{\partial \bar{U}}{\partial z} \end{aligned} \tag{2.136}$$

on taking the complex conjugate of the upper equation. Then from (2.133), on integrating with respect to z,

$$-2\mu\bar{U} = \bar{z}\phi_2'(z) + \phi_1'(z) + \overline{f(z)} \tag{2.137}$$

where $f(z)$ is an analytic function of z. Then from (2.135), (2.132) and (2.137),

$$f'(z) + \bar{f}'(\bar{z}) = -(1+2\kappa)[\phi_2'(z) + \bar{\phi}_2'(\bar{z})]$$

Comparing terms in z, $f'(z) = -(1+2\kappa)\phi_2'(z) + iA$ (A real)
and on integrating, $f(z) = -(1+2\kappa)\phi_2(z) + iAz + B$ (B complex)
Substituting this result into the conjugate of (2.137) gives

$$2\mu U = 2\mu(u_1 + iu_2) = -z\overline{\phi_2'(z)} - \overline{\phi_1'(z)} + (1+2\kappa)\phi_2(z) \qquad \textbf{(2.138)}$$

ignoring the rigid-body rotations and displacements given by A and B respectively. Thus displacements as well as stresses can be found from the stress functions. Also, it is possible to find stress functions for displacements prescribed on the surface of a body, rather than just tractions.

2.8.2.6 Coordinate transformation

Suppose that we wish to find the stresses $\tilde{\sigma}_K$ with respect to a coordinate system $\tilde{x}_L$ which is at an angle θ to the x_I coordinate system, as shown in Fig. 2.6. (The symbol ~ is used here instead of a straight bar to avoid confusion with complex conjugates.) These stresses $\tilde{\sigma}_{11}, \tilde{\sigma}_{12}, \tilde{\sigma}_{21}$ and $\tilde{\sigma}_{22}$ could be the physical components of stresses σ_{rr}, $\tau_{r\theta}$, $\tau_{\theta r}$ and $\sigma_{\theta\theta}$ associated with a polar coordinate system (r,θ) for example. In two dimensions, the required transformation matrix is given by the first two rows and columns of that in (1.33) and correspondingly, t_{ij} in (2.61) can be reduced to the form

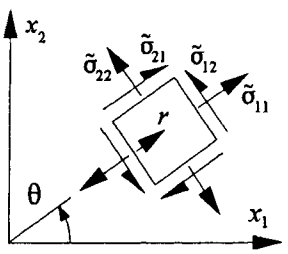


Figure 2.6 Rotated coordinates.

$$t_{IJ} = \frac{1}{2} i^{(I+J)}(i^{2J}e^{i\theta} + i^{2I}e^{-i\theta}) \qquad \textbf{(2.139)}$$

It follows from (2.42) that σ_{II} is invariant in the x_1x_2 plane, so that

$$\tilde{\sigma}_{11} + \tilde{\sigma}_{22} = \tilde{\sigma}_{JJ} = \sigma_{II} = \sigma_{11} + \sigma_{22} = 4\Re\{\phi_2'(z)\}$$

from (2.122). Also,

$$\sigma_{22} - \sigma_{11} - 2i\sigma_{12} = \Sigma_I\Sigma_J i^{(I+J)}\sigma_{IJ} \qquad \textbf{(2.140)}$$

where Σ_I indicates summation over the range of I (1 to 2) and other summations are similarly defined. With respect to the new coordinate system, from (2.15),

$$\begin{aligned}\tilde{\sigma}_{22} - \tilde{\sigma}_{11} - 2i\tilde{\sigma}_{12} &= \Sigma_K\Sigma_L i^{(K+L)}\tilde{\sigma}_{KL} = \Sigma_K\Sigma_L i^{(K+L)}t_{KI}t_{LJ}\sigma_{IJ} \\ &= \Sigma_I\Sigma_J\Sigma_K\Sigma_L\tfrac{1}{4}i^{2(K+L)}[i^{2I}e^{i\theta} + i^{2K}e^{-i\theta}][i^{2J}e^{i\theta} + i^{2L}e^{-i\theta}]i^{(I+J)}\sigma_{IJ} \qquad \textbf{(2.141)} \\ &= e^{-2i\theta}\Sigma_I\Sigma_J i^{(I+J)}\sigma_{IJ} = 2e^{-2i\theta}[z\bar{\phi}_2''(\bar{z}) + \bar{\phi}_1''(\bar{z})]\end{aligned}$$

using the results

$$\Sigma_K\Sigma_L i^{4(K+L)} = 4\ ,\ \Sigma_K\Sigma_L i^{2(K+L)} = \Sigma_K\Sigma_L i^{2K+4L} = \Sigma_K\Sigma_L i^{4K+2L} = 0.$$

On taking the complex conjugate of (2.141),

$$\tilde{\sigma}_{22} - \tilde{\sigma}_{11} + 2i\tilde{\sigma}_{12} = 2e^{2i\theta}[\bar{z}\phi_2''(z) + \phi_1''(z)] \qquad \textbf{(2.142)}$$

The strains transform in a similar fashion, so that from (2.132) and (2.133),

$$\tilde{e}_{11} + \tilde{e}_{22} = \tilde{e}_{II} = \frac{\kappa}{\mu}[\phi_2'(z) + \overline{\phi_2'(z)}]$$
$$\tilde{e}_{22} - \tilde{e}_{11} - 2i\tilde{e}_{12} = \frac{1}{\mu}e^{-2i\theta}[z\overline{\phi_2''(z)} + \overline{\phi_1''(z)}] \qquad (2.143)$$
$$\tilde{e}_{22} - \tilde{e}_{11} + 2i\tilde{e}_{12} = \frac{1}{\mu}e^{2i\theta}[\bar{z}\phi_2''(z) + \phi_1''(z)]$$

These results can also be obtained by treating the Mohr's circles for strain and stress in Figs. 1.6 and 1.9 as Argand diagrams.

2.8.3 The San Andreas Fault

(cf. 2.4, 2.5 and 2.7)

Two major plates of the lithosphere, the Pacific plate and the North American plate, meet along the line of the San Andreas fault. This runs from Cape Mendocino in the north of California to San Bernardino in the south. The relative velocity of the plates, V_p, is 55mm/year. The plates are locked together mostly in two zones, one in the region of San Francisco and the other near Los Angeles. The fault is shown schematically in Fig. 2.7, the shaded areas being the locked zones. In fact, the fault is somewhat curved, although for the purposes of analysis it will be taken as straight. Major earthquakes occur when these locked zones become unlocked. Thus there was a major earthquake in the Los Angeles area in 1857 when the southern zone became unlocked and one of magnitude 8.3 on the Richter scale in 1906 in the San Francisco area when the northern zone became unlocked. The seismic energy E of an earthquake is related to its magnitude M on the Richter scale by Bath's formula,

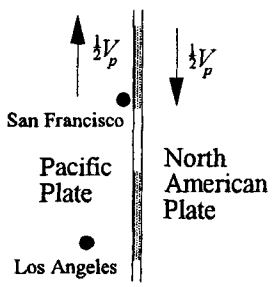


Figure 2.7 The San Andreas Fault.

$$\log_{10}E = 1.44M + 5.24 \qquad (2.144)$$

so that the energy associated with the 1906 earthquake was about 1.55×10^{17} joules. The following analyses have been employed in a number of research papers and the first two are also given by Turcotte and Schubert (1982).

2.8.3.1 A surface crack

An isotropic, linearly-elastic continuum occupies the half-space $x_2 > 0$ and has a crack of depth a in the plane $x_1 = 0$ as shown in Fig. 2.8. Skew-symmetric deformations about this plane, given by the displacement $\boldsymbol{u}_3 = u_3\,\boldsymbol{i}_3$, where u_3 is not a function of x_3, will be assumed to take place. From (2.34) and (2.64), it follows immediately that σ_{11}, σ_{12}, σ_{22} and σ_{33} are zero. Then in the absence of body forces and accelerations, the first two of equations (2.41) are satisfied identically and the third becomes

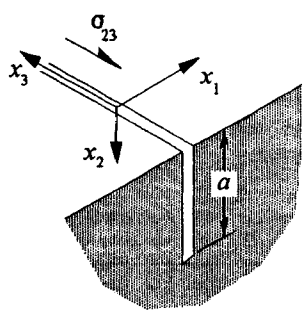


Figure 2.8 A surface crack.

$$\sigma_{13,1} + \sigma_{23,2} = 0 \qquad (2.145)$$

From (2.34) and (2.64),

$$\sigma_{13} = 2\mu e_{13} = \mu u_{3,1} \quad , \quad \sigma_{23} = 2\mu e_{23} = \mu u_{3,2} \tag{2.146}$$

Then (2.145) can be written in the form

$$u_{3,11} + u_{3,22} = \nabla^2 u_3 = 0 \tag{2.147}$$

From (2.119), (2.147) can be satisfied by taking u_3 as the real part of any analytic function $f_3(z)$. Let $f_3(z) = \frac{1}{2}A(a^2 + z^2)^{1/2}$ $(z = x_1 + ix_2)$ where A is real. Then

$$u_3 = \tfrac{1}{2}A\left((a^2 + z^2)^{1/2} + \overline{(a^2 + z^2)^{1/2}}\right) = A\Re\{(a^2 + z^2)^{1/2}\} \tag{2.148}$$

so that on $x_1 = 0$, $u_3 = A\Re\{(a^2 - x_2^2)^{1/2}\}$. This means that u_3 is zero for $x_2 > a$, because $(a^2 - x_2^2)^{1/2}$ is imaginary. From (2.114),

$$\begin{aligned} u_{3,1} &= \tfrac{1}{2}A\left(z(a^2 + z^2)^{-1/2} + \bar{z}(a^2 + \bar{z}^2)^{-1/2}\right) = A\Re\{z(a^2 + z^2)^{-1/2}\} \\ u_{3,2} &= \tfrac{1}{2}iA\left(z(a^2 + z^2)^{-1/2} - \bar{z}(a^2 + \bar{z}^2)^{-1/2}\right) = -A\Im\{z(a^2 + z^2)^{-1/2}\} \end{aligned} \tag{2.149}$$

On $x_2 = 0$, $\sigma_{23} = -\mu A\Im\{x_1(a^2 + x_1^2)^{-1/2}\} = 0$ and the other surface tractions σ_{22} and σ_{21} are also zero, so that this is a free surface.
On $x_1 = 0$, $\sigma_{13} = \mu A\Re\{ix_2(a^2 - x_2^2)^{-1/2}\}$ so that σ_{13} is zero for $x_2 < a$. Also, σ_{11} and σ_{12} are zero, so that the surface $x_1 = 0$, $0 < x_2 < a$ represents the unloaded crack surface. For very large x_1, from (2.146) and (2.149),

$$\sigma_{13} \approx \mu A\Re\{x_1/x_1\} = \mu A = \tau \quad \text{(say)} \tag{2.150}$$

In the absence of a crack, pure shear of the half-space under the action of the shear stress τ is given by the displacement field

$$u_3^0 = \frac{\tau}{\mu}x_1$$

so that from (2.148), on the surface $x_2 = 0$, the change in displacement which occurs when a crack develops is

$$\Delta u_3 = u_3 - u_3^0 = \frac{\tau}{\mu}[(a^2 + x_1^2)^{1/2} - x_1] \tag{2.151}$$

Thus, at a distance of $\frac{3}{4}a$ from the crack, the change of displacement falls to one half of that at the crack, and tends to zero as x_1 becomes very large. Turcotte and Schubert (ibid.) have shown that this distribution of displacement is in reasonable agreement with data from the 1906 earthquake in San Francisco, although they point out that it is likely that the crack would have extended through the full depth of the lithosphere during the earthquake.

2.8.3.2 An underground crack

Suppose that the lithosphere is of depth h and that a crack extends from its base to a distance d below the surface, as shown in Fig. 2.9. Again taking the displacements to be given by a function u_3 which is not a function of x_3, (2.147) implies that it can be given

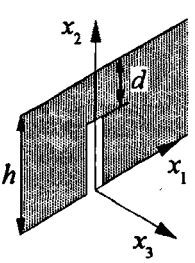


Figure 2.9 A buried crack.

by the real part of a harmonic function of z. Let

$$u_3 = B\Re\left\{ \ln\left(\left[\cosh\frac{\pi z}{2h} + \left(\cos^2\frac{\pi d}{2h} + \sinh^2\frac{\pi z}{2h} \right)^{1/2} \right] \Big/ \sin\frac{\pi d}{2h} \right) \right\} \qquad \textbf{(2.152)}$$

The boundary conditions are

On $x_1 = 0$: $u_{3,1} = 0$ $(0 < x_2 < h\text{-}d)$, $u_3 = 0$ $(h\text{-}d < x_2 < h)$. On $x_2 = 0$ or h: $u_{3,2} = 0$,

using (2.146) and assuming that there is no shear stress on the base of the lithosphere, as well as none on its top surface. These conditions are satisfied by the above expression for u_3. For example, on $x_1 = 0$, (2.152) takes the form

$$u_3 = B\Re\left\{ \ln\left(\left[\cos\frac{\pi x_2}{2h} + \left(\cos^2\frac{\pi d}{2h} - \sin^2\frac{\pi x_2}{2h} \right)^{1/2} \right] \Big/ \sin\frac{\pi d}{2h} \right) \right\} \qquad \textbf{(2.153)}$$

The expression is real, provided that $x_2 < h\text{-}d$. For the region $h\text{-}d < x_2 < h$, the expression in square brackets becomes

$$\cos\frac{\pi x_2}{2h} + i\left(\sin^2\frac{\pi x_2}{2h} - \cos^2\frac{\pi d}{2h} \right)^{1/2} = re^{i\theta} \quad \text{(say)} \equiv r\cos\theta + ir\sin\theta$$

where the term in round brackets is positive. Then

$$r^2 = r^2\cos^2\theta + r^2\sin^2\theta = \cos^2\frac{\pi x_2}{2h} + \sin^2\frac{\pi x_2}{2h} - \cos^2\frac{\pi d}{2h} = \sin^2\frac{\pi d}{2h} \qquad \textbf{(2.154)}$$

and so $$u_3 = B\Re\{\ln e^{i\theta}\} = B\Re\{i\theta\} = 0.$$

Similarly, the other boundary conditions are seen to be satisfied on using the same process of differentiation as in the previous subsection. Taking the crack to be at a depth d of 10km, the thickness h of the lithosphere to be 30km and the relative velocity V_p of the two plates to be reached at a distance of 37.5km on either side of it, Turcotte and Schubert (ibid.) have shown that this solution is in reasonable agreement with the measured shear strain rates $\dot{e}_{13}$ on the surface in the San Francisco area.

2.8.3.3 Major earth movements

A modified form of an analysis of the complete fault given by Spence and Turcotte† will now be examined. The plates will be taken to be in a state of plane stress so that the results of section 2.8.2 can be used.

The analytic function $\omega(z)$ is defined as

$$\omega(z) = z\bar{\phi}_2'(z) + \bar{\phi}_1'(z) \qquad \textbf{(2.155)}$$

Then (2.138) becomes

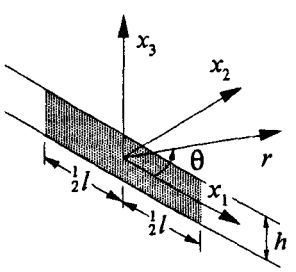


Figure 2.10 A locked zone.

†Spence, D.A. and Turcotte, D.L. (1976) An elastostatic model of stress accumulation on the San Andreas fault. *Proc. Roy. Soc. London Ser. A* **349** 319-341

$$2\mu(u_1 + iu_2) = (1 + 2\kappa)\phi_2(z) - \omega(\bar{z}) - (z - \bar{z})\overline{\phi_2'(z)} \qquad \textbf{(2.156)}$$

From (2.122), $$\sigma_{22} - i\sigma_{12} = \phi_2'(z) + \overline{\phi_2'(z)} + z\overline{\phi_2''(z)} + \overline{\phi_1''(z)} \qquad \textbf{(2.157)}$$

From (2.155), $$\omega'(z) = \overline{\phi_2'}(z) + z\overline{\phi_2''}(z) + \overline{\phi_1''}(z)$$

noting that $[\overline{f(z)}]'$ is equal to $\overline{f'(z)}$. Now (2.157) can be written as

$$\sigma_{22} - i\sigma_{12} = \phi_2'(z) + \omega'(\bar{z}) + (z - \bar{z})\overline{\phi_2''(z)} \qquad \textbf{(2.158)}$$

(In some other texts, ϕ_2, ϕ_2' and ω' are represented by ϕ, Φ and Ω.)

Fig. 2.10 shows a plate with a fault line on the plane $x_2 = 0$. The fault is locked in the region $-l/2 < x_1 < l/2$ where u_1 is zero. The rest of the fault is unlocked, so that σ_{21} is zero. Let

$$\phi_2 = -\omega = A\ln g(z), \quad \text{where} \quad g(z) = z + (z^2 - \tfrac{1}{4}l^2)^{1/2} = Ge^{i\gamma} \quad \text{(say)}$$

so that $$\phi_2' = -\omega' = A(z^2 - \tfrac{1}{4}l^2)^{-1/2}, \quad \phi_2'' = -Az(z^2 - \tfrac{1}{4}l^2)^{-3/2}$$

Let $$z \ (= x_1 + ix_2) = re^{i\theta}, \quad (z^2 - \tfrac{1}{4}l^2)^{1/2} = \rho e^{i\psi} \quad \text{(say)}$$

Then from (2.156),

$$\mu(u_1 + iu_2) = A\left[(\kappa + 1)\ln\left(2\frac{G}{l}\right) + i\kappa\left(\gamma - \frac{\pi}{2}\right) - i\frac{x_2 e^{i\psi}}{\rho} + \left\{(1 + \kappa)\ln\frac{l}{2} + \frac{i\kappa\pi}{2}\right\}\right] \qquad \textbf{(2.159)}$$

The terms in curly brackets represent a rigid-body motion and so can be left out, making the displacement at the origin zero. On $x_2 = 0$, if $x_1^2 < l^2/4$, then G is $l/2$ [cf. (2.154)] so that u_1 is zero along the locked section of the fault. From (2.158),

$$\sigma_{22} - i\sigma_{12} = -2iA\left(\rho^2 \sin\psi + x_2 re^{i(3\psi - \theta)}\right)\Big/\rho^3 \qquad \textbf{(2.160)}$$

On $x_2 = 0$, σ_{22} is zero and if $x_1^2 > l^2/4$ then ψ is zero so that the shear stress σ_{12} is also zero on the unlocked sections of the fault.

At very large distances from the origin $(r = R, R \gg l)$ then $\rho \approx R$, $\psi \approx \theta$, $G \approx 2R$ and $\gamma \approx \theta$ so that

$$\mu u_1 \approx A[(\kappa + 1)\ln(4R/l) + \sin^2\theta], \quad \mu u_2 \approx A[\kappa(\theta - \tfrac{\pi}{2}) - \sin\theta\cos\theta] \qquad \textbf{(2.161)}$$

and $$\sigma_{11} \approx 4A\cos^3\theta/R, \quad \sigma_{12} \approx 4A\sin\theta\cos^2\theta/R, \quad \sigma_{22} \approx 4A\sin^2\theta\cos\theta/R \qquad \textbf{(2.162)}$$

where σ_{11} is found from (2.122).

Similarly, a fault with two locked zones is given by taking

$$\phi_2 = -\omega = B\ln f(z) \quad \text{where} \quad f(z) = (z^2 - a^2)^{1/2} + (z^2 - b^2)^{1/2} = Fe^{i\alpha} \quad \text{(say)}$$

where $b > a$ so that

$$\mu(u_1 + iu_2) = B[(\kappa + 1)\ln\{F/(b^2 - a^2)^{1/2}\} + i\kappa(\alpha - \tfrac{\pi}{2}) - ix_2 re^{i(\beta - \theta)}/P^2] \qquad \textbf{(2.163)}$$

where $$P^2 e^{i\beta} = (z^2 - a^2)^{1/2}(z^2 - b^2)^{1/2}$$

and a rigid-body displacement has been added to eliminate the displacement at the origin. The above functions also give

$$\sigma_{22} - i\sigma_{12} = -2iB\left(P^2 r\sin(\beta-\theta) + x_2(r^4 e^{-4i\theta} - a^2b^2)e^{3i\beta}\right)\Big/P^6 \qquad (2.164)$$

On the plane $x_2 = 0$, σ_{22} is zero and

$$\sigma_{12} = 0 \quad (x_1 < -b,\ -a < x_1 < a,\ \text{or } x_1 > b),\ u_1 = 0 \quad (-b < x_1 < -a \text{ or } a < x_1 < b).$$

This corresponds to a fault on the plane $x_2 = 0$ with two locked zones ($-b < x_1 < -a$ and $a < x_1 < b$). At large distances from the origin ($r=R$, $R \gg a,b$)

$$\mu u_1 \approx B\left[(\kappa+1)\ln\left(2R/(b^2-a^2)^{1/2}\right) + \sin^2\theta\right],\ \mu u_2 \approx B\left[\kappa\left(\theta - \tfrac{\pi}{2}\right) - \sin\theta\cos\theta\right] \qquad (2.165)$$

$$\text{and}\quad \sigma_{11} \approx 4B\cos^3\theta/R,\ \sigma_{12} \approx 4B\sin\theta\cos^2\theta/R,\ \sigma_{22} \approx 4B\sin^2\theta\cos\theta/R \qquad (2.166)$$

During an earthquake, one of the zones becomes unlocked and then locks again. Further deformation due to the relative motion of the plates then takes place. Then, in general, the state of deformation is that given by that for a single locked zone, with a double locked zone deformation, appropriate for the further relative motion of the two plates, superimposed on it. The further relative motion of the plates at a time t after the most recent earthquake is $V_p t$. This relative motion is measured with respect to points at a large distance, R say, from the locked parts of the fault. Then from (2.165), for $\theta = \pi/2$,

$$\tfrac{1}{2}\mu V_p t \approx B\left[(\kappa+1)\ln\left(2R/(b^2-a^2)^{1/2}\right) + 1\right] \qquad (2.167)$$

The expression for the strain energy stored in the two plates is similar in form to that given by (1.129). However, in this case, it will be calculated from the work done on two semi-circular segments of radius R, ($0 \le \theta \le \pi$) by the tractions around their peripheries. Along the fault, no work is done by these tractions because either they are zero or the displacements through which they do work are zero. Around the semicircular arcs, the tractions are given by (2.40), where the unit normal is in the radial direction from the origin so that n_1 and n_2 are $\cos\theta$ and $\sin\theta$ respectively. Then the work done by the tractions is

$$\begin{aligned} W = 2h\int_0^{\pi} &\left[(\sigma_{11}n_1 + \sigma_{21}n_2)\left(\tfrac{1}{2}u_1 + u_1'\right) + (\sigma_{12}n_1 + \sigma_{22}n_2)\left(\tfrac{1}{2}u_2 + u_2'\right)\right. \\ &\left. + \tfrac{1}{2}(\sigma_{11}'n_1 + \sigma_{21}'n_2)u_1' + \tfrac{1}{2}(\sigma_{12}'n_1 + \sigma_{22}'n_2)u_2'\right]R\,d\theta \end{aligned} \qquad (2.168)$$

where the unprimed terms are the stresses and displacements immediately after an earthquake, given by (2.161) and (2.162), and the primed terms are the additional stresses and displacements, given by (2.165) and (2.166), arising from the regular motion of the plates. Substituting these expressions into (2.168) and integrating gives

$$W = \frac{\pi h}{\mu}\left(2(\kappa+1)\left[A^2\ln\frac{4R}{l} + (2A+B)B\ln\frac{2R}{(b^2-a^2)^{1/2}}\right] - \kappa(A+B)^2\right) \qquad (2.169)$$

Suppose that an earthquake occurs in which one zone (zone 1) remains locked and the other (zone 2) becomes unlocked. This state is described by taking $A = A_1$ in the first solution. Suppose further that both zones are locked again for a time t, after which a

second earthquake takes place in which zone 1 unlocks and zone 2 remains locked. This state is described by taking $A=A_2$ and making the appropriate shift of origin from zone 1 to zone 2. The energy released during this second earthquake is given by the difference between the energy found from (2.169) by taking $A=A_1$ and determining B from (2.167), and that found from (2.169) by taking $A=A_2$ and setting B to zero.

A relationship between A_1 and A_2 can be found from the condition that the distant point at radius R does not move relative to the zone which remains locked during the earthquake. Knowing the energy of the 1906 earthquake, the state following it can now be found. This can be used to predict the magnitude of an earthquake in the Los Angeles area as a function of the time of its occurrence after 1906 (see Fig. 2.11). This has been calculated from the data given above and taking $\mu = 30\text{GPa}$, $a = 110\text{km}$, $b = 450\text{km}$, $l = b-a = 340\text{km}$, $R = 3000\text{km}$, $h = 30\text{km}$ and $\kappa = 0.575$ as in some previous analyses.

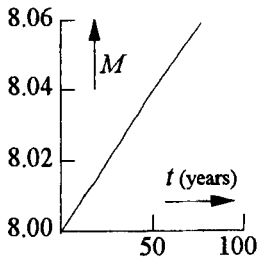


Figure 2.11 Earthquake magnitude.

It is not possible to predict the time of this earthquake, because the geological record indicates that earthquakes in the Los Angeles area take place at random intervals with a mean interval of 164 years (see Turcotte and Schubert ibid. pp. 363-365). This suggests that the shear resistance of the zone changes each time it becomes unlocked.

In the above solutions, the normal stress across the fault, σ_{22}, is zero. Because of the curvature of the fault, this is unlikely to be the case and there is reason to believe that a considerable portion of the stored strain energy is created by these stresses.

PROBLEMS

(2.1) Show that the coefficients 3/13, 4/13, 12/13; 4/5, -3/5, 0; 36/65, 48/65 and -5/13 satisfy the necessary conditions for them to form the elements of a transformation t_{11}, t_{12}, t_{13}; t_{21}, t_{22}, t_{23}; t_{31}, t_{32} and t_{33} respectively. (M)

(2.2) The coefficients 3, -2, 1; 1, 0, 2; 2, -1 and 1 form the elements A_{11}, A_{12}, A_{13}; A_{21}, A_{22}, A_{23}; A_{31}, A_{32} and A_{33} respectively of a second-order tensor A_{ij}. Find the coefficients $\bar{A}_{kl}$ of this tensor with respect to a second coordinate system, using the transformation given by problem 2.1. (M)

(2.3) For the above tensors, evaluate the invariants A_{ii} and $A_{ij}A_{ij}$, and show that they are equal to $\bar{A}_{kk}$ and $\bar{A}_{kl}\bar{A}_{kl}$ respectively. (S)

(2.4) Show that the second moments of area of an area A in the x_1x_2 plane, given by

$$I_{11} = \int_A x_2^2 \mathrm{d}A, \quad I_{12} = I_{21} = -\int_A x_1 x_2 \mathrm{d}A \quad \text{and} \quad I_{22} = \int_A x_1^2 \mathrm{d}A$$

form the elements of a second-order tensor in two dimensions. (M)

(2.5) Prove that $e_{iks}e_{mks} = 2\delta_{im}$ and that $\delta_{kk} = 3$, and hence evaluate $e_{ijk}e_{ijk}$. (M)

(2.6) Show that $e_{ijk}e_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$ and use this result to prove (1.25). (M)

(2.7) The displacements within a body are given by

$$u_1 = Ax_1x_2x_3 + B(x_1^2x_2 + \tfrac{1}{3}x_3^3),\ u_2 = Ax_1x_2x_3 + B(x_2^2x_3 + \tfrac{1}{3}x_1^3),\ u_3 = Ax_1x_2x_3 + B(x_3^2x_1 + \tfrac{1}{3}x_2^3).$$

Find the strains induced. If the body is isotropic and linearly elastic, and no body forces or accelerations are present, what relationship between A and B is implied by (2.86)? (M)

(2.8) Find the stresses induced in the body in problem 2.7 and confirm that (2.41) implies the same relationship between A and B. (M)

(2.9) A flat surface of the body in problem 2.7 intersects the x_1x_2 plane along the line $3x_1 + 4x_2 = 60$ and intersects the x_2x_3 plane along the line $2x_2 + 3x_3 = 30$. If the origin is within the body, find the components of the unit outward normal to the body. (A hint: use vector products.) If $\lambda = 56$, $\mu = 28$ and $A = 8$ in some consistent set of units, find the surface tractions at the point (3, 3, 2) on this surface. (M)

(2.10) Show that $\omega_{jk,l} = e_{jl,k} - e_{kl,j}$ [cf. (2.35)]. (S)

(2.11) Show that the invariant $e_{ij}e_{jk}e_{kl}e_{li}$ can be expressed in terms of the invariants e_{ii}, $e_{ij}e_{ji}$ and $e_{ij}e_{jk}e_{ki}$. Compare the form of the invariant with that given by problem 1.11. (L)

(2.12) Show that the components of traction p_i acting on any surface through an elementary volume of a body, relative to the principal (stress) coordinates, generate **Lamé's stress ellipsoid** given by $p_1^2/\sigma_1^2 + p_2^2/\sigma_2^2 + p_3^2/\sigma_3^2 = 1$. (S)

(2.13) For an isotropic incompressible material, show that the stress deviations given by (2.43) can be expressed completely in terms of the state of strain. (S)

(2.14) From (2.27) it follows that the most general isotropic fourth-order tensor has three independent coefficients. Prove that (2.54) implies that the isotropic stiffness tensor c_{ijkl} only has two independent coefficients. (S)

(2.15) For an anisotropic material, are the invariants of the stress tensor necessarily related to the invariants of the strain tensor? (S)

(2.16) The elastic constants λ, μ, d_1, d_2 and d_3 for a certain rock are 35GPa, 30GPa, -55GPa, -80GPa and -230GPa respectively. Its density is 2500kg/m^3. At a depth of 30km below the surface, the isotropic pressure from the self-weight produces a uniform compressive strain e (so that $e_{ij} = -e\delta_{ij}$). Ignoring terms smaller than $O(e)$, show that the equivalent values of the Lamé constants relating small increments in stress and strain are $\lambda' = \lambda - e(6d_1 + 4d_2)$ and $\mu' = \mu - e(3d_2 + d_3)$. Evaluate λ' and μ' for the above rock at this depth, taking the acceleration due to gravity to be 9.81m/s^2.

(2.17) The curl of a tensor A with elements A_{kl} has the elements $B_{il} = e_{ijk} A_{kl,j}$. A has the elements $A_{11} = \chi_1, A_{22} = \chi_2, A_{33} = \chi_3$ where χ_i ($i = 1$ to 3) are the **Maxwell stress functions** and $A_{ij} = 0$ ($i \neq j$). Then the stresses σ_{ij} are given by the tensor curl (curl A). Find expressions for these stresses in terms of the functions χ_i and show that in the absence of body forces and accelerations, (2.41) is satisfied automatically. (M)

(2.18) What conditions are imposed on the above Maxwell stress functions by the Beltrami-Michell equations? (S)

(2.19) **Gauss' divergence theorem** applied to a vector field related to a body of volume V and surface area S can be written as

$$\int_V v_{i,i} \, dV = \int_S v_i n_i \, dS .$$

For a body in static equilibrium, use this equation to prove that the internal strain energy of a linearly elastic body in equilibrium is equal to the work done by its body forces and surface tractions, taking these to be proportional to the displacements through which they move. That is,

$$\frac{1}{2}\int_V \sigma_{ij} e_{ij} \, dV = \frac{1}{2}\int_V \rho f_i u_i \, dV + \frac{1}{2}\int_S p_i u_i \, dS . \quad \text{(S)}$$

(2.20) If the stresses produced in a state of plane strain are given by (1.119) or (2.106), what form must (2.108) take to allow for thermoelastic effects? (M)

3

Curvilinear tensors

In general, the analysis of a body is facilitated by ensuring that its boundary is given by a simple function of the ordinates used. Then for curved bodies, curvilinear coordinates are often preferred. For a general curvilinear system, two possible sets of base vectors can be employed. This means that there is no unique description of a tensor quantity of any order (except zero). However, certain representations of tensor quantities are preferable to others, and these will be used in expressing the relationships for elastic continua.

This chapter deals with the general form of tensor analysis in relation to curvilinear coordinates. These will be denoted by (y^1, y^2, y^3) to distinguish them from the rectangular Cartesian coordinates denoted by (x_1, x_2, x_3) in the last chapter. Here, the indices of tensor elements will appear as both superscripts and subscripts to indicate their transformation properties. In Appendix A4, it will be seen that the position of an index is also associated with the type of base vector used in direct tensor notation. The indices 1 to 3 will relate tensor elements to the corresponding ordinates y^1 to y^3. However, where a specific coordinate system is used, such as the polar coordinates (r, θ, z), it is sometimes more convenient to use r, θ and z as the corresponding indices.

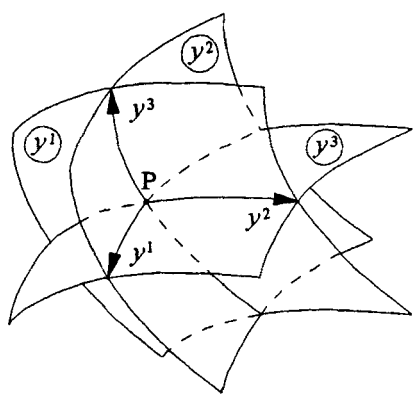


Figure 3.1 Curvilinear coordinate lines and surfaces.

3.1 BASE VECTORS

Consider a general curvilinear coordinate system in which each point is uniquely defined by three coordinates, (y^1, y^2, y^3). The set of points for which y^1 has a

common value forms a surface called the y^1 **coordinate surface**. Likewise, y^2 and y^3 coordinate surfaces are surfaces on which y^2 and y^3 are constant. The intersection of the y^2 and y^3 surfaces gives a line along which only y^1 varies. This is known as the y^1 **coordinate line** or curve. The y^2 and y^3 coordinate lines are similarly defined. The intersection of all three surfaces defines a unique point P and the three coordinate lines will pass through it (see Fig. 3.1). The arrows on the coordinate lines indicate the directions of increasing y^i (i = 1 to 3). It will be assumed throughout that the coordinate system is right-handed, that is, going from the y^1 direction to the y^2 direction is in the same sense as a clockwise rotation, as seen looking along the y^3 direction. These directions need not be mutually perpendicular.

Example 3.1 Oblique cylindrical polar coordinates

Suppose that (r,θ,z) are cylindrical polar coordinates, but that the polar axis is oblique to any plane on which z is constant, as shown in Figure 3.2a. (Such a plane is marked by an encircled z in the figure.) On any such z plane, the radial distance from the polar axis is given by r and a surface of constant r forms a cylinder about the polar axis. Such a cylinder is marked with an encircled r in the figure. The intersection of this cylinder with the z plane is circular. On any given z plane, the angle measured at the polar axis between a fixed direction and the direction of a point is denoted by θ. Then a θ surface is a plane through the polar axis, such as the one marked by an encircled θ. An intersection of z and θ surfaces gives an r coordinate line. Likewise, the intersection of r and z coordinate surfaces gives a θ coordinate line, and the intersection of θ and r coordinate surfaces gives a z coordinate line. As shown, the intersection of a set of three such coordinate lines defines a unique point P.

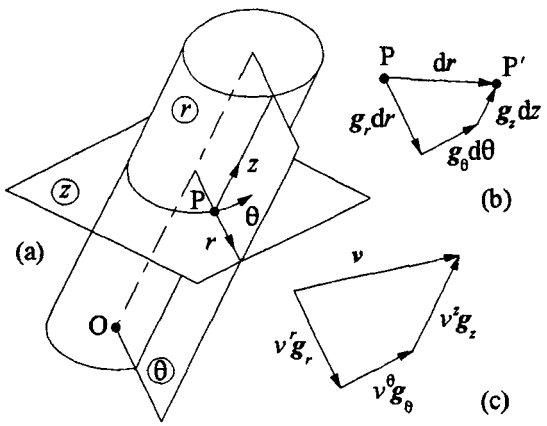


Figure 3.2 Oblique cylindrical polar coordinate lines and surfaces.

3.1.1 Covariant base vectors

In Fig. 3.2b, the coordinates of P are given by (r, θ, z) and those of an adjacent point P′ are $(r+dr, \theta+d\theta, z+dz)$. The relative position vector of P′ with respect to P can be expressed as

$$d\boldsymbol{r} = \boldsymbol{g}_r dr + \boldsymbol{g}_\theta d\theta + \boldsymbol{g}_z dz \qquad (3.1)$$

where the vectors $\boldsymbol{g}_r$, $\boldsymbol{g}_\theta$ and $\boldsymbol{g}_z$ are directed along the coordinate lines and are the local **covariant base vectors**. Notice that these base vectors are not an orthogonal set. Also, if r and z are taken as true increments in length, $\boldsymbol{g}_r$ and $\boldsymbol{g}_z$ are unit vectors. However, $d\theta$ is an angular increment and so $\boldsymbol{g}_\theta$ must be of length r. Likewise, any vector $\boldsymbol{v}$ can be

expressed as

$$\boldsymbol{v} = v^r \boldsymbol{g}_r + v^\theta \boldsymbol{g}_\theta + v^z \boldsymbol{g}_z \tag{3.2}$$

as shown in Figure 3.2c, where v^r, v^θ and v^z are known as the **contravariant components** of $\boldsymbol{v}$. More generally, for a coordinate system (y^1, y^2, y^3) the local covariant base vectors are denoted by $\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3$ and (3.1) becomes

$$\mathrm{d}\boldsymbol{r} = \boldsymbol{g}_1 \mathrm{d}y^1 + \boldsymbol{g}_2 \mathrm{d}y^2 + \boldsymbol{g}_3 \mathrm{d}y^3 = \boldsymbol{g}_i \mathrm{d}y^i \tag{3.3}$$

so that

$$\partial \boldsymbol{r} / \partial y^i \equiv \boldsymbol{r}_{,i} = \boldsymbol{g}_i \tag{3.4}$$

Likewise, the generalisation of (3.2) is

$$\boldsymbol{v} = v^1 \boldsymbol{g}_1 + v^2 \boldsymbol{g}_2 + v^3 \boldsymbol{g}_3 = v^i \boldsymbol{g}_i \tag{3.5}$$

using the summation convention.

3.1.2 Contravariant base vectors

It is convenient to define a second set of local base vectors $\boldsymbol{g}^1, \boldsymbol{g}^2, \boldsymbol{g}^3$ such that

$$\boldsymbol{g}^i \bullet \boldsymbol{g}_j = \delta^i_j \tag{3.6}$$

where

$$\delta^i_j = 0 \quad (i \neq j) \quad , \quad \delta^i_j = 1 \quad (i = j) \tag{3.7}$$

is the **Kronecker delta** in general tensor notation. Because $\boldsymbol{g}^1$ must be perpendicular to $\boldsymbol{g}_2$ and $\boldsymbol{g}_3$ it is normal to the local y^1 coordinate surface and because the dot product with $\boldsymbol{g}_1$ is positive, it has a component in the positive direction of y^1. Any vector $\boldsymbol{v}$ can also be expressed in terms of these local base vectors by

$$\boldsymbol{v} = v_1 \boldsymbol{g}^1 + v_2 \boldsymbol{g}^2 + v_3 \boldsymbol{g}^3 = v_i \boldsymbol{g}^i \tag{3.8}$$

where v_1, v_2 and v_3 are known as the **covariant resolved parts** of $\boldsymbol{v}$.

3.1.3 Contravariant transformation

Consider a second set of coordinates $(\bar{y}^1, \bar{y}^2, \bar{y}^3)$ with base vectors $\bar{\boldsymbol{g}}_1$, $\bar{\boldsymbol{g}}_2$, $\bar{\boldsymbol{g}}_3$. Then the relative position vector $\mathrm{d}\boldsymbol{r}$ of P′ adjacent to P is given by

$$\mathrm{d}\boldsymbol{r} = \bar{\boldsymbol{g}}_1 \mathrm{d}\bar{y}^1 + \bar{\boldsymbol{g}}_2 \mathrm{d}\bar{y}^2 + \bar{\boldsymbol{g}}_3 \mathrm{d}\bar{y}^3 = \bar{\boldsymbol{g}}_j \mathrm{d}\bar{y}^j \tag{3.9}$$

where

$$\mathrm{d}\bar{y}^j = \frac{\partial \bar{y}^j}{\partial y^1}\mathrm{d}y^1 + \frac{\partial \bar{y}^j}{\partial y^2}\mathrm{d}y^2 + \frac{\partial \bar{y}^j}{\partial y^3}\mathrm{d}y^3 = \frac{\partial \bar{y}^j}{\partial y^i}\mathrm{d}y^i \tag{3.10}$$

Comparing this with (3.3),

$$\bar{\boldsymbol{g}}_j \frac{\partial \bar{y}^j}{\partial y^i}\mathrm{d}y^i = \boldsymbol{g}_i \mathrm{d}y^i \tag{3.11}$$

for arbitrary $\mathrm{d}y^i$ so that

$$\bar{\boldsymbol{g}}_j \frac{\partial \bar{y}^j}{\partial y^i} = \boldsymbol{g}_i \tag{3.12}$$

In terms of the new coordinate system, (3.5) becomes

$$\boldsymbol{v} = \bar{v}^j \bar{\boldsymbol{g}}_j = v^i \boldsymbol{g}_i = v^i \frac{\partial \bar{y}^j}{\partial y^i} \bar{\boldsymbol{g}}_j \tag{3.13}$$

Comparing the coefficients of $\bar{\boldsymbol{g}}_j$ gives

$$\bar{v}^j = v^i \frac{\partial \bar{y}^j}{\partial y^i} \tag{3.14}$$

The same arguments lead to the inverse transformation

$$v^j = \bar{v}^i \frac{\partial y^j}{\partial \bar{y}^i} \tag{3.15}$$

Equation (3.14) is known as a **contravariant transformation** and any set of elements which transforms in this manner from one coordinate system to another forms a first-order contravariant tensor. From (3.10), the increments $\mathrm{d}y^i$ form a contravariant tensor. Likewise, if the velocity of a particle is expressed by the elements

$$v^i = \frac{\mathrm{d}y^i}{\mathrm{d}t}$$

then these are the components of a contravariant tensor, for in a different coordinate system

$$\bar{v}^j = \frac{\mathrm{d}\bar{y}^j}{\mathrm{d}t} = \frac{\partial \bar{y}^j}{\partial y^i}\frac{\mathrm{d}y^i}{\mathrm{d}t} = \frac{\partial \bar{y}^j}{\partial y^i} v^i$$

3.1.4 Scalar products

From (3.5), (3.6) and (3.8), the scalar product of two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ is given by

$$\boldsymbol{v}\cdot\boldsymbol{w} = v^i \boldsymbol{g}_i \cdot w_j \boldsymbol{g}^j = v^j w_j \tag{3.16}$$

and because the scalar product is independent of the coordinate system chosen,

$$\boldsymbol{v}\cdot\boldsymbol{w} = = \bar{v}^i \bar{w}_i \tag{3.17}$$

3.1.5 Covariant transformation

Comparing (3.16) and (3.17) and using (3.15),

$$\bar{v}^i \bar{w}_i = v^j w_j = \bar{v}^i \frac{\partial y^j}{\partial \bar{y}^i} w_j$$

Because this is true for an arbitrary choice of $\bar{v}^i$,

$$\bar{w}_i = \frac{\partial y^j}{\partial \bar{y}^i} w_j \tag{3.18}$$

This equation is known as a **covariant transformation** and the fact that a set of elements transforms in this way on changing the coordinate system is sufficient to ensure that they form a first-order covariant tensor. For example, if the elements v_j are derived from the gradient of some scalar potential ϕ, that is,

$$v_j = \frac{\partial \phi}{\partial y^j} \tag{3.19}$$

then in a different coordinate system,

$$\bar{v}_i = \frac{\partial \phi}{\partial \bar{y}^i} = \frac{\partial y^j}{\partial \bar{y}^i}\frac{\partial \phi}{\partial y^j} = \frac{\partial y^j}{\partial \bar{y}^i} v_j \tag{3.20}$$

thus showing that the elements of v_j form a covariant tensor. The inverse transformation can be written as

$$w_j = \frac{\partial \bar{y}^k}{\partial y^j} \bar{w}_k \tag{3.21}$$

so that
$$\bar{w}_i = \frac{\partial y^j}{\partial \bar{y}^i}\frac{\partial \bar{y}^k}{\partial y^j}\bar{w}_k = \frac{\partial \bar{y}^k}{\partial \bar{y}^i}\bar{w}_k = \delta_i^k \bar{w}_k$$

3.1.6 Covariance and contravariance of base vectors

From (3.5), the same vector $\boldsymbol{v}$ can be represented in two different coordinate systems by
$$\boldsymbol{v} = v^j \boldsymbol{g}_j = \bar{v}^i \bar{\boldsymbol{g}}_i \tag{3.22}$$

Using (3.15), this gives
$$\bar{v}^i \frac{\partial y^j}{\partial \bar{y}^i}\boldsymbol{g}_j = \bar{v}^i \bar{\boldsymbol{g}}_i$$

Because this is true for arbitrary $\bar{v}^i$,
$$\bar{\boldsymbol{g}}_i = \frac{\partial y^j}{\partial \bar{y}^i}\boldsymbol{g}_j \tag{3.23}$$

It can be seen that this is the same type of transformation as that given by (3.18), which is why $\boldsymbol{g}_j$ is called a covariant base vector. Likewise, from (3.8) and (3.21),
$$\boldsymbol{w} = \bar{w}_k \bar{\boldsymbol{g}}^k = w_j \boldsymbol{g}^j = \bar{w}_k \frac{\partial \bar{y}^k}{\partial y^j}\boldsymbol{g}^j$$

so that
$$\bar{\boldsymbol{g}}^k = \frac{\partial \bar{y}^k}{\partial y^j}\boldsymbol{g}^j \tag{3.24}$$

which is the same kind of transformation as that given by (3.14) and so $\boldsymbol{g}^j$ is referred to as a contravariant base vector.

Thus there are two ways of representing vectors with respect to curvilinear coordinates. With respect to rectangular Cartesian coordinates, the two kinds of base vector, and hence the two representations, are identical.

3.1.7 Magnitude of a vector

The square of the magnitude of a vector $\boldsymbol{v}$ is given by the scalar product of a vector with itself, so that from (3.16),
$$|\boldsymbol{v}|^2 = \boldsymbol{v}\cdot\boldsymbol{v} = v^i \boldsymbol{g}_i \cdot v_j \boldsymbol{g}^j = v^i v_i \tag{3.25}$$

3.2 METRIC TENSORS

Other ways of writing (3.25) would be
$$|\boldsymbol{v}|^2 = v^i \boldsymbol{g}_i \cdot v^j \boldsymbol{g}_j = v^i v^j g_{ij}$$
and
$$|\boldsymbol{v}|^2 = v_i \boldsymbol{g}^i \cdot v_j \boldsymbol{g}^j = v_i v_j g^{ij}$$

where
$$g_{ij} = \boldsymbol{g}_i \cdot \boldsymbol{g}_j \tag{3.26}$$
is known as the **covariant metric tensor** and
$$g^{ij} = \boldsymbol{g}^i \cdot \boldsymbol{g}^j \tag{3.27}$$
is known as the **contravariant metric tensor**.

The order of the dot product does not matter, so that g_{ij} and g^{ij} are symmetric, or
$$g_{ij} = g_{ji} \quad , \quad g^{ij} = g^{ji} \tag{3.28}$$

The square of the magnitude of a base vector is given by the dot product with itself, so that for example

$$|\boldsymbol{g}_1|^2 = \boldsymbol{g}_1 \cdot \boldsymbol{g}_1 = g_{11} \quad , \quad |\boldsymbol{g}^3|^2 = \boldsymbol{g}^3 \cdot \boldsymbol{g}^3 = g^{33} \tag{3.29}$$

3.2.1 Jacobian matrices

Equations (3.23) and (3.24) can be written in matrix form as

$$\begin{bmatrix} \bar{\boldsymbol{g}}_1 \\ \bar{\boldsymbol{g}}_2 \\ \bar{\boldsymbol{g}}_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial y^1}{\partial \bar{y}^1} & \frac{\partial y^2}{\partial \bar{y}^1} & \frac{\partial y^3}{\partial \bar{y}^1} \\ \frac{\partial y^1}{\partial \bar{y}^2} & \frac{\partial y^2}{\partial \bar{y}^2} & \frac{\partial y^3}{\partial \bar{y}^2} \\ \frac{\partial y^1}{\partial \bar{y}^3} & \frac{\partial y^2}{\partial \bar{y}^3} & \frac{\partial y^3}{\partial \bar{y}^3} \end{bmatrix} \begin{bmatrix} \boldsymbol{g}_1 \\ \boldsymbol{g}_2 \\ \boldsymbol{g}_3 \end{bmatrix} \equiv \mathbf{J}^{\mathrm{T}} \begin{bmatrix} \boldsymbol{g}_1 \\ \boldsymbol{g}_2 \\ \boldsymbol{g}_3 \end{bmatrix} \tag{3.30}$$

and

$$\begin{bmatrix} \bar{\boldsymbol{g}}^1 \\ \bar{\boldsymbol{g}}^2 \\ \bar{\boldsymbol{g}}^3 \end{bmatrix} = \begin{bmatrix} \frac{\partial \bar{y}^1}{\partial y^1} & \frac{\partial \bar{y}^1}{\partial y^2} & \frac{\partial \bar{y}^1}{\partial y^3} \\ \frac{\partial \bar{y}^2}{\partial y^1} & \frac{\partial \bar{y}^2}{\partial y^2} & \frac{\partial \bar{y}^2}{\partial y^3} \\ \frac{\partial \bar{y}^3}{\partial y^1} & \frac{\partial \bar{y}^3}{\partial y^2} & \frac{\partial \bar{y}^3}{\partial y^3} \end{bmatrix} \begin{bmatrix} \boldsymbol{g}^1 \\ \boldsymbol{g}^2 \\ \boldsymbol{g}^3 \end{bmatrix} \equiv \mathbf{K} \begin{bmatrix} \boldsymbol{g}^1 \\ \boldsymbol{g}^2 \\ \boldsymbol{g}^3 \end{bmatrix} \tag{3.31}$$

where $\mathbf{J}$ and $\mathbf{K}$ are **Jacobian matrices**.

Now

$$\frac{\partial y^i}{\partial \bar{y}^k} \frac{\partial \bar{y}^k}{\partial y^j} = \frac{\partial y^i}{\partial y^j} = \delta^i_j \tag{3.32}$$

so that

$$\mathbf{JK} = \mathbf{I} \tag{3.33}$$

where $\mathbf{I}$ is a unit matrix. Thus $\mathbf{K}$ can be obtained from the reciprocal of $\mathbf{J}$ and vice versa.

3.2.2 Relation of base vectors to Cartesian unit vectors

A case of particular interest is when (3.30) and (3.31) are used to express the base vectors $\boldsymbol{g}_i$ and $\boldsymbol{g}^i$ in terms of the unit vectors $\boldsymbol{i}_j$ of Cartesian coordinates x_j. Then the equations take the form

$$\begin{bmatrix} \boldsymbol{g}_1 \\ \boldsymbol{g}_2 \\ \boldsymbol{g}_3 \end{bmatrix} = \mathbf{J}_{\mathrm{c}}^{\mathrm{T}} \begin{bmatrix} \boldsymbol{i}_1 \\ \boldsymbol{i}_2 \\ \boldsymbol{i}_3 \end{bmatrix} \quad , \quad \begin{bmatrix} \boldsymbol{g}^1 \\ \boldsymbol{g}^2 \\ \boldsymbol{g}^3 \end{bmatrix} = \mathbf{K}_{\mathrm{c}} \begin{bmatrix} \boldsymbol{i}_1 \\ \boldsymbol{i}_2 \\ \boldsymbol{i}_3 \end{bmatrix} \tag{3.34}$$

The simplest way of finding $\mathbf{K}_{\mathrm{c}}$ is usually from the reciprocal of $\mathbf{J}_{\mathrm{c}}$.

Example 3.2 Base vectors

Suppose that the coordinate system of Example 3.1 is such that at the origin O, the z coordinate line lies in the plane containing the x_1 and x_3 axes of a Cartesian coordinate system, and is at an angle α to the x_1 axis. (In Fig. 3.3, the angle α is greater than 90°.) Any r coordinate line lies in a plane parallel to that containing the x_1 and x_2 axes and is at

an angle θ to an x_1 coordinate line. Then

$$x_1 = z\cos\alpha + r\cos\theta,\ x_2 = r\sin\theta,\ x_3 = z\sin\alpha \qquad (3.35)$$

giving

$$\begin{aligned} \boldsymbol{g}_r &= \boldsymbol{i}_1\cos\theta + \boldsymbol{i}_2\sin\theta \\ \boldsymbol{g}_\theta &= -\boldsymbol{i}_1 r\sin\theta + \boldsymbol{i}_2 r\cos\theta \\ \boldsymbol{g}_z &= \boldsymbol{i}_1\cos\alpha + \boldsymbol{i}_3\sin\alpha \end{aligned} \qquad (3.36)$$

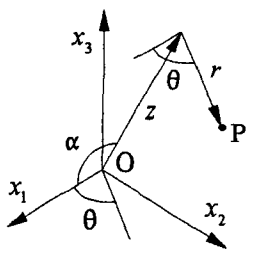


Figure 3.3 Base vectors.

and on inverting $\mathbf{J}_c$ to obtain $\mathbf{K}_c$,

$$\begin{aligned} \boldsymbol{g}^r &= \boldsymbol{i}_1\cos\theta + \boldsymbol{i}_2\sin\theta - \boldsymbol{i}_3\cos\theta\cot\alpha \\ \boldsymbol{g}^\theta &= \frac{1}{r}(-\boldsymbol{i}_1\sin\theta + \boldsymbol{i}_2\cos\theta + \boldsymbol{i}_3\sin\theta\cot\alpha) \\ \boldsymbol{g}^z &= \boldsymbol{i}_3\operatorname{cosec}\alpha \end{aligned} \qquad (3.37)$$

and

$$J_c \equiv \det(\mathbf{J}_c) = r\sin\alpha \qquad (3.38)$$

3.2.3 The metric equation

From (3.3), the incremental length ds between two adjacent points P and P′ is given by

$$ds^2 = d\boldsymbol{r}\bullet d\boldsymbol{r} = \boldsymbol{g}_i dy^i \bullet \boldsymbol{g}_j dy^j = g_{ij} dy^i dy^j \qquad (3.39)$$

This equation can be taken as a definition of the coordinate system in terms of the metric tensor g_{ij}. It is known as the **metric equation** or the **metric** of the coordinate system. Taking ds as a real length, ds^2 is always positive for any dy^i and dy^j. This means that the matrix of elements $[\,g_{ij}\,]$ is positive definite.

3.2.4 Relationship between base vectors and the metric tensor

Both the covariant and the contravariant base vectors can be used to describe any vector in the coordinate space, and so they can be used to describe each other. Suppose that $\boldsymbol{g}^i$ is described by a set of base vectors $\boldsymbol{g}_k$ by $\boldsymbol{g}^i = a^{ik}\boldsymbol{g}_k$ where a^{ik} is the appropriate set of coefficients. Then from (3.27) and (3.6),

$$g^{ij} = \boldsymbol{g}^i \bullet \boldsymbol{g}^j = a^{ik}\boldsymbol{g}_k \bullet \boldsymbol{g}^j = a^{ik}\delta^j_k = a^{ij}$$

so that

$$\boldsymbol{g}^i = g^{ij}\boldsymbol{g}_j \qquad (3.40)$$

and likewise

$$\boldsymbol{g}_i = g_{ij}\boldsymbol{g}^j \qquad (3.41)$$

and hence

$$\delta^j_i = \boldsymbol{g}_i \bullet \boldsymbol{g}^j = g_{ik}\boldsymbol{g}^k \bullet g^{jl}\boldsymbol{g}_l = g_{ik}g^{jl}\boldsymbol{g}_l = g_{ik}g^{jl}\delta^k_l = g_{ik}g^{jk} \qquad (3.42)$$

which can be written in matrix form as

$$[g_{ik}][g^{kj}] = \mathbf{I} \qquad (3.43)$$

where $\mathbf{I}$ is a unit matrix. This means that the matrix of coefficients $[\,g^{kj}\,]$ can be obtained by taking the inverse of $[\,g_{ik}\,]$.

Example 3.3 Metric tensors

Suppose that the oblique cylindrical coordinate system described in the previous examples

is to be used. Then the square of the distance between the points P and P′ shown in Fig. 3.4 is

$$ds^2 = (dr + dz\cos\alpha\cos\theta)^2 + (r\,d\theta - dz\cos\alpha\sin\theta)^2 + (dz\sin\alpha)^2 \tag{3.44}$$

Comparing this with (3.39) and noting that $g_{ij} = g_{ji}$,

$$g_{rr} = 1,\ g_{r\theta} = 0,\ g_{rz} = \cos\alpha\cos\theta,$$
$$g_{\theta\theta} = r^2,\ g_{\theta z} = -r\cos\alpha\sin\theta,\ g_{zz} = 1. \tag{3.45}$$

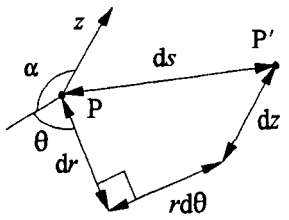


Figure 3.4 Incremental lengths.

Taking the inverse of the matrix $[g_{ij}]$ thus formed gives

$$g^{rr} = \operatorname{cosec}^2\alpha - \cot^2\alpha\sin^2\theta,\quad g^{r\theta} = -\frac{1}{r}\cot^2\alpha\sin\theta\cos\theta,$$
$$g^{rz} = -\operatorname{cosec}\alpha\cot\alpha\cos\theta,\quad g^{\theta\theta} = \frac{1}{r^2}(\operatorname{cosec}^2\alpha - \cot^2\alpha\cos^2\theta), \tag{3.46}$$
$$g^{\theta z} = \frac{1}{r}\operatorname{cosec}\alpha\cot\alpha\sin\theta,\quad g^{zz} = \operatorname{cosec}^2\alpha.$$

These results may also be derived from (3.26), (3.27), (3.36) and (3.37).

3.2.5 Relationship between the metric tensor and Jacobian matrices

Suppose that the coordinate space is also described by a Cartesian system. Then from (3.26) and (3.34), the matrix of the metric tensor may be written as

$$[g_{ij}] = \begin{bmatrix} \boldsymbol{g}_1 \\ \boldsymbol{g}_2 \\ \boldsymbol{g}_3 \end{bmatrix} \bullet [\boldsymbol{g}_1\ \boldsymbol{g}_2\ \boldsymbol{g}_3] = \mathbf{J}_c^{\mathrm{T}} \begin{bmatrix} \boldsymbol{i}_1 \\ \boldsymbol{i}_2 \\ \boldsymbol{i}_3 \end{bmatrix} \bullet [\boldsymbol{i}_1\ \boldsymbol{i}_2\ \boldsymbol{i}_3]\mathbf{J}_c = \mathbf{J}_c^{\mathrm{T}}\mathbf{J}_c \tag{3.47}$$

where the base vectors are treated as matrix elements, forming dot products in the matrix product. Similarly, from (3.27),

$$[g^{ij}] = \mathbf{K}_c\mathbf{K}_c^{\mathrm{T}} \tag{3.48}$$

A result of particular value is

$$g \equiv |g_{ij}| = |\mathbf{J}_c|^2 \equiv J_c^2 \tag{3.49}$$

where g is defined as the determinant of $[g_{ij}]$. In tensor analysis, $\sqrt{g}$ is commonly used when J_c might be expected. Similarly, from (3.48),

$$1/g = |g^{ij}| = |\mathbf{K}_c|^2 \tag{3.50}$$

Because $[g_{ij}]$ is the reciprocal of $[g^{ij}]$,

$$g^{ij} = \mathfrak{G}^{ij}/g\ ,\quad g_{ij} = g\mathfrak{G}_{ij} \tag{3.51}$$

where $\mathfrak{G}^{ij}$ and $\mathfrak{G}_{ij}$ are the cofactors of g_{ij} and g^{ij} respectively. These cofactors do not normally form tensors.

3.2.6 Relationship between types of first-order tensors

From (3.5), (3.8) and (3.40),

$$v^j \boldsymbol{g}_j = \boldsymbol{v} = v_i \boldsymbol{g}^i = v_i g^{ij} \boldsymbol{g}_j$$

so that on comparing the two sides of the equation,

$$v^j = g^{ij} v_i \qquad (3.52)$$

and similarly a covariant tensor can be derived from a contravariant tensor by the operation

$$v_j = g_{ij} v^i \qquad (3.53)$$

3.3 HIGHER ORDER TENSORS

Just as the transformations (3.14) and (3.18) define first-order tensors, additional operations of the same kind define the transformations for tensors of a higher order. A doubly-covariant tensor A_{ij} transforms according to the rule

$$\bar{A}_{kl} = \frac{\partial y^i}{\partial \bar{y}^k} \frac{\partial y^j}{\partial \bar{y}^l} A_{ij} \qquad (3.54)$$

The rule for a doubly-covariant tensor is

$$\bar{B}^{kl} = \frac{\partial \bar{y}^k}{\partial y^i} \frac{\partial \bar{y}^l}{\partial y^j} B^{ij} \qquad (3.55)$$

and for a mixed second-order tensor it is

$$\bar{C}^k_l = \frac{\partial \bar{y}^k}{\partial y^i} \frac{\partial y^j}{\partial \bar{y}^l} C^i_j \qquad (3.56)$$

From (3.26) and (3.23),

$$\bar{g}_{kl} = \bar{\boldsymbol{g}}_k \cdot \bar{\boldsymbol{g}}_l = \frac{\partial y^i}{\partial \bar{y}^k} \frac{\partial y^j}{\partial \bar{y}^l} \boldsymbol{g}_i \cdot \boldsymbol{g}_j = \frac{\partial y^i}{\partial \bar{y}^k} \frac{\partial y^j}{\partial \bar{y}^l} g_{ij}$$

so that g_{ij} is a doubly-covariant tensor. Likewise, g^{ij} is a doubly-contravariant tensor and from (3.6) it can be shown that δ^i_j is a mixed second-order tensor. It is also invariant, because from (3.56) it follows that it takes the same form in all coordinate systems.

Second-order tensors can also be generated from products of first-order tensors, for example

$$A_{ij} = v_i w_j \quad , \quad B^{ij} = v^i w^j \quad , \quad C^i_j = v^i w_j$$

can readily be shown to be doubly-covariant, doubly-contravariant and mixed second-order tensors respectively, using the transformations given by (3.14) and (3.18). The general rule of tensor transformation is an obvious extension of that for second-order tensors. It takes the form

$$\bar{D}^{pq..r}_{st..u} = \frac{\partial \bar{y}^p}{\partial y^i} \frac{\partial \bar{y}^q}{\partial y^j} .. \frac{\partial \bar{y}^r}{\partial y^k} \frac{\partial y^l}{\partial \bar{y}^s} \frac{\partial y^m}{\partial \bar{y}^t} .. \frac{\partial y^n}{\partial \bar{y}^u} D^{ij..k}_{lm..n} \qquad (3.57)$$

3.3.1 Tensor equations

It follows from (3.57) that if all the elements of any tensor $D^{ij..k}_{lm..n}$ are zero, then all the elements of its representation in any other coordinate system are also zero. Then, from the laws of transformation, if an equation relating tensors of the same kind can be established in one coordinate system, its form will be the same in all coordinate systems. For example, if $A^i_{jk} + B^i_{jk} = 0$, it being understood that such equations hold for all i, j and k, then

$$\frac{\partial \bar{y}^p}{\partial y^i}\frac{\partial y^j}{\partial \bar{y}^q}\frac{\partial y^k}{\partial \bar{y}^r}(A^i_{jk} + B^i_{jk}) = \bar{A}^p_{qr} + \bar{B}^p_{qr} = 0$$

This invariance of the form of equations in different coordinate systems is one of the most important properties of tensors.

3.3.2 Contraction

The third-order tensor A^{ij}_k transforms to give $$\bar{A}^{pq}_r = \frac{\partial \bar{y}^p}{\partial y^i}\frac{\partial \bar{y}^q}{\partial y^j}\frac{\partial y^k}{\partial \bar{y}^r}A^{ij}_k$$

Then the implied summation $\bar{A}^{sq}_s$ indicated by the repeated index is given by

$$\bar{A}^{sq}_s = \frac{\partial \bar{y}^s}{\partial y^i}\frac{\partial \bar{y}^q}{\partial y^j}\frac{\partial y^k}{\partial \bar{y}^s}A^{ij}_k = \frac{\partial \bar{y}^q}{\partial y^j}\delta^k_i A^{ij}_k = \frac{\partial \bar{y}^q}{\partial y^j}A^{kj}_k$$

Thus A^{kj}_k transforms to $\bar{A}^{sq}_s$ according to the rules for a first-order contravariant tensor. This summation over the same values of a subscript and a superscript is known as **contraction** because it reduces the order of a tensor by two. Contraction of a second-order mixed tensor produces a zero-order tensor, i.e. a scalar, which is invariant with respect to the coordinate system used. A simple example is given by the kinetic energy, T, of a particle of mass m which is $$T = \tfrac{1}{2}m\boldsymbol{v}\cdot\boldsymbol{v} = \tfrac{1}{2}mv^i\boldsymbol{g}_i\cdot v_j\boldsymbol{g}^j = \tfrac{1}{2}mv^iv_i$$

from (3.25), where $\boldsymbol{v}$ is the velocity vector of the particle. The term v^iv_i can be regarded as the contraction of a mixed second-order tensor v^iv_j and so is invariant. That is, it has the same value regardless of the coordinate system used. In particular, T can be evaluated in a Cartesian coordinate system and will be the same in all other coordinate systems. Note that in the notation used in this chapter, contraction only works for summation over an identical subscript and superscript pair. If both are either subscripts or superscripts, the result is not a tensor of two orders less.

3.3.3 The quotient rule

The tensor nature of a set of functions such as $A(i, j)$ can be determined if it can be established that, for example, $$A(i,j)B_j = C^i$$

where B_j and C^i are first-order tensors and summation over j is understood. In another coordinate system, this equation becomes $$\bar{A}(k,l)\bar{B}_l = \bar{C}^k$$

Using the transformations of the known tensors, this can be written as

$$\bar{A}(k,l)\frac{\partial y^j}{\partial \bar{y}^l}B_j = \frac{\partial \bar{y}^k}{\partial y^i}C^i = \frac{\partial \bar{y}^k}{\partial y^i}A(i,j)B_j$$

Thus $$\left[\bar{A}(k,l)\frac{\partial y^j}{\partial \bar{y}^l} - \frac{\partial \bar{y}^k}{\partial y^i}A(i,j)\right]B_j = 0$$

Because this is true for arbitrary B_j, the content of the square brackets must be zero. Then on multiplying the contents by $\partial \bar{y}^m/\partial y^j$ and carrying out the appropriate summation,

$$0 = \bar{A}(k,l)\frac{\partial y^j}{\partial \bar{y}^l}\frac{\partial \bar{y}^m}{\partial y^j} - \frac{\partial \bar{y}^k}{\partial y^i}\frac{\partial \bar{y}^m}{\partial y^j}A(i,j) = \bar{A}(k,l)\delta_l^m - \frac{\partial \bar{y}^k}{\partial y^i}\frac{\partial \bar{y}^m}{\partial y^j}A(i,j)$$

or

$$\bar{A}(k,m) = \frac{\partial \bar{y}^k}{\partial y^i}\frac{\partial \bar{y}^m}{\partial y^j}A(i,j)$$

Thus $A(i,j)$ proves to be a second-order contravariant tensor A^{ij}. The same method can be used to verify the tensor nature of other sets of functions and is known as the **quotient rule**. The particular case chosen here will be used later to establish that a certain definition of stress is expressed by a second-order contravariant tensor.

3.3.4 The relationship between types of higher order tensor

Consider the two representations of a relationship between two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ given by

$$w^i = c^{ij}v_j = c_k^i v^k$$

From (3.53), $c^{ij}v_j = c^{ij}g_{kj}v^k$ so that on comparing these two equations,

$$c_k^i = c^{ij}g_{kj} \tag{3.58}$$

Thus g_{kj} can be used to derive a mixed second-order tensor from a doubly contravariant one. More generally, g_{kj} can be used to replace a tensor superscript with a subscript and g^{kj} used to perform the reverse operation. Thus

$$c_l^k = c_{jl}g^{kj} \quad , \quad f_{ilj} = f_{il}^k g_{kj} \tag{3.59}$$

and so on. Note that a mixed tensor ς_k^i could also be formed from

$$\varsigma_k^i = c^{ji}g_{kj}$$

However, this will not be the same as the tensor c_k^i formed in (3.58) unless c^{ij} is symmetric, i.e. c^{ij} and c^{ji} are the same. In some texts, c_k^i is written as $c_{.k}^{i}$ and ς_k^i is written as $c_{k}^{.i}$

3.4 VECTOR PRODUCTS

The cross product of two vectors gives a vector which is mutually perpendicular to these two vectors. Then from (3.6) it follows that

$$\boldsymbol{g}_2 \times \boldsymbol{g}_3 = C\boldsymbol{g}^1 \tag{3.60}$$

where C is a scalar. From (3.23),

$$\boldsymbol{g}_i = \frac{\partial x_j}{\partial y^i}\boldsymbol{i}_j \tag{3.61}$$

and so from (2.26), (1.10), (3.30) and (3.34),

$$\begin{aligned}\boldsymbol{g}_1 \cdot (\boldsymbol{g}_2 \times \boldsymbol{g}_3) &= \frac{\partial x_j}{\partial y^1}\boldsymbol{i}_j \cdot \left(\frac{\partial x_k}{\partial y^2}\boldsymbol{i}_k \times \frac{\partial x_l}{\partial y^3}\boldsymbol{i}_l\right) = \frac{\partial x_j}{\partial y^1}\boldsymbol{i}_j \cdot \left(\frac{\partial x_k}{\partial y^2}\frac{\partial x_l}{\partial y^3}\boldsymbol{i}_m e_{mkl}\right) \\ &= e_{jkl}\frac{\partial x_j}{\partial y^1}\frac{\partial x_k}{\partial y^2}\frac{\partial x_l}{\partial y^3} = |\mathbf{J}_c^{\mathrm{T}}| = \mathrm{J}_c = \sqrt{g}\end{aligned} \tag{3.62}$$

Then from (3.60) and (3.6),

$$\sqrt{g} = \boldsymbol{g}_1 \cdot (\boldsymbol{g}_2 \times \boldsymbol{g}_3) = C\boldsymbol{g}_1 \cdot \boldsymbol{g}^1 = C$$

and similar relationships can be established for the other products of base vectors giving

$$\boldsymbol{g}_1 \times \boldsymbol{g}_2 = \sqrt{g}\,\boldsymbol{g}^3 \quad , \quad \boldsymbol{g}_2 \times \boldsymbol{g}_3 = \sqrt{g}\,\boldsymbol{g}^1 \quad , \quad \boldsymbol{g}_3 \times \boldsymbol{g}_1 = \sqrt{g}\,\boldsymbol{g}^2 \ . \tag{3.63}$$

Likewise, it can be shown that

$$\boldsymbol{g}^1 \times \boldsymbol{g}^2 = \boldsymbol{g}_3/\sqrt{g} \quad , \quad \boldsymbol{g}^2 \times \boldsymbol{g}^3 = \boldsymbol{g}_1/\sqrt{g} \quad , \quad \boldsymbol{g}^3 \times \boldsymbol{g}^1 = \boldsymbol{g}_2/\sqrt{g} \ . \tag{3.64}$$

3.4.1 Elementary coordinate surfaces

In Fig. 3.5, the vectors $dy^1\boldsymbol{g}_1$ and $dy^2\boldsymbol{g}_2$ describe the sides of an elementary area $dS(3)$ of the y^3 coordinate surface, which has a unit normal $\boldsymbol{n}(3)$ in the sense shown. Then

$$\begin{aligned} dy^1\boldsymbol{g}_1 \times dy^2\boldsymbol{g}_2 &= |dy^1\boldsymbol{g}_1||dy^2\boldsymbol{g}_2|\sin\phi\,\boldsymbol{n}(3) \\ &= dS(3)\boldsymbol{n}(3) = d\boldsymbol{S}(3) \quad \text{(say)} \end{aligned} \tag{3.65}$$

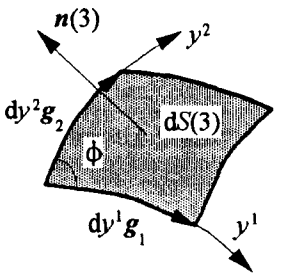


Figure 3.5 Coordinate surface element.

Then from (3.63),

$$d\boldsymbol{S}(3) = \sqrt{g}\,dy^1 dy^2 \boldsymbol{g}^3 = dS_3\,\boldsymbol{g}^3 \quad \text{(say)} \tag{3.66}$$

From (3.29), the true elementary area is given by

$$dS(3) = dy^1 dy^2 \sqrt{(g g^{33})} \tag{3.67}$$

and similar relationships hold for elementary areas of the other coordinate surfaces.

3.4.2 Arbitrary elementary surfaces

In Fig. 3.6, the area of the triangle RST will be denoted by $\frac{1}{2}d\boldsymbol{S}$ and the normal to this surface, in the direction shown, by $\boldsymbol{n}$. Then

$$\underline{\mathrm{RS}} \times \underline{\mathrm{RT}} = \boldsymbol{n}\,dS = n_i \boldsymbol{g}^i dS$$

where $\underline{\mathrm{RS}}$ and $\underline{\mathrm{RT}}$ are the position vectors of S and T relative to R. Now

$$\begin{aligned} \underline{\mathrm{RS}} \times \underline{\mathrm{RT}} &= (\underline{\mathrm{PS}} - \underline{\mathrm{PR}}) \times (\underline{\mathrm{PT}} - \underline{\mathrm{PR}}) \\ &= dy^2\boldsymbol{g}_2 \times dy^3\boldsymbol{g}_3 - dy^1\boldsymbol{g}_1 \times dy^3\boldsymbol{g}_3 - dy^2\boldsymbol{g}_2 \times dy^1\boldsymbol{g}_1 \\ &= d\boldsymbol{S}(1) + d\boldsymbol{S}(2) + d\boldsymbol{S}(3) \\ &= dS_1\boldsymbol{g}^1 + dS_2\boldsymbol{g}^2 + dS_3\boldsymbol{g}^3 \end{aligned}$$

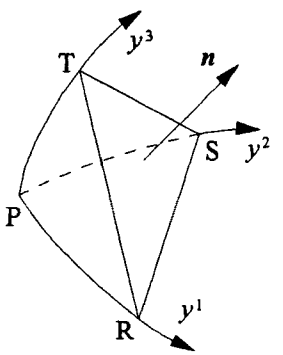


Figure 3.6 A triangular surface.

from the previous section. It follows that

$$n_i dS = dS_i \tag{3.68}$$

3.4.3 General vector products

From (3.63), a vector product can be written as

$$\begin{aligned} \boldsymbol{v} \times \boldsymbol{w} &= v^j \boldsymbol{g}_j \times w^k \boldsymbol{g}_k \\ &= [(v^2 w^3 - v^3 w^2)\boldsymbol{g}^1 + (v^3 w^1 - v^1 w^3)\boldsymbol{g}^2 + (v^1 w^2 - v^2 w^1)\boldsymbol{g}^3]\sqrt{g} \\ &= [e_{ijk} v^j w^k \boldsymbol{g}^i]\sqrt{g} \end{aligned} \tag{3.69}$$

where the permutation symbol e_{ijk} is defined by (2.19).

3.4.4 The permutation tensor

Let the right-hand side of (3.69) be denoted by the vector $a_i \boldsymbol{g}^i$ where the terms

$$a_i = (e_{ijk}\sqrt{g})v^j w^k$$

form a first-order covariant tensor. It follows from the quotient rule that

$$\varepsilon_{ijk} = e_{ijk}\sqrt{g} \tag{3.70}$$

is a third-order covariant tensor. This is known as a **permutation tensor**. Similarly, from (3.64), if the vectors in the cross product are expressed by the covariant terms v_j and w_k, the vector product can be written in a second form, giving the two expressions

$$\boldsymbol{v}\times\boldsymbol{w} = \varepsilon_{ijk}v^j w^k \boldsymbol{g}^i = \varepsilon^{ijk}v_j w_k \boldsymbol{g}_i \tag{3.71}$$

where

$$\varepsilon^{ijk} = e_{ijk}/\sqrt{g} \tag{3.72}$$

is the contravariant form of the permutation tensor. Equations (3.63) and (3.64) can be written as

$$\boldsymbol{g}_i \times \boldsymbol{g}_j = \varepsilon_{ijk}\boldsymbol{g}^k \tag{3.73}$$

and

$$\boldsymbol{g}^i \times \boldsymbol{g}^j = \varepsilon^{ijk}\boldsymbol{g}_k \tag{3.74}$$

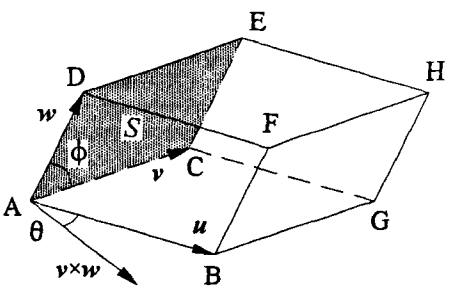


Figure 3.7 Volume of a parallelepiped.

3.4.5 Scalar triple products

In Fig 3.7,

$$\underline{AC}\times\underline{AD} = \boldsymbol{v}\times\boldsymbol{w} = |\boldsymbol{v}||\boldsymbol{w}|\sin\phi\,\boldsymbol{n} = S\boldsymbol{n}$$

where S is the area of the parallelogram ACDE and $\boldsymbol{n}$ is a unit normal to it. From (3.6) and (3.71), the volume V of the parallelepiped ABCDEFGH is given by

$$V = S|\underline{AB}|\cos\theta = \boldsymbol{u}\cdot(\boldsymbol{v}\times\boldsymbol{w}) = u^l \boldsymbol{g}_l \cdot \varepsilon_{ijk}v^j w^k \boldsymbol{g}^i = \varepsilon_{ijk}u^i v^j w^k$$

3.4.6 Elementary volumes

The elementary parallelepiped PRSTUVWP′ shown in Fig. 3.8 is formed from the volume dV between the coordinate surfaces through P and those through an adjacent point P′ with the incremental coordinates (dy^1, dy^2, dy^3) relative to P. From (3.62), this volume is given by

$$\begin{aligned} dV &= \underline{PR}\cdot(\underline{PS}\times\underline{PT}) \\ &= dy^1\boldsymbol{g}_1\cdot(dy^2\boldsymbol{g}_2\times dy^3\boldsymbol{g}_3) \\ &= \sqrt{g}\,dy^1 dy^2 dy^3 \end{aligned} \tag{3.75}$$

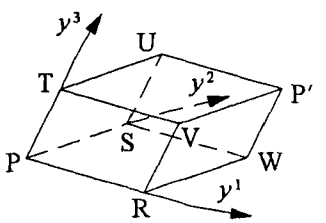


Figure 3.8 An elementary volume.

3.4.7 Elementary mass

If the local density of a body is ρ, then the mass dm of the elementary volume dV is

$$dm = \rho\,dV = \rho\sqrt{g}\,dy^1 dy^2 dy^3$$

Sometimes it is expressed as

$$dm = \rho^* dy^1 dy^2 dy^3$$

where ρ^* is known as the **tensor density**. This is not a true scalar but a **relative scalar**, see Fung (1965), p.475 for example.

3.5 ORTHOGONAL COORDINATE SYSTEMS

If the base vectors $\boldsymbol{g}_i$ are mutually perpendicular, then

$$\boldsymbol{g}_i \cdot \boldsymbol{g}_j = g_{ij} = 0 \qquad (i \neq j) \tag{3.76}$$

The only non-zero elements of the matrix $[\, g_{ij} \,]$ are then $g_{\alpha\alpha}$ ($\alpha = 1$ to 3). Because the matrix is positive-definite (see section 3.2.3), these elements must be positive and so

$$\boldsymbol{g}_\alpha \cdot \boldsymbol{g}_\alpha = g_{\alpha\alpha} = h_\alpha^2 \qquad \text{(say)} \tag{3.77}$$

where h_α is real. The metric equation (3.39) then becomes

$$ds^2 = (h_1 dy^1)^2 + (h_2 dy^2)^2 + (h_3 dy^3)^2 \tag{3.78}$$

and (3.53) becomes

$$v_\alpha = g_{\alpha\alpha} v^\alpha$$

It follows from (3.41) that the base vectors $\boldsymbol{g}^\alpha$ are now parallel to the base vectors $\boldsymbol{g}_\alpha$ and are related by

$$\boldsymbol{g}^\alpha = \boldsymbol{g}_\alpha / h_\alpha^2 \tag{3.79}$$

and so

$$g^{ij} = \boldsymbol{g}^i \cdot \boldsymbol{g}^j = 0 \qquad (i \neq j) \tag{3.80}$$

and

$$g^{\alpha\alpha} = \boldsymbol{g}_\alpha \cdot \boldsymbol{g}_\alpha / h_\alpha^4 = 1/h_\alpha^2 \tag{3.81}$$

The most commonly used coordinate systems are orthogonal and the above specifications make it particularly easy to relate tensor elements to the physical components of entities, as will be seen later.

For cylindrical polar coordinates, the angle α in Example 3.1 is 90° and

$$y^1 = r \;,\; y^2 = \theta \;,\; y^3 = z \tag{3.82}$$

and the metric equation takes the form

$$ds^2 = dr^2 + (r d\theta)^2 + dz^2 \tag{3.83}$$

so that from (3.78),

$$h_1^2 = g_{11} = 1 \tag{3.84}$$

$$h_2^2 = g_{22} = r^2 \tag{3.85}$$

$$h_3^2 = g_{33} = 1 \tag{3.86}$$

and from (3.49), the value of $\sqrt{g}$ is r.

The spherical polar coordinates shown in Fig. 3.9 are such that the point P lies in the shaded plane through the polar axis OQ. The orientation of this plane is given by some angle ϕ about this axis. The point P lies at a radius r from O, and OP is at an angle θ to OQ. The coordinates of P are given by these parameters, so that

$$y^1 = r \quad , \quad y^2 = \theta \quad , \quad y^3 = \phi$$

and the metric equation is

$$ds^2 = dr^2 + r^2 d\theta^2 + (r\sin\theta)^2 d\phi^2 \qquad (3.87)$$

giving

$$h_1^2 = g_{11} = 1 \qquad (3.88)$$

$$h_2^2 = g_{22} = r^2 \qquad (3.89)$$

$$h_3^2 = g_{33} = (r\sin\theta)^2 \qquad (3.90)$$

and $\sqrt{g}$ is equal to $r^2 \sin\theta$.

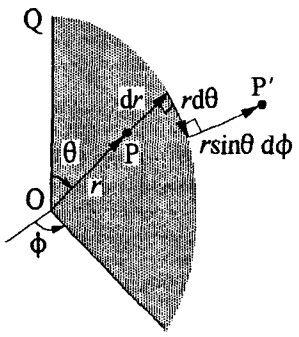


Fig. 3.9 Spherical polar coordinates.

3.5.1 Physical components

If a coordinate system is orthogonal, then in any small region of space a set of Cartesian axes can be found which coincide with the local coordinate lines. The local components of any tensor, expressed relative to this Cartesian coordinate system, are said to be **physical components**.

With respect to non-orthogonal coordinate systems, at least four different authors have produced distinct definitions of 'physical components' and so no attempt to give one will be made here. However, a tensor can always be transformed to refer to an orthogonal coordinate system, thus avoiding any ambiguity in the definition of physical components.

Consider the relative position vector $\underline{PP'}$ of two adjacent points such as those shown in Fig. 3.9. In terms of the local coordinate system,

$$\underline{PP'} = dy^i \boldsymbol{g}_i = \sum_{\alpha=1}^{3} \left(h_\alpha dy^\alpha\right) \hat{\boldsymbol{g}}_\alpha$$

where

$$\hat{\boldsymbol{g}}_\alpha = \boldsymbol{g}_\alpha / h_\alpha$$

are the unit vectors of the local Cartesian system. The coefficients $h_\alpha dy^\alpha$ are then the physical components of $\underline{PP'}$. More generally, the physical components† v_α^{P} of a vector

$$\boldsymbol{v} = v^i \boldsymbol{g}_i$$

are given by

$$v_\alpha^{\mathrm{P}} = h_\alpha v^\alpha \qquad (\alpha = 1 \text{ to } 3) \qquad (3.91)$$

The physical components can also be found from the covariant representation of $\boldsymbol{v}$. From (3.52), (3.80) and (3.81),

$$v_\alpha^{\mathrm{P}} = h_\alpha v^\alpha = h_\alpha g^{\alpha\alpha} v_\alpha = v_\alpha / h_\alpha \qquad (3.92)$$

The physical components of higher order tensors can be found from relationships expressed in terms of lower order tensors. For example, the tensor equation

$$w^i = c_j^i v^j \qquad (3.93)$$

has the physical equivalent

†Various forms of notation for physical components are given by Truesdell (*Zeitschrift für angewandte Mathematik und Mechanik* **33** 10/11 1953), Eringen (1962), Leigh (1968) and Malvern (1969).

$$w_\alpha^{\mathrm{P}} = \sum_{\beta=1}^{3} c_{\alpha\beta}^{\mathrm{P}} v_\beta^{\mathrm{P}} \tag{3.94}$$

but the physical components w_α^{P} and v_β^{P} can be found from (3.91) so that

$$h_\alpha w^\alpha = \sum_{\beta=1}^{3} c_{\alpha\beta}^{\mathrm{P}} h_\beta v^\beta \tag{3.95}$$

and on comparing (3.93) with (3.95) term by term,

$$c_{\alpha\beta}^{\mathrm{P}} = c_\beta^\alpha h_\alpha / h_\beta \tag{3.96}$$

From (3.58), (3.76) and (3.77),

$$c_\beta^\alpha = c^{\alpha j} g_{j\beta} = c^{\alpha\beta} g_{\beta\beta} = c^{\alpha\beta} h_\beta^2 \tag{3.97}$$

so that (3.96) can be written as

$$c_{\alpha\beta}^{\mathrm{P}} = c^{\alpha\beta} h_\alpha h_\beta \tag{3.98}$$

and similarly, from (3.59) and (3.96),

$$c_{\alpha\beta}^{\mathrm{P}} = c_{\alpha\beta} g^{\alpha\alpha} h_\alpha / h_\beta = c_{\alpha\beta} / h_\alpha h_\beta \tag{3.99}$$

From the above, the general rule for finding physical components from tensor components can be extrapolated, which is

$$c_{\alpha\beta\gamma\delta\ldots}^{\mathrm{P}} = c_{\gamma\delta\ldots}^{\alpha\beta\ldots} h_\alpha h_\beta \ldots / h_\gamma h_\delta \ldots \tag{3.100}$$

Note that mixed elements with identical upper and lower indices, such as c_α^α, are equal to their physical components. In engineering texts, stress and strain relative to curvilinear coordinates are usually expressed in terms of physical components. Further information on physical components and the differentials of vectors will be found in Appendix A1.

3.6 COVARIANT DIFFERENTIATION

If a curvilinear coordinate system is used, the variation of a vector field is not given directly by the variation of the tensor elements describing it. Consider for example the constant vector field $\boldsymbol{v}$ acting in two dimensions as shown in Fig. 3.10. In terms of polar coordinates (r,θ) this is described by the contravariant elements v^r and v^θ. However, although the field is constant, these elements change between P and Q. Conversely, if these elements are constant, it does not imply that the vector field is constant.

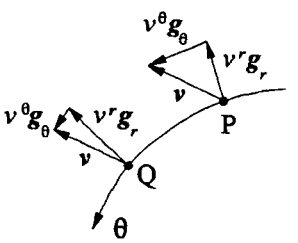


Figure 3.10 A constant vector field.

3.6.1 The derivative of a contravariant tensor

In a variable vector field, let $\boldsymbol{v}$ be given by $\quad \boldsymbol{v} = v^i \boldsymbol{g}_i$

and a small variation of $\boldsymbol{v}$ is then given by $\quad d\boldsymbol{v} = dv^i \boldsymbol{g}_i + v^i d\boldsymbol{g}_i.$

Then the rate of variation along the y^j coordinate line is

$$\frac{\partial \boldsymbol{v}}{\partial y^j} \equiv \boldsymbol{v}_{,j} = v^i{}_{,j}\boldsymbol{g}_i + v^i \boldsymbol{g}_{i,j} \tag{3.101}$$

The vector $\boldsymbol{g}_{i,j}$ can be expressed as a linear function of the base vectors $\boldsymbol{g}_k$ by

$$\boldsymbol{g}_{i,j} = \Gamma^k_{ij}\boldsymbol{g}_k \tag{3.102}$$

where Γ^k_{ij} is known as the **Christoffel three-index symbol**†. In 'flat' coordinate systems, such as Cartesian coordinates, where the base vectors do not vary, the Christoffel symbols are zero. In general, these symbols are not tensor elements.

Now (3.101) can be written as

$$\boldsymbol{v}_{,j} = v^i{}_{,j}\boldsymbol{g}_i + v^i\Gamma^k_{ij}\boldsymbol{g}_k = \left(v^i{}_{,j} + v^k\Gamma^i_{kj}\right)\boldsymbol{g}_i$$

or if we define the **covariant derivative** of v^i as $v^i{}_{|j}$ where

$$\boldsymbol{v}_{,j} = v^i_{|j}\boldsymbol{g}_i \tag{3.103}$$

then

$$v^i_{|j} = v^i{}_{,j} + v^k\Gamma^i_{kj} \tag{3.104}$$

It is important to remember that even if an element of a tensor is identically zero, it does not follow that its covariant derivative is also zero.

3.6.2 The derivative of a covariant tensor

From (3.6),

$$\boldsymbol{g}^i\cdot\boldsymbol{g}_j = \delta^i_j$$

at all points in space, so that

$$\mathrm{d}(\boldsymbol{g}^i\cdot\boldsymbol{g}_j) = \mathrm{d}\boldsymbol{g}^i\cdot\boldsymbol{g}_j + \boldsymbol{g}^i\cdot\mathrm{d}\boldsymbol{g}_j = 0$$

giving

$$\boldsymbol{g}^i{}_{,k}\cdot\boldsymbol{g}_j + \boldsymbol{g}^i\cdot\boldsymbol{g}_{j,k} = 0$$

Then from (3.102),

$$\boldsymbol{g}^i{}_{,k}\cdot\boldsymbol{g}_j + \boldsymbol{g}^i\cdot\Gamma^m_{jk}\boldsymbol{g}_m = 0$$

giving

$$\boldsymbol{g}^i{}_{,k}\cdot\boldsymbol{g}_j + \Gamma^i_{jk} = 0$$

Suppose that

$$\boldsymbol{g}^i{}_{,k} = G^i_{mk}\boldsymbol{g}^m$$

then

$$G^i_{mk}\boldsymbol{g}^m\cdot\boldsymbol{g}_j + \Gamma^i_{jk} = G^i_{jk} + \Gamma^i_{jk} = 0$$

so that

$$\boldsymbol{g}^i{}_{,k} = -\Gamma^i_{jk}\boldsymbol{g}^j \tag{3.105}$$

Carrying out the same process as before in terms of the covariant elements,

$$\boldsymbol{v}_{,j} = v_{i,j}\boldsymbol{g}^i + v_i\boldsymbol{g}^i{}_{,j} = (v_{i,j} - v_k\Gamma^k_{ij})\boldsymbol{g}^i$$

If the covariant derivative of v_i is defined as $v_{i|j}$ where

$$\boldsymbol{v}_{,j} = v_{i|j}\boldsymbol{g}^i \tag{3.106}$$

then

$$v_{i|j} = v_{i,j} - v_k\Gamma^k_{ij} \tag{3.107}$$

†In other texts, this form is denoted by $\left\{ \begin{matrix} k \\ ji \end{matrix} \right\}$ or $\left\{ \begin{matrix} ji \\ k \end{matrix} \right\}$ and is also referred to as the three-index symbol of the second kind.

3.6.3 Tensor properties of covariant derivatives

From (3.101) and (3.103),

$$\mathbf{dv} = v^{i}_{|j}\boldsymbol{g}_i \mathrm{d}y^j = \bar{v}^{k}_{\|l}\bar{\boldsymbol{g}}_k \mathrm{d}\bar{y}^l$$

where the double bar indicates covariant differentiation with respect to a second coordinate system $\bar{y}^l$. From (3.10) and (3.12),

$$\boldsymbol{g}_i \mathrm{d}y^j = \bar{\boldsymbol{g}}_k \frac{\partial \bar{y}^k}{\partial y^i}\frac{\partial y^j}{\partial \bar{y}^l}\mathrm{d}\bar{y}^l$$

Making this substitution in the previous equation and comparing coefficients of $\bar{\boldsymbol{g}}_k \mathrm{d}\bar{y}^l$,

$$\bar{v}^{k}_{\|l} = \frac{\partial \bar{y}^k}{\partial y^i}\frac{\partial y^j}{\partial \bar{y}^l} v^{i}_{|j}$$

so that from (3.56), $v^{i}_{|j}$ is a mixed second-order tensor. Likewise it can be shown that $v_{i|j}$ is a doubly-covariant tensor.

3.6.4 Covariant derivatives of higher order tensors

Suppose that

$$\boldsymbol{u} = v_{ij}w^j\boldsymbol{g}^i$$

Then from (3.106),

$$\boldsymbol{u}_{,k} = v_{ij,k}w^j\boldsymbol{g}^i + v_{ij}w^j{}_{,k}\boldsymbol{g}^i + v_{ij}w^j\boldsymbol{g}^i{}_{,k} = (v_{ij}w^j)_{|k}\boldsymbol{g}^i \tag{3.108}$$

Let the covariant derivative $v_{ij|k}$ be defined in such a way that

$$(v_{ij}w^j)_{|k} = v_{ij|k}w^j + v_{ij}w^j_{|k} \tag{3.109}$$

That is, the normal rule for differentiation of a product is taken to apply. It immediately follows from the tensor nature of the other terms in the equation that $v_{ij|k}$ is a third-order covariant tensor. Substituting this expression into (3.108), making use of (3.104) and (3.105) and comparing the coefficients of $w^j\boldsymbol{g}^i$ gives

$$v_{ij|k} = v_{ij,k} - v_{lj}\Gamma^l_{ik} - v_{il}\Gamma^l_{jk} \tag{3.110}$$

Similarly, the other derivatives of second-order tensors are taken to obey the normal rules of differentiation, which implies that they are given by

$$v^{i}_{j|k} = v^{i}_{j,k} + v^{l}_{j}\Gamma^i_{lk} - v^{i}_{l}\Gamma^l_{jk} \tag{3.111}$$

and

$$v^{ij}_{|k} = v^{ij}{}_{,k} + v^{lj}\Gamma^i_{lk} + v^{il}\Gamma^j_{lk} \tag{3.112}$$

The process extends to the derivatives of higher order tensors in a similar fashion, leading to the general rule

$$v^{ij..}_{pq..|k} = v^{ij..}_{pq..,k} + v^{lj..}_{pq..}\Gamma^i_{lk} + v^{il..}_{pq..}\Gamma^j_{lk} + .. - v^{ij..}_{lq..}\Gamma^l_{pk} - v^{ij..}_{pl..}\Gamma^l_{qk} .. \tag{3.113}$$

3.6.5 The derivative of a scalar

It was seen from (3.19) that the derivative of a scalar ϕ is a first-order covariant tensor and we can write

$$\phi_{,i} = \phi_{|i} \tag{3.114}$$

as a particular case of the general rule. It can readily be shown from the above results that if ϕ is given by the scalar product $v_j w^j$, then

$$v_{j,i} w^j + v_j w^j{}_{,i} = (v_j w^j)_{,i} = (v_j w^j)_{|i} = v_{j|i} w^j + v_j w^j_{|i}$$

3.6.6 Ricci's theorem

From (3.26) and (3.102),

$$g_{ij,k} = (\boldsymbol{g}_i \cdot \boldsymbol{g}_j)_{,k} = \boldsymbol{g}_{i,k} \cdot \boldsymbol{g}_j + \boldsymbol{g}_i \cdot \boldsymbol{g}_{j,k} = \Gamma^l_{ik} \boldsymbol{g}_l \cdot \boldsymbol{g}_j + \Gamma^l_{jk} \boldsymbol{g}_i \cdot \boldsymbol{g}_l = g_{lj} \Gamma^l_{ik} + g_{il} \Gamma^l_{jk}$$

So, because g_{ij} is a doubly-covariant tensor, from (3.110),

$$g_{ij|k} = g_{ij,k} - g_{lj} \Gamma^l_{ik} - g_{il} \Gamma^l_{jk} = 0 \tag{3.115}$$

Similarly, it can be shown that

$$g^{ij}_{|k} = 0 \tag{3.116}$$

This means that a tensor can be transformed into the same alternative form by means of metric tensors either before or after covariant differentiation. For example,

$$v^j_{|k} = (v_i g^{ij})_{|k} = v_{i|k} g^{ij} + v_i g^{ij}_{|k} = v_{i|k} g^{ij}$$

3.6.7 Symmetry of the Christoffel symbol

From (3.4), $\boldsymbol{r}_{,ij} = \boldsymbol{g}_{i,j}$

However, the order of differentiating $\boldsymbol{r}$ does not matter, so that from (3.102),

$$\Gamma^k_{ij} \boldsymbol{g}_k = \boldsymbol{g}_{i,j} = \boldsymbol{r}_{,ij} = \boldsymbol{r}_{,ji} = \boldsymbol{g}_{j,i} = \Gamma^k_{ji} \boldsymbol{g}_k$$

giving

$$\Gamma^k_{ij} = \Gamma^k_{ji} \tag{3.117}$$

3.6.8 Standard expressions for the Christoffel symbol

From (3.42) and (3.115),

$$g^{il}(g_{jl,k} + g_{kl,j} - g_{jk,l}) = \Gamma^i_{jk} + g^{il} g_{jm} \Gamma^m_{lk} + \Gamma^i_{kj} + g^{il} g_{km} \Gamma^m_{lj} - g^{il}(g_{mk} \Gamma^m_{jl} + g_{jm} \Gamma^m_{kl})$$

so that from the symmetry of g_{ij} and Γ^i_{jk},

$$\Gamma^i_{jk} = \frac{1}{2} g^{il}(g_{jl,k} + g_{kl,j} - g_{jk,l}) \tag{3.118}$$

From (3.6) and (3.102),

$$\Gamma^i_{jk} = \boldsymbol{g}^i \cdot \boldsymbol{g}_{j,k} \tag{3.119}$$

Suppose that the space is **Euclidean**, which means that it can be described by Cartesian coordinates x_l and base vectors $\boldsymbol{i}_l$. From (3.24) and (3.23),

$$\boldsymbol{g}^i = \frac{\partial y^i}{\partial x_l} \boldsymbol{i}_l \quad , \quad \boldsymbol{g}_j = \frac{\partial x_m}{\partial y^j} \boldsymbol{i}_m$$

so that

$$\Gamma^i_{jk} = \frac{\partial y^i}{\partial x_l} \frac{\partial^2 x_m}{\partial y^j \partial y^k} \boldsymbol{i}_l \cdot \boldsymbol{i}_m = \frac{\partial y^i}{\partial x_l} \frac{\partial^2 x_l}{\partial y^j \partial y^k} \tag{3.120}$$

Example 3.4 Christoffel symbols

For the coordinate system shown in Fig. 3.3, the only non-zero Christoffel symbols are

$$\Gamma^r_{\theta\theta} = -r \quad , \quad \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}$$

These can be derived from (3.45) and (3.46), using (3.118). Exactly the same result is

obtained for ordinary cylindrical polar coordinates ($\alpha = \pi/2$).

3.6.9 The differential of the metric determinant

Differentiating g, which is given as a function of g_{kj} by (3.49),

$$g_{,i} = \frac{\partial g}{\partial g_{kj}} g_{kj,i} = \frac{\partial g}{\partial g_{kj}} (g_{lj}\Gamma^l_{ki} + g_{kl}\Gamma^l_{ji}) \tag{3.121}$$

from (3.115). Now g can be expressed in terms of its elements $g_{\alpha j}$ and their cofactors $G^{\alpha j}$ (which do not contain terms in $g_{\alpha j}$) by $g = g_{\alpha j} G^{\alpha j}$ (summing over j only) so that

$$\frac{\partial g}{\partial g_{kj}} = G^{kj} = g g^{kj}$$

from (3.51). Then (3.121) becomes

$$g_{,i} = g g^{kj}(g_{lj}\Gamma^l_{ki} + g_{kl}\Gamma^l_{ji}) = 2g\Gamma^j_{ji}$$

from (3.42). This can be written as

$$\Gamma^j_{ji} = \tfrac{1}{2} g_{,i}/g = (\ln\sqrt{g})_{,i} = (\ln J_c)_{,i} = J_{c,i}/J_c \tag{3.122}$$

3.6.10 Covariant differentiation of the permutation tensor

From the rule for third-order covariant tensors,

$$\varepsilon_{ijk|l} = (e_{ijk}\sqrt{g})_{,l} - \sqrt{g}(e_{mjk}\Gamma^m_{il} + e_{imk}\Gamma^m_{jl} + e_{ijm}\Gamma^m_{kl})$$

making use of (3.70). The permutation symbol has constant values as defined by (2.19), so that from (3.122),

$$\varepsilon_{ijk|l} = \sqrt{g}(e_{ijk}\Gamma^m_{ml} - e_{mjk}\Gamma^m_{il} - e_{imk}\Gamma^m_{jl} - e_{ijm}\Gamma^m_{kl})$$

The right-hand side is zero if any two of i, j and k are the same. For example, if i and j are the same, the first and last terms in parenthesis are zero and the other two are equal and opposite. If i, j and k are all different, the second, third and fourth terms are only non-zero for m equal to i, j and k respectively. These three terms then cancel out those generated by the implied sum of the first term giving

$$\varepsilon_{ijk|l} = 0 \tag{3.123}$$

and it follows from Ricci's theorem that $\varepsilon^{ijk}_{|l}$ is also zero.

3.6.11 Higher order derivatives

The second-order covariant derivatives of v_i are denoted by

$$v_{i|jk} \equiv (v_{i|j})_{|k} = (v_{i|j})_{,k} - v_{l|j}\Gamma^l_{ik} - v_{i|l}\Gamma^l_{jk} \tag{3.124}$$

because $v_{i|j}$ is a second-order covariant tensor. Other higher order differentials can be found in a similar fashion. It is not necessarily true that $v_{i|jk}$ is the same as $v_{i|kj}$ unless the space is Euclidean or 'flat'. However, **Riemannian spaces** can be intrinsically curved and then Euclidean geometry no longer applies. For example, the surface of a sphere is an intrinsically curved two-dimensional Riemannian space. On this surface, the sum of the internal angles of a triangle is greater than 180°. Cartesian coordinates, although they can be used locally with reasonable accuracy, cannot be applied globally to such a surface.

Thus maps of small areas of the earth's surface can be drawn on a square grid with proportional side lengths, but this ceases to be a sufficiently accurate representation for larger areas.

3.6.12 The Riemann-Christoffel tensor

Writing (3.124) in full,

$$\begin{aligned} v_{i|jk} &= (v_{i,j} - v_l\Gamma^l_{ij})_{,k} - (v_{l,j} - v_m\Gamma^m_{lj})\Gamma^l_{ik} - (v_{i,l} - v_m\Gamma^m_{il})\Gamma^l_{jk} \\ &= v_{i,jk} - v_l(\Gamma^l_{ij})_{,k} - v_{l,k}\Gamma^l_{ij} - v_{l,j}\Gamma^l_{ik} - v_{i,l}\Gamma^l_{jk} + v_m(\Gamma^m_{lj}\Gamma^l_{ik} + \Gamma^m_{il}\Gamma^l_{jk}) \end{aligned}$$

and similarly, using the symmetry of $v_{i,jk}$ and Γ^i_{jk} with respect to j and k,

$$v_{i|kj} = v_{i,kj} - v_l(\Gamma^l_{ik})_{,j} - v_{l,j}\Gamma^l_{ik} - v_{l,k}\Gamma^l_{ij} - v_{i,l}\Gamma^l_{jk} + v_m(\Gamma^m_{lk}\Gamma^l_{ij} + \Gamma^m_{il}\Gamma^l_{jk})$$

Taking the difference between the two, and noting that l and m are dummy indices and so are interchangeable,

$$\begin{aligned} v_{i|jk} - v_{i|kj} &= v_l[(\Gamma^l_{ik})_{,j} - (\Gamma^l_{ij})_{,k}] + v_m(\Gamma^m_{lj}\Gamma^l_{ik} - \Gamma^m_{lk}\Gamma^l_{ij}) \\ &= v_l[(\Gamma^l_{ik})_{,j} - (\Gamma^l_{ij})_{,k} + \Gamma^l_{mj}\Gamma^m_{ik} - \Gamma^l_{mk}\Gamma^m_{ij}] \equiv v_l R^l_{ijk} \end{aligned} \qquad \textbf{(3.125)}$$

where R^l_{ijk}, which is given by the expression in the preceding square brackets, is known as the **mixed Riemann-Christoffel tensor** or the **Riemann-Christoffel tensor of the second kind**. Because the left-hand side of the equation is a triply-covariant tensor and v_l is a is a first-order tensor, it follows that R^l_{ijk} must be a fourth-order mixed tensor. It also follows from the expression that

$$R^l_{ijk} = -R^l_{ikj} \qquad \textbf{(3.126)}$$

and that

$$R^l_{ijk} + R^l_{kij} + R^l_{jki} = 0 \qquad \textbf{(3.127)}$$

It is possible to generate a fourth-order covariant tensor R_{hijk} from this mixed form, using the metric tensor.

$$R_{hijk} = g_{lh}R^l_{ijk} \qquad \textbf{(3.128)}$$

where R_{hijk} is known as the **covariant Riemann-Christoffel tensor** or the **Riemann-Christoffel tensor of the first kind**.

A Euclidean space is one which can be described by a coordinate system with constant metric tensors. Such a coordinate system could be Cartesian, and so a Euclidean space is also referred to as a 'flat' space. In any 'flat' space, it follows from (3.118) that the Christoffel symbols are identically zero and so from (3.125) the Riemann-Christoffel tensors are also zero. The order of covariant differentiation is unimportant if, and only if, the Riemann-Christoffel tensors are zero. Hence for any Euclidean or 'flat' space, the order of covariant differentiation does not matter.

Example 3.5 Riemann-Christoffel tensors

Consider the surface of a sphere of radius a described by the coordinate system $y^2 = \theta$, $y^3 = \phi$ (see Fig. 3.9). This gives the metric equation for the surface as

$$ds^2 = a^2d\theta^2 + a^2\sin^2\theta\, d\phi^2 = g_{22}dy^2dy^2 + g_{33}dy^3dy^3$$

so that in y^2y^3 space, the elements of the covariant metric tensor are

$$g_{22} = a^2 \quad , \quad g_{33} = a^2\sin^2\theta \quad , \quad g_{23}\ (= g_{32}) = 0$$

and from (3.42), the corresponding elements of the contravariant metric tensor are

$$g^{22} = 1/a^2 \quad , \quad g^{33} = 1/(a^2\sin^2\theta) \quad , \quad g^{23} \ (= g^{32}) = 0$$

so that from (3.118), the non-zero Christoffel three-index symbols are

$$\Gamma^2_{33} = -\sin\theta\cos\theta \quad , \quad \Gamma^3_{23} \ (= \Gamma^3_{32}) = \cot\theta \tag{3.129}$$

Then using (3.125) gives the following non-zero elements of the Riemann-Christoffel tensor of the second kind.

$$R^2_{323} = -R^2_{332} = \sin^2\theta \quad , \quad R^3_{232} = -R^3_{223} = 1 \tag{3.130}$$

From (3.128), the non-zero elements of the Riemann-Christoffel tensor of the first kind are

$$R_{2323} = R_{3232} = -R_{2332} = -R_{3223} = a^2\sin^2\theta \tag{3.131}$$

In fact, for any two-dimensional curved surface, these are the only non-zero elements of the covariant form and they are always related to one another in the above way, and so characterised by a single quantity. More generally, in n-dimensional Riemannian space, this tensor is characterised by $n^2(n^2\text{-}1)/12$ independent quantities. Equating these to zero gives an equal number of conditions on the form of the metric tensor, if it is to describe a Euclidean space.

If we now examine the Riemann-Christoffel tensor for the spherical polar coordinate system shown in Fig. 3.9, then in addition to the Christoffel three-index symbols given by (3.129), there are also the symbols

$$\Gamma^1_{22} = -r \quad , \quad \Gamma^1_{33} = -r\sin^2\theta \quad , \quad \Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = 1/r \tag{3.132}$$

These lead to the generation of extra terms in the expression for the Riemann-Christoffel tensor of the second kind which exactly cancel out those given by (3.130), and so the tensor is now zero. This illustrates that the tensor is related to the intrinsic curvature of the space considered and not that of the coordinate system employed.

A Euclidean space can be 'flat' in the mathematical sense without being 'flat' in common parlance. Thus the surface of a cone or cylinder is a two-dimensional Euclidean space. This can be reconciled with the normal concept of flatness by noting that these are developable surfaces. That is, they can be formed readily from a flat sheet of material without distorting it. Developable surfaces also have zero Gaussian curvature. Such curvature is directly related to the Riemann-Christoffel tensor, see for example Green and Zerna (1968), section 1.13.

3.7 STRAIN AND STRESS TENSORS

Equivalents of the strain and stress tensors used earlier will now be defined in terms of general curvilinear coordinates. These must become the tensors given in Chapter 2 when the coordinate system is Cartesian. This is not sufficient to define them uniquely, but it will be seen later that certain forms are preferable to others.

Suppose that the displacement field $\boldsymbol{u}$ of an elastic body is given by

$$\boldsymbol{u} = u_i\boldsymbol{g}^i = u^j\boldsymbol{g}_j$$

Then the equivalent of (2.34) can be written as

$$e_{jk} = \tfrac{1}{2}(u_{j|k} + u_{k|j}) = e_{kj} \tag{3.133}$$

so that

$$\bar{e}_{lm} = \tfrac{1}{2}(\bar{u}_{l\|m} + \bar{u}_{m\|l}) = \frac{1}{2}\left(\frac{\partial y^j}{\partial \bar{y}^l}\frac{\partial y^k}{\partial \bar{y}^m} u_{j|k} + \frac{\partial y^k}{\partial \bar{y}^m}\frac{\partial y^j}{\partial \bar{y}^l} u_{k|j} \right) = \frac{\partial y^j}{\partial \bar{y}^l}\frac{\partial y^k}{\partial \bar{y}^m} e_{jk}$$

showing that e_{jk} is a doubly-covariant tensor. It is not possible to define a mixed form of the strain tensor as $\tfrac{1}{2}(u^j_{|k} + u^k_{|j})$ because this expression does not transform as a single tensor. However, mixed and doubly-contravariant strain tensors can be obtained using the contravariant metric tensor.

$$e^i_j = e_{jk}g^{ki} = e_{kj}g^{ki} \quad , \quad e^{ij} = e_{kl}g^{ki}g^{lj} = e_{lk}g^{lj}g^{ki} = e^{ji} \tag{3.134}$$

Covariant differentials of strain can be related to covariant differentials of displacement, leading to compatibility conditions which are equivalent to those given by (2.37),

$$e_{jk|lm} + e_{lm|jk} = \tfrac{1}{2}(u_{j|klm} + u_{k|jlm} + u_{l|mjk} + u_{m|ljk}) = e_{ji|km} + e_{km|jl} \tag{3.135}$$

3.7.1 The stress tensor

For the purposes of this chapter, it will be assumed that the deflections of a body are sufficiently small for the geometrical changes between the stressed and unstressed states to be ignored while writing the equations of equilibrium. The area PRSW shown in Fig. 3.11 is the same as the area $dS(3)$ shown in Fig, 3.5. This is a small area of a y^3 coordinate surface which sections the body. Considering PRSW to be an upper face of the body at this section, the force $\boldsymbol{F}(3)$ is taken to act as shown. (Correspondingly, an equal and opposite force $-\boldsymbol{F}(3)$ will act on the adjoining lower face.) The force $\boldsymbol{F}(3)$ is taken to be related to the covariant representation of $dS(3)$, see (3.66), by

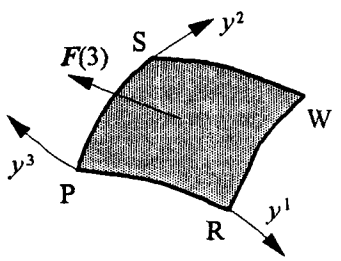


Figure 3.11 Traction on a coordinate surface.

$$\boldsymbol{F}(3) = (\sigma^{3i}\boldsymbol{g}_i)dS_3 = \sigma^{3i}\boldsymbol{g}_i\sqrt{g}\,dy^1dy^2 \tag{3.136}$$

and similar relationships are taken to hold for elementary areas of the other coordinate surfaces.

Consider the elementary tetrahedron shown in Fig. 3.6 loaded as shown in Fig. 3.12, where $\boldsymbol{p}$ is the vector force per unit area on RST. (The factors of ½ associated with the forces on the surfaces of the tetrahedron arise because the triangular areas are only one half of the elementary areas defined earlier. The minus signs occur because the faces are negative or 'lower' faces in the sense described above.) The force vectors shown in the figure are of second-order smallness. Body forces, being related to an elementary volume, are of third-order smallness and can be ignored in writing the equilibrium equation for the tetrahedron, which takes the form

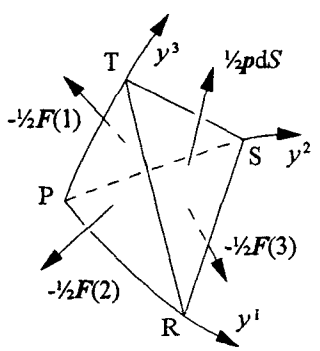


Figure 3.12 Equilibrium of a tetrahedron.

$$\tfrac{1}{2}\boldsymbol{p}\,\mathrm{d}S = \tfrac{1}{2}[\boldsymbol{F}(1) + \boldsymbol{F}(2) + \boldsymbol{F}(3)] \tag{3.137}$$

or, from the set of three equations of the form given by (3.136),

$$\boldsymbol{p}\,\mathrm{d}S = \sigma^{ji}\boldsymbol{g}_i\,\mathrm{d}S_j$$

Then taking

$$\boldsymbol{p} = p^i\boldsymbol{g}_i \tag{3.138}$$

and using (3.68),

$$p^i = \sigma^{ji}n_j \tag{3.139}$$

From the forms of the tensors p^i and n_j it follows that σ^{ji} is a second-order contravariant tensor.

3.7.2 Moment equilibrium

Suppose that a pair of forces $\boldsymbol{F}(3)$ act on the elementary parallelepiped considered in section 3.4.6, as shown in Fig. 3.13. These forces are of second-order smallness, being related to the surface area $\mathrm{d}S(3)$, and produce a couple $\boldsymbol{M}(3)$ of third-order smallness. If smaller order terms are ignored, the difference between PT and P′T′, and the variation in $\boldsymbol{F}(3)$ between faces are not significant so that

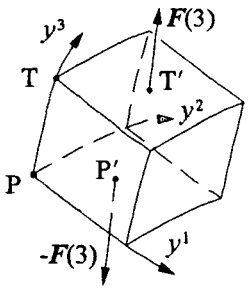


Figure 3.13 Moment equilibrium.

$$\begin{aligned}\boldsymbol{M}(3) &= \underline{\mathrm{PT}}\times\boldsymbol{F}(3)\\ &= \boldsymbol{g}_3\mathrm{d}y^3\times\sigma^{3i}\boldsymbol{g}_i\,\mathrm{d}S_3\\ &= \boldsymbol{g}_3\mathrm{d}y^3\times\sigma^{3i}\boldsymbol{g}_i\sqrt{g}\,\mathrm{d}y^1\mathrm{d}y^2\end{aligned} \tag{3.140}$$

from (3.136) and (3.66). The total moment $\boldsymbol{M}$ produced by couples on the three pairs of faces is

$$\boldsymbol{M} = \sigma^{ji}(\boldsymbol{g}_j\times\boldsymbol{g}_i)\sqrt{g}\,\mathrm{d}y^1\mathrm{d}y^2\mathrm{d}y^3 = \sigma^{ji}e_{jik}\boldsymbol{g}^k\sqrt{g}\,\mathrm{d}y^1\mathrm{d}y^2\mathrm{d}y^3 \tag{3.141}$$

from (3.73) and (3.70). Taking these to be the only moments of this order†, $\boldsymbol{M}$ must be zero for the element to be in equilibrium. Then on comparing coefficients of $\boldsymbol{g}^k$,

$$\sigma^{ij} = \sigma^{ji} \tag{3.142}$$

3.7.3 Equations of motion

The effects of body forces and variations in surface tractions on the equilibrium equations of the last two sections are an order of magnitude smaller than the most significant terms, and so were neglected. Likewise, the effects of mass accelerations are also be of a smaller order, and so these static equations also apply to dynamic states.

The conditions governing such smaller order terms can be found from the equations of motion for the elementary parallelepiped shown in Fig. 3.14. Suppose that the force per unit mass on the element shown is $\boldsymbol{f}$, its density is ρ and its acceleration is $\ddot{\boldsymbol{u}}$. Then its equation of motion is

$$\sum_{\alpha=1}^{3}\mathrm{d}\boldsymbol{F}(\alpha) + \rho\boldsymbol{f}\,\mathrm{d}V = \rho\ddot{\boldsymbol{u}}\,\mathrm{d}V \tag{3.143}$$

†It is possible that there could be a body moment per unit volume within the parallelepiped and couple stresses (moments per unit area) on its surfaces. Symmetry of the stress tensor is then lost. This is discussed in Appendix A6.

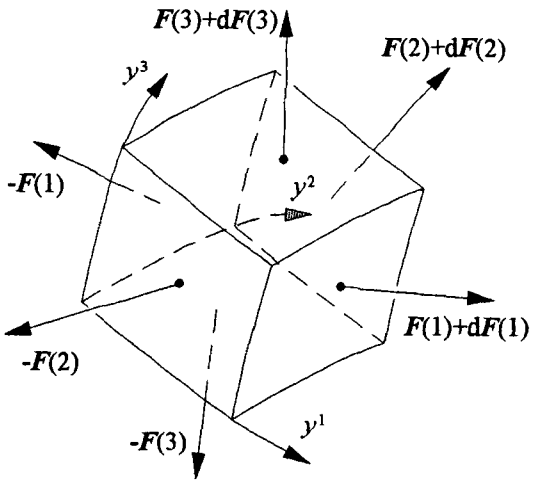


Figure 3.14 Surface forces on a parallelepiped.

where the volume dV of the element is given by (3.75). From (3.136),

$$dF(3) = (\sigma^{3j}\boldsymbol{g}_j\sqrt{g})_{,3}dy^1dy^2dy^3$$

Generalising this expression and using (3.102), (3.112) and (3.122),

$$\begin{aligned}\sum_{\alpha=1}^{3} dF(\alpha) &= (\sigma^{ij}\boldsymbol{g}_j\sqrt{g})_{,i}dy^1dy^2dy^3 \\ &= \{[\sigma^{ij}{}_{,i}\sqrt{g} + \sigma^{ij}(\sqrt{g})_{,i}]\boldsymbol{g}_j + \sigma^{ij}\sqrt{g}\,\boldsymbol{g}_{j,i}\}dy^1dy^2dy^3 \\ &= (\sigma^{ij}_{|i} - \sigma^{kj}\Gamma^i_{ki} - \sigma^{ik}\Gamma^j_{ki} + \sigma^{ij}\Gamma^k_{ki})\sqrt{g}\,\boldsymbol{g}_j + \sigma^{ij}\sqrt{g}\Gamma^k_{ij}\boldsymbol{g}_k\}dy^1dy^2dy^3 \\ &= \sigma^{ij}_{|i}\sqrt{g}\,\boldsymbol{g}_j dy^1dy^2dy^3\end{aligned}$$

so that in terms of the contravariant components of $\boldsymbol{X}$ and $\ddot{\boldsymbol{u}}$†, (3.143) becomes

$$\sigma^{ij}_{|i}\sqrt{g}\,\boldsymbol{g}_j dy^1dy^2dy^3 + \rho f^j\boldsymbol{g}_j\sqrt{g}\,dy^1dy^2dy^3 = \rho\ddot{u}^j\boldsymbol{g}_j\sqrt{g}\,dy^1dy^2dy^3$$

or

$$\sigma^{ij}_{|i} + \rho f^j = \rho\ddot{u}^j \qquad \textbf{(3.144)}$$

Example 3.6 Equations of motion

Suppose that the equation of motion given by (3.144) was required in terms of physical components, using the spherical coordinate system shown in Fig. 3.9. From (3.88) to (3.90), the h_α terms are

$$h_1 = 1 \quad , \quad h_2 = r \quad , \quad h_3 = r\sin\theta \qquad \textbf{(3.145)}$$

and the physical components are then found from (3.92) to (3.98), covariant differentiation of σ^{ij} is given by (3.112) and the Christoffel three-index symbols by (3.129) and (3.132), so that on dividing throughout by h_α, the equations of motion become

†In general, the components $\ddot{u}^j$ of $\ddot{\boldsymbol{u}}$ are not $\partial^2u^j/\partial t^2$, see section 3.10.2.

$$\sigma^{P}_{rr,r} + \frac{1}{r}(\sigma^{P}_{r\theta,\theta} + \operatorname{cosec}\theta\,\sigma^{P}_{r\phi,\phi} + 2\sigma^{P}_{rr} + \cot\theta\,\sigma^{P}_{r\theta} - \sigma^{P}_{\theta\theta} - \sigma^{P}_{\phi\phi}) + \rho f^{P}_{r} = \rho \ddot{u}^{P}_{r} \quad \textbf{(3.146)}$$

$$\sigma^{P}_{r\theta,r} + \frac{1}{r}(\sigma^{P}_{\theta\theta,\theta} + \operatorname{cosec}\theta\,\sigma^{P}_{\theta\phi,\phi} + 3\sigma^{P}_{r\theta} + \cot\theta\,\sigma^{P}_{\theta\theta} - \cot\theta\,\sigma^{P}_{\phi\phi}) + \rho f^{P}_{\theta} = \rho \ddot{u}^{P}_{\theta} \quad \textbf{(3.147)}$$

$$\sigma^{P}_{r\phi,r} + \frac{1}{r}(\sigma^{P}_{\theta\phi,\theta} + \operatorname{cosec}\theta\,\sigma^{P}_{\phi\phi,\phi} + 3\sigma^{P}_{r\phi} + 2\cot\theta\,\sigma^{P}_{\theta\phi}) + \rho f^{P}_{\phi} = \rho \ddot{u}^{P}_{\phi} \quad \textbf{(3.148)}$$

3.7.4 Invariants

A scalar, which may be thought of as a tensor of order zero, has the same value in all coordinate systems. Thus any process of contraction which reduces a tensor expression to a scalar produces an invariant. In particular, this process gives the invariants

$$e_i^{\ i} \quad , \quad e_j^{\ i} e_i^{\ j} \quad , \quad e_j^{\ i} e_k^{\ j} e_i^{\ k} \qquad \textbf{(3.149)}$$

in terms of the mixed strain tensor given by (3.134). These invariants can be related to the strain invariants of (1.72) by considering the particular case of Cartesian coordinates given by (2.36), giving

$$\begin{aligned} K_1 &= e_i^{\ i} \quad (= g^{ij} e_{ij}) \\ K_2 &= \tfrac{1}{2}(e_i^{\ i} e_j^{\ j} - e_j^{\ i} e_i^{\ j}) \\ K_3 &= \tfrac{1}{6}(e_i^{\ i} e_j^{\ j} e_k^{\ k} - 3 e_j^{\ i} e_i^{\ j} e_k^{\ k} + 2 e_j^{\ i} e_k^{\ j} e_i^{\ k}) \end{aligned} \qquad \textbf{(3.150)}$$

The invariants are by definition the same in any coordinate system. In particular, they can be evaluated in terms of Cartesian tensors. For example, the volumetric strain of an elementary cuboid of material, $\delta V/V$, is given by the Cartesian tensor sum e_{ii} which is K_1 (cf. sections 2.4 and 2.7.1). As illustrated in the above expression for K_1, these invariants can also be expressed in terms of other forms of the strain tensor relative to the same coordinate system, by means of the metric tensor. The stress invariants are given by similar expressions in terms of the mixed stress tensor, or conversely,

$$\begin{aligned} \sigma_i^{\ i} &= I_1 \\ \sigma_j^{\ i}\sigma_i^{\ j} &= I_1^2 - 2I_2 \\ \sigma_j^{\ i}\sigma_k^{\ j}\sigma_i^{\ k} &= I_1^3 - 3I_1 I_2 + 3I_3 \end{aligned} \qquad \textbf{(3.151)}$$

These three are the only independent stress invariants. An expression such as $\sigma_j^{\ i}\sigma_k^{\ j}\sigma_l^{\ k}\sigma_i^{\ l}$ is also invariant but is a function of the above three invariants in three-dimensional space, for the same reasons outlined earlier in section 2.4.

3.8 ELASTIC BEHAVIOUR

It will be assumed that the initial state of the body is one of zero strain, stress and internal energy. Also, deformation will be assumed to take place under isothermal conditions. This means that the total energy stored in the body is given by an elastic potential U per unit volume of (unstrained) material, and this is induced by the work of the loads on it.

Suppose that the body is subject to small changes $\delta\boldsymbol{u}$ in its displacement field $\boldsymbol{u}$. The elementary parallelepiped within the body, shown in Fig. 3.13, will be subject to rigid-body motion and variations of displacement between its faces. The net work done $\delta W(3)$ by the equal and opposite pair of forces $\boldsymbol{F}(3)$ is caused solely by the variation in displacement, $\mathrm{d}\delta\boldsymbol{u}(3)$, between the faces on which these forces act. It is given by the dot

product $\boldsymbol{F}(3)\bullet d\delta\boldsymbol{u}(3)$. The general tensor form of this is given by

$$\delta W(3) = \boldsymbol{F}(3)\cdot d\delta\boldsymbol{u}(3) = \sigma^{3i}\sqrt{g}\,dy^1dy^2\boldsymbol{g}_i\cdot\boldsymbol{g}^j\delta u_{j|3}dy^3 = \delta u_{i|3}\sigma^{3i}dV$$

using (3.136) and (3.75). Similar expressions can be written for the other two pairs of faces, so that the total work done on an elementary parallelepiped is

$$\delta W = \delta u_{i|j}\sigma^{ji}dV = \tfrac{1}{2}(\delta u_{i|j}\sigma^{ji} + \delta u_{j|i}\sigma^{ij})dV = \tfrac{1}{2}\delta(u_{i|j} + u_{j|i})\sigma^{ij}dV = \delta e_{ij}\sigma^{ij}dV$$

using (3.142) and (3.133). This work is equal to the gain in strain energy potential in the parallelepiped, $\delta U dV$, so that by the same arguments as before,

$$\delta U dV = \frac{\partial U}{\partial e_{ij}}\delta e_{ij}dV = \delta W = \delta e_{ij}\sigma^{ij}dV$$

for arbitrary δe_{ij}. Without loss of generality, U can be taken as symmetrical in e_{ij} and e_{ji} these two being by definition the same. Then

$$\frac{\partial U}{\partial e_{ij}} = \sigma^{ij} \qquad \left(= \frac{\partial U}{\partial e_{ji}} \right) \tag{3.152}$$

which may be compared with (2.52). Again, no assumption of *linear* elasticity has been made.

If the strain energy per unit volume is expressed as a power series in terms of strain of the kind given by (2.53), then because it is a scalar quantity, it takes the form

$$U(e_{ij}) = \tfrac{1}{2}c^{ijkl}e_{ij}e_{kl} + \tfrac{1}{3}d^{ijklmn}e_{ij}e_{kl}e_{mn} + O(e^4) \tag{3.153}$$

where c^{ijkl} and d^{ijklmn} are fourth-order and sixth-order contravariant tensors respectively. (The initial conditions assumed at the outset preclude any constant or linear terms.) The symmetry conditions given by (2.54) and (2.55) again apply. From (3.152),

$$\sigma^{ij} = c^{ijkl}e_{kl} + d^{ijklmn}e_{kl}e_{mn} + O(e^3) \tag{3.154}$$

This is the equivalent of (2.56) under the assumption of zero initial stress and strain conditions. The thermoelastic equations in terms of curvilinear coordinates can readily be deduced by the same method used in Chapter 2.

3.8.1 Isotropic elasticity

If the material is isotropic, the expression for the strain energy in terms of strain is identical in all coordinate systems and so can be formulated as a set of functions of the strain invariants given by (3.149). Taking the same initial conditions as those assumed previously for the anisotropic case, the form corresponding to (2.62) is

$$U = \tfrac{1}{2}\lambda e_i^{\,i}e_j^{\,j} + \mu e_j^{\,i}e_i^{\,j} + \tfrac{1}{3}d_1e_i^{\,i}e_j^{\,j}e_k^{\,k} + d_2e_i^{\,i}e_k^{\,j}e_j^{\,k} + \tfrac{1}{3}d_3e_j^{\,i}e_k^{\,j}e_i^{\,k} + O(e^4) \tag{3.155}$$

where λ, μ, d_1, d_2 and d_3 are independent scalar elastic constants. To find the stress-strain relationship from (3.152), the mixed strain tensors must be expressed in terms of doubly-covariant strain tensors. This can be done using the contravariant metric tensors, as for example in the alternative form of K_1 given by (3.150). Then

$$\sigma^{ij} = \frac{\partial U}{\partial e_{ij}} = [\lambda g^{ij}g^{kl} + \mu(g^{ik}g^{jl} + g^{il}g^{jk})]e_{kl} + [g^{ij}(d_1 g^{kl}g^{mn} + d_2 g^{lm}g^{kn}) + 2d_2 g^{ik}g^{jl}g^{mn} + d_3 g^{il}g^{jn}g^{km}]e_{kl}e_{mn} + O(e^3) \tag{3.156}$$

In the particular case of linear elasticity, this becomes

$$\sigma^{ij} = \lambda g^{ij}e_k^k + 2\mu e^{ij} \tag{3.157}$$

3.8.2 Equations of motion in terms of displacement

For a linearly-elastic isotropic material, the relationship between stress and displacement given by (3.157) and (3.133) is

$$\sigma^{ij} = [\lambda g^{ij}g^{kl} + \mu(g^{ik}g^{jl} + g^{il}g^{jk})]u_{k|l} \tag{3.158}$$

If the material is homogeneous then, using Ricci's theorem, (3.139) and (3.144) can be written in the covariant forms

$$g^{jk}[\mu(u_{i|j} + u_{j|i})n_k + \lambda u_{j|k}n_i] = p_i \tag{3.159}$$

and

$$g^{jk}[\mu u_{i|jk} + (\lambda + \mu)u_{j|ki}] + \rho f_i = \rho\ddot{u}_i \tag{3.160}$$

which is **Navier's equation** expressed in terms of curvilinear coordinates. These two equations are generalisations of (2.85) and (2.86). The corresponding relationships between the contravariant forms of the loads and displacements can be found by multiplying the above equations by contravariant metric tensors.

3.8.3 The wave equations

In the absence of body forces, (3.160) becomes

$$g^{jk}[\mu u_{i|jk} + (\lambda + \mu)u_{j|ki}] = \rho\ddot{u}_i \tag{3.161}$$

Two forms of $\boldsymbol{u}$ are of particular interest. If $\boldsymbol{u}$ is the gradient of a scalar potential ς, then

$$u_i = \varsigma_{,i} = \varsigma_{|i} \qquad \text{so that} \qquad u_{i|j} = \varsigma_{|ij} = u_{j|i}$$

Then (3.161) reduces to

$$c_P^2 g^{jk}u_{i|jk} = \ddot{u}_i \tag{3.162}$$

where

$$c_P^2 = (\lambda + 2\mu)/\rho \tag{3.163}$$

If $\boldsymbol{u}$ is given by the curl of a vector potential η, then

$$u_j = g_{jl}\varepsilon^{lmn}\eta_{n|m}$$

Then, using (3.42), (3.115), (3.123) and the symmetry properties of the permutation tensor,

$$\begin{aligned} g^{jk}u_{j|ki} &= g^{jk}g_{jl}\varepsilon^{lmn}\eta_{n|mki} = \delta_l^k\varepsilon^{lmn}\eta_{n|kmi} \\ &= \varepsilon^{kmn}\eta_{n|mki} = \tfrac{1}{2}\varepsilon^{kmn}(\eta_{n|mki} + \eta_{n|kmi}) \\ &= \tfrac{1}{2}\eta_{n|mki}(\varepsilon^{kmn} + \varepsilon^{mkn}) = 0 \end{aligned} \tag{3.164}$$

This means that (3.161) now reduces to

$$c_S^2 g^{jk}u_{i|jk} = \ddot{u}_i \tag{3.165}$$

where
$$c_S^2 = \mu/\rho \tag{3.166}$$

Now from (3.133) and (3.134),
$$e_k^k = u_{|k}^k = g^{jk}u_{j|k}$$

and by a similar proof to that used to obtain (3.164), it can be shown that when $\boldsymbol{u}$ is given by the curl of a vector potential, e_t^k is zero. As seen in the preceding chapters, this invariant gives the local volumetric strain of the material. Thus the displacement field corresponding to (3.162) is dilatational (giving pressure or P waves), but that corresponding to (3.165) is distortional without dilatation (giving shear or S waves). The vector form of these equations is examined in Appendix A3.

3.8.4 Travelling and stationary waves

Suppose that the displacement $\boldsymbol{u}$ of a point P always becomes that of an adjacent point P′ at a time δt later, where P′ is $\delta\boldsymbol{r}$ from P (see Fig. 3.15). Then

$$\boldsymbol{u}(\boldsymbol{r},t) = \boldsymbol{u}(\boldsymbol{r}+\delta\boldsymbol{r}, t+\delta t)$$

This is ensured by making $\boldsymbol{u}$ a function of the parameter $\boldsymbol{X}$ where
$$\boldsymbol{X} = (\boldsymbol{r} - c\boldsymbol{t}) = X^j\boldsymbol{g}_j \tag{3.167}$$

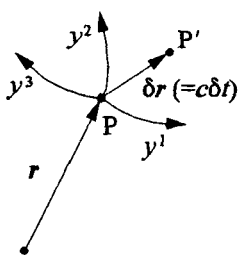


Figure 3.15 Wave motion.

and $\boldsymbol{c}$ is such that $\delta\boldsymbol{r} = c\delta t$ for then

$$\boldsymbol{u}[(\boldsymbol{r}+\delta\boldsymbol{r}) - c(t+\delta t)] = \boldsymbol{u}[\boldsymbol{r} - c\boldsymbol{t}]$$

This corresponds to the motion of a wave with a velocity of magnitude c ($= |\boldsymbol{c}|$) in the direction PP′.

Suppose that
$$\boldsymbol{r} = r^j\boldsymbol{g}_j \quad , \quad \boldsymbol{c} = c^j\boldsymbol{g}_j \tag{3.168}$$

Here it should be noted that r^j is not necessarily the same as y^j. For example, the values of r^j for the coordinate system shown in Fig.3.3 are

$$r^1 = r \quad , \quad r^2 = 0 \quad , \quad r^3 = z \tag{3.169}$$

This can be confirmed by inspection or from (3.35) and (3.36). These values also apply to the special case when the system is the normal cylindrical polar coordinate system given by (3.82). For the spherical polar coordinate system shown in Fig. 3.9,

$$r^1 = r \quad , \quad r^2 = r^3 = 0 \tag{3.170}$$

From (3.4) and (3.103),
$$\boldsymbol{g}_i = \boldsymbol{r}_{,i} = r_{|i}^j\boldsymbol{g}_j \tag{3.171}$$

so that
$$r_{|i}^j = \delta_i^j \tag{3.172}$$

Now
$$\dot{\boldsymbol{u}} = \frac{\partial}{\partial t}\{\boldsymbol{u}[r^j - c^j t]\} = -c^j\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}X^j} = -c^j\frac{\partial\boldsymbol{u}}{\partial r^j}$$

Considering the particular cases where the wave travels only along a y^α coordinate line which is such that $y^\alpha \equiv r^\alpha$,

$$\dot{\boldsymbol{u}} = -c^{\alpha}\boldsymbol{u}_{,\alpha} \quad , \quad \ddot{\boldsymbol{u}} = (c^{\alpha})^2\boldsymbol{u}_{,\alpha\alpha} \tag{3.173}$$

Furthermore, if $\boldsymbol{g}_{\alpha}$ is a unit vector, then from (3.168) c^{α} is equal to the wave velocity c. Then (3.162) and (3.165) take the form

$$c_*^2 g^{jk} u_{i|jk} = c^2 u_{i,\alpha\alpha} \qquad (c_* = c_P \text{ or } c_S) \tag{3.174}$$

Standing waves are given by

$$\boldsymbol{u} = \boldsymbol{U}(\boldsymbol{r})\cos(\omega t + \psi) \qquad (\boldsymbol{U} = U_j \boldsymbol{g}^j) \tag{3.175}$$

so that $\ddot{\boldsymbol{u}} = -\omega^2 \boldsymbol{u}$ where ω is the wave frequency.

Then (3.162) and (3.165) take the form

$$c_*^2 g^{jk} u_{i|jk} = -\omega^2 u_i \qquad (c_* = c_P \text{ or } c_S) \tag{3.176}$$

When $\boldsymbol{u}$ has a harmonic solution such that $u_{i,\alpha\alpha}$ is a constant multiple of u_i, then from (3.174) and (3.176) the solution can be applied to both travelling and stationary waves.

3.8.5 Pochhammer-Chree waves

Cylindrical polar coordinates are such that

$$y^1 = r \ , \ y^2 = \theta \ , \ y^3 = z \ ,$$
$$g^{11} = 1 \ , \ g^{22} = 1/r^2 \ , \ g^{33} = 1 \ ,$$
$$\Gamma^1_{22} = -r \ , \ \Gamma^2_{12} = \Gamma^2_{21} = 1/r \ .$$

(see Appendix A1). Relative to these coordinates, a solution of (3.176) is given by

$$\begin{aligned} U_1 &= R_1(r)\sin m\theta \sin pz \\ U_2 &= R_2(r)\cos m\theta \sin pz \\ U_3 &= R_3(r)\sin m\theta \cos pz \end{aligned} \tag{3.177}$$

Pressure waves† ($c_* = c_P$) are given by $U = \varsigma_{,i}$ where $\varsigma = AZ_m(r)\sin m\theta \sin pz$ and A is an arbitrary constant. Then

$$R_1 = AZ_{m,r} \ , \ R_2 = AmZ_m \ , \ R_3 = ApZ_m$$

Shear waves‡ ($c_* = c_S$) are given by

$$R_1 = \frac{1}{r}BmZ_m + CpZ_{m,r} \ , \ R_2 = BrZ_{m,r} + CmpZ_m \ , \ R_3 = C(p^2 - \omega^2/c_s^2)Z_m$$

where B and C are arbitrary constants. Using the vector notation of Appendix A3, this result can be expressed as $U = \nabla \times \eta$ where

$$\eta = -BZ_m(r)\cos m\theta \sin pz\, \boldsymbol{g}^3 - C\nabla \times [Z_m(r)\sin m\theta \cos pz\, \boldsymbol{g}^3]$$

For both pressure and shear waves, (3.176) implies that Z_m must satisfy the equation

$$Z_{m,rr} + \frac{1}{r}Z_{m,r} + (\omega^2/c_*^2 - p^2 - \frac{m^2}{r^2})Z_m = 0$$

†Also known as *dilatational* or *irrotational* waves.

‡Also known as *distortional*, *rotational* or *equivoluminal* waves.

This is satisfied by the functions

$$Z_m(r) = J_m[r\sqrt{(\omega^2/c_*^2 - p^2)}] \quad \text{or} \quad Y_m[r\sqrt{(\omega^2/c_*^2 - p^2)}] \ , \quad (\omega^2 > p^2 c_*^2)$$
$$= I_m[r\sqrt{(p^2 - \omega^2/c_*^2)}] \quad \text{or} \quad K_m[r\sqrt{(p^2 - \omega^2/c_*^2)}] \ , \quad (\omega^2 < p^2 c_*^2)$$

where J_m and Y_m are the mth Bessel functions of the first and second kind respectively and I_m and K_m are the mth modified Bessel functions of the first and second kind. A solution for waves travelling in the z direction is given by taking u_i as U_i but replacing the arguments pz in (3.177) with $p(z\text{-}ct)$. The appropriate expressions for R_i and Z_m are now given by substituting pc for ω.

A particular case of interest occurs when m is zero. The solutions then give longitudinal waves involving u_1 and u_3 alone and a torsional wave involving u_2 alone.

3.8.6 Radial waves

If $\boldsymbol{u}$ is derived from a scalar potential ς, (3.162) is satisfied if

$$c_P^2 g^{jk} \varsigma_{|jk} = \ddot{\varsigma} \tag{3.178}$$

A number of solutions for ς exist of the form

$$\varsigma = f(\boldsymbol{r})\, F(r^\alpha \pm c_P t)$$

where $r^\alpha \equiv y^\alpha$ and $F(r^\alpha \pm c_P t)$ is an arbitrary function. Substituting this form into (3.178) and using (3.107), (3.114) and (3.173) gives

$$c_P^2 g^{jk}[f_{,jk}F + f_{,j}F_{,k} + f_{,k}F_{,j} + fF_{,jk} - (f_{,l}F + fF_{,l})\Gamma^l_{jk}] = c_P^2 f F_{,\alpha\alpha}$$

Now F is an arbitrary function of only y^α and so $F_{,k}$ is zero unless $k = \alpha$. Also, because F is to be an entirely arbitrary function, the coefficients of F, $F_{,\alpha}$ and $F_{,\alpha\alpha}$ can be compared in the above equation. Making use of the symmetry of g^{jk}, this gives

$$g^{jk}(f_{,jk} - f_{,l}\Gamma^l_{jk}) = 0 \ , \quad 2g^{\alpha j} f_{,j} = f g^{jk} \Gamma^\alpha_{jk} \ , \quad g^{\alpha\alpha} = 1 \tag{3.179}$$

For cylindrical polar coordinates, taking y^α as r,

$$f(\boldsymbol{r}) = r^{-1/2} \cos\left(\tfrac{1}{2}\theta + \psi\right) \tag{3.180}$$

For spherical polar coordinates (see Appendix A1), taking y^α as r,

$$f(\boldsymbol{r}) = \frac{1}{r} \tag{3.181}$$

Propagation of an arbitrary wave form relative to spherical polar coordinates is also given by the potential

$$\varsigma = \cos\theta \frac{\partial}{\partial r}\left[\frac{1}{r} F(r \pm c_P t)\right] \tag{3.182}$$

This is known as **dipole radiation**.

Standing wave solutions can be generated from suitable sums of solutions of the above type, noting that

$$\cos\left(\frac{\omega}{c_P}(r + c_P t) + \psi\right) + \cos\left(\frac{\omega}{c_P}(r - c_P t) + \psi\right) = 2\cos\left(\frac{\omega r}{c_P} + \psi\right)\cos\omega t \tag{3.183}$$

3.8.7 Spherical vibrations

A solution of (3.178) in terms of spherical polar coordinates takes the form

$$\varsigma = Z(r)P(\theta)\cos(m\phi + \beta)\cos(\omega t + \gamma) \tag{3.184}$$

where, by the usual separation of variables,

$$r^2\left(\frac{\partial^2 Z}{\partial r^2} + \frac{2}{r}\frac{\partial Z}{\partial r} + \frac{\omega^2}{c_P^2}Z\right) = (n^2 + n)Z \tag{3.185}$$

$$\frac{\partial^2 P}{\partial \theta^2} + \cot\theta\frac{\partial P}{\partial \theta} - \frac{m^2}{\sin^2\theta}P = -(n^2 + n)P \tag{3.186}$$

Making the substitution $Z = r^{-\frac{1}{2}}J(R)$, $R = \omega r/c_P$ (3.185) becomes

$$\frac{\partial^2 J}{\partial R^2} + \frac{1}{R}\frac{\partial J}{\partial R} + \left(1 - \frac{1}{R^2}\left(n + \frac{1}{2}\right)^2\right)J = 0$$

so that J is given by the Bessel functions

$$J = J_{\pm(n+\frac{1}{2})}(R)$$

These are spherical Bessel functions.

Making the substitution $P = P(\psi)$, $\psi = \cos\theta$, (3.186) becomes

$$(1 - \psi^2)\frac{\partial^2 P}{\partial \psi^2} - 2\psi\frac{\partial P}{\partial \psi} + \left((n^2 + n) - \frac{m^2}{(1 - \psi^2)}\right)P = 0$$

This is known as Legendre's associated equation and becomes the ordinary Legendre equation when m is zero. The solutions

$$P = P_n^{\pm m}(\psi) \quad \text{or} \quad Q_n^{\pm m}(\psi)$$

are known as Legendre's associated functions and are related to the ordinary Legendre functions $P_n(\psi)$ and $Q_n(\psi)$ by

$$P_n^{\pm m} = (\psi^2 - 1)^{\pm m/2}\frac{d^m P_n(\psi)}{d\psi^m}$$

$$Q_n^{\pm m} = (\psi^2 - 1)^{\pm m/2}\frac{d^m Q_n(\psi)}{d\psi^m}$$

Linear combinations of the standing wave solutions given by (3.184) can be used to generate travelling wave solutions with an angular velocity of ω/m.

3.8.8 Displacement potentials

In the absence of body forces and accelerations, the form of Navier's equation given by (3.160) reduces to the homogeneous form

$$g^{jk}[\mu u_{i|jk} + (\lambda + \mu)u_{j|ki}] = 0 \tag{3.187}$$

Performing covariant differentiation of this with respect to y^l, multiplying it by g^{il} and taking into account the summation implied by repeated indices gives

$$g^{ij}g^{kl}(\lambda + 2\mu)u_{i|jkl} = 0 \tag{3.188}$$

Likewise, carrying out the harmonic operation $\{\ \}_{|lm} g^{lm}$ on (3.187) gives

$$g^{jk} g^{lm} [\mu u_{i|jklm} + (\lambda + \mu) u_{j|klmi}] = 0$$

Then from (3.188),

$$g^{jk} (g^{lm} u_{i|lm})_{|jk} = 0 \qquad \textbf{(3.189)}$$

This expresses the biharmonic equation (2.88) in terms of general coordinates.

Instead of seeking biharmonic forms of u_i which also satisfy (3.187), it is possible to use the general solution of this equation in terms of harmonic functions. This is given by the **Papkovich-Neuber equation**

$$\mu u_i = \Phi_i - \frac{\lambda + \mu}{2(\lambda + 2\mu)} [\Psi + r^j \Phi_j]_{|i} \qquad \textbf{(3.190)}$$

where Ψ is a harmonic scalar and the elements Φ_i form a harmonic vector potential. That is,

$$g^{lm} \Psi_{|lm} = g^{lm} \Phi_{i|lm} = 0 \qquad \textbf{(3.191)}$$

The elements r^j are the contravariant components of the position vector $\boldsymbol{r}$, as given by (3.168), and so its covariant derivatives are given by (3.172). Together with (3.191) and the symmetry of the metric tensor, this can be used to show that (3.190) is a solution of (3.187). The form of (3.190) can be expressed more simply in terms of Poisson's ratio, ν, by noting that $(\lambda+\mu)/2(\lambda+2\mu)$ is equal to $1/4(1-\nu)$. A further discussion of Papkovich-Neuber functions and differential operators will be found in Appendix A3.

Sternberg and Eubanks† have shown that the form given by (3.190) can also be used for problems where there are body forces and dynamic effects. The conditions which must be satisfied by the potentials are then

$$g^{jk} \Phi_{i|jk} - \frac{1}{c_S^2} \ddot{\Phi}_i + \rho f_i = 0 \qquad \textbf{(3.192)}$$

$$g^{jk} \Psi_{|jk} - \frac{1}{c_P^2} \ddot{\Psi} + r^i \left(g^{jk} \Phi_{i|jk} - \frac{1}{c_P^2} \ddot{\Phi}_i \right) = 0 \qquad \textbf{(3.193)}$$

where c_P and c_S are given by (3.163) and (3.166). These results were derived from the generalised definition of the Galerkin vector, here denoted by $G_i \boldsymbol{g}^i$, given by Iacovache‡. This must satisfy the conditions

$$2\mu u_i = 2(1 - \nu) \left(g^{kl} G_{i|kl} - \frac{1}{c_P^2} \ddot{G}_i \right) - g^{jk} G_{j|ki} \qquad \textbf{(3.194)}$$

$$g^{jk} g^{mn} G_{i|jkmn} - \left(\frac{1}{c_P^2} + \frac{1}{c_S^2} \right) g^{jk} \ddot{G}_{i|jk} + \frac{1}{c_P^2 c_S^2} \ddot{\ddot{G}}_i = - \frac{1}{1 - \nu} \rho f_i \qquad \textbf{(3.195)}$$

†Sternberg, E. and Eubanks, R.A. (1957) On stress functions for elastokinetics and the integration of the repeated wave equation. *Quart. Appl. Math.* **15** 149-153

‡Iacovache, M. (1949) O extindere a metodei lui Galerkin pentru sistemul ecuatiilor elastisticatii. *Acad. Romane. Bul. Sti* **A1** 593-596

(cf. (1.112) and (1.113)). As pointed out by Sternberg†, use of the Galerkin vector leads to expressions which are unwieldy in comparison with those derived from the Papkovich-Neuber functions. Thus the Boussinesq solution, which for cylindrical polar coordinates takes the form

$$\Psi = \Psi(r,z) \quad , \quad \Phi_1 = \Phi_2 = 0 \quad , \quad \Phi_3 = \Phi_3(r,z) \tag{3.196}$$

is preferable to the Love function

$$G_1 = G_2 = 0 \quad , \quad G_3 = G_3(r,z) \tag{3.197}$$

3.9 MEMBRANE THEORY OF THIN SHELLS

In analysing the behaviour of a membrane or a shell, it is convenient to define its middle surface as a coordinate surface. In Fig. 3.16, this is shown as a y^3 coordinate surface, that is, the surface on which y^3 takes a given constant value. Further, it will be taken that the y^3 coordinate lines are everywhere normal to this surface and that $\boldsymbol{g}_3$ (and hence $\boldsymbol{g}^3$) is a unit vector normal to the surface. The matrix of metric tensors g_{ij} (equal to $\boldsymbol{g}_i \cdot \boldsymbol{g}_j$) then takes the form

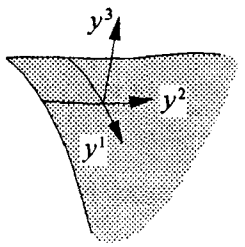


Figure 3.16 Membrane coordinates.

$$[g_{ij}] = \begin{bmatrix} g_{11} & g_{12} & 0 \\ g_{21} & g_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.198}$$

and from (3.43) its inverse is

$$[g^{ij}] = \begin{bmatrix} g^{11} & g^{12} & 0 \\ g^{21} & g^{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.199}$$

where, from the usual process of inverting a matrix,

$$g_{11} = g g^{22} \quad , \quad g_{12} = -g g^{12} \; (= g_{21} = -g g^{21}) \quad , \quad g_{22} = g g^{11} \tag{3.200}$$

The surface forms a subspace with the coordinates y^1 and y^2 only, and metric tensor matrices given by the first two rows and columns of (3.198) and (3.199), so that the value of g for the subspace is the same as that for the three-dimensional space, that is,

$$g = g_{11}g_{22} - g_{12}g_{21}$$

For the elementary triangle shown in Fig. 3.17,

$$\underline{QR} = \underline{PR} - \underline{PQ}$$

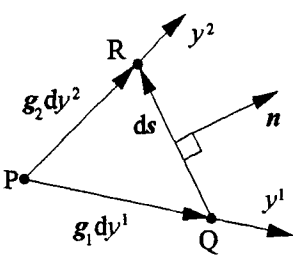


Figure 3.17 A membrane triangle.

†Sternberg, E. (1960) On some recent developments in the linear theory of elasticity. *Structural Mechanics*, (Goodier and Hoff eds.) Pergamon Press

or $\mathrm{d}\boldsymbol{s} = \boldsymbol{g}_2\mathrm{d}y^2 - \boldsymbol{g}_1\mathrm{d}y^1$ so that

$$\mathrm{d}s^2 = g_{22}\mathrm{d}y^2\mathrm{d}y^2 - (g_{12}+g_{21})\mathrm{d}y^1\mathrm{d}y^2 + g_{11}\mathrm{d}y^1\mathrm{d}y^1 \tag{3.201}$$

If $\boldsymbol{n}$ is a unit normal to this line and is given by $\boldsymbol{n} = n_1\boldsymbol{g}^1 + n_2\boldsymbol{g}^2$ then

$$\begin{aligned} 1 = \boldsymbol{n}\cdot\boldsymbol{n} &= (n_1)^2 g^{11} + n_1 n_2(g^{12}+g^{21}) + (n_2)^2 g^{22} \\ &= [(n_1)^2 g_{22} - n_1 n_2(g_{12}+g_{21}) + (n_2)^2 g_{11}]/g \end{aligned} \tag{3.202}$$

from (3.200) and

$$0 = \boldsymbol{n}\cdot\mathrm{d}\boldsymbol{s} = -n_1\mathrm{d}y^1 + n_2\mathrm{d}y^2$$

so that from (3.201) and (3.202),

$$\left(\frac{n_1}{\mathrm{d}y^2}\mathrm{d}s\right)^2 = \left(\frac{n_2}{\mathrm{d}y^1}\mathrm{d}s\right)^2 = g$$

and taking $\boldsymbol{n}$ to be in the outwards direction shown,

$$n_1\mathrm{d}s = \sqrt{g}\,\mathrm{d}y^2 \quad , \quad n_2\mathrm{d}s = \sqrt{g}\,\mathrm{d}y^1 \tag{3.203}$$

Now because $\boldsymbol{g}_3\cdot\boldsymbol{g}_3$ $(= \boldsymbol{g}^3\cdot\boldsymbol{g}^3) \equiv 1$ then from (3.119),

$$0 = \boldsymbol{g}_3\cdot\boldsymbol{g}_{3,i} = \boldsymbol{g}^3\cdot\boldsymbol{g}_{3,i} = \Gamma^3_{3i} \quad (= \Gamma^3_{i3})$$

and in terms of the b_{IJ} notation of Green and Zerna (1968),

$$b_{IJ} = b_{JI} \equiv \Gamma^3_{IJ} = \boldsymbol{g}^3\cdot\boldsymbol{g}_{I,J} \quad (I,J = 1 \text{ to } 2)$$

3.9.1 Membrane stresses

The membrane forces $\boldsymbol{F}(1)$ and $\boldsymbol{F}(2)$ shown in Fig. 3.18 are taken to act only in the local plane of the membrane. The analysis of an element of the membrane is then a quasi-two-dimensional version of the analysis of a three-dimensional continuum carried out earlier. Suppose that the element has a surface area $\mathrm{d}S$ as given by (3.66), which is subject to a loading $\boldsymbol{p}$ per unit area and that its density is ρ per unit area. Then the equivalent of (3.143) is

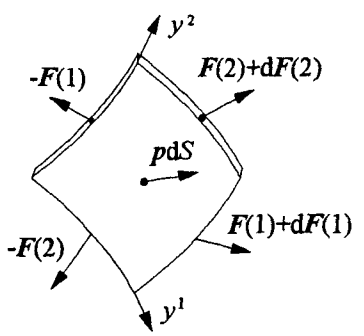


Figure 3.18 Equilibrium of a membrane element.

$$\mathrm{d}\boldsymbol{F}(1) + \mathrm{d}\boldsymbol{F}(2) + \boldsymbol{p}\mathrm{d}S = \rho\ddot{\boldsymbol{u}}\mathrm{d}S \tag{3.204}$$

where $\ddot{\boldsymbol{u}}$ is the local acceleration vector of the surface.

If we define

$$\boldsymbol{F}(1) = \boldsymbol{N}_1\mathrm{d}y^2 \quad , \quad \boldsymbol{F}(2) = \boldsymbol{N}_2\mathrm{d}y^1 \tag{3.205}$$

where

$$\boldsymbol{N}_I = \sqrt{g}\,N^{IJ}\boldsymbol{g}_J \qquad (I,J = 1 \text{ to } 2) \tag{3.206}$$

then, taking only terms of second-order smallness, the equivalent of the moment equation (3.141) is

$$\boldsymbol{0} = \boldsymbol{g}_1\mathrm{d}y^1\times\boldsymbol{F}(1) + \boldsymbol{g}_2\mathrm{d}y^2\times\boldsymbol{F}(2) = \sqrt{g}\,(N^{12}-N^{21})\mathrm{d}y^1\mathrm{d}y^2\boldsymbol{g}^3$$

or

$$N^{12} = N^{21} \tag{3.207}$$

Suppose that a membrane force $\boldsymbol{F}ds$ acts on the edge QR of the elementary triangle shown in Fig. 3.17. This must be balanced by the other forces acting, as shown in Fig. 3.19, ignoring smaller order effects as before. The equivalent of (3.137) is given by

$$\boldsymbol{F}(1) + \boldsymbol{F}(2) = \boldsymbol{F}ds \qquad \textbf{(3.208)}$$

If $\boldsymbol{F} = f^J \boldsymbol{g}_J$ $(J = 1 \text{ to } 2)$ then from (3.203), (3.205), (3.206) and (3.208),

$$N^{IJ} n_I = f^J \quad (I,J = 1 \text{ to } 2) \qquad \textbf{(3.209)}$$

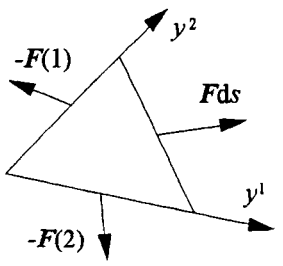


Figure 3.19 Edge equilibrium.

Because n_I and f^J are first-order tensors in the $y^3 = 0$ subspace, N^{IJ} is a doubly-contravariant tensor in this subspace. From (3.205) and (3.206),

$$d\boldsymbol{F}(\alpha) = \boldsymbol{N}_{\alpha,\alpha} dy^1 dy^2 = (\sqrt{g} N^{\alpha J} \boldsymbol{g}_J)_{,\alpha} dy^1 dy^2 \qquad (\alpha = 1 \text{ or } 2)$$

and from (3.67), $\quad dS = \sqrt{g}\, dy^1 dy^2.$

Let $\quad \boldsymbol{p} = p^i \boldsymbol{g}_i$, $\ddot{\boldsymbol{u}} = \ddot{u}^i \boldsymbol{g}_i$. Then (3.204) becomes

$$(\sqrt{g} N^{IJ} \boldsymbol{g}_J)_{,I} + \sqrt{g} p^i \boldsymbol{g}_i = \rho \sqrt{g} \ddot{u}^i \boldsymbol{g}_i \qquad (I,J = 1 \text{ to } 2,\ i = 1 \text{ to } 3)$$

Taking the dot product of this equation with $\boldsymbol{g}^j$ and using (3.112), (3.119) and (3.122) gives

$$N^{IK}_{|I} + p^K = \rho \ddot{u}^K \qquad (j = K) \qquad \textbf{(3.210)}$$

$$N^{IJ} \Gamma^3_{IJ} + p^3 = \rho \ddot{u}^3 \qquad (j = 3) \qquad \textbf{(3.211)}$$

where summation takes place over the repeated indices I and J, the indices I, J and K take the values 1 and 2 and covariant differentiation is with respect to the subspace only.

Example 3.7 A toroidal membrane under uniform pressure

The metric equation for the toroidal coordinates y^1, y^2 and y^3, given by θ, ϕ and r respectively, as shown in Fig. 3.20 is

$$ds^2 = r^2 d\theta^2 + (b + r\cos\theta)^2 d\phi^2 + dr^2$$

$(b > r)$. Then the coordinate system is orthogonal with the tensor coefficients

$$1/g^{11} = g_{11} = r^2,$$
$$1/g^{22} = g_{22} = (b + r\cos\theta)^2, \qquad \textbf{(3.212)}$$
$$g^{33} = g_{33} = 1 \ , \ \sqrt{g} = r(b + r\cos\theta).$$

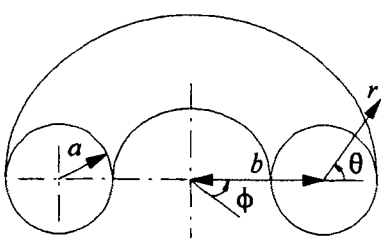


Figure 3.20 Toroidal coordinates.

and the Christoffel symbols

$$\Gamma^1_{22} = \frac{1}{r}\sin\theta(b + r\cos\theta) \ , \ \Gamma^2_{12} = \Gamma^2_{21} = \frac{-r\sin\theta}{b + r\cos\theta} \ , \ \Gamma^1_{31} = \Gamma^1_{13} = \frac{1}{r},$$
$$\Gamma^2_{23} = \Gamma^2_{32} = \frac{\cos\theta}{b + r\cos\theta} \ , \ b_{11} = \Gamma^3_{11} = -r \ , \ b_{22} = \Gamma^3_{22} = -\cos\theta(b + r\cos\theta) \qquad \textbf{(3.213)}$$

Consider the problem of a toroidal membrane of mean radius b and cross-sectional radius a. The Christoffel symbols then take values on the surface given by substituting $r=a$ in the above expressions. Suppose that the torus forms a complete ring and is subject to an internal pressure p. Then p^1 and p^2 are zero and p^3 is p. In the static case ($\ddot{u}=0$) examined here, (3.210) and (3.211) yield the equations

$$N^{11}_{|1} + N^{21}_{|2} = N^{11}{}_{,1}\{+N^{21}{}_{,2}\} - N^{11}\frac{a\sin\theta}{b+a\cos\theta} + N^{22}\frac{1}{a}\sin\theta(b+a\cos\theta) = 0 \quad \textbf{(3.214)}$$

$$N^{12}_{|1} + N^{22}_{|2} = N^{12}{}_{,1} - 3N^{12}\frac{a\sin\theta}{b+a\cos\theta}\{+N^{22}{}_{,2}\} = 0 \quad \textbf{(3.215)}$$

$$-aN^{11} - \cos\theta(b+a\cos\theta)N^{22} + p = 0 \quad \textbf{(3.216)}$$

Because the problem is rotationally symmetrical about the centre of the torus, the stresses are not functions of ϕ and so the terms in curly brackets in (3.214) and (3.215) disappear. (Note that the symmetry implies that partial differentiation, but not covariant differentiation, is zero with respect to ϕ.) From (3.214) and (3.216),

$$a[N^{11}{}_{,1}(b+a\cos\theta)\cos\theta - N^{11}(b+2a\cos\theta)\sin\theta] + p(b+a\cos\theta)\sin\theta = 0$$

or

$$a\frac{\partial}{\partial\theta}[N^{11}(b+a\cos\theta)\cos\theta] = -p(b+a\cos\theta)\sin\theta$$

giving

$$N^{11} = \frac{1}{a(b+a\cos\theta)}\left(\tfrac{1}{2}p(2b+a\cos\theta) + \frac{C}{\cos\theta}\right)$$

$$N^{22} = \frac{1}{\cos\theta(b+a\cos\theta)^2}\left(\tfrac{1}{2}pa\cos\theta - \frac{C}{\cos\theta}\right)$$

where C is an arbitrary constant. The corresponding physical forces per unit length can be found from (3.205) and (3.206) and are

$$N_\theta = N^{\mathrm{P}}_{11} = \sqrt{(gg_{11}/g_{22})}N^{11} = a^2N^{11}$$

$$N_\phi = N^{\mathrm{P}}_{22} = \sqrt{(gg_{22}/g_{11})}N^{22} = (b+a\cos\theta)^2N^{22}$$

using a common engineering notation for the membrane loads. Sectioning the torus in half, as shown in Fig. 3.21, each section must satisfy the equation

$$2\pi a^2 p = 2\int_0^{2\pi} N_\phi a\,d\theta$$

This condition is satisfied if C is zero, giving the physical components as

$$N_\theta = \frac{pa(2b+a\cos\theta)}{2(b+a\cos\theta)} \quad \textbf{(3.217)}$$

$$N_\phi = \tfrac{1}{2}pa \quad \textbf{(3.218)}$$

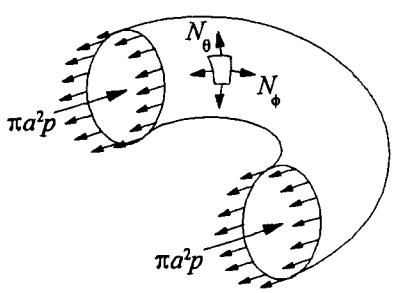


Figure 3.21 Loads on a torus.

Shear forces per unit length N^{P}_{12} can be derived from the independent equation (3.215).

However, this is related to the torsion of an incomplete torus and so N_{12}^{P} can be taken as zero in the above example. Membrane force analysis takes no account of the elastic behaviour of the membrane, which in certain cases can modify the results significantly.

3.10 APPLICATIONS AND WORKED EXAMPLES

3.10.1 Stress distribution around circular notches

(cf. sections 3.1, 3.2, 3.5, 3.6, 3.7, 3.8 and Appendix A1.)

To solve circular notch problems, Neuber (1946) used **ellipsoidal coordinates**, y^j, which can be related to Cartesian coordinates, x_i, by

$$x_1 = \sinh y^1 \cos y^2 \ , \ x_2 = \cosh y^1 \sin y^2 \cos y^3 \ , \ x_3 = \cosh y^1 \sin y^2 \sin y^3 . \quad \textbf{(3.219)}$$

The metric equation is then

$$\begin{aligned} ds^2 &= dx_1^2 + dx_2^2 + dx_3^2 \\ &= (\sinh^2 y^1 + \cos^2 y^2)[(dy^1)^2 + (dy^2)^2] + (\cosh y^1 \sin y^2 dy^3)^2 \end{aligned} \quad \textbf{(3.220)}$$

Because there are no terms $dy^i dy^j$ $(i \neq j)$ in this equation, the coordinate system is orthogonal, so that the formulae in Appendix A1 can be used. From (3.220) and (3.78),

$$\begin{aligned} h_1 &= h_2 = \sqrt{(\sinh^2 y^1 + \cos^2 y^2)} \quad (= h \ \text{ say}) \\ h_3 &= \cosh y^1 \sin y^2 \quad (\sqrt{g} = h_1 h_2 h_3) \end{aligned} \quad \textbf{(3.221)}$$

From (A1.9),

$$\nabla^2 \phi = \frac{1}{\sqrt{g}} [\sin y^2 (\phi_{,1} \cosh y^1)_{,1} + \cosh y^1 (\phi_{,2} \sin y^2)_{,2} + \operatorname{sech} y^1 \operatorname{cosec} y^2 (\sinh^2 y^1 + \cos^2 y^2) \phi_{,33}]$$

so that if ϕ is harmonic,

$$\operatorname{sech} y^1 (\phi_{,1} \cosh y^1)_{,1} + \operatorname{cosec} y^2 (\phi_{,2} \sin y^2)_{,2} + (\operatorname{cosec}^2 y^2 - \operatorname{sech}^2 y^1) \phi_{,33} = 0 \quad \textbf{(3.222)}$$

This can be solved by the method of separation of variables, taking

$$\phi = f_1(y^1) f_2(y^2) f_3(y^3) \quad \textbf{(3.223)}$$

Substituting this expression into (3.222) and dividing the resulting equation by $f_1 f_2 f_3$ gives

$$\frac{(f_{1,1} \cosh y^1)_{,1}}{f_1 \cosh y^1} + \frac{(f_{2,2} \sin y^2)_{,2}}{f_2 \sin y^2} + (\operatorname{cosec}^2 y^2 - \operatorname{sech}^2 y^1) \frac{f_{3,33}}{f_3} = 0 \quad \textbf{(3.224)}$$

Only $f_{3,33}/f_3$ is a function of y^3 in the above expression and so it must be a constant, $-n^2$ say, giving

$$f_3 = c_n \sin(n y^3) + d_n \cos(n y^3)$$

where c_n and d_n are constants. Separating the functions of y^1 and y^2 in (3.224) and equating them to $\pm m(m+1)$ gives

$$\frac{(f_{1,1} \cosh y^1)_{,1}}{f_1 \cosh y^1} + n^2 \operatorname{sech}^2 y^1 = m(m+1) = -\frac{(f_{2,2} \sin y^2)_{,2}}{f_2 \sin y^2} + n^2 \operatorname{cosec}^2 y^2 \quad \textbf{(3.225)}$$

Substituting either $\lambda = \cosh y^1$ or $\lambda = \sin y^2$ in (3.225) yields a differential equation for f_1 or f_2 of the form

$$(\lambda^2 - 1)\frac{d^2f}{d\lambda^2} + \left(2\lambda - \frac{1}{\lambda}\right)\frac{df}{d\lambda} + \left(\frac{n^2}{\lambda^2} - m(m+1)\right)f = 0 \qquad \textbf{(3.226)}$$

This has a power series solution

$$f = N_{n,m}(\lambda) = \lambda^m - \frac{(m^2-n^2)}{2(2m-1)}\lambda^{m-2} + \frac{(m^2-n^2)[(m-2)^2-n^2]}{2.4(2m-1)(2m-3)}\lambda^{m-4} - \ldots \qquad \textbf{(3.227)}$$

If m is replaced by $-m-1$ in (3.226), the equation is unchanged. Thus $N_{n,-m-1}(\lambda)$ is also a solution of (3.226).

Three series of particular interest reduce to the more compact forms

$$\begin{aligned} N_{0,1} &= \sqrt{(\lambda^2-1)} \\ N_{0,-1} &= \operatorname{arccot}\sqrt{(\lambda^2-1)} \equiv -i\ln\{\lambda/[1+\sqrt{(1-\lambda^2)}]\} + \tfrac{1}{2}\pi \\ N_{0,-2} &= -3[\sqrt{(\lambda^2-1)}\operatorname{arccot}\sqrt{(\lambda^2-1)} - 1] \end{aligned} \qquad \textbf{(3.228)}$$

and $N_{0,0}$ is unity. A further solution of (3.222) is given by

$$\phi = \ln\cosh y^1 + \ln\sin y^2 \qquad \textbf{(3.229)}$$

If α is defined in terms of the Lamé constants as $(\lambda+2\mu)/(\lambda+\mu)$, the Papkovich-Neuber equation (3.190) becomes

$$\mu u_i = \Phi_i - \frac{1}{2\alpha}[\Psi + r^j\Phi_j]_{|i} \qquad \textbf{(3.230)}$$

The above harmonic functions can be used to give the scalar Ψ. Here, the form

$$\begin{aligned} \Psi &= A[\ln\cosh y^1 + \ln\sin y^2] - iAN_{0,0}(\cosh y^1)N_{0,-1}(\sin y^2) \\ &\quad + \tfrac{1}{2}i\pi A + \tfrac{1}{3}iBN_{0,-2}(\cosh y^1)N_{0,1}(\sin y^2) \\ &= A[\ln\cosh y^1 + \ln(1+\cos y^2)] + B[T\sinh y^1 - 1]\cos y^2 \end{aligned} \qquad \textbf{(3.231)}$$

where $T = \operatorname{arccot}(\sinh y^1)$

will be used. If Φ_i^C are the elements Φ_i with respect to a Cartesian coordinate system, then from (3.191) each of them must be harmonic. The form

$$\Phi_1^C = CN_{0,-1}(\cosh y^1)N_{0,0}(\sin y^2) = CT \quad , \quad \Phi_2^C = \Phi_3^C = 0 \qquad \textbf{(3.232)}$$

will be used here. From (3.18), the elements of Φ_i with respect to the y^i coordinate system

are $\Phi_i = \dfrac{\partial x_j}{\partial y^i}\Phi_j^C$ so that from (3.219),

$$\Phi_1 = \cosh y^1 \cos y^2 CT \quad , \quad \Phi_2 = -\sinh y^1 \sin y^2 CT \quad , \quad \Phi_3 = 0.$$

Because $r^j\Phi_j$ is a scalar,

$$r^j\Phi_j = x_i\Phi_i^C = \sinh y^1 \cos y^2 CT$$

Then (3.230) gives

$$\mu u_1 = \cosh y^1 \cos y^2 CT - \frac{1}{2\alpha}[A\tanh y^1 + (B+C)(T\cosh y^1 - \tanh y^1)\cos y^2]$$
$$\mu u_2 = -\sinh y^1 \sin y^2 CT + \frac{1}{2\alpha}\left(\frac{A\sin y^2}{1+\cos y^2} + (B+C)T\sinh y^1 \sin y^2 - B\sin y^2\right) \quad (3.233)$$

using the relationship $\partial T/\partial y^1 = -\operatorname{sech} y^1$.

The non-zero Christoffel symbols can be found from (A1.2) to (A1.4) and are

$$\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^2_{21} = -\Gamma^1_{22} = \frac{1}{h^2}\sinh y^1 \cosh y^1 \ , \ \Gamma^1_{33} = -\frac{1}{h^2}\sin^2 y^2 \sinh y^1 \cosh y^1,$$
$$\Gamma^2_{22} = \Gamma^1_{12} = \Gamma^1_{21} = -\Gamma^2_{11} = -\frac{1}{h^2}\sin y^2 \cos y^2 \ , \ \Gamma^2_{33} = -\frac{1}{h^2}\cosh^2 y^1 \sin y^2 \cos y^2, \quad (3.234)$$
$$\Gamma^3_{13} = \Gamma^3_{31} = \tanh y^1 \ , \ \Gamma^3_{23} = \Gamma^3_{32} = \cot y^2.$$

The stresses induced can now be determined from (3.158) using (3.107) to evaluate $u_{i|j}$. After simplification, these take the form

$$h^4\alpha\sigma^{11} = A\tanh^2 y^1 + B\operatorname{sech}^2 y^1 \cos y^2 + C\cos y^2(\operatorname{sech}^2 y^1 - 2 - \alpha)$$
$$- \frac{1}{h^2}\cos y^2(A - B - C\cos^2 y^2)$$
$$h^4\alpha\sigma^{22} = -\frac{A\cos y^2}{1+\cos y^2} + C(\alpha - 1)\cos y^2 + \frac{1}{h^2}(A - B - C\cos^2 y^2)\cos y^2$$
$$h^2\alpha\cosh^2 y^1 \sin^2 y^2 \sigma^{33} = A\left(\frac{\cos y^2}{1+\cos y^2} - \tanh^2 y^1\right) \quad (3.235)$$
$$- B\operatorname{sech}^2 y^1 \cos y^2 + C(\alpha - 1 - \operatorname{sech}^2 y^1)\cos y^2$$
$$h^4\alpha\sigma^{12} = \tanh y^1 \sin y^2\left(\frac{-A}{1+\cos y^2} + C(\alpha - 1) - \frac{1}{h^2}(A - B - C\cos^2 y^2)\right)$$
$$\sigma^{13} = \sigma^{23} = 0$$

The corresponding physical components of these stresses can be found from (3.98) and (3.221).

A geometric interpretation of the coordinate system is shown in Fig. 3.22, where

$$r = \mathrm{OQ} = \cosh y^1 \sin y^2$$
$$x = x_1 = \mathrm{QP} = \sinh y^1 \cos y^2 \quad (3.236)$$

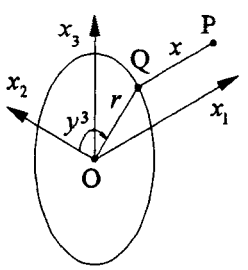


Figure 3.22 Coordinate systems.

Because the above solution is not a function of y^3, it is rotationally symmetric about the x_1 axis. The y^1 coordinate surfaces, on which y^1 is constant, or

$$\sinh y^1 = a \ , \ \cosh y^1 = b \quad \text{(say)}$$

are such that

$$x^2/a^2 + r^2/b^2 = 1$$

generating ellipsoids about the x_1 axis.

The y^2 coordinate surfaces, on which y^2 is constant, or $\cos y^2 = c$, $\sin y^2 = d$ (say) are such that

$$r^2/d^2 - x^2/c^2 = 1$$

generating hyperboloids about the x_1 axis. Fig. 3.23 shows the intersection of these surfaces with the r-x plane. The central ellipse is for y^1 equal to approximately 0.25, growing to approximately 1.25 at the outer ellipse. The hyperbolas are for y^2 equal to approximately -1.2 at the bottom to +1.2 at the top. (x and r lie in the range ±2.5.)

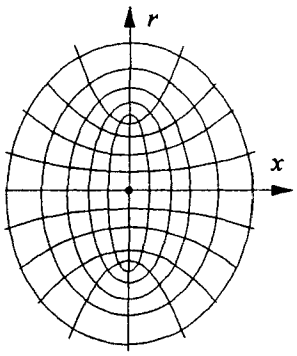


Figure 3.23 Coordinate lines.

The above solution can be used to find the stresses in a hyperboloid under axial loading. At the free surfaces of the hyperboloid, $y^2 = y$ say, the stresses σ^{21}, σ^{22} and σ^{23} must be zero. From (3.235), the last condition is satisfied immediately and the other two are satisfied by taking

$$A = C(\alpha - 1)(1 + \cos y), \quad B = A - C\cos^2 y \quad \textbf{(3.237)}$$

The axial force P exerted can be found by integrating over the coordinate surface $y^1=0$ within its intersection with the hyperboloid. This corresponds to the waist of the hyperboloid, where $x_1 = 0$. Using the equivalent of (3.136) for this surface,

$$\begin{aligned} P &= \int_0^{2\pi}\int_0^{y} \sigma^{11}|g_1|\sqrt{g}\,dy^2 dy^3 \\ &= \frac{1}{\alpha}\int_0^{2\pi}\int_0^{y} [B - C(1+\alpha) - (A - B - C\cos^2 y^2)\sec^2 y^2]\sin y^2\, dy^2 dy^3 \qquad \textbf{(3.238)} \\ &= \frac{2\pi}{\alpha} C(1 - \cos y)[(\alpha - 2)\cos y - 1 - \cos^2 y] \end{aligned}$$

from (3.235), (3.26), (3.77), (3.221) and (3.237). From (3.236), the radius r at the waist is $\sin y$ so that the mean axial stress, σ, is

$$\sigma = P/(\pi \sin^2 y) = 2C[(\alpha - 2)\cos y - 1 - \cos^2 y]/\alpha(1 + \cos y) \qquad \textbf{(3.239)}$$

The other stresses can be expressed in terms of σ using (3.235), (3.237) and (3.239). The physical stress σ^P_{11} at the edge of the waist ($y^1 = 0, y^2 = y$) can be much higher than the mean stress. The ratio of the two gives the **stress concentration factor**,

$$\frac{\sigma^P_{11}}{\sigma} = \frac{(1 + \cos y)[2 + (1 - \alpha)\cos y + \cos^2 y]}{2\cos y[1 + (2 - \alpha)\cos y + \cos^2 y]} \qquad \textbf{(3.240)}$$

using (3.235) and (3.98). This can be related to the radius of curvature, ρ, of the hyperbola at this point which is given by

$$\rho = \cos y \cot y$$

Notch curvature is defined as half the waist width divided by the curvature, ρ, of the waist at this point, i.e. $\tan^2 y$. Stress concentration factors can then be determined for various circumferential notch curvatures using (3.240). These rise from unity at zero notch curvature to over six at a notch curvature of forty [see Neuber (1946) Fig. 48]. Neuber also examines the bending and shear of such hyperboloids and the stresses around ellipsoidal holes, using similar solutions to those above. Some of the results are given by

Young (1989).

3.10.2 Velocity and acceleration (cf. 3.5.1, 3.6.1 and Appendix A1.2)

The rate of change of a vector $\boldsymbol{v}$ with time in a moving coordinate system is given by

$$\dot{\boldsymbol{v}} = \frac{\partial \boldsymbol{v}}{\partial y^j}\frac{\mathrm{d}y^j}{\mathrm{d}t} = (v^i{}_{,j}\boldsymbol{g}_i + v^i\Gamma^k_{ij}\boldsymbol{g}_k)\frac{\mathrm{d}y^j}{\mathrm{d}t} = \frac{\mathrm{d}v^i}{\mathrm{d}t}\boldsymbol{g}_i + v^i\frac{\mathrm{d}y^j}{\mathrm{d}t}\Gamma^k_{ij}\boldsymbol{g}_k \qquad \textbf{(3.241)}$$

from (3.101) and (3.102). For example, if $\boldsymbol{v}$ is the position vector $\boldsymbol{r}$ expressed in terms of spherical polar coordinates (cf. Appendix A1.2), then $\boldsymbol{v} = \boldsymbol{r} = r\boldsymbol{g}_r$ and the velocity is given by $\dot{\boldsymbol{r}} = \dot{r}\boldsymbol{g}_r + \dot{\theta}\boldsymbol{g}_\theta + \dot{\phi}\boldsymbol{g}_\phi$, the first term arising from the first expression on the right-hand side of (3.241)and the other two from the second expression.

To find the acceleration, (3.241) is now applied to $\dot{\boldsymbol{r}}$ where now

$$v^r = \dot{r}\ ,\quad v^\theta = \dot{\theta}\ ,\quad v^\phi = \dot{\phi}$$

giving
$$\ddot{\boldsymbol{r}} = (\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta)\boldsymbol{g}_r + (\ddot{\theta} + 2\dot{r}\dot{\theta}/r - \dot{\phi}^2\sin\theta\cos\theta)\boldsymbol{g}_\theta + (\ddot{\phi} + 2\dot{r}\dot{\phi}/r + 2\dot{\theta}\dot{\phi}\cot\theta)\boldsymbol{g}_\phi \qquad \textbf{(3.242)}$$

or in terms of physical components,

$$\ddot{\boldsymbol{r}} = (\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta)\hat{\boldsymbol{g}}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)\hat{\boldsymbol{g}}_\theta + [(r\ddot{\phi} + 2\dot{r}\dot{\phi})\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta]\hat{\boldsymbol{g}}_\phi \qquad \textbf{(3.243)}$$

3.10.3 Generalised plane stress (cf. 3.7.3 and 3.8.2)

Suppose that two-dimensional curvilinear coordinates are used to describe a flat plane. From (3.158), (2.96) becomes

$$\sigma^{IJ} \approx [\lambda^* g^{IJ}g^{KL} + \mu(g^{IK}g^{JL} + g^{IL}g^{JK})]u_{K|L} \quad (I,J,K,L = 1 \text{ to } 2) \qquad \textbf{(3.244)}$$

where λ^* is given by (2.97). From (3.144), (2.99) becomes

$$\sigma^{IJ}_{|I} + \rho f^J = \rho\ddot{u}^J \qquad \textbf{(3.245)}$$

The two-dimensional form of Navier's equation (3.160)

$$g^{JK}[\mu u_{I|JK} + (\lambda^* + \mu)u_{J|KI}] + \rho f_I \approx \rho\ddot{u}_I \qquad \textbf{(3.246)}$$

can be deduced from (3.244) and (3.245). In solving plane stress problems, the above equations are treated as equalities rather than approximations.

3.10.4 A spinning disc (cf. 3.6.2, 3.6.4, 3.7.1 and Appendix A1)

The problem of a thin, flat disc will be examined relative to plane polar coordinates (r, θ). The disc and its coordinate system are taken to spin about an axis normal to the plane of the disc, through $r = 0$, at an angular velocity $\dot{\theta}$ and an angular acceleration $\ddot{\theta}$. A 'steady state' solution will be sought in which the points on the disc appear to be stationary relative to the coordinate system. The acceleration of a material point at a radius r is then

$$\ddot{\boldsymbol{u}} = -\dot{\theta}^2 r\boldsymbol{g}^1 + \ddot{\theta}r^2\boldsymbol{g}^2$$

(cf. section 3.10.2). The disc will be taken to be fixed to a rigid shaft at $r = a$ and to have a free edge at $r = b$. The boundary conditions are then

$$u_1 = u_2 = 0 \qquad (r = a) \tag{3.247}$$

and from (3.139),

$$\sigma^{11} = \sigma^{12} = 0 \qquad (r = b) \tag{3.248}$$

Because the problem is rotationally symmetrical, all partial derivatives of the parameters with respect to θ are zero, and from (3.107) and section A1.1, the covariant derivatives of u_I are

$$u_{1|1} = u_{1,1}\ ,\ u_{1|2} = -u_2/r\ ,\ u_{2|1} = u_{2,1} - u_2/r\ ,\ u_{2|2} = ru_1 \tag{3.249}$$

so that from (3.110),

$$\begin{aligned} &u_{1|11} = u_{1,11}\ ,\ u_{1|12} = u_{1|21} = 2u_2/r^2 - u_{2,1}/r, \\ &u_{1|22} = ru_{1,1} - u_1\ ,\ u_{2|11} = u_{2,11} - 2u_{2,1}/r + 2u_2/r^2, \\ &u_{2|12} = u_{2|21} = ru_{1,1} - u_1\ ,\ u_{2|22} = ru_{2,1} - 2u_2. \end{aligned} \tag{3.250}$$

Substituting these results into (3.246) gives

$$(\lambda^* + 2\mu)(u_{1,11} + u_{1,1}/r - u_1/r^2) = -\rho\dot{\theta}^2 r$$

and

$$\mu(u_{2,11} - u_{2,1}/r) = \rho\ddot{\theta}r^2$$

The form of these equations indicates that solutions in terms of powers of r should be sought. Substituting expressions of the form Cr^m for u_I and solving these equations yields the general forms

$$u_1 = A_1/r + A_2 r - \rho\dot{\theta}^2 r^3/8(\lambda^* + 2\mu) \tag{3.251}$$

$$u_2 = B_1 + B_2 r^2 + \rho\ddot{\theta}r^4/8\mu \tag{3.252}$$

where A_1, A_2, B_1 and B_2 are constants. From (3.244),

$$\begin{aligned} \sigma^{11} &= (\lambda^* + 2\mu)u_{1,1} + \lambda^* u_1/r \\ \sigma^{12} &= \mu(ru_{2,1} - 2u_2)/r^3 \end{aligned}$$

The boundary conditions given by (3.247) and (3.248) can now be used to evaluate the constants in (3.251) and (3.252) giving

$$u_r^{\,P} = u_1 = \frac{\rho\dot{\theta}^2(r^2 - a^2)[b^4(2\lambda^* + 3\mu) - \mu a^2 r^2 - b^2(a^2 + r^2)(\lambda^* + \mu)]}{8r(\lambda^* + 2\mu)[\mu a^2 + (\lambda^* + \mu)b^2]} \tag{3.253}$$

$$ru_\theta^{\,P} = u_2 = \frac{\rho\ddot{\theta}}{8\mu a^2}(r^2 - a^2)(r^2 a^2 - b^4) \tag{3.254}$$

where $u_r^{\,P}$ and $u_\theta^{\,P}$ are the physical components of displacement in the radial and tangential directions respectively.

3.10.5 A gravitating sphere

(cf. 3.5.1, 3.6.2, 3.6.4, 3.8.2 and A.1.2)

The body force per unit mass exerted by a uniform gravitating sphere of radius a on its own material is

$$f = (-\varrho r/a)g^r \tag{3.255}$$

where g is the acceleration due to gravity at its surface and r is the radius from the centre of the sphere to the material point and $\boldsymbol{g}^r$ is the radial contravariant base vector, using spherical polar coordinates. Taking the material to be isotropic, the deformations will also be uniform, so that

$$\boldsymbol{u} = U(r)\boldsymbol{g}^r \tag{3.256}$$

Then from (3.107) the covariant derivatives of $\boldsymbol{u}$ are

$$u_{r|r} = U_{,r} \quad , \quad u_{\theta|\theta} = rU \quad , \quad u_{\phi|\phi} = rU\sin^2\theta \tag{3.257}$$

and from (3.110) the second derivatives are

$$\begin{aligned} &u_{r|rr} = U_{,rr} \quad , \quad u_{r|\theta\theta} = u_{\theta|r\theta} = u_{\theta|\theta r} = rU_{,r} - U \ , \\ &u_{r|\phi\phi} = u_{\phi|r\phi} = u_{\phi|\phi r} = (rU_{,r} - U)\sin^2\theta \ . \end{aligned} \tag{3.258}$$

Then Navier's equation (3.160) gives two identities and

$$(\lambda + 2\mu)[U_{,rr} + 2(rU_{,r} - U)/r^2] - \rho g r/a = 0$$

This can be rearranged to give the form

$$(\lambda + 2\mu)\frac{d}{dr}\left(\frac{1}{r^2}\frac{d}{dr}(r^2U)\right) = \frac{\rho g r}{a}$$

On integrating, this gives

$$U = \frac{\rho g r^3}{10a(\lambda + 2\mu)} + Ar + Br^{-2}$$

As the displacement is zero at the centre of the body, the constant B is zero. The tractions $\sigma^{r\theta}$ and $\sigma^{r\phi}$ on the surface are identically zero, and equating σ^{rr} to zero at $r = a$ gives

$$0 = \sigma^{rr} = (\lambda + 2\mu)U_{,r} + \frac{2\lambda}{r}U = \frac{(5\lambda + 6\mu)\rho g a}{10(\lambda + 2\mu)} + (3\lambda + 2\mu)A$$

from (3.158). Then

$$U = \frac{\rho g r}{10a(\lambda + 2\mu)}\left(r^2 - \frac{(5\lambda + 6\mu)a^2}{(3\lambda + 2\mu)}\right) \tag{3.259}$$

PROBLEMS

Solutions can make use of the results in this chapter and Appendix A1.

(3.1) A vector is given with respect to a Cartesian coordinate system by $\boldsymbol{v} = a\boldsymbol{i}_1 + b\boldsymbol{i}_2 + c\boldsymbol{i}_3$. Find the covariant and contravariant elements v_i and v^i of $\boldsymbol{v}$ with respect to the coordinate system shown in Fig. 3.3. Show that $v_i v^i$ is invariant. (M)

(3.2) Relative to a spherical polar coordinate system, at a point where $r = 2$ and $\theta = 45°$, the elements of a doubly-contravariant tensor $A^{11}, A^{12}, A^{13}; A^{21}, \ldots A^{33}$ are respectively 3, -2, 1; 1, 0, 2; 2, -1, 1. Find the corresponding elements of the doubly-covariant and mixed tensors. (S)

(3.3) For the previous example, evaluate the invariants $A_k^{\ k}$, $A^{ij}A_{ij}$ and $A_j^{\ i}A_i^{\ j}$. (S)

(3.4) The surface of a paraboloid is generated by rotating the curve $z = ar^2$ about the z axis of a cylindrical polar coordinate system. Find the metric equation for the points on this surface in terms of r and θ only and hence the metric tensors for this surface. (S)

(3.5) For the above problem, find the Christoffel symbols and the Riemann-Christoffel tensors. (M)

(3.6) Find the metric equation for the surface of a cone with half-angle α in terms of cylindrical coordinates (θ, z) with the origin at the apex. Find the Christoffel symbols and hence show that this surface is a Euclidean or 'flat' space. (M)

(3.7) Find the derivatives $\boldsymbol{g}_{I,J}$ in terms of the base vectors $\boldsymbol{g}_K$ for the above cone by examining the geometry of the coordinate lines on its developed surface. (A developed surface is one which has been unrolled and laid flat.) Confirm that (3.102) is satisfied. (M)

(3.8) The coordinate system used in problem (3.6) is now extended by introducing a third ordinate, y, which is the perpendicular distance above the surface of a point in space. Find the metric equation for the three-dimensional space and hence the Christoffel symbols Γ_{IJ}^3. (S)

(3.9) For the Schwarzschild metric (A7.10), show that the Christoffel symbols $\Gamma_{r\theta}^{\theta}$, $\Gamma_{\phi\phi}^{\theta}$, $\Gamma_{r\phi}^{\phi}$ and $\Gamma_{\theta\phi}^{\phi}$ are the usual symbols for spherical polar coordinates. Find the remaining non-zero symbols and the Riemann-Christoffel element R_{rtrt}. (L)

(3.10) Show that for orthogonal coordinates $\Gamma_{jk}^{i} = 0 \quad (i \neq j \neq k \neq i)$ and that

$$\Gamma_{\alpha\beta}^{\alpha} = \tfrac{1}{2}(\ln g_{\alpha\alpha})_{,\beta} \quad \text{and} \quad \Gamma_{\beta\beta}^{\alpha} = -\tfrac{1}{2}g^{\alpha\alpha}g_{\beta\beta,\alpha} . \qquad \text{(M)}$$

(3.11) Show that for three-dimensional orthogonal coordinates

$$\nabla^2\phi = \sum_{\alpha=1}^{3}(\phi_{,\alpha}\sqrt{g}/g_{\alpha\alpha})_{,\alpha}/\sqrt{g} \qquad \text{(M)}$$

(3.12) Using tensor notation, show that curl(grad ϕ) and div(curl $\boldsymbol{v}$) are both zero. (M)

(3.13) The general form of (2.106) is $\sigma^{IJ} = \varepsilon^{IK}\varepsilon^{JL}\phi_{|KL}$, where ε^{IK} and ε^{JL} are two-dimensional contravariant permutation tensors [cf. (3.72)]. Show that for plane polar coordinates, the physical components of stress are then given by the expressions

$$\sigma_{rr}^{P} = \phi_{,\theta\theta}/r^2 + \phi_{,r}/r \ , \quad \sigma_{\theta\theta}^{P} = \phi_{,rr} \quad \text{and} \quad \sigma_{r\theta}^{P} = -\phi_{,r\theta}/r + \phi_{,\theta}/r^2 . \qquad \text{(M)}$$

(3.14) A load of intensity p per unit length acts on an elastic half-space $x_2 \geq 0$ on the line $x_1 = x_2 = 0$. With respect to cylindrical polar coordinates such that the x_3 and z axes are coincident and θ is measured from the x_1 axis, this induces stresses

$$\sigma_{11} = -2p\sin\theta\cos^2\theta/\pi r \ , \quad \sigma_{22} = -2p\sin^3\theta/\pi r \ , \quad \sigma_{12} = -2p\sin^2\theta\cos\theta/\pi r .$$

Find the contravariant components of stress with respect to these coordinates and verify that (3.144) is satisfied in the absence of body forces and accelerations. (M)

(3.15) Find the physical components of stress within the gravitating sphere examined in section 3.10.5. (M)

(3.16) An isotropic elastic body is in a spherically-symmetrical state of stress so that relative to spherical polar coordinates, the induced displacements are given by $u_r = u_r(r)$, $u_\theta = u_\phi = 0$. Show that Navier's equation in the form given by (3.187) yields a single differential equation in terms of u_r. (Time can be saved by noting that $g^{jk} u_{j|k}$ is a scalar.) Find the general solution of this differential equation. (M)

(3.17) A thick spherical shell is subject to an internal pressure p at a radius a and has no tractions on its outer surface, of radius b. Use the results of the previous question to find the stresses induced in the shell. (M)

(3.18) Show that $\Phi_r = \Phi_\theta = 0$, $\Phi_z = Q/R$ and $\Psi = [\mu Q \ln(R+z)]/(\lambda+\mu)$ where $R^2 = r^2 + z^2$ and $Q = P(\lambda+2\mu)/2\pi(\lambda+\mu)$ are suitable functions for the Papkovich-Neuber equation (3.190) relative to cylindrical polar coordinates. Find the displacements given by these functions. (M)

(3.19) For the above problem, find the stresses induced and the traction p^z on a surface where R is constant. Hence show that the stress distribution is that for an elastic half-space $z \geq 0$ with a point force P at the origin. (L)

(3.20) A conical membrane contains a liquid with a density ρ. The apex of the cone is at a depth h below the surface of the liquid and the cone is supported above this surface. In terms of the coordinate system used in problems (3.6) to (3.8), find the membrane stresses induced below the surface, taking the acceleration due to gravity to be $\mathbf{g}$. (M)

(3.21) Find the contravariant components of acceleration of a point relative to cylindrical polar coordinates (r, θ, z) in terms of the rate of change of these coordinates. (M)

4

Large deformation theory

The use of large deformation theory becomes necessary when the geometry of a body in its deformed state is significantly different to that before deformation, so that the usual approximations of linear theory no longer apply. This may be because the deformations are large or because a more precise analysis of the body is required.

Stability problems lie in the second category. Instability may occur after large or small deformations but, as will be seen later in this chapter, the governing equations can only be derived correctly by means of large deformation theory. An appropriate approximation to these exact equations can be made on the assumption that the initial deformations are small (although not infinitesimal). However, this does not yield the same equations as those derived from a generalisation of small-deflection theory. None the less, both sets of equations can give similar results for some common problems.

Because the form of the body changes significantly with deformation, it is necessary to choose a configuration of the body in which its geometry is defined. This is known as the **reference configuration**. This need not be one which actually occurs during the deformation being examined, but in this chapter it will be taken as the initial unstressed and unstrained state of the body. In conjunction with this, the notation of Green and Zerna (1968) will be used. This means that lower case letters are used where upper case letters are employed in some other texts and vice versa.

4.1 LAGRANGEAN AND EULERIAN STRAIN

If the reference configuration of a body is taken to be that of its original undeformed state, then the coordinates (y^1, y^2, y^3) of a material point are still used to define the point, even after deformation and displacement of the body. Such coordinates are known as **material coordinates**†. It is possible to think of these coordinates as deforming and displacing with

†Also known as 'convected' or 'intrinsic' coordinates.

the body, or alternatively as viewing the behaviour of the body as it is mapped onto the initial state. The advantage of this approach is that the surface of the body is always defined by the surface coordinates in the initial state. Strain as measured relative to such a coordinate system is known as **Lagrangean strain.** It is also possible to express the behaviour of the body relative to coordinates (Y^1, Y^2, Y^3) which are permanently fixed in space. Strain as measured relative to these coordinates is known as **Eulerian strain**.

A simple illustration of this is shown in Fig. 4.1. Here, the material coordinates in the unstrained state are the usual Cartesian coordinates (x, y, z). Thus the distance apart of the two adjacent points p and p′ shown in Fig. 4.1a, ds, is also dx. If the body is now given a uniform stretch in the x direction, these two points become the points P and P′, dS apart, shown in Figure 4.1b, where it will be taken that

$$dS = \lambda\, ds \qquad (4.1)$$

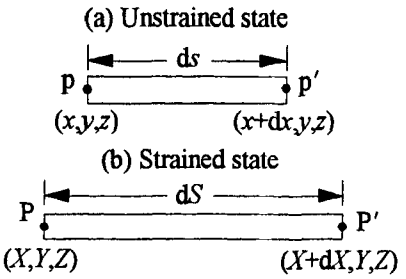


Figure 4.1 Definitions of strain.

Although the *material* coordinates of P and P′ are the same as those of p and p′,the distance dS is not dx but λdx. Let the fixed coordinates (Y^1, Y^2, Y^3) be a Cartesian system (X, Y, Z) so that the coordinates of P and P′ are as shown in Fig. 4.1b. Then dS is dX but ds is dX/λ.

In terms of the initial reference state, $\boldsymbol{u} = (\lambda - 1)x\boldsymbol{i} = u_1\boldsymbol{i}$ which gives

$$u_{1|1} = u_{1,1} = (\lambda - 1) \qquad (4.2)$$

and so

$$dS = \lambda\, ds = \lambda\, dx = (1 + u_{1|1})dx$$

The **Lagrangean strain** γ_{ij} is defined in terms of the general material coordinates y^i by the equation

$$dS^2 - ds^2 = 2\gamma_{ij}dy^i dy^j \qquad (4.3)$$

and so in the above particular case $dS^2 - ds^2 = (\lambda^2 - 1)dx\,dx$

From (4.2) and (4.3), this gives $\gamma_{11} = \tfrac{1}{2}(\lambda^2 - 1) = u_{1|1} + \tfrac{1}{2}(u_{1|1})^2$ and all other strains are zero.

Taking the (X, Y, Z) coordinates to have the same origin and orientation as the (x, y, z) system, the displacement field can also be written as

$$\boldsymbol{u} = (1 - 1/\lambda)X\boldsymbol{i} = U_1\boldsymbol{i} \qquad (X = \lambda x)$$

where U_1 is the only covariant element of the displacement tensor relative to (X, Y, Z). Then

$$U_{1\|1} = (1 - 1/\lambda)$$

the double line indicating covariant differentiation with respect to (X, Y, Z). From the above,

$$ds = dS/\lambda = dX/\lambda = (1 - U_{1\|1})dX \qquad (4.4)$$

The **Eulerian strain** η_{ij} is defined in terms of the fixed coordinates (Y^1, Y^2, Y^3) by the relationship

$$dS^2 - ds^2 = 2\eta_{ij}dY^i dY^j \tag{4.5}$$

and in the particular case considered here,

$$dS^2 - ds^2 = (1 - 1/\lambda^2)dXdX$$

so that from (4.4) and (4.5),

$$\eta_{11} = \tfrac{1}{2}(1 - 1/\lambda^2) = U_{1\|1} - \tfrac{1}{2}(U_{1\|1})^2$$

and all other strains are zero.

Note that Lagrangean and Eulerian strains are different not only in form but in value. However, if the displacement rate is sufficiently small for the quadratic terms to be ignored in comparison with the linear terms, the two become equal to one another and to the linear strain of small-deflection theory. For the particular case examined above, the strains are then

$$e_{11} = \dot{\lambda} - 1 \approx \gamma_{11} \approx \eta_{11} .$$

Because the definitions of strain are related to changes of material length, their values are unaltered by rigid-body motions of the medium.

4.2 MATERIAL COORDINATES

In this chapter, the material coordinates (y^1, y^2, y^3) described in the previous section will be used. This is known as the Lagrangean or material description. As already pointed out, in this system, the coordinates of a material point do not change with deformation of the body. Then let the difference in coordinates between two adjacent material points p and p′ be dy^i before deformation. If, after deformation, these become the points P and P′ as shown in Fig. 4.2, the difference in their coordinates is still dy^i. However, the base vectors, and hence the metric tensors of the coordinate system, will change as the body deforms. Wherever appropriate, lower-case letters will be used to refer to the undeformed state and capitals to refer to the deformed state. Thus the base vectors $\boldsymbol{g}_i$ and $\boldsymbol{g}^j$ and the metric tensors g_{ij} and g^{ij} refer to the material coordinates in the undeformed state and these become $\boldsymbol{G}_i$, $\boldsymbol{G}^j$, G_{ij} and G^{ij} respectively after deformation. Once the necessary relationships have been established, all the tensor equations can be written relative to the coordinate system of the undeformed (reference) state.

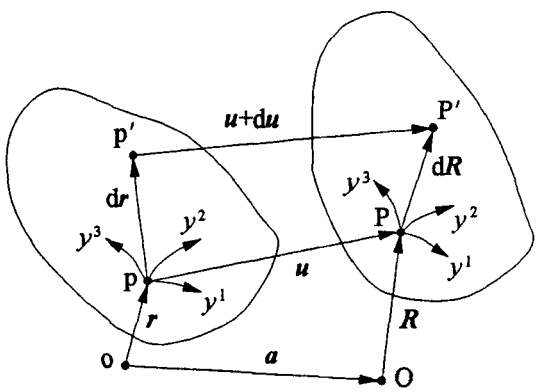


Figure 4.2 Relationships for material coordinates.

In Fig. 4.2, the position of a material point p is given by the vector $\boldsymbol{r}$ from the origin o in the undeformed state. This origin may or may not lie within the body. Because

the coordinate system deforms with the body, this origin may move to some new point O after deformation, where $\underline{oO}$ is given by the vector $\boldsymbol{a}$. The material point at p will have moved to the point P given by the position vector $\boldsymbol{R}$ relative to the origin O. Then the displacement $\boldsymbol{u}$ of the material point is given by

$$\boldsymbol{u} = \boldsymbol{R} - \boldsymbol{r} + \boldsymbol{a} \tag{4.6}$$

The adjacent material point, originally at p′, $d\boldsymbol{r}$ from p, moves through a displacement $\boldsymbol{u}+d\boldsymbol{u}$ to P′, which is $d\boldsymbol{R}$ from P, so that

$$d\boldsymbol{u} = d\boldsymbol{R} - d\boldsymbol{r} \tag{4.7}$$

The relative position vectors of the adjacent points can be expressed in terms of the base vectors of the initial and deformed states by

$$d\boldsymbol{r} = dy^i \boldsymbol{g}_i \quad , \quad d\boldsymbol{R} = dy^i \boldsymbol{G}_i \tag{4.8}$$

so that

$$d\boldsymbol{u} = (\boldsymbol{G}_i - \boldsymbol{g}_i)dy^i \tag{4.9}$$

The change in the square of the length of the line joining these adjacent material points is then given by

$$\begin{aligned} d\boldsymbol{R}\cdot d\boldsymbol{R} - d\boldsymbol{r}\cdot d\boldsymbol{r} &= (\boldsymbol{G}_i\cdot\boldsymbol{G}_j - \boldsymbol{g}_i\cdot\boldsymbol{g}_j)dy^i dy^j = (G_{ij} - g_{ij})dy^i dy^j \\ &= 2\gamma_{ij}dy^i dy^j \qquad \text{from (4.3)}, \\ \text{so that } \gamma_{ij} &= \tfrac{1}{2}(G_{ij} - g_{ij}). \end{aligned} \tag{4.10}$$

The displacement $\boldsymbol{u}$ can be expressed in terms of the original coordinate system, or the coordinate system in the deformed state (which may be regarded as coincident with the fixed coordinate system used to define the Eulerian strain). Then

$$\boldsymbol{u} = u_i \boldsymbol{g}^i = U_i \boldsymbol{G}^i \tag{4.11}$$

From (3.106) {using the definition in (3.101)} the increment in displacement is given by

$$d\boldsymbol{u} = u_{i|j}dy^j \boldsymbol{g}^i = U_{i\|j}dy^j \boldsymbol{G}^i \tag{4.12}$$

where the single bar is used to indicate covariant differentiation with respect to the coordinate system in its initial configuration and the double bar indicates covariant differentiation relative to its deformed configuration.

The exact strain can now be expressed in terms of displacement, using (4.7) and (4.12). With reference to the coordinate system in its undeformed state,

$$\begin{aligned} d\boldsymbol{R}\cdot d\boldsymbol{R} - d\boldsymbol{r}\cdot d\boldsymbol{r} &= (d\boldsymbol{r} + d\boldsymbol{u})\cdot(d\boldsymbol{r} + d\boldsymbol{u}) - d\boldsymbol{r}\cdot d\boldsymbol{r} \\ &= 2d\boldsymbol{r}\cdot d\boldsymbol{u} + d\boldsymbol{u}\cdot d\boldsymbol{u} \\ &= 2dy^j \boldsymbol{g}_j\cdot u_{k|i}dy^i \boldsymbol{g}^k + u_{l|j}dy^j \boldsymbol{g}^l\cdot u_{k|i}dy^i \boldsymbol{g}^k \\ &= (2u_{k|i}\delta^k_j + u_{l|j}u_{k|i}g^{lk})dy^i dy^j \\ &= (u_{i|j} + u_{j|i} + u_{l|j}u_{k|i}g^{lk})dy^i dy^j \end{aligned} \tag{4.13}$$

The exact strain given by (4.10) is thus

$$\gamma_{ij} = \tfrac{1}{2}(u_{i|j} + u_{j|i} + u_{l|j}u_{k|i}g^{lk}) \tag{4.14}$$

which is the Lagrangean form of the strain tensor. Likewise, with reference to the coordinate system in the deformed state,

$$\begin{aligned} d\boldsymbol{R}\cdot d\boldsymbol{R} - d\boldsymbol{r}\cdot d\boldsymbol{r} &= d\boldsymbol{R}\cdot d\boldsymbol{R} - (d\boldsymbol{R} - d\boldsymbol{u})\cdot(d\boldsymbol{R} - d\boldsymbol{u}) \\ &= 2d\boldsymbol{R}\cdot d\boldsymbol{u} - d\boldsymbol{u}\cdot d\boldsymbol{u} \\ &= (U_{i\|j} + U_{j\|i} - U_{l\|j}U_{k\|i}G^{lk})\,dy^i dy^j \end{aligned} \tag{4.13a}$$

so that the exact strain can also be expressed as

$$\eta_{ij} = \tfrac{1}{2}(U_{i\|j} + U_{j\|i} - U_{l\|j}U_{k\|i}G^{lk}) \tag{4.15}$$

which is the Eulerian form of the strain tensor†. From (3.28), the metric tensors are symmetrical, so that it follows from (4.14) and (4.15) that the strain tensors are symmetrical. Note that from the above equations it follows that g_{ij}, G_{ij}, γ_{ij} and η_{ij} are doubly-covariant tensors with respect to the material coordinates in both configurations. When the displacements and their derivatives are small, the last terms in (4.14) and (4.15) are of second-order smallness and can be ignored. The forms of γ_{ij} and η_{ij} are then the same as that of e_{ij} as given by (3.133).

4.2.1 Compatibility

The vanishing of γ_{ij} locally is sufficient to ensure that this neighbourhood of the body has moved as a rigid body. Where the body does deform, the material coordinate system will continue to lie in a Euclidean three-dimensional space so that the Riemann-Christoffel tensor elements are always zero. Because there are six independent elements of this tensor in three dimensions, this gives six independent compatibility conditions.

4.2.2 The relationship between base vectors

From (4.7),

$$\boldsymbol{u}_{,i} = \boldsymbol{G}_i - \boldsymbol{g}_i$$

and from (3.103),

$$\boldsymbol{u}_{,i} = u^j_{\,|i}\,\boldsymbol{g}_j$$

so that

$$\boldsymbol{G}_i = \boldsymbol{g}_i + u^j_{\,|i}\boldsymbol{g}_j = (\delta^j_i + u^j_{\,|i})\boldsymbol{g}_j \tag{4.16}$$

4.3 THE STATE OF STRESS

The equations of equilibrium are derived in a similar fashion to those of Chapter 3, using the same elementary volumes of material but allowing for their deformation under load. Thus the points P, R, S and T in Fig. 3.12 which were treated as points in an undeformed body are now correctly taken to be the points in the deformed body shown in Fig. 4.3. Nevertheless, as will be seen later, it is convenient to relate the forces acting to the undeformed state of the areas on which they act. The state being analysed can then be mapped onto the undeformed (reference) configuration more

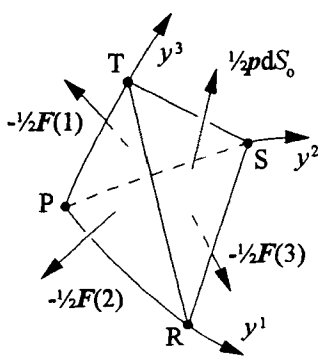


Figure 4.3 Equilibrium of a tetrahedron.

†By comparing (4.13) and (4.13a), it can be seen that this is identical in value to γ_{ij}. In the example used in section 4.1, this was not true because the system (X,Y,Z) was not identical to the deformed material coordinates (x,y,z).

readily. The force per unit *undeformed* area of the face RST is taken to be $\boldsymbol{p}$. As given by (3.137), the sum of the forces on the faces of the tetrahedron must be zero for equilibrium, or

$$\tfrac{1}{2}\boldsymbol{p}\,\mathrm{d}S_o = \tfrac{1}{2}[\boldsymbol{F}(1) + \boldsymbol{F}(2) + \boldsymbol{F}(3)] \tag{4.17}$$

where $\frac{1}{2}\mathrm{d}S_o$ is the undeformed area of the triangle RST. The surface traction $\boldsymbol{p}$ expressed in terms of the base vectors of the undeformed coordinate system is

$$\boldsymbol{p} = p^j \boldsymbol{g}_j \tag{4.18}$$

The forces acting on the other faces of the tetrahedron can also be related to these surfaces in their undeformed states, but will be expressed in terms of the base vectors in the deformed state. Thus (3.136) becomes

$$\boldsymbol{F}(3) = (s^{3k}\boldsymbol{G}_k)\mathrm{d}S_{o3} = s^{3k}\boldsymbol{G}_k\sqrt{g}\,\mathrm{d}y^1\mathrm{d}y^2 \tag{4.19}$$

where $\frac{1}{2}\mathrm{d}S_{o3}$ is the undeformed area of the triangle PRS. From (4.16), this becomes

$$\boldsymbol{F}(3) = (s^{3i} + s^{3k}u^i_{|k})\boldsymbol{g}_i\,\mathrm{d}S_{o3} \tag{4.20}$$

in terms of the base vectors of the undeformed state. Equation (3.68), which refers to the undeformed state, becomes in the new notation

$$n_{oi}\,\mathrm{d}S_o = \mathrm{d}S_{oi}$$

where again the subscript o refers to the undeformed configuration. Using this, (4.18) and the three equations for $\boldsymbol{F}(\alpha)$ of the type given by (4.20), (4.17) gives

$$p^i = (s^{ji} + s^{jk}u^i_{|k})n_{oj} = t^{ji}n_{oj} \tag{4.21}$$

where

$$t^{ji} = s^{ji} + s^{jk}u^i_{|k} \tag{4.22}$$

Because p^i and n_{oj} are first-order tensors with respect to the undeformed coordinate system, t^{ji} is a doubly-contravariant tensor with respect to this system and forms the **first Piola-Kirchhoff stress tensor**. The elements s^{ji} form the **second Piola-Kirchhoff stress tensor**. Further possible definitions of stress will be found in Green and Adkins (1970) for example.

4.3.1 Moment equilibrium

The arguments concerning moment equilibrium of the elementary parallelepiped shown in Fig. 4.4 follow the same lines as those in section 3.7.2. Ignoring smaller-order quantities, the moment produced by the $\boldsymbol{F}(3)$ couple shown in Fig. 4.4 is given by

$$\boldsymbol{M}(3) = \boldsymbol{G}_3\mathrm{d}y^3 \times s^{3k}\boldsymbol{G}_k\sqrt{g}\,\mathrm{d}y^1\mathrm{d}y^2$$

using (4.19), {cf. (3.140}. The total moment $\boldsymbol{M}$ produced by all three such couples is zero, where

$$\begin{aligned}\boldsymbol{M} &= s^{jk}\boldsymbol{G}_j \times \boldsymbol{G}_k\sqrt{g}\,\mathrm{d}y^1\mathrm{d}y^2\mathrm{d}y^3 \\ &= s^{jk}e_{jki}\boldsymbol{G}^i\sqrt{(gG)}\,\mathrm{d}y^1\mathrm{d}y^2\mathrm{d}y^3\end{aligned}$$

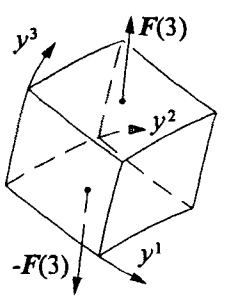


Figure 4.4 Moment equilibrium.

and G is $|G_{ij}|$, cf. (3.49), (3.71) and (3.141). On comparing the coefficients of $\boldsymbol{G}^i$,

$$s^{jk} = s^{kj} \tag{4.23}$$

Thus s^{ij} is symmetric although t^{ij} will only be symmetric under special circumstances.

4.3.2 Equations of motion

The analysis is again similar to that in section 3.7.3. The mass of a deformed element is the same as it was in the undeformed state. Then if the volume of the deformed element is dV and its density is ρ, and in its undeformed state its volume and density are dV_o and ρ_o respectively,

$$\rho dV = \rho_o dV_o = \rho_o \sqrt{g}\, dy^1 dy^2 dy^3 \tag{4.24}$$

If the force per unit mass in the loaded state is $\boldsymbol{f}$ and the local acceleration is $\ddot{\boldsymbol{u}}$, then the equation of motion of the parallelepiped shown in Fig. 4.5 is

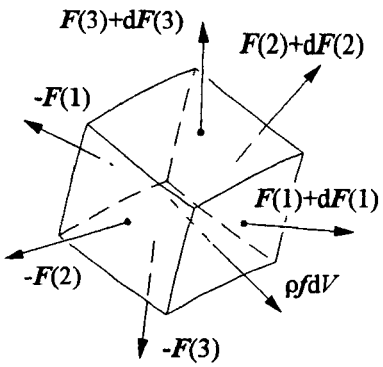


Figure 4.5 Equilibrium of a parallelepiped.

$$\sum_{\alpha=1}^{3} d\boldsymbol{F}(\alpha) + \rho \boldsymbol{f} dV = \rho \ddot{\boldsymbol{u}} dV \tag{4.25}$$

cf. (3.143). From (4.19), (4.20), (4.22) and (4.24),

$$\sum_{\alpha=1}^{3} d\boldsymbol{F}(\alpha) = (t^{ji}\boldsymbol{g}_i\sqrt{g})_{,j} dy^1 dy^2 dy^3 = t^{ji}_{|j}\sqrt{g}\,\boldsymbol{g}_i dy^1 dy^2 dy^3 = t^{ji}_{|j}\boldsymbol{g}_i dV_o$$

cf. section 3.7.3. Expressing $\boldsymbol{f}$ and $\ddot{\boldsymbol{u}}$ in terms of the base vectors of the undeformed system,

$$\boldsymbol{f} = f^i \boldsymbol{g}_i \quad , \quad \ddot{\boldsymbol{u}} = \ddot{u}^i \boldsymbol{g}_i \tag{4.26}$$

Equation (4.25) now reduces to the form

$$t^{ji}_{|j} + \rho_o f^i = \rho_o \ddot{u}^i \tag{4.27}$$

cf. (3.144), or in terms of s^{ji},

$$(s^{ji} + s^{jk} u^i_{|k})_{|j} + \rho_o f^i = \rho_o \ddot{u}^i \tag{4.28}$$

4.3.3 Invariants

By choosing the deformed (Lagrangean or material) coordinate system to be the Eulerian or fixed spatial system, it was seen that the strain tensors γ_{ij} and η_{ij} were identical. However, the mixed (and contravariant) forms of the strain tensor must be formed relative to each coordinate system, using the appropriate metric tensors. Thus the mixed form of the Lagrangean strain tensor is given by

$$\gamma^i_j = \gamma_{jk} g^{ik} \tag{4.29}$$

and the mixed form of the Eulerian strain tensor is

$$\eta^i_j = \eta_{jk} G^{ik} = \gamma_{jk} G^{ik} \tag{4.30}$$

This means that the mixed (and contravariant) forms of the strain tensors are not identical. For the example given at the beginning of this chapter,

$$\gamma^1_1 = \tfrac{1}{2}(\lambda^2 - 1) \quad , \quad \eta^1_1 = \tfrac{1}{2}(1 - 1/\lambda^2)$$

Invariants can be formed with respect to either mixed form, but here they will be expressed in terms of the Lagrangean strains. Three independent invariants can be formed by the process of contraction as before, cf. (3.150). These are related to the invariants J_1, J_2 and J_3 given by Green and Adkins [(1970, equations (1.1.20)] by

$$\begin{aligned} \tfrac{1}{2}J_1 &= K_1 = \gamma_i^i \\ \tfrac{1}{4}J_2 &= K_2 = \tfrac{1}{2}(\gamma_i^i\gamma_j^j - \gamma_j^i\gamma_i^j) \\ \tfrac{1}{8}J_3 &= K_3 = \tfrac{1}{6}(\gamma_i^i\gamma_j^j\gamma_k^k - 3\gamma_j^i\gamma_i^j\gamma_k^k + 2\gamma_j^i\gamma_k^j\gamma_i^k) \end{aligned} \tag{4.31}$$

Similar stress invariants can also be formed.

4.3.4 Work and energy

Consider a pair of equal and opposite forces $\boldsymbol{F}$ acting at points A and B as shown in Fig. 4.6. The increment in work done by the pair when their points of application move to adjacent points A′ and B′ is given by

$$\delta W = \boldsymbol{F}\cdot\delta\boldsymbol{r}_B - \boldsymbol{F}\cdot\delta\boldsymbol{r}_A = \boldsymbol{F}\cdot\delta\boldsymbol{s}$$

where $\delta\boldsymbol{s}$ is the change in the relative position vector $\underline{AB}$.

The change in the relative position of the forces $\boldsymbol{F}(3)$ shown in Fig. 4.4 during a small change in the displacements of the parallelepiped is $\delta(\boldsymbol{G}_3\,dy^3)$ so that the work done by the pair is

$$\begin{aligned} \delta W(3) &= \boldsymbol{F}(3)\cdot\delta(\boldsymbol{G}_3\,dy^3) \\ &= s^{3k}\sqrt{g}\,dy^1dy^2\boldsymbol{G}_k\cdot\delta(\boldsymbol{G}_3\,dy^3) \end{aligned}$$

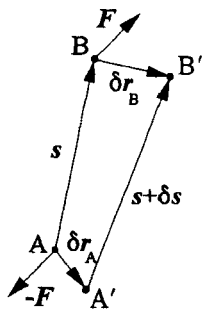


Figure 4.6 Force pair work.

ignoring fourth-order terms in dy^i and using (4.19). The total work done by all three force pairs is

$$\begin{aligned} \delta W &= s^{jk}\boldsymbol{G}_k\cdot\delta\boldsymbol{G}_j\sqrt{g}\,dy^1dy^2dy^3 \\ &= s^{jk}\tfrac{1}{2}(\boldsymbol{G}_k\cdot\delta\boldsymbol{G}_j + \boldsymbol{G}_j\cdot\delta\boldsymbol{G}_k)\sqrt{g}\,dy^1dy^2dy^3 \end{aligned}$$

using the symmetry of s^{jk}. From (4.10), the increment in strain is given by

$$2\delta\gamma_{ij} = \boldsymbol{G}_i\cdot\delta\boldsymbol{G}_j + \delta\boldsymbol{G}_i\cdot\boldsymbol{G}_j$$

so that

$$\delta W = s^{ij}\delta\gamma_{ij}\sqrt{g}\,dy^1dy^2dy^3 = s^{ij}\delta\gamma_{ij}dV_o \tag{4.32}$$

If the body is perfectly elastic, this increment can be related to a change in the elastic potential per unit original volume, $U(\gamma_{ij})$. This may be taken as symmetrical in γ_{ij} and γ_{ji} without loss of generality. That is, starting from some general asymmetric form $U(\gamma_{ij})$, we can write

$$U(\gamma_{ij}) \equiv U[\tfrac{1}{2}(\gamma_{ij} + \gamma_{ji})]$$

because, from the definition given by (4.10), γ_{ij} and γ_{ji} are identical under all circumstances. Relating the increment in work given by (4.32) to the change in potential,

$$\delta U dV_o = s^{ij}\delta\gamma_{ij}dV_o \qquad \text{or} \qquad \frac{1}{2}\left(\frac{\partial U}{\partial\gamma_{ij}} + \frac{\partial U}{\partial\gamma_{ji}}\right) = \frac{\partial U}{\partial\gamma_{ij}} = s^{ij} \tag{4.33}$$

where γ_{ij} is treated as independent of γ_{ji} for the purposes of partial differentiation. This

symmetric form of U satisfies the condition that s^{ij} is symmetric. Because $\delta\gamma_{ij}$ is a doubly-covariant tensor with respect to both states of the coordinate system and δU is a scalar, s^{ij} must be a doubly-contravariant tensor with respect to both states.

For isotropic materials, U is often expressed in terms of strain invariants and differentiated with respect to these invariants (see Green and Adkins (1970) section 1.15 and Hunter (1983) section 8.5 for example). This approach will be used in examining incompressible materials.

If it is assumed that U can be expressed as a power series in terms of strain, similar to that given by (3.153),

$$U(\gamma_{ij}) = \tfrac{1}{2}c^{ijkl}\gamma_{ij}\gamma_{kl} + \tfrac{1}{3}d^{ijklmn}\gamma_{ij}\gamma_{kl}\gamma_{mn} + O(\gamma^4) \tag{4.34}$$

where the symmetry conditions given by (2.54) and (2.55) again apply. Then from (4.33),

$$s^{ij} = c^{ijkl}\gamma_{kl} + d^{ijklmn}\gamma_{kl}\gamma_{mn} + O(\gamma^3) \tag{4.35}$$

For isotropic elasticity, U can be expressed entirely in terms of strain invariants in a form similar to (3.155),

$$U(\gamma_j^i) = \tfrac{1}{2}\lambda\gamma_i^i\gamma_j^j + \mu\gamma_j^i\gamma_i^j + \tfrac{1}{3}d_1\gamma_i^i\gamma_j^j\gamma_k^k + d_2\gamma_i^i\gamma_k^j\gamma_j^k + \tfrac{1}{3}d_3\gamma_j^i\gamma_k^j\gamma_i^k + O(\gamma^4) \tag{4.36}$$

giving

$$s^{ij} = [\lambda g^{ij}g^{kl} + \mu(g^{ik}g^{jl} + g^{il}g^{jk})]\gamma_{kl}$$
$$+ [g^{ij}(d_1 g^{kl}g^{mn} + d_2 g^{lm}g^{kn}) + 2d_2 g^{ik}g^{jl}g^{mn} + d_3 g^{il}g^{jn}g^{km}]\gamma_{kl}\gamma_{mn} + O(\gamma^3) \tag{4.37}$$

Unless the strain energy potential is completely expressed by a finite series, the use of such expressions will be limited to cases where the strains are sufficiently small for the series to be adequately convergent. Engineering materials usually remain elastic only for small strains; in terms of Cartesian coordinates, these are typically no more than 0.007 for metals. Over such a limited range, such power series in terms of strain are likely to be rapidly convergent.

4.3.5 Elastic constants

The coefficients d_1, d_2 and d_3 correspond to the coefficients C, B and A respectively, proposed by Landau and Lifshitz (1970). Their relationship to other third-order elastic constants, as given by Green (1973), is shown in the following table.

Table 4.1 Relationships between various definitions of third-order elastic constants

(4.36)	Brugger[8]	Toupin and Bernstein[9]	Murnaghan (1951)
d_1	$\frac{1}{2}c_{123}$	$\frac{1}{2}\nu_1$	$l - m + \frac{1}{2}n$
d_2	c_{144}	ν_2	$m - \frac{1}{2}n$
d_3	$4c_{456}$	$4\nu_3$	n
$2d_1 + 6d_2 + 2d_3$	c_{111}	$\nu_1 + 6\nu_2 + 8\nu_3$	$2l + 4m$
$2(d_1+d_2)$	c_{112}	$\nu_1 + 2\nu_2$	$2l$

(4.36)	Brugger§	Toupin and Bernstein¶	Murnaghan (1951)
$2d_1$	c_{123}	ν_1	$2l - 2m + n$
$d_2+\frac{1}{2}d_3$	c_{155}	$\nu_2 + \nu_3$	m
$\frac{1}{4}\, d_3$	c_{456}	ν_3	$\frac{1}{4}\, n$
d_1+d_2	$\frac{1}{2}\, c_{112}$	$\frac{1}{2}\nu_1 + \nu_2$	l

§ Brugger, K., (1964) *Phys. Rev.* **133** A1604

¶ Toupin, R.A. and Bernstein, B. (1961) *J. Acoust. Soc. Amer.* **33** 216

Bridgman† measured the change in volume of a sodium aggregate under high pressure. Linear theory was inadequate to explain his results, but taking $(3\lambda+2\mu)$ as 1.9 GPa and $(9d_1+9d_2+d_3)$ as -400GPa, the theoretical prediction derived from (4.37) was found to lie within 1% of the experimental data.

More recently, elastic constants have been determined by acoustic methods. The values given in the following table are derived from those given by Pollard (1977). The steel Hecla 37 has 0.4% carbon, 0.3% silicon and 0.8% manganese; Hecla ATV has 36% nickel, 10% chromium and 1% manganese.

Table 4.2 Elastic constants for isotropic materials (in gigapascals)

	Steel Hecla 37	Steel Hecla ATV	Molybdenum	Tungsten
λ	111^a	87^a	157^a	75^a
μ	82^a	72^a	110^a	73^a
d_1	-179^c	17^b	-25^c	-107^b
d_2	-282^c	-552^d	-283^c	-143^b
d_3	-708^c	-400^c	-772^b	-496^b

$(1\text{Gpa} = 10^9\ \text{N/M}^2 = 10^{10}\ \text{dyn/cm}^2 = 10^4\ \text{bar} \approx 145{,}038\ \text{p.s.i.})$

The superscripts $(\,)^a$ to $(\,)^d$ indicate the degree of precision of the values listed. Thus $(\,)^a$ indicates that the correct value can be expected to lie within the range ± 3GPa of the listed value. Similarly $(\,)^b$, $(\,)^c$ and $(\,)^d$ indicate ranges of ±(10 to 20)GPa, ±(25 to 40)GPa and ±80GPa respectively.

Values have also been obtained for anisotropic materials. For these, Brugger's coefficients correspond to the third-order coefficients in (4.34), when the reference coordinate system is Cartesian. Then

†Bridgman, P.W., (1948) *Proc. Am. acad. Arts Sci.* 76 55-70

$$c_{PQR} = d^{ijklmn}$$

where there is a correspondence between the indices P and ij, Q and kl, R and mn respectively. The compressed Voigt notation is used for the Brugger indices, so that the values 1, 2, 3, 4, 5 and 6 correspond to the pairs 11, 22, 33, 23 (or 32), 31 (or 13), and 12 (or 21) respectively, cf. (1.122).

For cubic crystals,

$$\begin{aligned} &c_{111} = c_{222} = c_{333} , \\ &c_{112} = c_{113} = c_{221} = c_{223} = c_{331} = c_{332} , \\ &c_{144} = c_{255} = c_{366} , \\ &c_{155} = c_{166} = c_{244} = c_{266} = c_{344} = c_{355} . \end{aligned} \tag{4.38}$$

Also, in accordance with (2.55), coefficients with the same set of subscripts in a different order are equal. Ledbetter and Kim† list the third-order stiffnesses for the cubic crystals of 25 elements and 62 compounds. Their values are used in all but the last line of the following table. The values on the last line are derived from Breazeale and Jacob Philip‡ and may be in error by 20GPa to 50GPa.

Table 4.3 Third-order elastic constants for cubic constants (in gigapascals)

(4.34)	Brugger	Silicon	Copper	Germanium
d^{111111}	c_{111}	-825	-1271	-732
d^{111122}	c_{112}	-451	-814	-290
d^{112233}	c_{123}	-64	-50	216
d^{112323}	c_{144}	12	-3	-8
d^{113131}	c_{155}	-310	-780	-304
d^{233112}	c_{456}	(-7)	(-55)	(-71)

Prosser§ gives the third-order elastic constants for graphite/epoxy composites exhibiting transverse isotropy or orthotropy.

Holder and Granato¶ have estimated fourth-order elastic constants for copper, silver and gold. These are all positive and of the order of 10TPa (10^{13} N/m^2). They point out that this indicates a rule that each successive order of elastic stiffnesses tends to be of the opposite sign and an order of magnitude greater than its predecessor. From this, they

†See Levy and Furr (2001) Chapter 8, Table 8.1.

‡Breazeale, M.A. and Jacob Philip (1984) *Physical Acoustics* **17** 1-60

§Prosser, W.H. (1987) Ultrasonic Characterisation of the Nonlinear Elastic Properties of Unidirectional Graphite/Epoxy Composites. *MS Thesis* Johns Hopkins University, Baltimore Md.

¶Holder, J. and Granato, A.V. (1971) *Physical Acoustics* **8** 237-277

conclude that the Taylor's expansion of strain energy in terms of strains will cease to converge at strains of around 0.1. This seems to be a reasonable conjecture, although gold, silver and copper all belong to Group 1B of the periodic table and so may not be representative of metals as a whole.

4.4 ELEMENTARY SOLUTIONS

For these solutions, it will be assumed that the material is isotropic and that the elastic potential is a quadratic function, giving the linear stress-strain relationship

$$s^{ij} = [(\lambda g^{ij} g^{kl} + \mu(g^{ik} g^{jl} + g^{il} g^{jk})]\gamma_{kl} = \lambda g^{ij} K_1 + \mu(g^{ik} g^{jl} + g^{il} g^{jk})\gamma_{kl} \quad \textbf{(4.39)}$$

from (4.37) and (4.31). If the embedded coordinate system is initially Cartesian, this reduces to the Cartesian tensor form

$$s_{pq} = \lambda \delta_{pq} \gamma_{rr} + 2\mu \gamma_{pq} \quad \textbf{(4.40)}$$

4.4.1 Uniaxial tension

A simple tensile state is represented by the displacement field

$$\boldsymbol{u} = e_1 x\boldsymbol{i} + e_2 y\boldsymbol{j} + e_3 z\boldsymbol{k}$$

Then for $p \neq q$, γ_{pq} and hence from (4.40) s_{pq}, is zero. The remaining strains are

$$\gamma_{\alpha\alpha} = e_\alpha + \tfrac{1}{2}e_\alpha^2 \qquad (\alpha = 1 \text{ to } 3)$$

which are induced by the stresses

$$s_{\alpha\alpha} = \lambda \gamma_{rr} + 2\mu \gamma_{\alpha\alpha} \qquad (\alpha = 1 \text{ to } 3)$$

If s_{22} and s_{33} are zero, from the second and third of these equations

$$\mu\gamma_{11} = (\lambda + \mu)\gamma_{kk}$$

giving

$$s_{11} = \mu[2 + \lambda/(\lambda + \mu)]\gamma_{11} = E\gamma_{11}$$

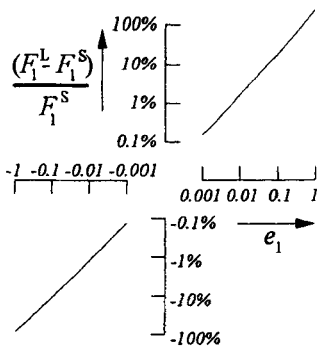


Figure 4.7 Variation from standard theory.

where E is Young's modulus. The only force acting is then in the x direction and the force per unit original (unstrained) area of the x face is given by

$$\boldsymbol{F}_1 = s_{11}\boldsymbol{G}_1 = E\gamma_{11}(\boldsymbol{g}_1 + \boldsymbol{u}_{,1}) = Ee_1(1 + \tfrac{1}{2}e_1)(1 + e_1)\boldsymbol{i} \quad \textbf{(4.41)}$$

using (4.19) and (4.9). Standard small deflection theory gives $\boldsymbol{F}_1 = Ee_1\boldsymbol{i}$ **(4.42)**

The results from the two theories are compared in Fig. 4.7, where $\boldsymbol{F}_1^L$ is the force given by (4.41), using large deflection theory, and $\boldsymbol{F}_1^S$ is the force given by (4.42), using small deflection theory. It can be seen that the variation from the force predicted by small deflection theory is less than 1% over the elastic range of most engineering materials.

4.4.2 Simple shear

A simple state of shear in the x-y plane is given by $\boldsymbol{u} = \gamma y\boldsymbol{i}$

The only non-zero strains given by (4.14) are

$$\gamma_{12} = \gamma_{21} = \tfrac{1}{2}\gamma \;\; ; \;\; \gamma_{22} = \tfrac{1}{2}\gamma^2$$

Then from (4.40),

$$s_{12} = s_{21} = \mu\gamma$$

as in the case of small deflection shear, and

$$s_{11} = s_{33} = \tfrac{1}{2}\lambda\gamma^2 \quad ; \quad s_{22} = (\tfrac{1}{2}\lambda + \mu)\gamma^2$$

The existence of these unequal normal stresses is known as the **Poynting effect**. The need for the stress s_{33} normal to the plane, which could be thought of as maintaining constant volume, is known as the **Kelvin effect**. It will be seen that if γ is sufficiently small, these normal stresses are negligible in comparison with the shear stress. Indeed, the ratio of the normal stresses to the shear stress is less than 1% over the elastic range of engineering materials.

4.4.3 Cylindrical symmetry

The behaviour of a hollow circular cylinder made of a homogeneous, isotropic material will be examined. Uniform radial pressures can be applied to the internal and external surfaces of the cylinder, and axial pressure distribution constrains plane cross-sections of the cylinder to remain plane. The cylinder will then deform in a rotationally symmetric fashion, given in cylindrical polar coordinates by

$$\boldsymbol{u} = \rho(r)\boldsymbol{g}^r + cz\boldsymbol{g}^z$$

where c is a constant.

If the elastic properties of the material are given by (4.39), the non-zero elements of stress are

$$\begin{aligned} s^{rr} &= \lambda K_1 + 2\mu\rho_{,r}(1 + \tfrac{1}{2}\rho_{,r}) \\ r^2 s^{\theta\theta} &= \lambda K_1 + 2\mu(\rho/r)(1 + \tfrac{1}{2}\rho/r) \\ s^{zz} &= \lambda K_1 + 2\mu c(1 + \tfrac{1}{2}c) \end{aligned}$$

where the first invariant of the strain tensor is given by

$$K_1 = \rho_{,r} + \tfrac{1}{2}\rho_{,r}^2 + \rho/r + \tfrac{1}{2}(\rho/r)^2 + c + \tfrac{1}{2}c^2$$

For static equilibrium in the absence of body forces, two of equations (4.28) are automatically satisfied and the remaining condition is

$$\begin{aligned} &r\rho_{,rr}\{3(\lambda + 2\mu)(1 + \rho_{,r})^2 + \lambda[(1 + \rho/r)^2 + (1 + c)^2] - 3\lambda - 2\mu\} \\ &\quad + (\rho_{,r} - \rho/r)\{(\lambda + 2\mu)[(1 + \rho_{,r})^2 + (1 + \rho/r)^2] \\ &\quad + 2(\lambda + \mu)(1 + \rho_{,r})(1 + \rho/r) + \lambda(1 + c)^2 - 3\lambda - 2\mu\} = 0 \end{aligned} \tag{4.43}$$

The solution of the equivalent problem in small deflection theory takes the form

$$\rho = ar + b/r \tag{4.44}$$

where a and b are independent constants. This solution allows independent boundary conditions of pressure or displacement to be imposed at the internal and external surfaces. One solution of (4.43) is also given by taking ρ as ar. A second solution can be obtained from a generalisation of the form of (4.44) given by the series

$$\rho = ar + \frac{1}{r}\sum_{i=0}^{\infty} b_i r^{-2i} \tag{4.45}$$

where b_0 is independent of a but otherwise b_i will depend on both b_0 and a. (It should be noted that in large deflection theory, independent solutions cannot be added together to give a general solution, because of the non-linearity of the problem.) In the case of plane strain given by taking c as zero, on substituting (4.45) into (4.43) and equating the coefficients of $1/r^4$ to zero,

$$b_1 = \frac{b_0^2(1 + a)(\lambda + 3\mu)}{\lambda(4a^2 + 8a + 2) + 2\mu(3a^2 + 6a + 2)}$$

Thus if the first two terms each give deflections of order $r\varepsilon$, the term associated with b_1 gives deflections of order $r\varepsilon^2$. An examination of (4.43) shows that each subsequent term diminishes by at least a further order ε. A detailed discussion of convergence will be found in Chapter 5 of Green and Adkins (1970) for example. Murnaghan [(1951) section 7.2] gives a solution for a generalised isotropic material which includes all terms no smaller than $r\varepsilon^2$.

4.4.4 Spherical symmetry

The behaviour of a hollow sphere, made from a homogeneous isotropic material, under the action of uniform radial pressure can be analysed in a similar fashion to that in the previous section. In terms of spherical polar coordinates, the displacement field can be taken as

$$\boldsymbol{u} = \rho(r)\boldsymbol{g}^r \tag{4.46}$$

Taking the elastic properties of the material to be such that (4.39) holds, the non-zero components of stress are

$$s^{rr} = \lambda K_1 + 2\mu(\rho_{,r} + \tfrac{1}{2}\rho_{,r}^2)$$
$$r^2 s^{\theta\theta} = \lambda K_1 + 2\mu[\rho/r + \tfrac{1}{2}(\rho/r)^2] = r^2\sin^2\theta\, s^{\phi\phi}$$

where

$$K_1 = \rho_{,r} + \tfrac{1}{2}\rho_{,r}^2 + 2\rho/r + (\rho/r)^2$$

As before, two of the equations of static equilibrium given by (4.28) are identically satisfied and the remaining condition is

$$r\rho_{,rr}\{3(\lambda + 2\mu)(1 + \rho_{,r}) + 2\lambda(1 + \rho/r)^2 - 3\lambda - 2\mu\} + 2(\rho_{,r} - \rho/r)\times$$
$$\{2(\lambda + \mu)[(1 + \rho/r)^2 + (1 + \rho_{,r})(1 + \rho/r)] + (\lambda + 2\mu)(1 + \rho_{,r})^2 - 3\lambda - 2\mu\} = 0 \tag{4.47}$$

The solution of the equivalent problem in small deflection theory is

$$\rho = ar + b/r^2 \tag{4.48}$$

Again, generalising this as a power series,

$$\rho = ar + \frac{1}{r^2}\sum_{i=0}^{\infty} b_i r^{-3i} \tag{4.49}$$

where a and b_0 are independent constants and b_i can be found on substituting this expression into (4.47) and equating the coefficients of each power of r to zero. Thus

$$b_1 = \frac{2b_0^2(1 + a)(\lambda + 3\mu)}{\lambda(5a^2 + 10a + 2) + 2\mu(3a^2 + 6a + 2)}$$

is given by equating the coefficients of $1/r^6$ to zero. On the same basis as in the previous section, if the first two terms give displacements of order $r\,\varepsilon$, the third term associated with b_1 gives displacements of order $r\,\varepsilon^2$. A solution by Murnaghan [(1951) section 7.1] includes terms of this order of smallness in the analysis of a spherical shell made from a general isotropic material. He asserts that these second-order terms are particularly significant for very thin shells.

Fig. 4.8 shows results obtained for a shell made from a homogeneous isotropic material with a Poisson's ratio of ⅓. Its initial outer radius is 1.1 times its initial inner radius and it is subject to an internal pressure p. The first six terms of the series given by (4.49) were used to derive the curves. These show the percentage increase in the values of $\rho_{,r}$ and ρ/r at the inner radius above those predicted by small deflection theory with increasing pressure. Most engineering metals will yield at a value of p/μ no greater than 0.002. The pressure can be increased up to a local maximum of 0.099μ after which it has to be reduced, falling to a minimum of 0.034μ at much higher strains. This inflation instability has been discussed by Adkins and Rivlin† but it is not shown on the graph because it may lie beyond the limit of convergence of the series.

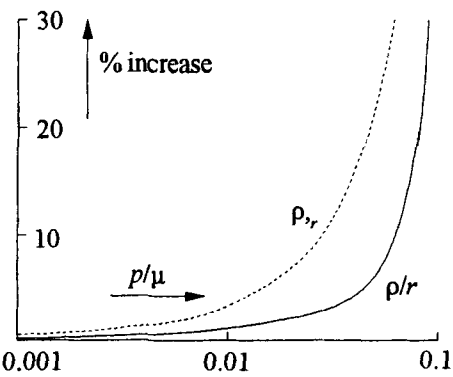


Figure 4.8 Large deformation theory for a spherical shell.

4.5 INCOMPRESSIBLE MATERIALS

In dealing with problems of finite strain, a great deal of attention has been given to incompressible materials. By definition, these undergo no change in volume under stress, so that each elementary volume is constant, or from (3.75),

$$\sqrt{g}\,dy^1dy^2dy^3 = \sqrt{G}\,dy^1dy^2dy^3 \qquad \text{or} \qquad g = G. \tag{4.50}$$

For such materials, the state of stress cannot be completely determined from the state of strain because there may also be an isotropic or 'hydrostatic' stress given by

$$s^{ij} = pG^{ij} \tag{4.51}$$

where p is a scalar function of the coordinates. Such a stress system does no work during deformation and hence is independent of the stored strain energy.

Proof From (4.32), (4.51) and (4.10),

$$\delta W = pG^{ij}\delta\gamma_{ij}dV = \tfrac{1}{2}pG^{ij}\delta G_{ij}dV$$

From (3.51), GG^{ij} is the cofactor of G_{ij}, so that

$$3\delta G = GG^{ij}\delta G_{ij}$$

†Adkins, J.E. and Rivlin, R.S. (1952) *Phil. Trans. Roy. Soc. (A)* **244** 505-531

Then because G is constant for incompressible materials, $G^{ij}\delta G_{ij}$ an hence δW is zero.

For isotropic materials, it is often convenient to define a set of strain invariants which are related to the metric tensors of the initial and deformed states. These are

$$\begin{aligned} I_1 &= g^{ij}G_{ij} = 3 + 2\gamma_i^i = 3 + J_1 \\ I_2 &= g_{ij}G^{ij}I_3 = 3 + 4\gamma_i^i + 2(\gamma_i^i\gamma_j^j - \gamma_j^i\gamma_i^j) = 3 + 2J_1 + J_2 \\ I_3 &= G/g = |[G_{jk}][g^{ki}]| = |\delta_j^i + 2\gamma_j^i| = 1 + J_1 + J_2 + J_3 \end{aligned} \tag{4.52}$$

For incompressible materials, I_2 becomes $g_{ij}G^{ij}$ because I_3 is equal to unity. Instead of (4.36), the strain energy per unit volume in an elastic isotropic material can now be expressed as

$$U(I_1,I_2,I_3) = \sum_{p,q,r=0}^{\infty} c_{pqr}(I_1 - 3)^p(I_2 - 3)^q(I_3 - 3)^r \tag{4.53}$$

where c_{000} is zero, see for example R.W. Ogden [(1984) equation (4.3.52)]. For incompressible materials, this becomes

$$U(I_1,I_2) = \sum_{p,q=0}^{\infty} c_{pq}(I_1 - 3)^p(I_2 - 3)^q \tag{4.54}$$

where c_{00} is zero. **Mooney material** is given by the particular form

$$U(I_1,I_2) = C_1(I_1 - 3) + C_2(I_2 - 3) \tag{4.55}$$

This expression is considered to be satisfactory for rubber-like materials, provided that the deformations are not too large. Hunter [(1983) section 8.5] states that there is no molecular basis for the constant C_2, although there has been some success in correlating it with behaviour in plane extension. For example, experiments by Rivlin and Saunders† on vulcanised rubber suggest values of 1.8GPa for C_1 and 0.23GPa for C_2. If C_2 is zero, the above expression reduces to the **neo-Hookean** form

$$U(I_1) = C_1(I_1 - 3) \tag{4.56}$$

For incompressible isotropic materials, from (4.33) and (4.51),

$$\begin{aligned} s^{ij} &= pG^{ij} + \frac{1}{2}\sum_{k=1}^{2}\frac{\partial U}{\partial I_k}\left(\frac{\partial I_k}{\partial \gamma_{ij}} + \frac{\partial I_k}{\partial \gamma_{ji}}\right) \\ &= pG^{ij} + 2g^{ij}\frac{\partial U}{\partial I_1} + 2(g^{ij}g^{kl} - g^{ik}g^{jl})G_{kl}\frac{\partial U}{\partial I_2} \end{aligned} \tag{4.57}$$

For example, the neo-Hookean material given by (4.56) has a stress-strain relationship given by

$$s^{ij} = pG^{ij} + 2C_1g^{ij} \tag{4.58}$$

or, after operating on both sides of the equation with $G_{jk}g^{kl}$,

$$s^{il} + 2\gamma_j^l s^{ij} = \bar{p}g^{il} + 4C_1\gamma^{il} \qquad (\bar{p} = p + 2C_1)$$

Taking the displacement rates to be small and the terms on the right-hand side of the equation to be of the same order, this becomes

†Rivlin, R.S. and Saunders, D.W. (1951) *Phil. Trans. Roy. Soc. (A)* **243** 251-288

$$\sigma^{il} = 4C_1 e^{il} + \bar{p} g^{il} \tag{4.59}$$

In small deflection theory, incompressibility implies that e_k^k becomes zero. By considering the strains produced by uniaxial stress, it can be shown that Poisson's ratio must become 0.5 and hence that λ becomes infinity. The product λe_k^k is then an indeterminate scalar, q say, so that (3.157) can be written in the form

$$\sigma^{il} = 2\mu e^{il} + q g^{il} \tag{4.60}$$

On comparing this with (4.59),

$$2C_1 = \mu \quad , \quad \bar{p} = q \tag{4.61}$$

4.5.1 Principal stretches

An elementary sphere of material within the body will become an ellipsoid after deformation. The axes of the ellipsoid can be taken as coincident with a mutually orthogonal set of coordinate lines which were also mutually orthogonal before the deformation, each of which has simply stretched. The ratios of the lengths of the semi-axes of the ellipsoid to the radius of the sphere are known as the **principal stretches** $\lambda_1, \lambda_2, \lambda_3$. Taking the original coordinate system to be Cartesian, it follows from (4.8) and (4.10) that the local values of the metric tensors are

$$\begin{gathered} g_{11} = g_{22} = g_{33} = 1 \;\; ; \;\; G_{11} = \lambda_1^2 \, , \;\; G_{22} = \lambda_2^2 \, , \;\; G_{33} = \lambda_3^2 \, ; \\ g_{ij} = G_{ij} = 0 \quad (i \neq j). \end{gathered} \tag{4.62}$$

and the non- zero contravariant terms are the reciprocals of the non-zero covariant terms. The local values of the invariants given by (4.52) are then

$$\begin{aligned} I_1 &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \\ I_2 &= (\lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2})\lambda_1^2\lambda_2^2\lambda_3^2 = \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 + \lambda_1^2\lambda_2^2 \\ I_3 &= \lambda_1^2\lambda_2^2\lambda_3^2 \end{aligned} \tag{4.63}$$

Thus these invariants are necessarily positive. Other invariants can also be expressed in terms of the principal stretches. For example, from (4.10), (4.29) and (4.62),

$$\gamma_i^j \gamma_j^i = \left[\sum_{k=1}^{3} (\lambda_k^2 - 1) \right]^2 \, , \quad \gamma_j^i \gamma_i^j = \sum_{k=1}^{3} (\lambda_k^2 - 1)^2 \tag{4.64}$$

and so the potential function used earlier, consisting of the first two terms of the expansion given by (4.36), is positive definite, provided that the Lamé constants are positive.

Example 4.1 Torsion and extension of a circular cylinder

Suppose that the position of a material point p is given initially in cylindrical polar coordinates by (r, θ, z) and after deformation by (R, Θ, Z) where

$$R = R(r) \, , \quad \Theta = \theta + \lambda\psi z \, , \quad Z = \lambda z .$$

Then the displacement of the point to P is given by

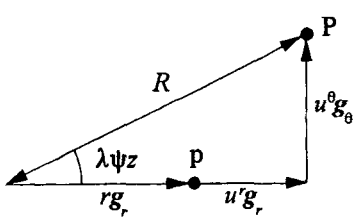


Figure 4.9 Change of polar coordinates.

$$\boldsymbol{u} = u^r \boldsymbol{g}_r + u^\theta \boldsymbol{g}_\theta + u^z \boldsymbol{g}_z = (R\cos\lambda\psi z - r)\boldsymbol{g}_r + \frac{R}{r}\sin\lambda\psi z\,\boldsymbol{g}_\theta + (\lambda - 1)z\boldsymbol{g}_z$$

(see Fig. 4.9). From (3.103) and (4.9),

$$u^j_{|i}\boldsymbol{g}_j dy^i = (\boldsymbol{G}_i - \boldsymbol{g}_i)dy^i$$

Hence $\boldsymbol{G}_i$ is given by

$$\begin{aligned}
\boldsymbol{G}_R &= R_{,r}(\cos\lambda\psi z\boldsymbol{g}_r + \tfrac{1}{r}\sin\lambda\psi z\boldsymbol{g}_\theta)\\
\boldsymbol{G}_\Theta &= R(-\sin\lambda\psi z\boldsymbol{g}_r + \tfrac{1}{r}\cos\lambda\psi z\boldsymbol{g}_\theta)\\
\boldsymbol{G}_Z &= R\lambda\psi(-\sin\lambda\psi z\boldsymbol{g}_r + \tfrac{1}{r}\cos\lambda\psi z\boldsymbol{g}_\theta) + \lambda\boldsymbol{g}_z
\end{aligned}$$

and from the scalar products of these vectors,

$$[G_{ij}] = \begin{bmatrix} R_{,r}^2 & 0 & 0 \\ 0 & R^2 & R^2\lambda\psi \\ 0 & R^2\lambda\psi & R^2\lambda^2\psi^2 + \lambda^2 \end{bmatrix} \tag{4.65}$$

giving

$$G = R_{,r}^2 R^2 \lambda^2$$

and because the value of g is r^2, the incompressibility condition is

$$r = R_{,r}R\lambda \tag{4.66}$$

or

$$R^2 = r^2/\lambda + k \tag{4.67}$$

where k is a constant. The inverse of (4.65) is

$$[G^{ij}] = \begin{bmatrix} 1/R_{,r}^2 & 0 & 0 \\ 0 & \psi^2 + 1/R^2 & -\psi/\lambda \\ 0 & -\psi/\lambda & 1/\lambda^2 \end{bmatrix}$$

The invariants given by (4.52) are then

$$I_1 = R_{,r}^2 + \left(\frac{R}{r}\right)^2 + \lambda^2(R^2\psi^2 + 1) = \left(\frac{r}{\lambda}\right)^2\left(\frac{r^2}{\lambda} + k\right)^{-1} + \left(\frac{r^2}{\lambda} + k\right)\left(\frac{1}{r^2} + \lambda^2\psi^2\right) + \lambda^2$$

$$I_2 = \left\{\frac{1}{R_{,r}^2} + r^2\left(\psi^2 + \frac{1}{R^2}\right) + \frac{1}{\lambda^2}\right\}I_3 = r^2\left(\frac{r^2}{\lambda} + k\right)^{-1} + \left(\frac{\lambda}{r}\right)^2\left(\frac{r^2}{\lambda} + k\right) + r^2\psi^2 + \frac{1}{\lambda^2}$$

making use of the incompressibility condition.

Considering a neo-Hookean material, from (4.22), (4.58) and (4.61),

$$\begin{aligned}
s^{rr} &= p/R_{,r}^2 + \mu & t^{rr} &= (\mu R_{,r} + p/R_{,r})\cos\lambda\psi z\\
s^{r\theta} &= 0 & rt^{r\theta} &= (\mu R_{,r} + p/R_{,r})\sin\lambda\psi z\\
s^{rz} &= 0 & t^{rz} &= 0\\
s^{\theta r} &= 0 & t^{\theta r} &= -R(p/R^2 + \mu/r^2)\sin\lambda\psi z\\
s^{\theta\theta} &= p(\psi^2 + 1/R^2) + \mu/r^2 & rt^{\theta\theta} &= R(p/R^2 + \mu/r^2)\cos\lambda\psi z
\end{aligned}$$

$$s^{\theta z} = -p\psi/\lambda \qquad t^{\theta z} = -p\psi$$
$$s^{zr} = 0 \qquad t^{zr} = -\mu R\lambda\psi \sin\lambda\psi z$$
$$s^{z\theta} = -p\psi/\lambda \qquad rt^{z\theta} = \mu R\lambda\psi \cos\lambda\psi z$$
$$s^{zz} = p/\lambda^2 + \mu \qquad t^{zz} = p/\lambda + \mu\lambda$$

In the absence of body forces and accelerations, one of equations (4.27) is identically satisfied and the other two give the same condition, which on using (4.66) reduces to

$$p_{,r} = \frac{\mu}{\lambda}\left(r\lambda^2\psi^2 + \frac{1}{r} + \frac{r^3}{R^4\lambda^2} - \frac{2r}{\lambda R^2} \right)$$

On making use of (4.67), this gives

$$p = \frac{\mu}{2\lambda}\left\{ r^2\lambda^2\psi^2 + \frac{k\lambda}{r^2+k\lambda} - \ln\left(1 + \frac{k\lambda}{r^2} \right) \right\} + c$$

where c is a constant. If the surfaces of the cylinder at radii a and b in the unloaded state remain free from surface tractions, then s^{rr}, $s^{r\theta}$ and s^{rz} are zero on these surfaces. Because $s^{r\theta}$ and s^{rz} are identically zero, this leaves two conditions from which the constants k and c can be determined. In the particular case of a solid cylinder, no radial motion of the point at the origin is possible, because this would give rise to a circular hole. In this case, from (4.67), k is zero.

The axial force F and the torque T on the cylinder are given by

$$F = \int_a^b t^{zz} 2\pi r\,dr$$

$$T = \int_a^b (rt^{z\theta} R\cos\lambda\psi z - t^{zr} R\sin\lambda\psi z) 2\pi r\,dr = \tfrac{1}{2}\pi\mu\psi[b^4 - a^4 + 2k\lambda(b^2 - a^2)]$$

Note that for a solid circular section, the relationship between torque and rate of twist per unit (deformed) length is exactly the same as that given by small deflection theory.

Example 4.2 Flexure of a cuboid

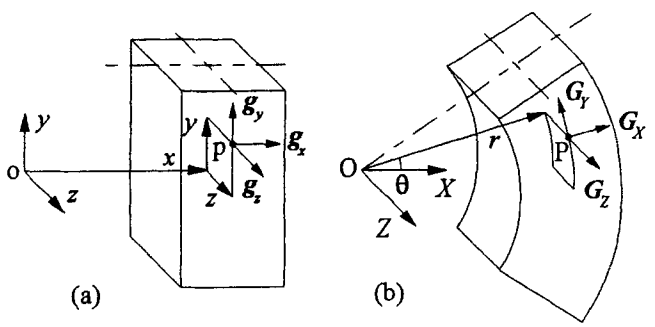


Figure 4.10 Deformation of a cuboid to form a circular arc.

The cuboid shown in Fig. 4.10a is distorted into the form shown in Fig. 4.10b. Planes originally perpendicular to the x axis become cylindrical surfaces generated about the z

(or Z) axis. Cross-sectional planes, originally perpendicular to the y axis, pass through the Z axis after deformation. There is a uniform stretch λ in the z direction. Apart from this last condition, these assumptions are related to the engineering theory of bending.

If a point p with coordinates (x, y, z) moves to a point P after deformation, its coordinates are then $(r\cos\theta, r\sin\theta, \lambda z)$ where r is the radius of the cylindrical surface on which P lies, and θ is the angle between the cross-sectional plane on which P lies, and the X axis. The radius r will be some function of x, and θ will be proportional to y (cy say). The displacement of the point is given by

$$\boldsymbol{u} = (r\cos cy - x)\boldsymbol{g}_x + (r\sin cy - y)\boldsymbol{g}_y + (\lambda - 1)z\boldsymbol{g}_z$$

By the same processes as before,

$$\boldsymbol{G}_X = (r_{,x}\cos cy)\boldsymbol{g}_x + (r_{,x}\sin cy)\boldsymbol{g}_y$$
$$\boldsymbol{G}_Y = (-rc\sin cy)\boldsymbol{g}_x + (rc\cos cy)\boldsymbol{g}_y$$
$$\boldsymbol{G}_Z = \lambda\boldsymbol{g}_z$$

$$[G_{ij}] = \begin{bmatrix} r_{,x}^2 & 0 & 0 \\ 0 & r^2c^2 & 0 \\ 0 & 0 & \lambda^2 \end{bmatrix}$$

Hence $$G = c^2\lambda^2 r^2 r_{,x}^2$$

and $$[G^{ij}] = \begin{bmatrix} 1/r_{,x}^2 & 0 & 0 \\ 0 & 1/r^2c^2 & 0 \\ 0 & 0 & 1/\lambda^2 \end{bmatrix}$$

The incompressibility condition then gives

$$rr_{,x} = 1/\lambda c \quad \text{or} \quad r = \sqrt{(2x/\lambda c + K)}$$

where K is a constant. Again considering a neo-Hookean material,

$$s^{xx} = p/r_{,x} + \mu, \quad t^{xx} = \cos cy(p/r_{,x} + \mu r_{,x}), \quad t^{xy} = \sin cy(p/r_{,x} + \mu r_{,x}),$$
$$s^{yy} = p/r^2c^2 + \mu, \quad t^{yy} = \cos cy(p/rc + \mu rc), \quad t^{yx} = -\sin cy(p/rc + \mu rc),$$
$$s^{zz} = p/\lambda^2 + \mu, \quad t^{zz} = p/\lambda + \mu\lambda,$$

all other components of these stresses being zero. In the absence of body forces and accelerations, the equilibrium equations are satisfied by taking p as a function of x only and by the condition

$$p/r_{,x} + \mu r_{,x} = k$$

where k is a constant, or

$$p = 2\{k\sqrt{[\lambda c(2x + \lambda cK)]} - 2\mu\}/\lambda c(2x + \lambda cK)$$

The surfaces of a cuboid initially perpendicular to the x axis can be taken as stress free by making k zero. However, those surfaces perpendicular to the z axis cannot be made stress

free. This mode of deformation is therefore an inadequate representation of beam bending. This is because no attempt has been made to introduce a large-deformation equivalent of anticlastic curvature†.

General forms of the above examples and many others will be found in Green and Adkins (1970).

4.6 STABILITY OF CONTINUA

A common approach to the analysis of stability problems is to seek a critical load for which there is no unique elastic response. At such a load, an adjacent equilibrium state is taken to exist which is infinitesimally different from the elastic response up to that point. This adjacent state arises at a point of bifurcation and is associated with a separate load-deformation characteristic known as the buckling mode. However, ordinary linear small deflection theory does not permit such a lack of uniqueness and so the equilibrium equations are modified to allow for additional effects produced by large loads. Such an approach is only an approximation to the exact equations of large deflection theory and suffers from a lack of consistency because not all terms of the same magnitude are included. It would be convenient to retain some of the simplifying features of small deflection theory while retaining the consistency of large deflection theory. To do this, it will be necessary to distinguish between small (but finite) initial deflections, of order ε say, and infinitesimal additional deflections, of order ι say, undergone in passing to an immediately adjacent state. The terms of order ε may be such that by comparison with them, terms of order ε^2 and higher powers of ε may be neglected. An infinitesimal quantity is smaller than any given finite quantity, so that ι is necessarily negligible in comparison with ε and ι^2 is negligible in comparison with ι. It is thus possible to set up a pecking order of magnitudes, giving results such as

$$\iota \ll \varepsilon^n \ll 1 \quad (n \text{ finite and positive}), \quad \varepsilon^m \iota^r \ll \varepsilon^n \iota^s \quad (r > s, \text{ or } m > n \text{ and } r = s)$$

Here, the examination of the stability problem will be confined to that of an infinitesimal deflection superposed on a small but finite deflection. The problem of small deformations superposed on a general finite deformation is examined in Green and Zerna [(1968), Chapter 4] for example.

4.6.1 The governing equations

Under certain critical applied loading, a displacement field $\boldsymbol{u}$ is produced which has displacement gradients of order ε. Under the same loading, a second state with a displacement field of $\boldsymbol{u}+\boldsymbol{v}$ may also occur, where the gradients of $\boldsymbol{v}$ are of order ι. For this second state, the metric tensors found from (4.16) become

$$\bar{\boldsymbol{G}}_i = (\delta_i^j + u_{|i}^j + v_{|i}^j)\boldsymbol{g}_j = \boldsymbol{G}_i + v_{|i}^j \boldsymbol{g}_j \tag{4.68}$$

where the bar over a symbol will be used to refer to this second state. The strains found from (4.14) become

$$\bar{\gamma}_{ij} = \gamma_{ij} + \tfrac{1}{2}[v_{i|j} + v_{j|i} + (u_{l|j}v_{k|i} + u_{k|i}v_{l|j} + v_{l|j}v_{k|i})g^{lk}] \tag{4.69}$$

The equation of equilibrium at a surface, (4.21), now takes the form

†See for example Timoshenko and Goodier (1970) section 102.

$$\bar{t}^{ji} n_{oj} = [\bar{s}^{ji} + \bar{s}^{jk}(u^{i}_{|k} + v^{i}_{|k})] n_{oj} = \bar{p}^{i} \tag{4.70}$$

and in the interior, from (4.28),

$$\bar{t}^{ji}_{|j} + \rho_o \bar{f}^{i} = [\bar{s}^{ji} + \bar{s}^{jk}(u^{i}_{|k} + v^{i}_{|k})]_{|j} + \rho_o \bar{f}^{i} = \rho_o(\ddot{u}^{i} + \ddot{v}^{i}) \tag{4.71}$$

From (4.35), the stress-strain relationship is

$$\bar{s}^{ij} = c^{ijkl}\bar{\gamma}_{kl} + d^{ijklmn}\bar{\gamma}_{kl}\bar{\gamma}_{mn} + O(\bar{\gamma}^3) \tag{4.72}$$

Use will be made of the following tensor differences

$$\begin{gathered} \hat{\gamma}_{ij} = \bar{\gamma}_{ij} - \gamma_{ij} \\ \hat{s}^{ji} = \bar{s}^{ji} - s^{ji} \quad , \quad \hat{t}^{ji} = \bar{t}^{ji} - t^{ji} \\ \hat{p}^{i} = \bar{p}^{i} - p^{i} \quad , \quad \hat{f}^{i} = \bar{f}^{i} - f^{i} \end{gathered} \tag{4.73}$$

The differences between the equations for the primary and secondary loaded states are given by

$$\hat{\gamma}_{ij} = \tfrac{1}{2}[v_{i|j} + v_{j|i} + (u_{l|j}v_{k|i} + u_{k|i}v_{l|j})g^{lk}] + O(\iota^2) \tag{4.74}$$

from (4.69),

$$\hat{t}^{ji} n_{oj} = [\hat{s}^{ji} + \hat{s}^{jk}(u^{i}_{|k} + v^{i}_{|k}) + s^{jk}v^{i}_{|k}] n_{oj} = \hat{p}^{i} \tag{4.75}$$

from (4.21) and (4.70), and

$$\hat{t}^{ji}_{|j} + \rho_o \hat{f}^{i} = [\hat{s}^{ji} + \hat{s}^{jk}(u^{i}_{|k} + v^{i}_{|k}) + s^{jk}v^{i}_{|k}]_{|j} + \rho_o \hat{f}^{i} = \rho_o \ddot{v}^{i} \tag{4.76}$$

from (4.27), (4.28) and (4.71). From (4.35), (4.72), (4.14) and (4.69), if all terms no smaller than $O(\varepsilon\iota)$ are included,

$$\begin{aligned} \hat{s}^{ij} \approx\ & \tfrac{1}{2}c^{ijkl}[v_{k|l} + v_{l|k} + (u^{m}_{|k}v_{m|l} + u^{m}_{|l}v_{m|k})] \\ & + \tfrac{1}{4}d^{ijklmn}[(u_{k|l} + u_{l|k})(v_{m|n} + v_{n|m}) + (u_{m|n} + u_{n|m})(v_{k|l} + v_{l|k})] \end{aligned} \tag{4.77}$$

(Including the next term in the strain power series adds terms of no greater magnitude than $O(\varepsilon^2\iota)$.)

If the terms in (4.75) and (4.76) are also confined to those no smaller than $O(\varepsilon\iota)$, it is sufficient to make the substitution

$$s^{jk}v^{i}_{|k} \approx \tfrac{1}{2}c^{kjmn}(u_{m|n} + u_{n|m})v^{i}_{|k} \tag{4.78}$$

The solution of the stability problem is given by satisfying (4.75) and (4.76) in which s^{jk} only appears as part of this product, so that the accuracy to which it needs to be determined is not greater than that given by small deflection theory. From (4.77) and (4.78),

$$\begin{aligned} \hat{t}^{ji} &= [\hat{s}^{ji} + \hat{s}^{jk}(u^{i}_{|k} + v^{i}_{|k}) + s^{jk}v_{|k}] \\ &= \tfrac{1}{2}c^{ijmn}(\{v_{m|n} + v_{n|m}\} + u^{l}_{|m}v_{l|n} + u^{l}_{|n}v_{l|m}) \\ &+ \tfrac{1}{2}c^{kjmn}[\{v^{i}_{|k}(u_{m|n} + u_{n|m})\} + u^{i}_{|k}(v_{m|n} + v_{n|m})] \\ &+ \tfrac{1}{4}d^{ijmnpq}[(u_{m|n} + u_{n|m})(v_{p|q} + v_{q|p}) + (u_{p|q} + u_{q|p})(v_{m|n} + v_{n|m})] + O(\varepsilon^2\iota) \end{aligned} \tag{4.79}$$

Bolotin [(1964) section 52] gives continuum equations for a linear isotropic medium which are the equivalents of (4.75) and (4.76). These are derived from a simple extension of small deflection theory to allow for large initial stresses. If his approach is generalised to encompass the behaviour of anisotropic media with respect to curvilinear coordinates, $\hat{t}^{ji}$

is given only by the terms in curly brackets in (4.79). However, these include terms of order $\varepsilon\iota$ so that for consistency all the terms in (4.79) should be included. In fact, a stability problem only arises if terms at least as small as this are included in the analysis, so that the above approach gives the necessary first approximation for cases when ε is small.

The following examples will be confined to the examination of homogeneous isotropic bodies for which the stress-strain relationship is given by (4.37), and there are no body forces or accelerations acting. Equations (4.75), (4.76) and (4.79) then become

$$\hat{t}^{ji} n_{oj} = \hat{p}^i \tag{4.80}$$

and

$$\hat{t}^{ji}_{|j} = 0 \tag{4.81}$$

$$\begin{aligned}\hat{t}^{ji} &= g^{ij}[v_{|k}^{k}(\lambda + 2d_1 u_l^{l}) + (\lambda + d_2) g^{kl} u_{|k}^{m} v_{m|l} + d_2 u_{|l}^{k} v_{|k}^{l}] + \mu(g^{ik} v_{|k}^{j} + g^{kj} v_{|k}^{i}) \\ &+ (\lambda + d_2) g^{jk}(u_{|k}^{i} v_{|l}^{l} + u_{|l}^{l} v_{|k}^{i}) + d_2 g^{il}(u_{|k}^{k} v_{|l}^{j} + u_{|l}^{j} v_{|k}^{k}) \\ &+ (\mu + \tfrac{1}{4} d_3)[g^{ik} g^{jl}(u_{|l}^{m} v_{m|k} + u_{|k}^{m} v_{m|l}) + g^{kl}(u_{|k}^{i} v_{|l}^{j} + u_{|l}^{j} v_{|k}^{i}) \\ &+ g^{jk}(u_{|l}^{i} v_{|k}^{l} + u_{|k}^{l} v_{|l}^{i})] + \tfrac{1}{4} d_3 g^{ik}(u_{|l}^{j} v_{|k}^{l} + u_{|k}^{l} v_{|l}^{j}) + O(\varepsilon^2 \iota)\end{aligned} \tag{4.82}$$

Example 4.3 Buckling of an axially-loaded cylindrical rod under axial compression
The behaviour of an isotropic rod with a circular cross-section of initial diameter $2r$ under an axial load will be examined with respect to cylindrical polar coordinates (r, θ, z), the z axis being coincident with the axis of the rod in the unloaded state. The stress-strain relationship will be taken as that given by (4.39). If the rod has undergone an axial compression $u_{z|z}$ equal to $-\varepsilon$ under the action of a uniaxial stress s^{zz} at the point of bifurcation, then from section 4.4.1,

$$s^{zz} = E(-\varepsilon + \tfrac{1}{2}\varepsilon^2) \approx -E\varepsilon \qquad \left(E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}\right)$$

$$u_{r|r} = \frac{u_{\theta|\theta}}{r^2} = -1 + \sqrt{[1 + \nu(2\varepsilon - \varepsilon^2)]} \approx \nu\varepsilon \qquad \left(\nu = \frac{\lambda}{2(\lambda + \mu)}\right)$$

where the approximations are to the degree of accuracy required for substitution in (4.79) and are of course the results obtained from small deflection theory. A buckling mode v will be examined which has the contravariant elements

$$\begin{aligned} v^1 &= f(R)\cos\theta\sin kz \\ Rv^2 &= kg(R)\sin\theta\sin kz \\ v^3 &= h(R)\cos\theta\cos kz \end{aligned} \tag{4.83}$$

where

$$\begin{aligned} R &= kr \\ f(R) &= \sum_{i=0}^{\infty} f_i R^{2i} \\ g(R) &= \sum_{i=0}^{\infty} g_i R^{2i} \\ h(R) &= \sum_{i=0}^{\infty} h_i R^{2i+1} \end{aligned} \tag{4.84}$$

On substituting these functions into (4.82), the internal conditions given by (4.81) can be expressed as

$$
\begin{aligned}
&(1+2\nu\varepsilon)[(2-2\nu)(R^2f''+Rf')-(3-4\nu)(f+g)+Rg'] \\
&\quad -(1-2\nu)(1-2\varepsilon)R^2f-[1-(1-\nu)\varepsilon]R^2h'=0 \\
&(1+2\nu\varepsilon)[(1-2\nu)(R^2g''+Rg')-(3-4\nu)(f+g)-Rf'] \\
&\quad -(1-2\nu)(1-2\varepsilon)R^2g+[1-(1-\nu)\varepsilon]Rh=0 \\
&(1-2\varepsilon)(1-2\nu)(R^2h''+Rh'-h)+R[1-(1-\nu)\varepsilon](Rf'+f+g) \\
&\quad -2R^2[(1-\nu)(1-3\varepsilon)+2\nu^2\varepsilon]h=0
\end{aligned}
\tag{4.85}
$$

where a prime indicates differentiation with respect to R. Three solutions of these equations are possible. One takes the form

$$
f=\frac{A}{R}I_1(KR)\quad,\quad g=-A\frac{\mathrm{d}}{\mathrm{d}R}I_1(KR)\quad,\quad h=0\qquad\text{where}\quad K^2=\frac{1-2\varepsilon}{1+2\nu\varepsilon}
$$

A is an arbitrary constant and I_1 is a modified Bessel function of the first kind. The other two solutions cannot be expressed in terms of known functions. They are given by taking the initial coefficients of the series as

$$
\begin{aligned}
f_o=-g_o=B\quad,\quad f_1=-3g_1&=\frac{3}{8(1-2\varepsilon)}B\left(1-3\varepsilon+\frac{2\nu^2\varepsilon}{1-\nu}\right) \\
\text{or}\quad f_o=-g_o=C\quad,\quad f_1=-3g_1&=\tfrac{3}{8}CK^2
\end{aligned}
$$

where B and C are arbitrary constants and the other coefficients in the power series are determined from the recurrence relationships. These are found by substituting (4.84) into (4.85) and comparing the coefficients of given powers of R.

If the surface of the rod at $r=a$ is always stress free, $\hat{p}^i$ is zero and (4.80) gives

$$
\begin{aligned}
(1+2\nu\varepsilon)[(1-\nu)Rf'+\nu(f+g)]-\nu[1-\varepsilon(1-\nu)]Rh&=0 \\
Rg'-f-g&=0 \\
[1-\varepsilon(1-\nu)]f+(1-2\varepsilon)h'&=0
\end{aligned}
$$

On substituting the above solutions into these equations, the determinant of the coefficients of A, B and C must be zero for buckling to take place. This condition gives the axial buckling load, P_c. The Euler buckling load is given by

$$
P_e=\frac{\pi^3a^4}{4l^2}E
$$

where l is the half-wave length (π/k). A more accurate result, incorporating a shear correction, was derived by Engesser† and takes the form

$$
\frac{P_c}{P_e}=\frac{1}{1+k}
$$

where for a circular section,

$$
k=0.555(1+\nu)\left(\frac{\pi a}{l}\right)^2
$$

Fig. 4.11 shows the results obtained for an isotropic circular rod for which

†Engesser, F. (1891) *Zentr. Bauverwaltung* **11** 483

Poisson's ratio is 0.25. Curve **A** is obtained from the analysis given above and curve **B** is derived from Engesser's formula. The curve given by the generalisation of Bolotin's equations lies between the two. Timoshenko and Gere (1961) derive another expression for the critical load which also takes account of shear deformation, but the predicted values of P_c are considerably higher than those shown in Fig. 4.11, except for small values of a/l. Wilkes† gives an exact solution of the problem for Mooney material. For the neo-Hookean case, this can be compared with the above solution. The results are virtually coincident with curve **A** for values of a/l up to 0.2 and Wilkes' value for P_c is approximately 25% higher when a/l is 0.5.

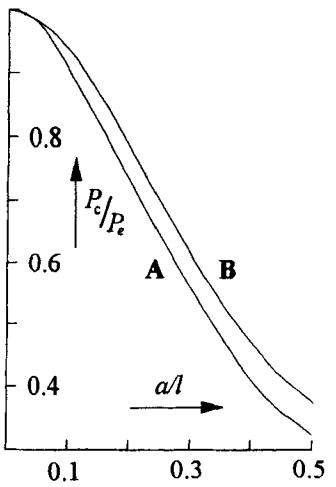


Figure 4.11 Buckling of a cylindrical rod.

It should be noted that the results given by Wilkes are expressed relative to the dimensions of the rod in the deformed state, immediately before buckling. This is a result of choosing the deformed state as the reference configuration, which is quite a common practice.

Example 4.4 Buckling of a thick cylindrical shell under lateral pressure

A long isotropic cylinder will be taken to be subject to an internal critical pressure p_i at its inner radius $r=a$ and an external critical pressure p_o at its outer radius $r=b$ at the point of buckling. Again, the problem is expressed in terms of an embedded coordinate system (r, θ, z). The cylinder will be taken to be in a state of plane strain, and to the degree of accuracy required by the analysis, the small deflection solution is sufficient to give the initial deflected form $\boldsymbol{u}$ so that the contravariant components of $\boldsymbol{u}$ are given by

$$u^r = r(-Ps^2 + Q) \quad , \quad u^\theta = u^z = 0$$

where
$$P = (p_o - p_i)/2\mu[1 - (a/b^2)]$$
$$Q = [p_i(a/b)^2 - p_o]/\{2(\lambda + \mu)[1 - (a/b)^2]\}$$
$$s = a/r$$

A buckling mode $\boldsymbol{v}$ will be chosen with contravariant components

$$v^r = U(s)\cos m\theta \quad , \quad rv^\theta = V(s)\sin m\theta \quad , \quad v^z = 0 \quad . \tag{4.86}$$

In this case, (4.81) imposes the conditions

$$\begin{aligned} &\alpha(s^2U'' + sU' - U - 2mV) - \beta m^2U + m(\alpha - \beta)(V - sV') \\ &\quad + \gamma s^2[s^2U'' + 3sU' + U + mV] = 0 \\ &\beta[s^2V'' + sV' - 2mU - (m^2 + 1)V] + m(\alpha - \beta)(sU' - U - mV) \\ &\quad + \gamma s^2 m(U + mV) = 0 \end{aligned} \tag{4.87}$$

where

†Wilkes, E.W. (1955) *Q. Jl. Mech. appl. Math.* **8** 88-100

$$\alpha = \lambda + 2\mu + 2Q(2\lambda + 3\mu + 2d_1 + 4d_2 + d_3)$$
$$\beta = \mu + Q(2\lambda + 4\mu + 2d_2 + d_3)$$
$$\gamma = 2P(\lambda + 3\mu + 2d_2 + d_3)$$

and primes indicate differentiation with respect to s. Four solutions of these equations are possible. These take the form

$$U_k = s^{-(m+1)}\sum_{i=r}^{\infty} u_{ik}s^{2i} + (c_{3k}U_3 + c_{4k}U_4)\ln s$$
$$V_k = s^{-(m+1)}\sum_{i=r}^{\infty} v_{ik}s^{2i} + (c_{3k}V_3 + c_{4k}V_4)\ln s$$

where k takes the values 1, 2, 3 and 4, and the corresponding values of r are 0, 1, m and $m+1$. The coefficients of u_{ik} and v_{ik} are found from the recurrence relationships given by comparing the powers of s when these expressions are substituted into (4.87). The coefficients c_{3k} and c_{4k} are zero for k equal to 3 or 4. For the two other series, problems arise with the recurrence relationships when i is equal to m and $m+1$ and it necessary to introduce the extra terms with the coefficients c_{3k} and c_{4k} in order to determine u_{ik} and v_{ik}. Even then, the choice of these coefficients is not unique, because any arbitrary multiple of the third and fourth series can be added to the first and second. However, the general solution is the sum of four independent solutions multiplied by arbitrary constants A, B, C and D. The above lack of uniqueness does not affect the nature of the general solution.

Even though the pressures are assumed to remain constant during buckling, the increments in surface traction expressed by $\hat{p}^i$ are not zero. This is because the surface area and the direction of the surface normal change during buckling. If the surface pressure at a radius r is p, then

$$\hat{p}^i \mathbf{g}_i r\,d\theta\,dz = -p[\bar{\mathbf{G}}_\theta \times \bar{\mathbf{G}}_r - \mathbf{G}_\theta \times \mathbf{G}_r]\,d\theta\,dz \qquad \textbf{(4.88)}$$

or, in terms of the mode given by (4.86),

$$\hat{p}^r \approx -\frac{p}{r}(U + mV)\cos m\theta,$$
$$\hat{p}^\theta \approx -\frac{p}{r^2}(mU + V)\sin m\theta\ ,\quad \hat{p}^z = 0.$$

to the degree of accuracy required in the analysis. The surface conditions deduced from (4.80) are then

$$(\alpha - 2\beta - 2\mu P s^2)(U + mV) + (\alpha + \gamma s^2)sU' = 0$$
$$(\beta + 2\mu P s^2)(mU + V) + \beta s V' = 0$$

which apply at s equal to 1 and a/b. Imposing these four conditions on the general solution yields the condition that the determinant of the coefficients of A, B, C and D in these equations must be zero for buckling to take place.

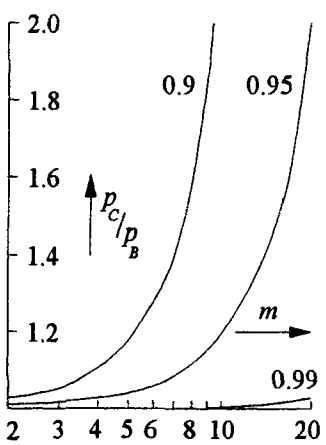


Figure 4.12 Buckling of a cylinder.

Bryan† derived an equation for the

†Bryan, G.H. (1888) *Proc. Camb. Phil. Soc.* **6** 287

buckling pressure, p_B, of a thin cylinder under external pressure only. This can be written in the form

$$p_B = \frac{4\mu(m^2-1)}{3(1-\nu)}\left(\frac{b-a}{b+a}\right)^3$$

The ratio of the critical pressure p_C found by the above analysis to p_B is plotted in Fig. 4.12. Curves are shown for the ratios a/b equal to 0.9, 0.95 and 0.99. The elastic data used is for Hecla 37 steel (see Table 4.2). If the third-order elastic constants d_1, d_2 and d_3 are taken as zero, p_C is always less than p_B.

An approximate buckling pressure for rings can be obtained from the above results by taking μ as common to both the ring and the cylinder and using the relationship

$$\nu = \frac{\nu^*}{1+\nu^*}$$

where ν and ν^* are the values of Poisson's ratio for the cylinder and ring respectively.

Example 4.5 Buckling of a thick spherical shell under pressure

The stability of an isotropic spherical shell with inner and outer radii a and b under internal and external pressures p_i and p_o respectively can be analysed in a similar fashion. The initial deflection u given by small deflection theory, relative to a spherical coordinate system (r, θ, ϕ), has contravariant components

$$u^r = r(-Js^3 + K) \quad , \quad u^\theta = u^\phi = 0$$

where

$$J = (p_o - p_i)\Big/4\mu\left[1-\left(\frac{a}{b}\right)^3\right]$$

$$K = \frac{1-2\nu}{2+2\nu}\left[p_i\left(\frac{a}{b}\right)^3 - p_o\right]\Big/\mu\left[1-\left(\frac{a}{b}\right)^3\right]$$

and

$$s = a/r.$$

The buckling mode will be taken to have the contravariant components

$$v^r = U(s)P_m(\cos\theta) \quad , \quad rv^\theta = V(s)\frac{\mathrm{d}}{\mathrm{d}\theta}[P_m(\cos\theta)] \quad , \quad v^\phi = 0$$

where $P_m(\cos\theta)$ is the mth Legendre polynomial. The equilibrium conditions found from (4.81) using (4.82) can be written in the form

$$\alpha[s^2U'' - 2U + (m^2+m)(V+sV')] - (m^2+m)(\beta+\gamma s^3)[U - V + sV'] + \delta s^3[4U + 12sU' + 4s^2U'' + (m^2+m)(sV' - 2V)] = 0$$

$$\alpha[2U - sU' - (m^2+m)V] + (\beta+\gamma s^3)[sU' + s^2V''] + 3\gamma s^3[U - V + sV'] - \delta s^3[4U + sU' - 2(m^2+m)V] = 0$$

where

$$\alpha = \lambda + 2\mu + K[5\lambda + 6\mu + 6d_1 + 10d_2 + 2d_3]$$
$$\beta = \mu + K[3\lambda + 4\mu + 3d_2 + d_3]$$
$$\gamma = J[2\mu + \tfrac{1}{2}d_3]$$
$$\delta = J[\lambda + 3\mu + 2d_2 + d_3]$$

and primes indicate differentiation with respect to s. Four solutions of these equations are possible and take the form

$$U_k = s^r \sum_{i=0}^{\infty} u_{ik} s^{3i} \quad \{ + c_{jk} U_j \ln s\}$$
$$V_k = s^r \sum_{i=0}^{\infty} v_{ik} s^{3i} \quad \{ + c_{jk} V_j \ln s\}$$

where k takes the values 1, 2, 3 and 4 and the corresponding values of r are $-m-1$, $-m+1$, m and $m+2$. The terms in curly brackets are required for the same reasons as in the previous example and only exist under the following circumstances.

(a) $m = 3n$: $j = 4$, $k = 1$
(b) $m = 3n+1$: $j = 3$, $k = 1$ or $j = 4$, $k = 2$
(c) $m = 3n+2$: $j = 3$, $k = 2$

where n is an integer.

The conditions at the surfaces found from (4.80) are

$$[\alpha - 2\beta + s^3(\delta - 2\gamma - 3\mu J)][2U - (m^2+m)V] - (\alpha + 4s^3\delta)sU' = 0$$
$$[\beta + s^3(\gamma + 3\mu J)](U - V) - (\beta + \gamma s^3)sV' = 0$$

where s is equal to 1 and a/b at the inner and outer surfaces respectively and $\hat{p}^i$ has been obtained in a manner similar to that in the previous example. The sum of the four solutions, multiplied by arbitrary constants, can be used to satisfy these four equations. For a non-zero buckling mode, the determinant of the coefficients of these constants must be zero, giving the buckling criterion. Zoelly† derived an expression for the external buckling pressure p_z on a thin spherical shell. This takes the form

$$p_Z = 16\mu \left(\frac{3\lambda + 2\mu}{3(\lambda + 2\mu)} \right)^{1/2} \left(\frac{b-a}{b+a} \right)^2$$

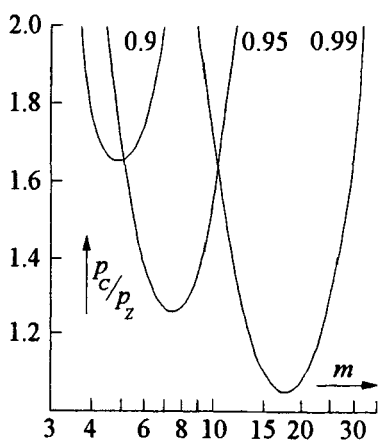


Figure 4.13 Buckling of a spherical shell.

where the buckling mode is again expressed in terms of a Legendre polynomial $P_m(\cos\theta)$, the value of m being chosen to minimise p_z. Fig. 4.13 shows the ratio of p_c found from the above analysis to p_z, for spherical shells made from Hecla 37 steel (see Table 4.2). Results for three ratios of the internal and external radii, a/b, equal to 0.9, 0.95 and 0.99 are shown. If the third-order elastic constants are taken as zero, the minima on these curves are at the same values of m, but the corresponding ratios p_c/p_z are less than unity.

†Zoelly, R. (1915) Über ein Knickungsproblem an die Kugelschale (Dissertation) Zurich

The buckling pressures of thin shells are considerably reduced by any imperfections they may have. A full discussion of this is given by Thompson and Hunt [(1984) sections 8.6 and 8.7].

PROBLEMS

Many of these problems can be solved more readily by arranging the tensor elements in matrix form.

(4.1) The displacement field in an elastic body is given by

$$\boldsymbol{u} = a_1(x_2 + x_3)\boldsymbol{i}_1 + a_2(x_3 + x_1)\boldsymbol{i}_2 + a_3(x_1 + x_2)\boldsymbol{i}_3$$

relative to the initial Cartesian coordinates of a material point. Find the Lagrangean strains induced, directly from (4.3) and also from (4.14). (M)

(4.2) A material coordinate system, initially coincident with the Cartesian system, is embedded in the above body. Find the base vectors $\boldsymbol{G}_i$ and $\boldsymbol{G}^j$ of this system in the deformed state and the covariant elements of displacement U_i relative to this state. (M)

(4.3) Using the above results and taking $a_1=1$, $a_2=2$ and $a_3=3$, confirm that (4.15) gives the same strains as those found in problem (4.1). (L)

(4.4) With respect to fixed spatial Cartesian coordinates X_I, coincident with the above coordinates x_i in the initial state, find the Eulerian strains η_{ij}. (M)

(4.5) Again taking $a_1=1$, $a_2=2$ and $a_3=3$, evaluate the mixed forms of the strain tensor γ^{α}_{α} and η^{α}_{α} with respect to the spatial coordinates. Hence find the invariants γ^i_i and η^i_i. (M)

(4.6) Find the numerical values of the invariants J_i, K_i and I_i for the above problem and check that the different expressions for I_i give the same result. (L)

(4.7) Show that the principal stretches are given by the roots of the equation

$$\lambda^6 - I_1\lambda^4 + I_2\lambda^2 - I_3 = 0$$

and hence find their numerical values for the above problem (cf. section 1.6.6). (M)

(4.8) Of what matrix is the equation in problem (4.7) the characteristic equation? (S)

(4.9) Derive an expression for the strain invariant I_3 in terms of the scalar functions of γ^i_j. (M)

(4.10) Find the expression for the strain energy potential U of an isotropic elastic material in terms of Lamé's constants, Murnaghan's coefficients l, m n and the invariants K_i. (M)

(4.11) Find the form of (4.37) if the coordinate system is Cartesian. (S)

(4.12) The elastic body examined in problems (4.1) to (4.7) is isotropic and its elastic constants λ, μ, d_1, d_2 and d_3 are 35, 30, -55, -80 and -230 GPa respectively. Find the

stresses s^{ij} predicted from (4.37) and comment on your answer. (M)

(4.13) What are the corresponding stresses t^{ij} in the above problem? (M)

(4.14) In the above problem, what is the true force $\boldsymbol{p}$ per unit area acting on a plane perpendicular to the (spatial) X_1 coordinate line? (S)

(4.15) What relationships between the constants a_i in problem (4.1) must hold for the deformation to be **isochoric** (i.e. constant volume)? (S)

(4.16) A physical definition of large strain given by Novozhilov (1961) is that $\gamma^N_{\alpha\alpha}$ is the extension of a fibre, initially in the direction of the x_α axis, divided by its original length, and $\gamma^N_{\alpha\beta}$ $(\alpha \neq \beta)$ is the change in angle between fibres initially in the directions of the x_α and x_β axes. (The initial coordinates are taken to be Cartesian.) Obtain expressions for these in terms of the displacement rates. Do these strains form the elements of a second-order tensor? (M)

(4.17) An isotropic elastic body of initial volume V_o has an energy potential given by the first five terms on the right-hand side of (4.36). Under a uniform pressure p its volume becomes V. Show that if $\alpha = (V_o/V)^3$, then

$$\alpha p = \tfrac{1}{2}(3\lambda + 2\mu)(1 - \alpha^2) - \tfrac{1}{4}(9d_1 + 9d_2 + d_3)(1 - \alpha^2)^2. \text{ (M)}$$

(4.18) A thick spherical shell of initial internal and external radii a and b is made from an isotropic incompressible material. It is subject to uniform internal and external pressures, so that so that its internal radius becomes A after deformation. Find its external radius B after deformation and the principal stretches in the radial and tangential directions at a point initially at a radius r. (S)

(4.19) An isotropic elastic body undergoes an initial uniform compression e so that relative to Cartesian coordinates, $u^i_{|j} = -e\delta^i_j$. Find the relationship between the increments of stress $\hat{t}^{ji}$ and the increments of displacement gradient $v^i_{|j}$. Under what conditions is this relationship the same as that for a linearly-elastic isotropic body? What are then the apparent values λ', μ' of the Lamé constants? Compare those found in this way with those found in problem (2.16). (M)

(4.20) Explain qualitatively why the solutions to stability problems, taking into account the negative third-order elastic constants, might be expected to give higher buckling loads than the solutions omitting these terms. (S)

A1

Formulae for orthogonal coordinate systems

For orthogonal coordinate systems, h_α is expressed in terms of metric tensors as

$$h_\alpha^2 = g_{\alpha\alpha} = 1/g^{\alpha\alpha}$$

from (3.77) and (3.81), where an element of the metric tensor is zero unless its indices are equal. The Christoffel three-index symbol given by (3.118) is

$$\Gamma_{jk}^{i} = \tfrac{1}{2}g^{il}(g_{jl,k} + g_{kl,j} - g_{jk,l}) \qquad \textbf{(A1.1)}$$

It follows that for orthogonal coordinate systems, the non-zero Christoffel symbols fall into three categories:

$$\Gamma_{\alpha\alpha}^{\alpha} = \tfrac{1}{2}g_{\alpha\alpha,\alpha}/g_{\alpha\alpha} = h_{\alpha,\alpha}/h_\alpha \qquad \textbf{(A1.2)}$$

$$\Gamma_{\alpha\beta}^{\alpha} = \Gamma_{\beta\alpha}^{\alpha} = \tfrac{1}{2}g_{\alpha\alpha,\beta}/g_{\alpha\alpha} = h_{\alpha,\beta}/h_\alpha \qquad \textbf{(A1.3)}$$

$$\Gamma_{\beta\beta}^{\alpha} = -\tfrac{1}{2}g_{\beta\beta,\alpha}/g_{\alpha\alpha} = -h_\beta h_{\beta,\alpha}/h_\alpha^2 \qquad \textbf{(A1.4)}$$

where $\alpha \neq \beta$.

From (3.119),

$$\Gamma_{\beta\gamma}^{\alpha} = \boldsymbol{g}^{\alpha}\cdot\boldsymbol{g}_{\beta,\gamma} = \frac{1}{h_\alpha}\hat{\boldsymbol{g}}_\alpha\cdot(h_\beta\hat{\boldsymbol{g}}_\beta)_{,\gamma} \qquad \textbf{(A1.5)}$$

or

$$\Gamma_{\beta\gamma}^{\alpha} = \frac{h_\beta}{h_\alpha}\hat{\boldsymbol{g}}_\alpha\cdot\hat{\boldsymbol{g}}_{\beta,\gamma} \qquad (\alpha \neq \beta) \qquad \textbf{(A1.6)}$$

where $\hat{\boldsymbol{g}}_\alpha$ is a unit vector in the direction of $\boldsymbol{g}_\alpha$ (or $\boldsymbol{g}^\alpha$). Taking $\alpha=\beta$ in (A1.5), it follows from (A1.3) that $\hat{\boldsymbol{g}}_\alpha\cdot\hat{\boldsymbol{g}}_{\alpha,\gamma}$ is zero.

The divergence of a vector in terms of physical components can now be expressed by means of the above equations in the form

$$\nabla\cdot\boldsymbol{v} = \Sigma_\alpha\Sigma_\beta\left(\hat{\boldsymbol{g}}_\alpha\frac{1}{h_\alpha}\frac{\partial}{\partial y^\alpha}\right)\cdot(v_\beta^{\mathrm{P}}\hat{\boldsymbol{g}}_\beta) = \Sigma_\alpha(v_{\alpha,\alpha}^{\mathrm{P}} + \Sigma_\beta v_\beta^{\mathrm{P}}\hat{\boldsymbol{g}}_\alpha\cdot\hat{\boldsymbol{g}}_{\beta,\alpha})/h_\alpha$$
$$= \Sigma_\alpha(v_{\alpha,\alpha}^{\mathrm{P}} + \Sigma_\beta v_\beta^{\mathrm{P}}h_{\beta,\alpha}/h_\beta)/h_\alpha$$

where Σ_α implies summation over all values of α and so on. Because

$$h_1 h_2 h_3 = \sqrt{g}$$

the above expression for the divergence of a vector can be reduced to the compact form

$$\nabla \cdot \boldsymbol{v} = \frac{1}{\sqrt{g}} \Sigma_\alpha \left(\sqrt{g} \frac{v_\alpha^{\mathrm{P}}}{h_\alpha} \right)_{,\alpha} \tag{A1.7}$$

From (3.114), the derivatives of a scalar ϕ form a first-order covariant tensor. Thus from (3.92), the physical components of grad ϕ are

$$\nabla \phi_\alpha^{\mathrm{P}} = \phi_{,\alpha} / h_\alpha \tag{A1.8}$$

The Laplacian of a scalar, $\nabla^2 \phi$, can now be found from (A1.7) and (A1.8) and is given by

$$\nabla^2 \phi = \frac{1}{\sqrt{g}} \Sigma_\alpha \left(\sqrt{g} \frac{\phi_{,\alpha}}{h_\alpha^2} \right)_{,\alpha} \tag{A1.9}$$

The curl of a vector, in terms of physical components, is

$$\begin{aligned} \nabla \times \boldsymbol{v} &= \Sigma_\alpha \Sigma_\beta \left(\frac{\hat{\boldsymbol{g}}_\alpha}{h_\alpha} \frac{\partial}{\partial y^\alpha} \right) \times \left(h_\beta v_\beta^{\mathrm{P}} \frac{\hat{\boldsymbol{g}}_\beta}{h_\beta} \right) \\ &= \Sigma_\alpha \Sigma_\beta \left[\Sigma_\gamma \left\{ \hat{\boldsymbol{g}}_\gamma e_{\gamma\alpha\beta} \frac{1}{h_\alpha h_\beta} (h_\beta v_\beta^{\mathrm{P}})_{,\alpha} \right\} + h_\beta v_\beta^{\mathrm{P}} \frac{\hat{\boldsymbol{g}}_\alpha}{h_\alpha} \times \left(\frac{\hat{\boldsymbol{g}}_\beta}{h_\beta} \right)_{,\alpha} \right] \end{aligned} \tag{A1.10}$$

From (3.105),

$$(\hat{\boldsymbol{g}}_\beta / h_\beta)_{,\alpha} = - \Sigma_\gamma \Gamma^\beta_{\gamma\alpha} \hat{\boldsymbol{g}}_\gamma / h_\gamma$$

and from (A1.2) to (A1.4) it then follows that the sum of the last cross products in (A1.10) is zero, giving

$$\nabla \times \boldsymbol{v} = \Sigma_\alpha \Sigma_\beta \Sigma_\gamma \left\{ e_{\alpha\beta\gamma} \frac{1}{h_\alpha h_\beta} (h_\beta v_\beta^{\mathrm{P}})_{,\alpha} \right\} \hat{\boldsymbol{g}}_\gamma \tag{A1.11}$$

From (3.92), (3.107), (3.133) and (A1.2) to (A1.4) it is possible to deduce that the physical components of the small strain tensor are

$$\begin{aligned} e_{\alpha\alpha}^{\mathrm{P}} &= (u_{\alpha,\alpha}^{\mathrm{P}} + u_\beta^{\mathrm{P}} h_{\alpha,\beta} / h_\beta + u_\gamma^{\mathrm{P}} h_{\alpha,\gamma} / h_\gamma) / h_\alpha \\ e_{\alpha\beta}^{\mathrm{P}} &= \tfrac{1}{2} (h_\alpha u_{\alpha,\beta}^{\mathrm{P}} + h_\beta u_{\beta,\alpha}^{\mathrm{P}} - h_{\alpha,\beta} u_\alpha^{\mathrm{P}} - h_{\beta,\alpha} u_\beta^{\mathrm{P}}) / h_\alpha h_\beta \quad (\alpha \neq \beta \neq \gamma \neq \alpha) \end{aligned} \tag{A1.12}$$

A1.1 Cylindrical coordinates

The indices 1, 2 and 3 correspond to the ordinates r, θ and z respectively, shown in Fig. A1.1.

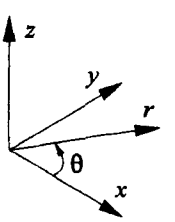


Figure A1.1
Cylindrical polar coordinates.

Base vectors:

$$\begin{aligned} \boldsymbol{g}^r &= \boldsymbol{g}_r = \boldsymbol{i}\cos\theta + \boldsymbol{j}\sin\theta \\ r^2 \boldsymbol{g}^\theta &= \boldsymbol{g}_\theta = r(-\boldsymbol{i}\sin\theta + \boldsymbol{j}\cos\theta) \\ \boldsymbol{g}^z &= \boldsymbol{g}_z = \boldsymbol{k} \end{aligned}$$

Metric tensors:

$$g^{rr} = g^{zz} = 1 \ , \ g^{\theta\theta} = 1/r^2$$
$$g_{rr} = g_{zz} = 1 \ , \ g_{\theta\theta} = r^2$$
$$h_r = h_z = 1 \ , \ h_\theta = r$$

Physical components:

$$v_r^{\mathrm{P}} = v^r = v_r$$
$$v_\theta^{\mathrm{P}} = v^\theta r = v_\theta / r$$
$$v_z^{\mathrm{P}} = v^z = v_z$$

Non-zero Christoffel symbols:

$$\Gamma^r_{\theta\theta} = -r \ , \ \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = 1/r$$

Vector expressions:

$$\nabla\psi = \mathrm{grad}\,\psi = \psi_{,r}\hat{\boldsymbol{g}}_r + \frac{1}{r}\psi_{,\theta}\hat{\boldsymbol{g}}_\theta + \psi_{,z}\hat{\boldsymbol{g}}_z$$

$$\nabla\cdot\boldsymbol{v} = \mathrm{div}\,\boldsymbol{v} = \frac{1}{r}(rv_r^{\mathrm{P}})_{,r} + \frac{1}{r}v_\theta^{\mathrm{P}}{}_{,\theta} + v_z^{\mathrm{P}}{}_{,z}$$

$$\nabla\times\boldsymbol{v} = \mathrm{curl}\,\boldsymbol{v} = \left(\frac{1}{r}v_z^{\mathrm{P}}{}_{,\theta} - v_\theta^{\mathrm{P}}{}_{,z}\right)\hat{\boldsymbol{g}}_r + (v_r^{\mathrm{P}}{}_{,z} - v_z^{\mathrm{P}}{}_{,r})\hat{\boldsymbol{g}}_\theta + \frac{1}{r}[(rv_\theta^{\mathrm{P}})_{,r} - v_r^{\mathrm{P}}{}_{,\theta}]\hat{\boldsymbol{g}}_z$$

Tensor expressions:

$$g^{jk}v^1_{|jk} = \nabla^2 v^r - \frac{1}{r^2}v^r - \frac{2}{r}v^\theta{}_{,\theta}$$
$$g^{jk}v^2_{|jk} = \nabla^2 v^\theta + \frac{2}{r}v^\theta{}_{,r} + \frac{2}{r^3}v^r{}_{,\theta}$$
$$g^{jk}v^3_{|jk} = \nabla^2 v^z$$
$$g^{jk}\psi_{|jk} = \nabla^2\psi$$
$$\text{where} \quad \nabla^2 f = f_{,rr} + \frac{1}{r}f_{,r} + \frac{1}{r^2}f_{,\theta\theta} + f_{,zz} \qquad \textbf{(A1.13)}$$

Small strains:

$$e_{rr}^{\mathrm{P}} = u_r^{\mathrm{P}}{}_{,r} \ , \ e_{r\theta}^{\mathrm{P}} = \frac{1}{2r}(u_r^{\mathrm{P}}{}_{,\theta} + r\,u_\theta^{\mathrm{P}}{}_{,r} - u_\theta^{\mathrm{P}})$$
$$e_{rz}^{\mathrm{P}} = \frac{1}{2}(u_r^{\mathrm{P}}{}_{,z} + u_z^{\mathrm{P}}{}_{,r}) \ , \ e_{\theta\theta}^{\mathrm{P}} = \frac{1}{r}(u_\theta^{\mathrm{P}}{}_{,\theta} + u_r^{\mathrm{P}})$$
$$e_{\theta z}^{\mathrm{P}} = \frac{1}{2}\left(\frac{1}{r}u_z^{\mathrm{P}}{}_{,\theta} + u_\theta^{\mathrm{P}}{}_{,z}\right) \ , \ e_{zz}^{\mathrm{P}} = u_z^{\mathrm{P}}{}_{,z}$$

Equations of linear motion:

For small deflection theory, allowing for the asymmetry of stress found in Cosserat continua, see Appendix A6.

$$\sigma_{rr,r}^{\mathrm{P}} + \frac{1}{r}\sigma_{\theta r,\theta}^{\mathrm{P}} + \sigma_{zr,z}^{\mathrm{P}} + \frac{1}{r}(\sigma_{rr}^{\mathrm{P}} - \sigma_{\theta\theta}^{\mathrm{P}}) + \rho f_r^{\mathrm{P}} = \rho\ddot{u}_r^{\mathrm{P}}$$
$$\sigma_{r\theta,r}^{\mathrm{P}} + \frac{1}{r}\sigma_{\theta\theta,\theta}^{\mathrm{P}} + \sigma_{z\theta,z}^{\mathrm{P}} + \frac{1}{r}(\sigma_{r\theta}^{\mathrm{P}} + \sigma_{\theta r}^{\mathrm{P}}) + \rho f_\theta^{\mathrm{P}} = \rho\ddot{u}_\theta^{\mathrm{P}}$$
$$\sigma_{rz,r}^{\mathrm{P}} + \frac{1}{r}\sigma_{\theta z,\theta}^{\mathrm{P}} + \sigma_{zz,z}^{\mathrm{P}} + \frac{1}{r}\sigma_{rz}^{\mathrm{P}} + \rho f_z^{\mathrm{P}} = \rho\ddot{u}_z^{\mathrm{P}}$$

A1.2 Spherical coordinates

The indices 1, 2 and 3 correspond to the ordinates r, θ and ϕ respectively, shown in Fig. A1.2.

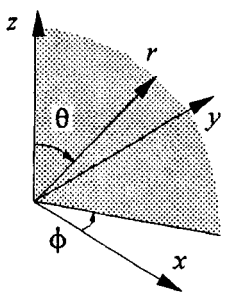


Figure A1.2
Spherical coordinates.

Base vectors:

$$\boldsymbol{g}^r = \boldsymbol{g}_r = \boldsymbol{i}\sin\theta\cos\phi + \boldsymbol{j}\sin\theta\sin\phi + \boldsymbol{k}\cos\theta$$
$$r^2\boldsymbol{g}^\theta = \boldsymbol{g}_\theta = r(\boldsymbol{i}\cos\theta\cos\phi + \boldsymbol{j}\cos\theta\sin\phi - \boldsymbol{k}\sin\theta)$$
$$r^2\sin^2\theta\,\boldsymbol{g}^\phi = \boldsymbol{g}_\phi = r\sin\theta(-\boldsymbol{i}\sin\phi + \boldsymbol{j}\cos\phi)$$

Metric tensors:

$$g^{rr} = 1 \;,\; g^{\theta\theta} = 1/r^2 \;,\; g^{\phi\phi} = 1/r^2\sin^2\theta$$
$$g_{rr} = 1 \;,\; g_{\theta\theta} = r^2 \;,\; g_{\phi\phi} = r^2\sin^2\theta$$
$$h_r = 1 \;,\; h_\theta = r \;,\; h_\phi = r\sin\theta$$

Physical components:

$$v_r^{\mathrm{P}} = v^r = v_r$$
$$v_\theta^{\mathrm{P}} = v^\theta r = v_\theta/r$$
$$v_\phi^{\mathrm{P}} = v^\phi r\sin\theta = v_\phi/r\sin\theta$$

Non-zero Christoffel symbols:

$$\Gamma^r_{\theta\theta} = -r \;,\; \Gamma^r_{\phi\phi} = -r\sin^2\theta \;,\; \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = 1/r$$
$$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta \;,\; \Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = 1/r \;,\; \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta$$

Vector expressions:

$$\nabla\psi = \mathrm{grad}\,\psi = \psi_{,r}\hat{\boldsymbol{g}}_r + \frac{1}{r}\psi_{,\theta}\hat{\boldsymbol{g}}_\theta + \frac{1}{r}\mathrm{cosec}\,\theta\,\psi_{,\phi}\hat{\boldsymbol{g}}_\phi$$
$$\nabla\cdot\boldsymbol{v} = \mathrm{div}\,\boldsymbol{v} = \frac{1}{r^2}(r^2v_r^{\mathrm{P}})_{,r} + \frac{1}{r}\mathrm{cosec}\,\theta[(v_\theta^{\mathrm{P}}\sin\theta)_{,\theta} + v_\phi^{\mathrm{P}}{}_{,\phi}]$$
$$\nabla\times\boldsymbol{v} = \mathrm{curl}\,\boldsymbol{v} = \frac{1}{r}\mathrm{cosec}\,\theta[(v_\phi^{\mathrm{P}}\sin\theta)_{,\theta} - v_\theta^{\mathrm{P}}{}_{,\phi}]\hat{\boldsymbol{g}}_r + \frac{1}{r}[\mathrm{cosec}\,\theta\,v_r^{\mathrm{P}}{}_{,\phi} - (rv_\phi^{\mathrm{P}})_{,r}]\hat{\boldsymbol{g}}_\theta + \frac{1}{r}[(rv_\theta^{\mathrm{P}})_{,r} - v_r^{\mathrm{P}}{}_{,\theta}]\hat{\boldsymbol{g}}_\phi$$

Tensor expressions:

$$g^{jk}v^1_{|jk} = \nabla^2v^r - \frac{2}{r}\left(\frac{1}{r}v^r + v^\theta{}_{,\theta} + v^\theta\cot\theta + v^\phi{}_{,\phi}\right)$$
$$g^{jk}v^2_{|jk} = \nabla^2v^\theta + \frac{2}{r^3}[v^r{}_{,\theta} + r^2v^\theta{}_{,r} + \tfrac{1}{2}rv^\theta(1-\cot^2\theta) - rv^\phi{}_{,\phi}\cot\theta]$$
$$g^{jk}v^3_{|jk} = \nabla^2v^\phi + \frac{2}{r^3}[\mathrm{cosec}^2\theta(v^r{}_{,\phi} + rv^\theta{}_{,\phi}\cot\theta) + r^2v^\phi{}_{,r} + rv^\phi{}_{,\theta}\cot\theta]$$
$$g^{jk}\psi_{|jk} = \nabla^2\psi$$

where $\nabla^2 f = \frac{1}{r^2}[(r^2f_{,r})_{,r} + f_{,\theta\theta} + f_{,\theta}\cot\theta + f_{,\phi\phi}\mathrm{cosec}^2\theta]$ **(A1.14)**

Small strains:

$$e^{\mathrm{P}}_{rr} = u^{\mathrm{P}}_{r,r} \quad , \quad e^{\mathrm{P}}_{r\theta} = \tfrac{1}{2r}(u^{\mathrm{P}}_{r,\theta} + r u^{\mathrm{P}}_{\theta,r} - u^{\mathrm{P}}_{\theta})$$

$$e^{\mathrm{P}}_{r\phi} = \tfrac{1}{2r}(u^{\mathrm{P}}_{r,\phi}\operatorname{cosec}\theta + r u^{\mathrm{P}}_{\phi,r} - u^{\mathrm{P}}_{\phi}) \quad , \quad e^{\mathrm{P}}_{\theta\theta} = \tfrac{1}{r}(u^{\mathrm{P}}_{\theta,\theta} + u^{\mathrm{P}}_{r})$$

$$e^{\mathrm{P}}_{\theta\phi} = \tfrac{1}{2r}(u^{\mathrm{P}}_{\theta,\phi}\operatorname{cosec}\theta + u^{\mathrm{P}}_{\phi,\theta} - u^{\mathrm{P}}_{\phi}\cot\theta)$$

$$e^{\mathrm{P}}_{\phi\phi} = \tfrac{1}{r}(u^{\mathrm{P}}_{\phi,\phi}\operatorname{cosec}\theta + u^{\mathrm{P}}_{\theta}\cot\theta + u^{\mathrm{P}}_{r})$$

Equations of linear motion:

For small deflection theory, allowing for the asymmetry of stress found in Cosserat continua, see Appendix A6.

$$\sigma^{\mathrm{P}}_{rr,r} + \tfrac{1}{r}[\sigma^{\mathrm{P}}_{\theta r,\theta} + \sigma^{\mathrm{P}}_{\phi r,\phi}\operatorname{cosec}\theta + 2\sigma^{\mathrm{P}}_{rr} - \sigma^{\mathrm{P}}_{\theta\theta} - \sigma^{\mathrm{P}}_{\phi\phi} + \sigma^{\mathrm{P}}_{\theta r}\cot\theta] + \rho f^{\mathrm{P}}_{r} = \rho\ddot{u}^{\mathrm{P}}_{r}$$

$$\sigma^{\mathrm{P}}_{r\theta,r} + \tfrac{1}{r}[\sigma^{\mathrm{P}}_{\theta\theta,\theta} + \sigma^{\mathrm{P}}_{\phi\theta,\phi}\operatorname{cosec}\theta + 2\sigma^{\mathrm{P}}_{r\theta} + \sigma^{\mathrm{P}}_{\theta r} + (\sigma^{\mathrm{P}}_{\theta\theta} - \sigma^{\mathrm{P}}_{\phi\phi})\cot\theta] + \rho f^{\mathrm{P}}_{\theta} = \rho\ddot{u}^{\mathrm{P}}_{\theta}$$

$$\sigma^{\mathrm{P}}_{r\phi,r} + \tfrac{1}{r}[\sigma^{\mathrm{P}}_{\theta\phi,\theta} + \sigma^{\mathrm{P}}_{\phi\phi,\phi}\operatorname{cosec}\theta + 2\sigma^{\mathrm{P}}_{r\phi} + \sigma^{\mathrm{P}}_{\phi r} + (\sigma^{\mathrm{P}}_{\phi\theta} + \sigma^{\mathrm{P}}_{\theta\phi})\cot\theta] + \rho f^{\mathrm{P}}_{\phi} = \rho\ddot{u}^{\mathrm{P}}_{\phi}$$

A1.3 Curvilinear anisotropy

Curvilinear anisotropy is said to occur when the elastic properties of the material appear to be uniform with respect to the local directions of a curvilinear coordinate system. The partial differentials of a quantity indicate the change in the quantity with respect to the local coordinate system. Thus uniformity of a curvilinearly anisotropic material is indicated by the partial differentials of the stiffness and compliance tensors being zero with respect to the appropriate coordinate system. Some natural materials exhibit curvilinear anisotropy, but the subject has become more relevant with the introduction of strong, fibre-reinforced materials. Possibly the earliest work on the subject was by Saint-Venant† who examined cases of cylindrical and spherical anisotropy. In these cases, uniform general anisotropy cannot exist if the polar axis of the coordinate system passes through the material.

For cylindrical polar coordinates (r, θ, z) where the z axis is along the beam, (1.154) becomes

$$u^{\mathrm{P}}_{r} = \tfrac{1}{2}z^2(\vartheta_y\cos\theta - \vartheta_x\sin\theta) + U_r(r,\theta)$$

$$u^{\mathrm{P}}_{\theta} = rz\vartheta_z - \tfrac{1}{2}z^2(\vartheta_y\sin\theta + \vartheta_x\cos\theta) + U_\theta(r,\theta)$$

$$u^{\mathrm{P}}_{z} = z(\varepsilon_z - r\vartheta_y\cos\theta + r\vartheta_x\sin\theta) + W(r,\theta)$$

giving

$$e^{\mathrm{P}}_{rr} = U_{r,r} \quad , \quad e^{\mathrm{P}}_{\theta\theta} = \tfrac{1}{r}(U_{\theta,\theta} + U_r) \quad , \quad e^{\mathrm{P}}_{zz} = \varepsilon_z - r\vartheta_y\cos\theta + r\vartheta_x\sin\theta$$

$$\gamma_{\theta z} = 2e^{\mathrm{P}}_{\theta z} = \tfrac{1}{r}W_{,\theta} + r\vartheta_z \quad , \quad \gamma_{rz} = 2e^{\mathrm{P}}_{rz} = W_{,r} \quad , \quad \gamma_{r\theta} = 2e^{\mathrm{P}}_{r\theta} = \tfrac{1}{r}U_{r,\theta} + U_{\theta,r} - \tfrac{1}{r}U_\theta$$

Because there are no variations of stress along the z axis, the equations of linear motion in terms of physical components of stress become

†Saint-Venant, B. (1865) *J. math. pures et appl. (Liouville) Ser. II* **10** 297-349

$$\sigma^P_{rr,r} + \frac{1}{r}(\sigma^P_{r\theta,\theta} + \sigma^P_{rr} - \sigma^P_{\theta\theta}) = 0$$
$$\sigma^P_{r\theta,r} + \frac{1}{r}(\sigma^P_{\theta\theta,\theta} + 2\sigma^P_{r\theta}) = 0$$
$$\sigma^P_{rz,r} + \frac{1}{r}(\sigma^P_{\theta z,\theta} + \sigma^P_{rz}) = 0$$

in the absence of body forces and accelerations.

The lateral surfaces of a circular rod are automatically stress free if

$$\sigma^P_{rr} = \sigma^P_{r\theta} = \sigma^P_{rz} = 0$$

From the above equations of linear motion, it then follows that

$$\sigma^P_{\theta\theta} = 0 \quad , \quad \sigma^P_{\theta z} = f(r)$$

and that σ^P_{zz} can be some general function of r and θ. For the compliances given by

$$\begin{bmatrix} e^P_{rr} \\ e^P_{\theta\theta} \\ e^P_{zz} \\ \gamma_{\theta z} \\ \gamma_{zr} \\ \gamma_{r\theta} \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} & s_{15} & s_{16} \\ s_{21} & s_{22} & s_{23} & s_{24} & s_{25} & s_{26} \\ s_{31} & s_{32} & s_{33} & s_{34} & s_{35} & s_{36} \\ s_{41} & s_{42} & s_{43} & s_{44} & s_{45} & s_{46} \\ s_{51} & s_{52} & s_{53} & s_{54} & s_{55} & s_{56} \\ s_{61} & s_{62} & s_{63} & s_{64} & s_{65} & s_{66} \end{bmatrix} \begin{bmatrix} \sigma^P_{rr} \\ \sigma^P_{\theta\theta} \\ \sigma^P_{zz} \\ \sigma^P_{\theta z} \\ \sigma^P_{zr} \\ \sigma^P_{r\theta} \end{bmatrix}$$

where the matrix is symmetric, as in (1.122), a simple solution exists if

$$s_{34} = s_{35} = s_{36} = s_{46} = 0 \quad , \quad s_{13} = s_{23} = 0 \quad , \quad s_{14} = 2s_{24}$$

Then

$$\sigma^P_{\theta z} = \frac{1}{s_{44}}(r\vartheta_z) \quad , \quad \sigma^P_{zz} = \frac{1}{s_{33}}(\varepsilon_z - r\vartheta_y\cos\theta + r\vartheta_x\sin\theta)$$

$$U_r = \frac{rs_{13}}{2s_{33}}(2\varepsilon_z - r\vartheta_y\cos\theta + r\vartheta_x\sin\theta) + \frac{r^2 s_{14}}{2s_{44}}\vartheta_z$$

$$U_\theta = -\frac{r^2 s_{13}}{2s_{33}}(\vartheta_y\sin\theta + \vartheta_x\cos\theta) \quad , \quad W = \frac{r^2 s_{45}}{2s_{44}}\vartheta_z$$

This behaviour is very similar to that of an isotropic beam. Lekhnitskii [(1981) sections 10, 17, 42, 43 and 60] has examined problems of curvilinear anisotropy. He derives solutions to problems of cylindrical anisotropy for bars under the action of bending moments, torques and axial loads and for tubes under lateral pressure. The bending of cylindrically orthotropic plates has been analysed by Rubin†.

†Rubin, C. (1983) *Proc. A.S.C.E. EMI* **109** 168-174

A2

Harmonic and biharmonic functions

A2.1 The two-dimensional case

A general differentiable function $F(z,\bar{z})$ can be regarded as a function expressed in terms of a complex variable z and its complex conjugate $\bar{z}$ where

$$z = x + iy \quad , \quad z = x - iy$$

Any differentiation of $F(z,\bar{z})$ can be expressed in terms of the partial differentials $\partial F/\partial z$ and $\partial F/\partial \bar{z}$ so that

$$\frac{\partial F}{\partial x} = \frac{\partial z}{\partial x}\frac{\partial F}{\partial z} + \frac{\partial \bar{z}}{\partial x}\frac{\partial F}{\partial \bar{z}} = \left(\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}\right)F$$

$$\frac{\partial F}{\partial y} = \frac{\partial z}{\partial y}\frac{\partial F}{\partial z} + \frac{\partial \bar{z}}{\partial x}\frac{\partial F}{\partial \bar{z}} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right)F$$

It follows that

$$\nabla^2 F = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)F = 4\frac{\partial^2}{\partial z \partial \bar{z}}F$$

An **analytic function** $f(z)$ is a differentiable function which can be expressed in terms of z alone, so that it satisfies **Laplace's** equation

$$\nabla^2 f = 0$$

because it is not a function of $\bar{z}$. Moreover, if it is expressed in terms of its real and imaginary parts,

$$f(z) = g(x,y) + ih(x,y)$$

then each must satisfy Laplace's equation separately, that is

$$\nabla^2 g = \nabla^2 h = 0$$

Functions which satisfy Laplace's equation are known as **harmonic functions**. Some harmonic functions found in this way are given in the following table.

Table A2.1 Harmonic functions $g(x,y)$, $h(x,y)$ derived from analytic functions $f(z)$.

$f(z)$	$g(x,y)$	$h(x,y)$
z^2	$x^2 - y^2$	$2xy$
z^3	$x^3 - 3xy^2$	$3x^2y - y^3$
z^n	$x^n - {}^nC_2 x^{n-2}y^2 + {}^nC_4 x^{n-4}y^4 \ldots = r^n \cos n\theta$†	$r^n \sin n\theta$†
$\ln z$	$\ln r$†	θ†
e^{mz}	$e^{mx} \cos my$	$e^{mx} \sin my$
$\cos mz$	$\cos mx \cosh my$	$-\sin mx \sinh my$
$\sin mz$	$\sin mx \cosh my$	$\cos mx \sinh my$

A **biharmonic function** $B(x,y)$ is one which satisfies the biharmonic equation

$$\nabla^4 B = \nabla^2(\nabla^2 B) = 0$$

All harmonic functions are necessarily biharmonic functions. Also, if $f(z)$ is an analytic function, then from (A2.1),

$$\nabla^2[\bar{z}f(z)] = 4\frac{\partial f}{\partial z}$$

Then

$$\nabla^4[\bar{z}f(z)] = \nabla^2\left(4\frac{\partial f}{\partial z}\right) = 4\frac{\partial}{\partial z}(\nabla^2 f) = 0$$

Thus $\bar{z}f(z)$ is a biharmonic function, and if it is expressed in terms of its real and imaginary parts,

$$\bar{z}f(z) = p(x,y) + iq(x,y)$$

it follows that these too must be biharmonic functions. Some biharmonic functions thus found are given in the following table.

Table A2.2 Biharmonic functions $p(x,y)$, $q(x,y)$ derived from analytic functions $f(z)$.

$f(z)$	$p(x,y)$	$q(x,y)$
z^3	$x^4 - y^4$	$2(x^3y + xy^3)$
z^3	$x^5 - 2x^3y^2 - 3xy^4$	$3x^4y + 2x^2y^3 - y^5$
z^n	$r^{n+1}\cos(n-1)\theta$†	$r^{n+1}\sin(n-1)\theta$†
$\ln z$	$r \ln r \cos\theta + r\theta \sin\theta$†	$-r \ln r \sin\theta + r\theta \cos\theta$†

†These results are expressed in terms of plane polar coordinates.

$f(z)$	$p(x,y)$	$q(x,y)$
e^{mz}	$e^{mx}(x\cos my + y\sin my)$	$e^{mx}(x\sin my - y\cos my)$
$\cos mz$	$x\cos mx\cosh my - y\sin mx\sinh my$	$-x\sin mx\sinh my - y\cos mx\cosh my$
$\sin mz$	$x\sin mx\cosh my + y\cos mx\sinh my$	$x\cos mx\sinh my - y\sin mx\cosh my$

The Cartesian forms of z^n are special cases of the general polynomial form of a biharmonic function proposed by Neou†,

$$B(x,y) = \Sigma_m \Sigma_n K_{m,n} x^m y^n$$

where summation is over a set of values of m and n, and the coefficients $K_{m,n}$ must satisfy the equation

$$3({}^mC_4 K_{m,n-4} + {}^nC_4 K_{m-4,n}) + {}^{m-2}C_2 {}^{n-2}C_2 K_{m-2,n-2} = 0$$

The above lists of harmonic and biharmonic functions are by no means exhaustive; further biharmonic functions will be found in Table A2.4 for example.

Biharmonic functions can be used as Airy stress functions to generate solutions to problems of plane strain or generalised plane stress. These satisfy the conditions of compatibility and equilibrium in the absence of body forces, provided that

$$\sigma_{xx} = \frac{\partial^2 B}{\partial y^2}, \quad \sigma_{yy} = \frac{\partial^2 B}{\partial x^2}, \quad \sigma_{xy} = -\frac{\partial^2 B}{\partial x \partial y}$$

This gives rise to the possible stress fields listed in Table A2.3.

Table A2.3 Stress fields relative to Cartesian coordinates

$B(x,y)$	σ_{xx}	σ_{yy}	σ_{xy}	$B(x,y)$	σ_{xx}	σ_{yy}	σ_{xy}
x^2	0	2	0	$x^4-3x^2y^2$	$-6x^2$	$12x^2-6y^2$	$12xy$
xy	0	0	-1	$y^4-3x^2y^2$	$12y^2-6x^2$	$-6y^2$	$12xy$
y^2	2	0	0	$x^5-5x^3y^2$	$-10x^3$	$20x^3-30xy^2$	$30x^2y$
x^3	0	$6x$	0	$x^4y-x^2y^3$	$-6x^2y$	$12x^2y-2y^3$	$-4x^3+6xy^2$
x^2y	0	$2y$	$-2x$	$x^3y^2-xy^4$	$2x^3-12xy^2$	$6xy^2$	$-6x^2y+4y^3$
xy^2	$2x$	0	$-2y$	$y^5-5x^2y^3$	$20y^3-30x^2y$	$-10y^3$	$30xy^2$
y^3	$6y$	0	0	$2x^6-15x^4y^2+y^6$	$30(y^4-x^4)$	$60x^2(x^2-3y^2)$	$120x^3y$
x^3y	0	$6xy$	$-3x^2$	$3x^5y-5x^3y^3$	$-30x^3y$	$30xy(2x^2-y^2)$	$15x^2(3y^2-x^2)$
xy^3	$6xy$	0	$-3y^2$	$3xy^5-5x^3y^3$	$30xy(2y^2-x^2)$	$-30xy^3$	$15y^2(3x^2-y^2)$
				$x^6-15x^2y^4+2y^6$	$60y^2(y^2-3x^2)$	$30(x^4-y^4)$	$120xy^3$

†Neou, C.Y. (1957) *J. Appl. Mech.* **24** 387-390

Physical components of stress relative to plane polar coordinates satisfy the conditions of equilibrium, in the absence of body forces, and the compatibility conditions, provided that

$$\sigma^{\mathrm{P}}_{rr} = \frac{1}{r}\frac{\partial B}{\partial r} + \frac{1}{r^2}\frac{\partial^2 B}{\partial \theta^2} \quad , \quad \sigma^{\mathrm{P}}_{\theta\theta} = \frac{\partial^2 B}{\partial r^2} \quad , \quad \sigma^{\mathrm{P}}_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial B}{\partial \theta}\right)$$

Possible stress fields thus generated by biharmonic functions are given in Table A2.4.

Table A2.4 Stress fields relative to plane polar coordinates

$B(r, \theta)$	σ^{P}_{rr}	$\sigma^{\mathrm{P}}_{\theta\theta}$	$\sigma^{\mathrm{P}}_{r\theta}$
r^2	2	2	0
$\ln r$	$1/r^2$	$-1/r^2$	0
$r^2 \ln r$	$1 + 2 \ln r$	$3 + 2 \ln r$	0
θ	0	0	$1/r^2$
θr^2	2θ	2θ	-1
$\theta \ln r$	θ/r^2	$-\theta/r^2$	$(\ln r - 1)/r^2$
$\theta r^2 \ln r$	$\theta(1 + 2 \ln r)$	$\theta(3 + 2 \ln r)$	$-\ln r - 1$
$r\theta \sin\theta$	$(2 \cos\theta)/r$	0	0
$r\theta \cos\theta$	$-(2 \sin\theta)/r$	0	0
$r \ln r \sin\theta$	$(\sin\theta)/r$	$(\sin\theta)/r$	$-(\cos\theta)/r$
$r \ln r \cos\theta$	$(\cos\theta)/r$	$(\cos\theta)/r$	$(\sin\theta)/r$
$r^m \sin n\theta$†	$(m-n^2)r^{m-2} \sin n\theta$	$(m^2-m)r^{m-2} \sin n\theta$	$n(1-m)r^{m-2}\cos n\theta$
$r^m \cos n\theta$†	$(m-n^2)r^{m-2} \cos n\theta$	$(m^2-m)r^{m-2} \cos n\theta$	$n(m-1)r^{m-2}\sin n\theta$

† $m = \pm n$ or $2 \pm n$

A2.2 The three-dimensional case

In three dimensions, Laplace's equation is again of the form $\nabla^2 f = 0$

and the biharmonic equation is again $\nabla^4 B = \nabla^2(\nabla^2 B) = 0$

where now $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$

relative to Cartesian coordinates. The harmonic functions given for the two-dimensional case are then particular cases of three-dimensional harmonic functions, and the parameters x and y may be replaced by any pair of x, y and z to generate more three-dimensional

harmonic functions.

The form of ∇^2 for cylindrical polar coordinates is given by (A1.13). Suppose that

$$f = R(r)\Theta(\theta)Z(z)$$

Then Laplace's equation becomes

$$\left(R_{,rr} + \frac{1}{r}R_{,r}\right)\Theta Z + R\left(\frac{1}{r^2}\Theta_{,\theta\theta}Z + \Theta Z_{,zz}\right) = 0$$

The function $\Theta(\theta)$ must have a period of 2π if the elastic continuum surrounds the polar axis. Using this condition and the method of separation of variables, the solution either takes the form

$$f = [J_0(kr/m) + AY_0(kr/m)]\cos m(\theta+\alpha)[B\cos kz + C\sin kz]$$

where J_0 and Y_0 are Bessel functions of the first and second kind of order zero, or it takes the form

$$f = [I_0(kr/m) + AK_0(kr/m)]\cos m(\theta+\alpha)[B\cosh kz + C\sinh kz]$$

where I_0 and K_0 are modified Bessel functions of the first and second kind of order zero. The coefficients A, B, C, k and α are constants and m is an integer.

The form of ∇^2 for spherical polar coordinates is given by (A1.14). Suppose that

$$f = R(r)\Psi(\theta,\phi)$$

Then Laplace's equation becomes

$$\left(R_{,rr} + \frac{2}{r}R_{,r}\right)\Psi + \frac{R}{r^2}(\Psi_{,\theta\theta} + \cot\theta\,\Psi_{,\theta} + \operatorname{cosec}^2\theta\,\Psi_{,\phi\phi}) = 0$$

The function $\Psi(\theta, \phi)$ must have a period of 2π in both its arguments if the elastic continuum surrounds the origin. Imposing this condition and using the method of separation of variables, leads to the general solution

$$f = (Ar^n + Br^{-n-1})X_{n;m}(\theta,\phi) + (Cr^n + Dr^{-n-1})Y_{n;m}(\theta,\phi)$$

where A, B, C and D are constants, m and n are integers and $X_{n,m}$ and $Y_{n,m}$ are **surface spherical harmonics**, see for example Sneddon [(1961) section 23].

The same general approach may be used to find solutions of the wave equation for cylindrical and spherical problems. These harmonic functions are also solutions of the biharmonic equation, of course.

A3

Equations in vector form

If the body forces and the material displacements and accelerations are expressed in vector form as

$$\boldsymbol{f} = f_j \boldsymbol{i}_j \quad , \quad \boldsymbol{u} = u_j \boldsymbol{i}_j \quad , \quad \ddot{\boldsymbol{u}} = \ddot{u}_j \boldsymbol{i}_j \tag{A3.1}$$

respectively, then Navier's equation (2.86) can be represented in vector form as

$$\mu(\nabla \cdot \nabla)\boldsymbol{u} + (\lambda + \mu)\nabla(\nabla \cdot \boldsymbol{u}) + \rho \boldsymbol{f} = \rho \ddot{\boldsymbol{u}} \tag{A3.2}$$

where

$$\nabla = \boldsymbol{i}_1 \frac{\partial}{\partial x_1} + \boldsymbol{i}_2 \frac{\partial}{\partial x_2} + \boldsymbol{i}_3 \frac{\partial}{\partial x_3} \tag{A3.3}$$

relative to Cartesian coordinates, [cf. (1.108) and (1.45)]. These vectors can also be related to curvilinear coordinates, but careful consideration must be given to the form and effect of the operator ∇ (see Appendix A1 for example). From Helmholtz's decomposition theorem, $\boldsymbol{u}$ can be written as

$$\boldsymbol{u} = \nabla \varsigma + \nabla \times \boldsymbol{\eta} \tag{A3.4}$$

where ς is a scalar potential and $\boldsymbol{\eta}$ is a vector potential. (Without loss of generality, it can be taken that $\nabla \cdot \boldsymbol{\eta}$ is zero. This can be seen by applying the decomposition theorem to $\boldsymbol{\eta}$ itself, and noting that the terms giving non-zero values to $\nabla \cdot \boldsymbol{\eta}$ vanish on evaluating $\nabla \times \boldsymbol{\eta}$. However, this condition does not need to be applied in the following derivations.) Substituting (A3.4) into (A3.2) gives

$$(\nabla \cdot \nabla)[(\lambda + 2\mu)\nabla \varsigma + \mu \nabla \times \boldsymbol{\eta}] = \rho(\nabla \ddot{\varsigma} + \nabla \times \ddot{\boldsymbol{\eta}}) - \rho \boldsymbol{f} \tag{A3.5}$$

A3.1 The Papkovich-Neuber functions

If there are no body forces or accelerations†, the right-hand side of (A3.5) is zero and so it can be written in the form

$$(\lambda + 2\mu)\nabla \varsigma + \mu \nabla \times \boldsymbol{\eta} = \mu \boldsymbol{u} + (\lambda + \mu)\nabla \varsigma = \boldsymbol{\Phi} \tag{A3.6}$$

where $\boldsymbol{\Phi}$ is a harmonic vector, that is,

†Malvern [(1969) section 8.4] gives the solution allowing for body forces.

$$(\nabla\cdot\nabla)\boldsymbol{\Phi} = \mathbf{0} \tag{A3.7}$$

Then

$$\nabla\cdot\boldsymbol{\Phi} = (\lambda + 2\mu)(\nabla\cdot\nabla)\varsigma \tag{A3.8}$$

and the most general form of the solution of this is

$$(\lambda + 2\mu)\varsigma = \tfrac{1}{2}(\Psi + \boldsymbol{r}\cdot\boldsymbol{\Phi}) \tag{A3.9}$$

where Ψ is a harmonic scalar. Then from (A3.6),

$$\begin{aligned}\mu\boldsymbol{u} &= \boldsymbol{\Phi} - \frac{\lambda+\mu}{2(\lambda+2\mu)}[\nabla\Psi + \nabla(\boldsymbol{r}\cdot\boldsymbol{\Phi})] \\ &= \boldsymbol{\Phi} - \frac{1}{4(1-\nu)}\nabla[\Psi + \boldsymbol{r}\cdot\boldsymbol{\Phi}]\end{aligned} \tag{A3.10}$$

[cf. (3.190)]. Thus $\boldsymbol{u}$ can be expressed in terms of four harmonic functions, namely Ψ and the three functions composing the harmonic vector $\boldsymbol{\Phi}$. Mindlin† has shown that these are related to the Galerkin vector. The need for four *independent* harmonic functions has been discussed by Neuber (1946) and Sokolnikoff (1956) amongst others. Eubanks and Sternberg‡ have shown that, without loss of generality, any one of the Cartesian components Φ_x, Φ_y or Φ_z of $\boldsymbol{\Phi}$ can be set to zero or, provided that ν is not ¼, Ψ can be set to zero in each case.

A3.2 The wave equations

When the body force $\boldsymbol{f}$ is zero, two particular solutions of (A3.5) are of interest in studying wave propagation. If η is zero, (A3.5) becomes

$$c_P^2(\nabla\cdot\nabla)\boldsymbol{u} = \ddot{\boldsymbol{u}} \tag{A3.11}$$

where

$$c_P^2 = (\lambda + 2\mu)/\rho \tag{A3.12}$$

Solutions of (A3.11) give dilatational waves. In this case, from (A3.4), (A3.11) can be written as

$$c_P^2(\nabla\cdot\nabla)\nabla\varsigma = \nabla\ddot{\varsigma} \tag{A3.13}$$

and since for any scalar ϕ

$$(\nabla\cdot\nabla)\nabla\phi = \nabla(\nabla\cdot\nabla\phi)$$

(A3.13) can be written as

$$\nabla(c_P^2\nabla\cdot\nabla\varsigma) = \nabla\ddot{\varsigma}$$

so that any solution of

$$c_P^2\nabla\cdot\nabla\varsigma = \ddot{\varsigma} \tag{A3.14}$$

is a dilatational wave solution.

If ς is zero, (A3.5) becomes

$$c_S^2(\nabla\cdot\nabla)\boldsymbol{u} = \ddot{\boldsymbol{u}} \tag{A3.15}$$

where

$$c_S^2 = \mu/\rho \tag{A3.16}$$

Solutions of (A3.15) give distortional waves.

†Mindlin, R.D. (1936) *Bull. Am. Math. Soc.* **42** 373-376
‡Eubanks, R.A. and Sternberg, E. (1956) *J. Rational Mech. Anal.* **5** 737-746

A solution of (A3.11) and (A3.15) takes the form

$$u_j = a_j \cos(\omega t - k_m x_m + \alpha_j) \qquad (k_m k_m = \omega^2/c_P^2 \text{ or } \omega^2/c_S^2)$$

where a_j and α_j are constants, j and m take values in the range 1 to 3, and summation over the repeated index m is implied. The form of u_j is unchanged by increasing x_m by δx_m in time δt provided that

$$\delta \boldsymbol{r} \cdot \boldsymbol{k} = \omega \delta t \qquad (\delta \boldsymbol{r} = \delta x_m \boldsymbol{i}_m \ , \ \boldsymbol{k} = k_m \boldsymbol{i}_m)$$

This implies that the wave front is normal to the direction of the vector $\boldsymbol{k}$, and travelling in that direction.

For standing waves, the solution can be assumed to take the form

$$\boldsymbol{u} = \boldsymbol{U} \cos(\omega t + \alpha)$$

This reduces (A3.11) and (A3.15) to the Helmholtz equation

$$(\nabla \cdot \nabla)\boldsymbol{U} + k^2 \boldsymbol{U} = \boldsymbol{0} \qquad (k = \omega/c_P \text{ or } \omega/c_S) \tag{A3.17}$$

In Cartesian coordinates, $\boldsymbol{U} = U_j \boldsymbol{i}_j$ and three particular solutions of (A3.17) are

$$U_j = a e^{i(k_m x_m)} \qquad (k_m k_m = k^2)$$
$$U_j = (a + b x_1) e^{i(k_2 x_2 + k_3 x_3)} \qquad (k_2 k_2 + k_3 k_3 = k^2)$$
$$U_j = (a + b x_1)(d + f x_2) e^{i k x_3}$$

where the coefficients a, b, d, f and k_m can be different for each U_j.

A3.3 Gradient, divergence and curl for curvilinear coordinates

The operator ∇ can be represented† in a general tensor form as

$$\nabla\{\ \} = \{\ \}_{|i} \boldsymbol{g}^i \tag{A3.18}$$

where the covariant differentiation operates on some scalar or tensor coefficient in the curly brackets. Suppose that it operates on a scalar, ϕ, and on a vector, $\boldsymbol{u}$, where

$$\boldsymbol{u} = u_j \boldsymbol{g}^j = u^k \boldsymbol{g}_k$$

It then gives the following forms for gradient, divergence and curl.

$$\nabla\phi = \phi_{|i} \boldsymbol{g}^i = \phi_{,i} \boldsymbol{g}^i \ , \ \nabla \cdot \boldsymbol{u} = u^k_{\ |i} \boldsymbol{g}_k \cdot \boldsymbol{g}^i = u^k_{\ |k} \ ,$$
$$\nabla \times \boldsymbol{u} = \varepsilon^{ijk} u_{k|j} \boldsymbol{g}_i = \varepsilon^{ijk} g_{il} u_{k|j} \boldsymbol{g}^l \tag{A3.19}$$

[cf. (2.31) and (2.32)]. The operation $\nabla^2\{\ \}$ is given by

$$\nabla \cdot \nabla\{\ \} = \{\ \}_{|ij} \boldsymbol{g}^i \cdot \boldsymbol{g}^j = \{\ \}_{|ij} g^{ij} \tag{A3.20}$$

Properties of ∇ can be deduced from these definition. For example

$$\begin{aligned} \nabla \times (\nabla \times \boldsymbol{u}) &= \varepsilon_{pqi} g^{qm} (\varepsilon^{ijk} u_{k|j})_{|m} \boldsymbol{g}^p \\ &= \varepsilon_{pqi} g^{qm} \varepsilon_{ijk} u_{k|jm} \boldsymbol{g}^p \\ &= u^m_{\ |jm} \boldsymbol{g}^j - u_{k|jm} g^{jm} \boldsymbol{g}^k \\ &= \nabla(\nabla \cdot \boldsymbol{u}) - (\nabla \cdot \nabla)\boldsymbol{u} \end{aligned}$$

†See Leigh (1968) page 90 for a discussion of the Cartesian tensor form.

using metric tensors to interchange subscripts and superscripts, using the expressions for permutation symbols given by (3.70) and (3.72) and noting that the only non-zero products of $e_{pqi}\, e_{ijk}$ are +1 ($p=j$, $q=k$) and -1 ($p=k$, $q=j$).

A3.4 The cone problem

The Papkovich-Neuber function Φ in (A3.10) can be conveniently applied to the analysis of solid and hollow cones. Let Φ be the harmonic vector $\{\Phi_r, \Phi_\theta, \Phi_\phi\}$, relative to the spherical polar coordinates shown in Fig. A1.2, so that for $\nabla^2\Phi$ to be zero,

$$\begin{aligned}
&r^2\sin\theta\nabla^2\Phi_r - 2\left[\Phi_r\sin\theta + \frac{\partial}{\partial\theta}(\Phi_\theta\sin\theta) + \frac{\partial\Phi_\phi}{\partial\phi}\right] = 0\\
&r^3\sin^2\theta\nabla^2\left(\frac{1}{r}\Phi_\theta\right) + 2\left[\left(\frac{\partial\Phi_r}{\partial\theta} + r\frac{\partial\Phi_\theta}{\partial r}\right)\sin^2\theta - \frac{1}{2}\Phi_\theta - \frac{\partial\Phi_\phi}{\partial\phi}\cos\theta\right] = 0 \qquad \textbf{(A3.21)}\\
&r^3\sin^2\theta\nabla^2\left(\frac{\Phi_\phi}{r\sin\theta}\right) + 2\left[\frac{\partial\Phi_r}{\partial\phi} + \frac{\partial\Phi_\theta}{\partial\phi}\cot\theta + r\frac{\partial\Phi_\phi}{\partial r}\sin\theta - \Phi_\phi\,\mathrm{cosec}\,\theta + \frac{\partial\Phi_\phi}{\partial\theta}\cos\theta\right] = 0
\end{aligned}$$

and $\boldsymbol{r}$ is the radius vector $\{r,0,0\}$ of the material point.

Solutions in terms of the harmonic scalar Ψ lead to no solutions which are independent of those found in terms of Φ and so will be ignored.

A3.4.1 The General Method

Solutions for a solid cone under axial loading or shear[†], torsion[‡] and flexure[§] were found separately but a general method[¶] will be described here which includes the responses of hollow cones. Using the above spherical polar coordinates, the origin will be taken at the apex of the cone. The solutions of the individual problems will now be considered. In all cases, the faces of the cone will be taken as generated by surfaces of constant θ, where $\theta=\alpha$ for the outer surface and $\theta=\beta$ for the inner one. Föppl's solution for the torsion of a cone applies equally well to hollow cones and solid cones, because no tractions are induced on any conical surface of constant θ. In principle, a total of six independent expressions for the material displacements (and hence the stresses) are required to satisfy the conditions of zero traction on the inner and outer surfaces of the cone. The form of these expressions as functions of r and ϕ can be determined from the vector nature of the applied load. The solution is facilitated by using t, the tangent of the half-angle $\theta/2$, instead of θ. The hollow cone problem can be dealt with by expressing the equations in terms of T (or t^{-1}), the tangent of the complement of the half-angle (i.e. the cotangent of $\theta/2$). If this is combined with redefining u_θ^P as $-u_\theta^P$, the equations governing the form of the deflections are invariant under such a change of parameter. The physical significance of this transformation can be seen by noting that the angle of the radius vector to the cone axis can be defined as θ or $(\pi-\theta)$. The sense of positive u_θ^P changes between the two cases, otherwise, the governing equations remain the same. This invariance can be seen from the original form of the equations and noting the relationship between the differentials given by

[†] Michell, J.H. (1900) *London Math. Soc. Proc.* **32** 29

[‡] Föppl, A. (1905) *Sitzungsberichte Bayer, Akademie Wissenschaften* **35** 249.

[§] Renton, J.D. (1997) *J. Mech. Phys. Solids* **45** 753

[¶] Renton, J.D. (1998) *Journal of Elasticity* **49** 101

$$t\frac{d}{dt} = -T\frac{d}{dT} \quad \left(= \sin\theta \frac{d}{d\theta} \right) \tag{A3.22}$$

It means that solutions in terms of t immediately yield solutions in terms of T. In some cases, these two solutions are identical and then the method of Frobenius is applied. In all cases, this yields a sufficient number of independent solutions to describe the response to the tip loading when all the surface tractions are zero.

A3.4.2 An Axially-loaded Hollow Cone

For this case, the harmonic vector Φ takes the form

$$\Phi = \frac{1}{r}\begin{bmatrix} f(t) \\ g(t) \\ 0 \end{bmatrix} \quad \text{where} \quad \nabla^2\Phi = 0 \tag{A3.23}$$

After some re-arrangement, the harmonic conditions to be satisfied by this vector are

$$\begin{aligned} (1+2t^2+t^4)t\frac{d}{dt}\left(t\frac{df}{dt}\right) - 8t^2 f - 4(1+t^2)t^2\frac{dg}{dt} - 4t(1-t^2)g = 0 \\ 4t^2\frac{df}{dt} + (1+t^2)t\frac{d}{dt}\left(t\frac{dg}{dt}\right) - (1+t^2)g = 0 \end{aligned} \tag{A3.24}$$

Taking $f(t)$ and $g(t)$ to be expressed by series in terms positive powers of t yields the solutions

$$\begin{aligned} &\text{A:} \quad f(t) = 1 \;,\; g(t) = -t \\ &\text{B:} \quad f(t) = \frac{1-t^2}{1+t^2} = \cos\theta \;,\; g(t) = \frac{-2t}{1+t^2} = -\sin\theta \end{aligned} \tag{A3.25}$$

These correspond to the solutions used by Michell for an axially-loaded solid cone. Rewriting (A3.24) in terms of T and redefining g as $-g$ yields the same form. Solution A now becomes

$$\text{C:} \quad f(T) = f(t) = 1 \;,\; g(T) = +T \;\text{ or }\; g(t) = 1/t \tag{A3.26}$$

This means that the solutions given by (A3.25) yield new solutions in terms of T. However, the new solution derived from B is the same as the old one. The method of Frobenius is now applied to find a fourth independent solution of the form

$$f(t) = \cos\theta \ln t + \sum_{i=0}^{\infty} f_i t^{2i} \;,\; g(t) = -\sin\theta \ln t + \sum_{i=0}^{\infty} g_i t^{2i+1} \tag{A3.27}$$

In this particular case, both power series are zero. The general form of the material deformation found from (A3.10) is then

$$G\boldsymbol{u} = \frac{1}{r}\begin{bmatrix} 4(1-\nu)(A + B + C\cos\theta + D\cos\theta\ln t) \\ 4(1-\nu)(-At + \frac{B}{t}) - (3-4\nu)(C\sin\theta + D\sin\theta\ln t) - D\cot\theta \\ 0 \end{bmatrix} \tag{A3.28}$$

This gives the physical stresses

$$\sigma_{rr} = -\frac{2}{r^2}[4(1-\nu)(A+B)+2(2-\nu)\cos\theta(C+D\ln t)+2\nu D]$$
$$\sigma_{\theta\theta} = \frac{2}{r^2}[2(1-\nu)\{A(1-t^2)+B(1-t^{-2})-D\}+(1-2\nu)\cos\theta(C+D\ln t)+D\cot^2\theta]$$
$$\sigma_{\phi\phi} = \frac{2}{r^2}[2(1-\nu)\{A(1+t^2)+B(1+t^{-2})\}+(1-2\nu)\cos\theta(C+D\ln t)-D\cot^2\theta-2\nu D]$$
$$\tau_{r\theta} = \frac{2}{r^2}[2(1-\nu)(2At-2Bt^{-1})+(1-2\nu)\sin\theta(C+D\ln t)+(3-2\nu)D\cot\theta] \quad \textbf{(A3.29)}$$
$$\tau_{r\phi} = \tau_{\theta\phi} = 0$$

where the physical components of stress $\{\sigma^P_{rr}, \sigma^P_{\theta\theta}, \sigma^P_{\phi\phi}, \sigma^P_{r\theta}, \sigma^P_{r\phi}, \sigma^P_{\theta\phi}\}$ are expressed in terms of the more common notation $\{\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{\phi\phi}, \tau_{r\theta}, \tau_{r\phi}, \tau_{\theta\phi}\}$. Given that these stresses are inversely proportional to r^2 and are not functions of ϕ, the equations of linear motion given in section A1.2 (for zero body forces and accelerations) yield

$$\frac{d}{d\theta}(\bar{\tau}_{r\theta}\sin\theta) = (\bar{\sigma}_{\theta\theta}+\bar{\sigma}_{\phi\phi})\sin\theta \quad , \quad \frac{d}{d\theta}(\bar{\sigma}_{\theta\theta}\sin\theta) = \bar{\sigma}_{\phi\phi}\cos\theta - \bar{\tau}_{r\theta}\sin\theta \quad \textbf{(A3.30)}$$

where the bar indicates that part of the expression for the stress given by (A3.29) which is in terms of θ only (i.e. the term r^2 is omitted). From these equations it follows that

$$\bar{\sigma}_{\theta\theta}\sin\theta - \bar{\tau}_{r\theta}\cos\theta = K\operatorname{cosec}\theta \quad \textbf{(A3.31)}$$

where K is a constant. This indicates that a suitable linear relationship between the constants A to D will make K zero. In fact, it is only necessary to make D zero. Then if $\sigma_{\theta\theta}$ is zero for a particular value of θ, $\tau_{r\theta}$ will be too. This reduces the conditions imposed on the constants A to C to two, namely that $\sigma_{\theta\theta}$ should be zero at θ equal to α and β. This gives two conditions relating these constants and the remaining condition is given by evaluating the axial tensile force P:

$$\begin{aligned} P &= \int_\beta^\alpha [\sigma_{rr}\cos\theta - \tau_{r\theta}\sin\theta]2\pi r\sin\theta\, r\,d\theta \\ &= 4\pi\left|{}_\beta^\alpha \cos\theta[2(1-\nu)(2A-2B+C) - C\sin^2\theta]\right| \end{aligned} \quad \textbf{(A3.32)}$$

This outlines the general method applied to the other problems, although it could be expressed more simply in this particular case. However, the inclusion of terms in D permits the solution of all problems associated with the displacement field given by (A3.23). The other cases can be dealt with similarly, so that only the outline solutions will be given.

A3.4.3 A Hollow Cone with a Tip Shear Force

A displacement field of the same form as that used by Michell for the problem of shear on a solid cone will be sought. This means that the harmonic vector $\boldsymbol{\Phi}$ is expressed by

$$\boldsymbol{\Phi} = \frac{1}{r}\begin{bmatrix}\cos\phi f(t)\\ \cos\phi g(t)\\ \sin\phi h(t)\end{bmatrix} \quad \textbf{(A3.33)}$$

In terms of t, the harmonic conditions give rise to the equations

$$\begin{aligned}(1+t^2)^2\left[t\frac{d}{dt}\left(t\frac{df}{dt}\right)-f\right]-8ft^2-4t\left[(1+t^2)\left(t\frac{dg}{dt}+h\right)+(1-t^2)g\right]=0\\(1+t^2)\left[t\frac{d}{dt}\left(t\frac{dg}{dt}\right)-2g\right]+4t^2\frac{df}{dt}-2h(1-t^2)=0\\(1+t^2)\left[t\frac{d}{dt}\left(t\frac{dh}{dt}\right)-2h\right]-4ft-2g(1-t^2)=0\end{aligned} \quad \textbf{(A3.34)}$$

These yield the solutions

$$\begin{aligned}&\text{A: } f(t)=t\ ,\ g(t)=1\ ,\ h(t)=-1\\&\text{B: } f(t)=0\ ,\ g(t)=(1+t^2)\ ,\ h(t)=-(1+t^2)\\&\text{C: } f(t)=\sin\theta\ ,\ g(t)=\cos\theta\ ,\ h(t)=-1\\&\text{D: } f(t)=t^{-1}\ ,\ g(t)=-1\ ,\ h(t)=-1\\&\text{E: } f(t)=0\ ,\ g(t)=-(1+t^{-2})\ ,\ h(t)=-(1+t^{-2})\\&\text{F: } f(t)=\sin\theta\ln t\ ,\ g(t)=\cos\theta\ln t\ ,\ h(t)=-\ln t\end{aligned} \quad \textbf{(A3.35)}$$

The general form of the displacement field in this case is

$$Gu=\frac{1}{r}\begin{bmatrix}\cos\phi[4(1-\nu)\sin\theta(A+F\ln t)+Ct+Et^{-1}]\\\cos\phi[(3-4\nu)\cos\theta(A+F\ln t)+B(1+t^2)-D(1+t^{-2})+C-E-F]\\\sin\phi[-(3-4\nu)(A+F\ln t)-B(1+t^2)-D(1+t^{-2})-C-E]\end{bmatrix} \quad \textbf{(A3.36)}$$

This gives the stresses

$$\begin{aligned}\sigma_{rr}&=-\frac{2\cos\phi}{r^2}[(4-2\nu)\sin\theta(A+F\ln t)+Ct+Et^{-1}-2\nu F\cot\theta]\\\sigma_{\theta\theta}&=\frac{2\cos\phi}{r^2}[(1-2\nu)\sin\theta(A+F\ln t)+B(t+t^3)+Ct+D(t^{-1}+t^{-3})+Et^{-1}+(3-2\nu)F\cot\theta]\\\sigma_{\phi\phi}&=\frac{2\cos\phi}{r^2}[(1-2\nu)\sin\theta(A+F\ln t)-B(t+t^3)-D(t^{-1}+t^{-3})-(1-2\nu)F\cot\theta]\\\tau_{r\theta}&=\frac{\cos\phi}{r^2}[-(2-4\nu)\cos\theta(A+F\ln t)-2B(1+t^2)+\tfrac{1}{2}C(t^2-3)+2D(1+t^{-2})\\&\qquad-\tfrac{1}{2}E(t^{-2}-3)+(6-4\nu)F]\\\tau_{\theta\phi}&=\frac{\sin\phi}{r^2}[-2B(t+t^3)-Ct+2D(t^{-1}+t^{-3})+Et^{-1}-2(1-2\nu)F\operatorname{cosec}\theta]\\\tau_{r\phi}&=\frac{\sin\phi}{r^2}[(2-4\nu)(A+F\ln t)+2B(1+t^2)+\tfrac{1}{2}C(3-t^2)+2D(1+t^{-2})+\tfrac{1}{2}E(3-t^{-2})]\end{aligned} \quad \textbf{(A3.37)}$$

The constants A to F are determined from the surface tractions. If these are zero, F is zero and the other surface conditions determine the ratios of the constants A to E. Their magnitudes are determined from the tip shear force S. Taking this to act in the sense of positive θ in the plane $\phi=0$:

$$S = -\int_{\beta}^{\alpha}\int_{0}^{2\pi}[(\tau_{r\theta}\cos\theta + \sigma_{rr}\sin\theta)\cos\phi - \tau_{r\phi}\sin\phi]r^2\sin\theta\, d\phi\, d\theta$$
$$= -\pi\left|_{\beta}^{\alpha}\cos\theta\{A[8(1-\nu)+2\sin^2\theta] + 4B + 3C + 4D + 3E\}\right| \quad \textbf{(A3.38)}$$

A3.4.4 A Hollow Cone in Torsion

For this case, the appropriate form of the harmonic vector is given by

$$\boldsymbol{\Phi} = \frac{1}{r^2}\begin{bmatrix} 0 \\ 0 \\ h(t) \end{bmatrix} \quad \textbf{(A3.39)}$$

The first two harmonic conditions given by (A3.21) are satisfied automatically and the third is

$$2h\sin^2\theta + \sin\theta\left[\frac{d}{d\theta}\left(\sin\theta\frac{dh}{d\theta}\right)\right] - h = 0 \quad \textbf{(A3.40)}$$

One solution† of this is

$$h = A\sin\theta \quad \textbf{(A3.41)}$$

which gives rise to only one stress,

$$\tau_{r\phi} = -\frac{3A}{r^3}\sin\theta \quad \textbf{(A3.42)}$$

This means that all conical surfaces of constant θ are stress-free. The constant A is found from the torque T applied:

$$T = \int_{\beta}^{\alpha} 2\pi\tau_{r\phi}r^3\sin^2\theta\, d\theta = 2A\pi[3(\cos\alpha - \cos\beta) + \cos^3\beta - \cos^3\alpha] \quad \textbf{(A3.43)}$$

A3.4.5 A Hollow Cone in Flexure

For this case, the appropriate form of the harmonic vector is given by

$$\boldsymbol{\Phi} = \frac{1}{r^2}\begin{bmatrix} \sin\phi\, f(t) \\ \sin\phi\, g(t) \\ \cos\phi\, h(t) \end{bmatrix} \quad \textbf{(A3.44)}$$

In terms of t, the harmonic conditions are

$$\begin{aligned}
&(1+t^2)^2\left(t^2\frac{d^2f}{dt^2} + t\frac{df}{dt} - f\right) - 4\left((1+t^2)t^2\frac{dg}{dt} + t(1-t^2)g\right) + 4t(1+t^2)h = 0 \\
&4t^2(1+t^2)\frac{df}{dt} + (1+t^2)^2\left(t^2\frac{d^2g}{dt^2} + t\frac{dg}{dt}\right) - 2(1-t^2)^2g + 2(1-t^4)h = 0 \\
&2t(1+t^2)f + (1-t^4)g + \tfrac{1}{2}(1+t^2)^2\left(t^2\frac{d^2h}{dt^2} + t\frac{dh}{dt}\right) - (1-t^2)^2h = 0
\end{aligned} \quad \textbf{(A3.45)}$$

† The other solution can be formulated as a power series in t or $\sin\theta$.

The solutions found are

$$
\begin{aligned}
&f(t) = 2t \;,\; g(t) = h(t) = -(1+t^2)\;;\\
&f(t) = 0 \;,\; g(t) = 1 \;,\; h(t) = \frac{2}{1+t^2} - 1 = \cos\theta\;;\\
&f(t) = \frac{t(1-t^2)}{(1+t^2)^2} = \tfrac{1}{2}\sin\theta\cos\theta \;,\; g(t) = \frac{-2t^2}{(1+t^2)^2} = -\tfrac{1}{2}\sin^2\theta \;,\; h(t) = 0\;; \qquad \textbf{(A3.46)}\\
&f(t) = 2t^{-1} \;,\; g(t) = (1+t^{-2}) \;,\; h(t) = -(1+t^{-2})\;;\\
&f(t) = \operatorname{cosec}\theta \;,\; g(t) = \ln t \;,\; h(t) = 1 + \cos\theta\ln t\;;\\
&f(t) = \tfrac{1}{2}(\sin\theta - \operatorname{cosec}\theta + \sin\theta\cos\theta\ln t) \;,\; g(t) = \tfrac{1}{2}(\cos\theta - \sin^2\theta\ln t) \;,\; h(t)
\end{aligned}
$$

These yield the general form of the displacement field given by

$$
\begin{aligned}
Gu_r^{\,P} &= \frac{\sin\phi}{r^2}[2At + 2Dt^{-1} + (5-4\nu)\{\sin\theta\cos\theta(C + F\ln t) + (E-F)\operatorname{cosec}\theta + F\sin\theta\}]\\
Gu_\theta^{\,P} &= \frac{\sin\phi}{r^2}[-A(1+t^2) + B - C + D(1+t^{-2}) - (2-4\nu)\{\sin^2\theta(C + F\ln t) - F\cos\theta\}\\
&\quad + (E-F)\cot\theta\operatorname{cosec}\theta + \{E(4-4\nu) - F\}\ln t] \qquad \textbf{(A3.47)}\\
Gu_\phi^{\,P} &= \frac{\cos\phi}{r^2}[-A(1+t^2) + (B-C)\cos\theta - D(1+t^{-2}) + (1+\cos\theta\ln t)\{E(4-4\nu) - F\}\\
&\quad + (F-E)\operatorname{cosec}^2\theta]
\end{aligned}
$$

The stresses resulting from this displacement field are

$$
\begin{aligned}
\sigma_{rr} &= -\frac{4\sin\phi}{r^3}[2At + 2Dt^{-1} + (5-\nu)\sin\theta\{\cos\theta(C + F\ln t) + F\}\\
&\quad + \{(E-F)(5-4\nu) - \nu F\}\operatorname{cosec}\theta]\\
\sigma_{\theta\theta} &= \frac{2\sin\phi}{r^3}[A(t - t^3) + (1-2\nu)\{\sin\theta\cos\theta(C + F\ln t) + F\sin\theta\}\\
&\quad + D(t^{-1} - t^{-3}) + \operatorname{cosec}\theta\{(1-\nu)(8E-6F) + 2\cot^2\theta(F-E) + F\}]\\
\sigma_{\phi\phi} &= \frac{2\sin\phi}{r^3}[A(3t + t^3) + D(3t^{-1} + t^{-3}) + 2(E-F)\operatorname{cosec}^3\theta\\
&\quad + (1-2\nu)\{3\sin\theta(C\cos\theta + F + F\cos\theta\ln t) - F\operatorname{cosec}\theta\}] \qquad \textbf{(A3.48)}\\
\tau_{r\theta} &= \frac{\sin\phi}{r^3}[4A(1+t^2) - 4D(1+t^{-2}) - 3B + (8-4\nu)\{C + (F-E)\cot\theta\operatorname{cosec}\theta + F\ln t\\
&\quad - (4+4\nu)\{\sin^2\theta(C + F\ln t) - F\cos\theta\} - 12E(1-\nu)\ln t]\\
\tau_{r\phi} &= \frac{\cos\phi}{r^3}[4A(1+t^2) + 4D(1+t^{-2}) - 3B\cos\theta - 12E(1-\nu)(1+\cos\theta\ln t)\\
&\quad + (8-4\nu)\{\cos\theta(C + F\ln t) - F\cot^2\theta + E\operatorname{cosec}^2\theta\}]\\
\tau_{\theta\phi} &= -\frac{2\cos\phi}{r^3}[A(t + t^3) - D(t^{-1} + t^{-3}) + (1-2\nu)\{\sin\theta(C + F\ln t) - F\cot\theta\}\\
&\quad - 2(E-F)\cot\theta\operatorname{cosec}^2\theta]
\end{aligned}
$$

The conditions of zero surface traction require that F be taken as equal to $2E$. There then remain four independent conditions of zero surface traction to be satisfied, giving relationships between the constants A to E. The bending moment about an axis in the plane $\phi=0$ and normal to that of the cone is then given by

$$\begin{aligned} M &= \int_0^{2\pi}\int_\beta^\alpha (r\cos\theta\,\tau_{r\phi}\cos\phi + r\tau_{r\theta}\sin\phi)r^2\sin\theta\,d\theta\,d\phi \\ &= \pi\,|_\beta^\alpha \cos\theta\{8(D-A)+(B-4C)(3+\cos^2\theta)+(1+\nu)[8C+4E(\sin^2\theta\ln t-\cos\theta)]\} \\ &\quad +2E(1+\nu)| \end{aligned} \qquad \textbf{(A3.49)}$$

A3.4.6 The Unloaded Cone Paradox

Carothers† found a solution for the bending of a plane wedge by a couple applied about its vertex. If this solution is taken to apply to wedge angles in excess of π radians, the problem becomes that of a notch. In particular, for a wedge angle of about 4.492 radians, a state of stress can exist when no couple is applied to the tip. In the case of the cone loaded by a bending couple M at its tip, analysed in section A3.4.5, A, E and F must be taken as zero to avoid singularities at $\theta=\pi$. The condition that M is zero is found from (A3.49) to be

$$[3B - 4C(1 - 2\nu) + 8D](1 + \cos\beta) + (B - 4C)(1 + \cos^3\beta) = 0 \qquad \textbf{(A3.50)}$$

The conditions that $\sigma_{\theta\theta}$ and $\tau_{r\theta}$ are zero on $\theta=\beta$ are given by

$$\begin{aligned} (1 - 2\nu)C\sin\beta\cos\beta + D(t^{-1} - t^{-3}) &= 0 \\ 3A + C[4(1 + \nu)\sin^2\beta - 8 + 4\nu] + 4D(1 + t^{-2}) &= 0 \end{aligned} \qquad \textbf{(A3.51)}$$

The condition that $\tau_{\theta\phi}$ is zero on this surface is then automatically satisfied, because E and F are zero. For a non-zero state of stress, either t is unity or

$$9(1 + \nu)t^6 + (9 - 3\nu)t^4 + (7 - 5\nu)t^2 - (1 + \nu) = 0 \qquad \textbf{(A3.52)}$$

This has one positive real root for t which depends on the value of Poisson's ratio. When ν is zero, the corresponding conical notch angle, 2β, is 1.342 radians and when ν is 0.5, 2β is 1.755 radians. The solution for t equal to unity implies that the ratios of the constants are

$$B = 4C \quad , \quad D = -(1 + \nu)C \qquad \textbf{(A3.53)}$$

Arguing along the lines of Sternberg and Koiter‡, the solution of section A3.4.5 should not be used beyond the first pathological case, i.e. when the cone becomes a half-space ($t=1$).

† Carothers, S.D. (1912) Plane strain in a wedge. *Proc. Roy. Soc. Edinburgh* **23** 292

‡ Sternberg, E. and Koiter, W.T. (1958) The wedge under a concentrated couple: a paradox in the two-dimensional theory of elasticity. *J. Appl. Mech.* **25** 575

A4

Direct tensor notation

In this book, a vector $\boldsymbol{v}$ may be represented by a column vector $\mathbf{v}$ in matrix notation, or by its tensor components v_i or v^j where $\boldsymbol{v} = v_i \boldsymbol{g}^i = v^j \boldsymbol{g}_j$

In many texts, all tensors are likewise expressed in a form which is free from their component representation. This is done by expressing them as an ordered assembly of vectors. Thus a second-order tensor could be denoted by $\boldsymbol{A}$ where

$$\boldsymbol{A} = A_{ij}\boldsymbol{g}^i\boldsymbol{g}^j = A^i_{\ j}\boldsymbol{g}_i\boldsymbol{g}^j = A^{\ j}_i \boldsymbol{g}^j\boldsymbol{g}_i = A^{ij}\boldsymbol{g}_i\boldsymbol{g}_j$$

and similar definitions hold for higher order tensors. It will be seen that the order of the base vectors following the tensor components strictly follows the order of the indices. This order is necessary in defining the result of a tensor product such as

$$\boldsymbol{A} = \boldsymbol{u}\boldsymbol{v} \quad \text{where} \quad \boldsymbol{u} = u_i\boldsymbol{g}^i \ , \ \boldsymbol{v} = v_j\boldsymbol{g}^j$$

which gives $\boldsymbol{A} = u_i v_j \boldsymbol{g}^i\boldsymbol{g}^j$ or $A_{ij} = u_i v_j \ (= v_j u_i)$

but $\boldsymbol{B} = \boldsymbol{v}\boldsymbol{u} = v_i u_j \boldsymbol{g}^i\boldsymbol{g}^j$ implies $B_{ij} = v_i u_j$

which is not the same tensor unless $\boldsymbol{A}$, and therefore $\boldsymbol{B}$, is symmetric. In some texts, this tensor product is written as

$$\boldsymbol{A} = \boldsymbol{u}\otimes\boldsymbol{v} = u_i v_j \boldsymbol{g}^i\otimes\boldsymbol{g}^j$$

Such representations of second-order tensors are known as **dyadics**. Higher order tensor products (triadics, tetradics etc.) can be formed in a similar fashion.

Addition of tensors of the same order and multiplication by a scalar, c, follow in a straightforward manner. Thus for two second-order tensors $\boldsymbol{A}$ and $\boldsymbol{B}$,

$$\boldsymbol{A} + \boldsymbol{B} = (A_{ij} + B_{ij})\boldsymbol{g}^i\boldsymbol{g}^j$$

$$c\boldsymbol{A} = \boldsymbol{A}c = cA_{ij}\boldsymbol{g}^i\boldsymbol{g}^j$$

for example. It readily follows that tensors of the same order satisfy the following axioms for addition.

(a) $\boldsymbol{A}+\boldsymbol{B}=\boldsymbol{B}+\boldsymbol{A}$ and the sum is a tensor of the same order as $\boldsymbol{A}$ and $\boldsymbol{B}$.

(b) $\boldsymbol{A}+(\boldsymbol{B}+\boldsymbol{C})=(\boldsymbol{A}+\boldsymbol{B})+\boldsymbol{C}$

(c) There is a unique tensor $\boldsymbol{0}$ such that $\boldsymbol{0}+\boldsymbol{A}=\boldsymbol{A}$

(d) There is a unique tensor $-\boldsymbol{A}$ such that $-\boldsymbol{A}+\boldsymbol{A}=\boldsymbol{0}$

The axioms of multiplication by scalars c and d are

(e) $c(\boldsymbol{A}+\boldsymbol{B})=c\boldsymbol{A}+c\boldsymbol{B}$

(f) $(c+d)\boldsymbol{A}=c\boldsymbol{A}+d\boldsymbol{A}$

(g) $c(d\boldsymbol{A})=(cd)\boldsymbol{A}$

These conditions are necessary for tensors to form a generalised linear vector space.

The definition of tensor products requires additional notation. A **scalar inner product** or **double dot product** $\boldsymbol{A}:\boldsymbol{B}$ of tensors $\boldsymbol{A}$ and $\boldsymbol{B}$ can be defined, where

$$\boldsymbol{A}:\boldsymbol{B} = A_{ij}B^{ij} = A^{ij}B_{ij} = A^{i}_{\cdot j}B_{i}^{\cdot j} = A_{i}^{\cdot j}B^{i}_{\cdot j}$$

Such a definition satisfies the additional axioms

(h) $\boldsymbol{A}:\boldsymbol{B} = \boldsymbol{B}:\boldsymbol{A}$

(i) $\boldsymbol{A}:(\boldsymbol{B}+\boldsymbol{C}) = \boldsymbol{A}:\boldsymbol{B}+\boldsymbol{A}:\boldsymbol{C}$

(j) $\boldsymbol{A}:(c\boldsymbol{B}) = c(\boldsymbol{A}:\boldsymbol{B})$

(k) $\boldsymbol{A}:\boldsymbol{A} > 0$ unless $\boldsymbol{A} = \boldsymbol{0}$

These conditions are necessary to define an inner product or unitary vector space. This definition of a scalar inner product can be extended to higher order tensors, the process of contraction taking place for superscript and subscript pairs of indices in sequence.

Products of vectors can be generalised to give products of tensors. The rule is that the product is between the vector component immediately preceding the operator and that immediately succeeding it. Thus the contraction formed by a scalar (dot) product is given by

$$\boldsymbol{A}\cdot\boldsymbol{u} = (A^{ij}\boldsymbol{g}_i\boldsymbol{g}_j)\cdot(u_k\boldsymbol{g}^k) = A^{ij}u_j\boldsymbol{g}_i$$

and

$$\boldsymbol{A}\cdot\boldsymbol{B} = (A^{ij}\boldsymbol{g}_i\boldsymbol{g}_j)\cdot(B_{kl}\boldsymbol{g}^k\boldsymbol{g}^l) = A^{ij}B_{jl}\boldsymbol{g}_i\boldsymbol{g}^l$$

for example. In some texts, this product is represented as in matrix notation, with the dot omitted.

Similarly, the concept of a cross product can be extended to tensors. Thus

$$\boldsymbol{A}\times\boldsymbol{B} = (A^{ij}\boldsymbol{g}_i\boldsymbol{g}_j)\times(B^{kl}\boldsymbol{g}_k\boldsymbol{g}_l) = A^{ij}B^{kl}\boldsymbol{g}_i(\boldsymbol{g}_j\times\boldsymbol{g}_k)\boldsymbol{g}_l = C^{i\cdot l}_{\cdot m}\boldsymbol{g}_i\boldsymbol{g}^m\boldsymbol{g}_l$$

where $C^{i\cdot l}_{\cdot m} = A^{ij}B^{kl}\varepsilon_{mjk}$ [cf. (3.71)].

The concepts of trace and transpose are also borrowed from matrix notation. Thus

$$\mathrm{tr}\,\boldsymbol{A} = A^{k}_{\cdot k} \quad (= A_{k}^{\cdot k}) \quad , \quad \boldsymbol{A}^{\mathrm{T}} = A^{ji}\boldsymbol{g}_i\boldsymbol{g}_j$$

Covariant differentiation can be performed using the operator $\nabla = \{\ \}_{|k}\boldsymbol{g}^k$

the arrow indicating that the operation is on what immediately precedes it. Thus

$$\boldsymbol{A}\nabla = A_{ij|k}\boldsymbol{g}^i\boldsymbol{g}^j\boldsymbol{g}^k$$

The operator $\nabla\cdot$ operates on what immediately succeeds it. Thus

$$\nabla\cdot\boldsymbol{u} = \nabla\cdot u_j\boldsymbol{g}^j = u_{j|i}\boldsymbol{g}^i\boldsymbol{g}^j$$

For example, the small strain tensor can be written as

$$\boldsymbol{E} = e_{kl}\boldsymbol{g}^k\boldsymbol{g}^l = \tfrac{1}{2}(u_{k|l} + u_{l|k})\boldsymbol{g}^k\boldsymbol{g}^l = \tfrac{1}{2}(\boldsymbol{u}\cdot\nabla + \nabla\cdot\boldsymbol{u})$$

cf. (1.60).

A fairly straightforward extension of the double dot product could be used to express the linear relationship between stress and strain in an anisotropic medium in the form

$$\boldsymbol{T} = \boldsymbol{C}:\boldsymbol{E} = c^{ijkl}e_{kl}\boldsymbol{g}_i\boldsymbol{g}_j$$

where $\quad \boldsymbol{T} = \sigma^{ij}\boldsymbol{g}_i\boldsymbol{g}_j \quad , \quad \boldsymbol{C} = c^{ijkl}\boldsymbol{g}_i\boldsymbol{g}_j\boldsymbol{g}_k\boldsymbol{g}_l$

cf. (3.154). However, the form of the notation and its significance varies considerably between texts.

These concepts can be expressed more simply using Cartesian tensors, see for example Malvern [(1969) Chapter 2]. It then becomes possible to describe the matrix $[\boldsymbol{A}]$ of a second-order tensor $\boldsymbol{A}$ in an unambiguous manner as follows.

$$\mathbf{A} = [\boldsymbol{A}] = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = [A_{ij}]$$

This is of course related to the matrix forms used in Chapter 1.

A5

Polar decomposition

The initial and final relative position vectors, $d\boldsymbol{r}$ and $d\boldsymbol{R}$, of adjacent material points as shown in Fig. 4.2, may be related by

$$d\boldsymbol{R} = \boldsymbol{F}\cdot d\boldsymbol{r} \qquad \textbf{(A5.1)}$$

in direct tensor notation, where $\boldsymbol{F}$ is known as the **deformation gradient tensor**. The polar decomposition theorem shows that $\boldsymbol{F}$ can be thought of as a local deformation $\boldsymbol{U}$ followed by a local rigid-body rotation $\boldsymbol{T}$, or as a local rotation $\boldsymbol{T}$ followed by a local deformation $\boldsymbol{V}$ where

$$\boldsymbol{F} = \boldsymbol{T}\cdot\boldsymbol{U} = \boldsymbol{V}\cdot\boldsymbol{T} \qquad \textbf{(A5.2)}$$

$\boldsymbol{U}$ and $\boldsymbol{V}$ are known as the right and left **stretch tensors** and are symmetric and positive-definite.

Proof With respect to a Cartesian coordinate system in the undeformed state, let

$$\boldsymbol{F} = F_{jk}\boldsymbol{i}_j\boldsymbol{i}_k \quad \text{and} \quad \mathbf{F} = [F_{jk}] \qquad \textbf{(A5.3)}$$

Let the symmetric matrix $\mathbf{C}$ be defined as

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} \quad \text{or} \quad [C_{ij}] = [F_{ki}F_{kj}]$$

and let $\bar{\mathbf{x}} = \mathbf{F}\mathbf{x}$, where $\mathbf{x}$, and hence $\bar{\mathbf{x}}$, are real. The quadratic form

$$\mathbf{x}^{\mathrm{T}}\mathbf{C}\mathbf{x} = \mathbf{x}^{\mathrm{T}}\mathbf{F}^{\mathrm{T}}\mathbf{F}\mathbf{x} = \bar{\mathbf{x}}^{\mathrm{T}}\bar{\mathbf{x}}$$

is positive definite because the right-hand side is a sum of squares. This means that $\mathbf{C}$ has real positive eigenvalues, λ_1^2, λ_2^2 and λ_3^2 say. If the matrix $\mathbf{P}$ is formed by the normalised row eigenvectors of $\mathbf{C}$, then

$$\mathbf{PCP}^{\mathrm{T}} = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix} = \mathbf{\Lambda\Lambda} \quad \text{where} \quad \mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Let $\mathbf{U} = \mathbf{P}^{\mathrm{T}}\mathbf{\Lambda P}$ so that $\mathbf{U}$ is symmetric. Then

$$\mathbf{U}^2 = \mathbf{UU} = \mathbf{P}^{\mathrm{T}}\mathbf{\Lambda P P}^{\mathrm{T}}\mathbf{\Lambda P} = \mathbf{P}^{\mathrm{T}}\mathbf{\Lambda\Lambda P} = \mathbf{P}^{\mathrm{T}}\mathbf{PCP}^{\mathrm{T}}\mathbf{P} = \mathbf{C} \qquad \textbf{(A5.4)}$$

Let $\mathbf{F} = \mathbf{TU}$. Then

$$\mathbf{T}^{\mathrm{T}}\mathbf{T} = (\mathbf{U}^{-1})^{\mathrm{T}}\mathbf{F}^{\mathrm{T}}\mathbf{FU}^{-1} = \mathbf{U}^{-1}\mathbf{CU}^{-1} = \mathbf{U}^{-1}\mathbf{U}^2\mathbf{U}^{-1} = \mathbf{I}$$

because $\mathbf{U}^{-1}$ is also symmetric. Then $\mathbf{T}$ is an orthonormal matrix and corresponds to a rotation. If $\mathbf{V} = \mathbf{TUT}^{\mathrm{T}}$ then $\mathbf{V}$ is symmetric and

$$\mathbf{F} = \mathbf{TUT}^{\mathrm{T}}\mathbf{T} = \mathbf{VT}$$

The terms in (A5.2) are then defined by

$$\begin{aligned} \mathbf{T} &= [T_{jk}] \quad , \quad \boldsymbol{T} = T_{jk}\boldsymbol{i}_j\boldsymbol{i}_k \ ; \\ \mathbf{U} &= [U_{jk}] \quad , \quad \boldsymbol{U} = U_{jk}\boldsymbol{i}_j\boldsymbol{i}_k \ ; \\ \mathbf{V} &= [V_{jk}] \quad , \quad \boldsymbol{V} = V_{jk}\boldsymbol{i}_j\boldsymbol{i}_k \ . \end{aligned}$$

Equation (4.10) can now be written in terms of the deformation gradient tensor as

$$\begin{aligned} d\boldsymbol{R}\cdot d\boldsymbol{R} - d\boldsymbol{r}\cdot d\boldsymbol{r} &= (\boldsymbol{F}\cdot d\boldsymbol{r})\cdot(\boldsymbol{F}\cdot d\boldsymbol{r}) - d\boldsymbol{r}\cdot d\boldsymbol{r} \\ &= (F_{jk}\boldsymbol{i}_j\boldsymbol{i}_k\cdot dx_l\boldsymbol{i}_l)\cdot(F_{mn}\boldsymbol{i}_m\boldsymbol{i}_n\cdot dx_p\boldsymbol{i}_p) - dx_k\boldsymbol{i}_k\cdot dx_n\boldsymbol{i}_n \\ &= F_{jk}\boldsymbol{i}_j dx_k\cdot F_{mn}\boldsymbol{i}_m dx_n - \delta_{kn}dx_k dx_n \\ &= (F_{jk}F_{jn} - \delta_{kn})dx_k dx_n \\ &= 2\gamma_{kn}dx_k dx_n \end{aligned}$$

If $\boldsymbol{C} = C_{kn}\boldsymbol{i}_k\boldsymbol{i}_n$, $\boldsymbol{I} = \delta_{kn}\boldsymbol{i}_k\boldsymbol{i}_n$, $\boldsymbol{\gamma} = \gamma_{kn}\boldsymbol{i}_k\boldsymbol{i}_n$ then from the above

$$\boldsymbol{C} - \boldsymbol{I} = F_{jk}F_{jn}\boldsymbol{i}_k\boldsymbol{i}_n - \delta_{kn}\boldsymbol{i}_k\boldsymbol{i}_n = 2\boldsymbol{\gamma} \qquad \textbf{(A5.5)}$$

Suppose that the strain energy per unit mass is a function of $\mathbf{F}$ and is independent of the local rigid-body rotations. It then follows that the energy is unchanged by expressing it as the same function in terms of $\mathbf{T}^{\mathrm{T}}\mathbf{F}$ and hence, from the above, the same function in terms of $\mathbf{U}$. But $\mathbf{U}$ is symmetric and so from (A5.4) there is a one-to-one correspondence between the elements of $\mathbf{U}$ and those of $\mathbf{C}$, and hence of $\boldsymbol{\gamma}$, from (A5.5). Hence the strain energy can be expressed in terms of $\boldsymbol{\gamma}$ only (see also Spencer (1980) section 10.2).

Having distilled from $\boldsymbol{F}$ a local rotation $\boldsymbol{T}$, it is necessary to find in what sense the material can be said not to rotate under the local deformation $\boldsymbol{U}$ only. If a line element $d\boldsymbol{r}$ does not change direction during local deformation, then it can only change in length to some value $\lambda d\boldsymbol{r}$ so that

$$\boldsymbol{U}\cdot d\boldsymbol{r} = \lambda\, d\boldsymbol{r}$$

giving an eigenvalue problem of the form $\mathbf{Ux} = \lambda\mathbf{x}$

The eigenvalues of $\mathbf{U}$ are already known to be λ_1, λ_2 and λ_3 where these are real

quantities. They are known as the local **principal stretches** of the material. As **U** is a real symmetric matrix, the corresponding eigenvectors are orthogonal, so that the three possible directions of $d\boldsymbol{r}$ are normal to one another. A physical interpretation of this result can be given as follows. Imagine a small unit sphere of the material. After deformation, the unit sphere becomes an ellipsoid with semi-axes λ_1, λ_2 and λ_3 and the lines forming these axes do not rotate during deformation. The local deformation given by $\boldsymbol{V}$ produces the same ellipsoid but, in general, with a different orientation.

A6

Cosserat continua and micropolar elasticity

The displacement field $\boldsymbol{u}$ may not always fully describe the state of deformation of an elastic medium. It may have a fine structure which distorts in a manner which is independent of the displacement gradient. The simplest case is where the local orientation of the fine structure can be represented by a triad of orthogonal directors, d_1, d_2 and d_3 which may rotate locally during deformation of the medium through some angle $\boldsymbol{\phi}$ ($= \phi_j \, \boldsymbol{i}_j$) where the components are related to the directors as shown in Fig. A6.1.

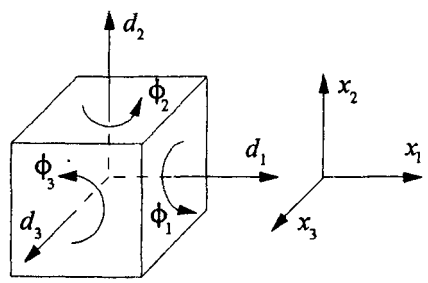


Figure A6.1 Micro rotations of Cosserat media.

The definition of axial strain is not affected in this simple model, but asymmetric 'shear' strains are given by the relative rotation of the continuum and the fine structure, as shown in Fig. A6.2. These give the **Cosserat strains** e_{ij}^{C}

$$e_{11}^{C} = u_{1,1} \quad , \quad e_{22}^{C} = u_{2,2}$$

$$e_{12}^{C} = u_{1,2} + \phi_3 \quad , \quad e_{21}^{C} = u_{2,1} - \phi_3$$

and in general, for a Cartesian system,

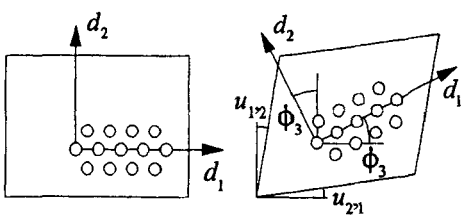


Figure A6.2 Distortion of a Cosserat medium.

$$e_{ij}^{C} = u_{i,j} + e_{ijk}\phi_k \qquad \textbf{(A6.1)}$$

It will be seen that the normal definition of a small strain e_{ij} is given by the average of e_{ij}^{C} and e_{ji}^{C}. Also, for the Cosserat strains to be different from the more usual small strains e_{ij}

$\boldsymbol{\phi}$ must be different from the local continuum rotation, $\frac{1}{2}\nabla\times\boldsymbol{u}$.

In such a medium, **micromoments** can arise which are analogous to stresses and body forces. The resultant micromoment $\boldsymbol{M}(1)$, on the right-hand face of the element shown in Fig. A6.1, is shown in Fig. A6.3. This has components per unit area m_{11}, m_{12} and m_{13} in the directions of the coordinate axes x_1, x_2 and x_3 respectively. These are known as **couple stresses**. They are related to $\boldsymbol{M}(1)$ by

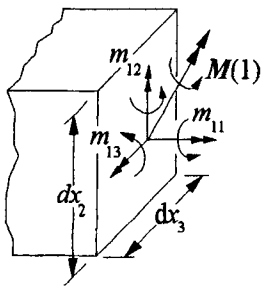


Figure A6.3 Couple stresses.

$$\begin{aligned}\boldsymbol{M}(1) &= (m_{11}\boldsymbol{i}_1 + m_{12}\boldsymbol{i}_2 + m_{13}\boldsymbol{i}_3)\,\mathrm{d}x_2\,\mathrm{d}x_3 \\ &= m_{1k}\boldsymbol{i}_k\,\mathrm{d}S_1\end{aligned} \qquad \textbf{(A6.2)}$$

where $\mathrm{d}S_1$ is the area of the face on which they act. Similar expressions can be written for the couple stresses on the other faces.

A micromoment $\boldsymbol{m}$ per unit area may act on the surface of the body. This can be expressed in terms of the micromoments acting on faces normal to the axes by considering the equilibrium of the elementary tetrahedron shown in Fig. A6.4. A couple stress acting on a positive face must be matched by a couple stress acting on the adjoining negative face, just as is the case with ordinary stresses. The faces of the tetrahedron, PST, PRT and PRS, are triangular negative faces and so have negative micromoments acting on them, as shown. Taking the surface area of the triangle RST to be $\frac{1}{2}\mathrm{d}S$, the equilibrium of the tetrahedron is given by

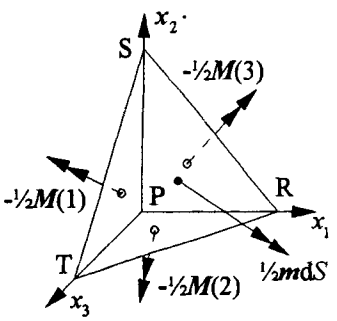


Figure A6.4 Micromoment equilibrium.

$$\tfrac{1}{2}\boldsymbol{m}\,\mathrm{d}S = \tfrac{1}{2}[\boldsymbol{M}(1) + \boldsymbol{M}(2) + \boldsymbol{M}(3)] \qquad \textbf{(A6.3)}$$

If the unit normal to the surface, $\boldsymbol{n}$, has a projected length n_j in the direction of the x_j axis, ($j = 1$ to 3), cf. Fig. 1.8b, then

$$\tfrac{1}{2}n_j\,\mathrm{d}S = \tfrac{1}{2}\mathrm{d}S_j$$

so that from (A6.2) and (A6.3),

$$\tfrac{1}{2}\boldsymbol{m}\,\mathrm{d}S = \tfrac{1}{2}m_{jk}\boldsymbol{i}_k n_j\,\mathrm{d}S$$

or

$$m_k = m_{jk}n_j \quad \text{where} \quad \boldsymbol{m} = m_k\boldsymbol{i}_k \qquad \textbf{(A6.4)}$$

The Cosserat stresses σ^{C}_{jk} are related to the related to the surface tractions as before, so that

$$p_k = \sigma^{\mathrm{C}}_{jk}n_j \qquad \textbf{(A6.5)}$$

cf. (2.40), although these stresses no longer necessarily form a symmetric tensor, as will be seen in (A6.8).

The equation of linear motion (Cauchy's first law) can again be derived in the form given by (2.41),

$$\sigma^{\mathrm{C}}_{ji,j} + \rho f_i = \rho\ddot{u}_i \qquad \textbf{(A6.6)}$$

However, the equations of rotary motion, (Cauchy's second law), no longer give rise to the

symmetry of the stress tensor given by (2.39).

Fig. A6.5 shows the stresses and couple stresses related to the rotation of the element. This is about an axis through the centre of the element, parallel to the x_1 axis. Let the element have an angular momentum per unit mass of l_1 about this axis, and assume that there is a body couple M_1 per unit mass about this axis. Then on equating the resultant moment to the change in angular momentum per unit time, $\dot{l}_1$,

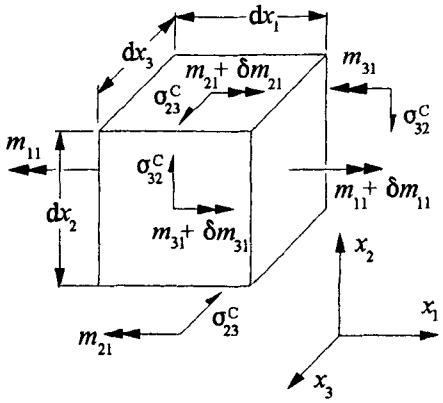


Figure A6.5 Couple stress equilibrium.

$$\begin{aligned}&\delta m_{11}\,dx_2dx_3 + \delta m_{21}\,dx_3dx_1 + \delta m_{31}\,dx_1dx_2 + \sigma^C_{23}\,dx_1dx_3dx_2\\ &-\sigma^C_{32}\,dx_1dx_2dx_3 + \rho M_1\,dx_1dx_2dx_3 \;=\; \rho\dot{I}_1\,dx_1dx_2dx_3\end{aligned} \qquad \textbf{(A6.7)}$$

or since $\delta m_{\alpha 1} = m_{\alpha 1,\alpha}dx_\alpha$ ($\alpha = 1$ to 3), $m_{j1,j} + \sigma^C_{23} - \sigma^C_{32} + \rho M_1 = \rho\dot{I}_1$.
This result can be generalised to give the form

$$m_{ji,j} + e_{ijk}\sigma^C_{jk} + \rho M_i = \rho\dot{I}_i \qquad \textbf{(A6.8)}$$

This is the equivalent of (2.39) for a Cosserat continuum. In a normal continuum, body couples are assumed not to arise, and the angular momentum term in (A6.7) would be of the order $(dx)^5$ rather than $O(dx^3)$ and so is normally neglected.

For a homogeneous, isotropic, centro-symmetric, linearly-elastic Cosserat body which is unstressed in its unstrained state, Nowacki (1962) and Jaunzemis (1967) give stress-strain relationships for the material of the form

$$\sigma^C_{ij} = (\mu-\alpha)e^C_{ij} + (\mu+\alpha)e^C_{ji} + \lambda\delta_{ij}e^C_{kk} = \lambda\delta_{ij}e_{kk} + 2\mu e_{ij} + \alpha[(u_{j,i} - u_{i,j}) - 2e_{ijk}\phi_k] \qquad \textbf{(A6.9)}$$

$$m_{ij} = (\gamma-\varepsilon)\phi_{i,j} + (\gamma+\varepsilon)\phi_{j,i} + \beta\delta_{ij}\phi_{k,k} \qquad \textbf{(A6.10)}$$

If the stiffness α is zero or ϕ is the local macroscopic rotation, (A6.9) reduces to the equation for an isotropic, linearly-elastic body, so that λ and μ can be thought of as generalised Lamé constants. Substituting (A6.1), (A6.9) and (A6.10) into (A6.6) and (A6.8) gives

$$(\mu+\alpha)u_{i,jj} + (\lambda+\mu-\alpha)u_{j,ji} + 2\alpha e_{ijk}\phi_{k,j} + \rho f_i = \rho\ddot{u}_i$$

or

$$(\mu+\alpha)\nabla^2\boldsymbol{u} + (\lambda+\mu-\alpha)\nabla(\nabla\cdot\boldsymbol{u}) + 2\alpha\nabla\times\boldsymbol{\phi} + \rho\boldsymbol{f} = \rho\ddot{\boldsymbol{u}}$$

cf. (2.86), and

$$(\gamma+\varepsilon)\phi_{i,jj} + (\beta+\gamma-\varepsilon)\phi_{j,ji} + 2\alpha(e_{ijk}u_{k,j} - 2\phi_i) + \rho M_i = \rho\dot{I}_i$$

or

$$(\gamma+\varepsilon)\nabla^2\boldsymbol{\phi} + (\beta+\gamma-\varepsilon)\nabla(\nabla\cdot\boldsymbol{\phi}) + 2\alpha(\nabla\times\boldsymbol{u} - 2\boldsymbol{\phi}) + \rho\boldsymbol{M} = \rho\dot{\boldsymbol{I}}$$

Further discussion of such continua is given by Jaunzemis [(1967) sections 11 and 16], Nowacki [(1962) pp. 39-57] and Malvern [(1969) Chapter 5]. The original work of E. and F. Cosserat was published on pages 953 to 1173 of *Traité de physique* Volume 2 (2nd edition) edited by O.D. Khvol'son, [Hermann & Cie. Paris, (1909)].

A7

Minimal curves and geodesics

A7.1 Minimal curves

Suppose that the coordinates y^i ($i = 1$ to n) of a family of curves in n-dimensional space are expressed in terms of a parameter u by some functions $y^i(u)$ which are continuously variable. Further, suppose that all the curves in the family pass through points A and B with coordinates y_A^i and y_B^i ($i = 1$ to n) respectively, as shown in Fig. A7.1. Let the value of u be u_A and u_B at these two points. Suppose that a curve in this family is sought which is such that the integral along it,

$$I = \int_{u_A}^{u_B} f(y^i, \dot{y}^i)\,\mathrm{d}u$$

is a minimum where

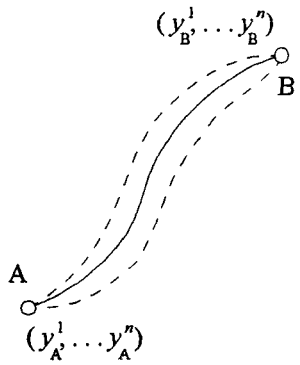


Figure A7.1 Minimising curves.

$$\dot{y}^i = \frac{\mathrm{d}y^i}{\mathrm{d}u} = \lim_{\Delta u \to 0}\{(y^i + \Delta y^i - y^i)/\Delta u\}$$

Then the change of I, δI say, in moving from the minimal curve to an immediately adjacent member of the family with coordinates $y^i + \delta y^i$ will be zero, or

$$\delta I = \int_{u_A}^{u_B} \delta f\,\mathrm{d}u = \int_{u_A}^{u_B} \left(\frac{\partial f}{\partial y^i}\delta y^i + \frac{\partial f}{\partial \dot{y}^i}\delta \dot{y}^i \right) \mathrm{d}u = 0$$

Now

$$\delta\dot{y}^i = \delta\frac{dy^i}{du}$$
$$= \lim_{\Delta u\to 0}\{[(y^i + \delta y^i + \Delta(y^i + \delta y^i) - (y^i + \delta y^i)]/\Delta u - (y^i + \Delta y^i - y^i)/\Delta u\}$$
$$= \lim_{\Delta u\to 0}\{\Delta(\delta y^i)/\Delta u\} = \frac{d}{du}(\delta y^i)$$

so that

$$\delta I = \int_{u_A}^{u_B}\left(\frac{\partial f}{\partial y^i}\delta y^i + \frac{\partial f}{\partial \dot{y}^i}\frac{d}{du}(\delta y^i)\right)du$$
$$= \left|\frac{\partial f}{\partial \dot{y}^i}\delta y^i\right|_{u_A}^{u_B} + \int_{u_A}^{u_B}\left(\frac{\partial f}{\partial y^i} - \frac{d}{du}\frac{\partial f}{\partial \dot{y}^i}\right)\delta y^i du = 0$$

As all the curves pass through A and B, there can be no variation in y^i at these points. This means that δy^i is zero in the first term in the final expression for δI, leaving just the integral. This integral must be zero for any arbitrary variation δy^i from the path of the minimal curve. This can only be true if the rest of the integrand is zero, that is if

$$\frac{\partial f}{\partial y^i} - \frac{d}{du}\frac{\partial f}{\partial \dot{y}^i} = 0 \qquad \textbf{(A7.1)}$$

This equation can be used in determining the minimal curve. It is known as **Euler's equation** and the method used to derive it is known as the **calculus of variations**. This method can be used to find minimal curves for functions which are expressed in terms of higher derivatives of y^i, see for example Sokolnikoff and Redheffer [(1966) Chapter 5 section 15].

A7.2 Geodesics

Geodesics are minimal-length curves. That is, the integral I of the previous section is the length of the curve between A and B. For flat spaces, geodesics are straight lines. More generally, using the metric of the space given by (3.39), we seek to minimise s where

$$s = \int_A^B ds = \int_A^B \sqrt{(g_{jk}dy^j dy^k)} = \int_A^B \frac{ds}{du}du = \int_A^B \sqrt{(g_{jk}\dot{y}^j\dot{y}^k)}\,du$$

Equation (A7.1) can be used, where now

$$\frac{\partial f}{\partial y^i} = \tfrac{1}{2}(g_{jk,i}\dot{y}^j\dot{y}^k)\Big/\frac{ds}{du}$$
$$\text{and} \qquad \frac{d}{du}\frac{\partial f}{\partial \dot{y}^i} = \tfrac{1}{2}\frac{d}{du}\left((g_{ik}\dot{y}^k + g_{ji}\dot{y}^j)\Big/\frac{ds}{du}\right) \qquad \textbf{(A7.2)}$$

If the parameter u is chosen to be s, then

$$\dot{y}^i = \frac{dy^i}{ds} \quad , \quad \frac{ds}{du} = 1$$

so that (A7.2) becomes

$$\frac{d}{ds}\frac{\partial f}{\partial \dot{y}^i} = \tfrac{1}{2}(g_{ik,j}\dot{y}^j\dot{y}^k + g_{ik}\ddot{y}^k + g_{ji,k}\dot{y}^k\dot{y}^j + g_{ji}\ddot{y}^j)$$

and (A7.1) can be written as

$$\tfrac{1}{2}(g_{jk,i} - g_{ik,j} - g_{ji,k})\dot{y}^j\dot{y}^k - g_{ij}\ddot{y}^j = 0$$

Taking the product of this with g^{li} and using (3.42) and (3.118) gives

$$\ddot{y}^l + \Gamma^l_{jk}\dot{y}^j\dot{y}^k = 0 \qquad \textbf{(A7.3)}$$

This equation can be used to determine the path of a geodesic. From the properties of differential equations, it can be shown that the path is unique for a given initial point on it, y_0^i, and an initial direction given by $\dot{y}_0^i$. For **null geodesics**, all path lengths are zero. In such cases, s must be treated as an arbitrary parameter in (A7.3). Such null geodesics give light paths in space-time for example, see section A7.3.2.

Example A7.1

For the surface of a sphere of radius a described in Example 3.5, the metric equation gives

$$1 = a^2\left(\frac{d\theta}{ds}\right)^2 + a^2\sin^2\theta\left(\frac{d\phi}{ds}\right)^2 \qquad \textbf{(A7.4)}$$

and (A7.3) gives

$$\frac{d^2\theta}{ds^2} - \sin\theta\cos\theta\left(\frac{d\phi}{ds}\right)^2 = 0 \qquad \textbf{(A7.5)}$$

and

$$\frac{d^2\phi}{ds^2} + 2\cot\theta\,\frac{d\theta}{ds}\frac{d\phi}{ds} = 0 \qquad \textbf{(A7.6)}$$

Equation (A7.5) can be derived from (A7.6) and the differential of (A7.4). Also, (A7.6) can be written as

$$\frac{d}{ds}\left(\sin^2\theta\,\frac{d\phi}{ds}\right) = 0$$

giving

$$\sin^2\theta\,\frac{d\phi}{ds} = \text{a constant} = \frac{1}{a}\cos\beta \quad \text{(say)}$$

Substituting this into (A7.4) gives

$$a\frac{d\theta}{ds} = \pm\left(1 - \frac{\cos^2\beta}{\sin^2\theta}\right)^{1/2}$$

Making the substitution $\cos\theta = \sin\alpha\sin\beta$ gives

$$a\frac{d\alpha}{ds} = \pm 1 \quad \text{so that} \quad s = \pm a\alpha + b$$

This result corresponds to a great circle on a plane at an angle β to the equatorial plane.

A7.3 Relativity

Tensor calculus has an important application in the theory of relativity. These last few pages are intended to indulge the curious reader who wishes to find out how the mathematical concepts given here are used. In its original form, the theory is concerned

with the nature of a four-dimensional space, namely the usual three spatial dimensions plus a time dimension.

A7.3.1 Special Relativity

In the special theory of relativity, it is assumed that constant linear motion cannot be detected and that the speed of light in vacuo, c, is constant (2.998×10^8 m/s) regardless of the relative motion of the source and the observer. The first condition means that there is no background 'ether' against which velocity can be measured absolutely, although the relative velocity of two bodies may be found. The second condition followed from the result of the Michelson-Morley experiment in 1887 which purported to show that the velocity of light was unaffected by the motion of the earth.

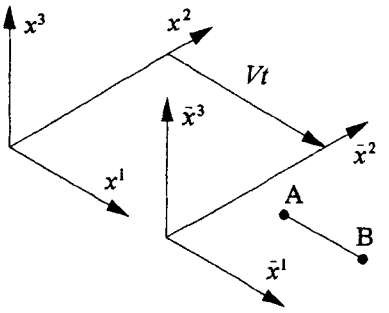


Figure A7.2 Constant relative velocity.

For special relativity, the 'flat' Minkowski space-time metric is used. This is given by

$$dS^2 = -(dx^1)^2 - (dx^2)^2 - (dx^3)^2 + c^2(dt)^2 \qquad \textbf{(A7.7)}$$

Then dS is real for all for all increments in space-time corresponding to velocities less than the speed of light and zero for light velocities. That is, in space-time, light paths in vacuo follow null geodesics.

Consider the transformation to a second coordinate system given by

$$\bar{x}^1 = \beta(x^1 - Vt) \quad , \quad \bar{x}^2 = x^2 \quad , \quad \bar{x}^3 = x^3 \quad , \quad \bar{t} = \beta(t - Vx^1/c^2) \qquad \textbf{(A7.8)}$$

where $$\beta = (1 - V^2/c^2)^{-\frac{1}{2}}$$

Inverting this relationship gives

$$x^1 = \beta(\bar{x}^1 + V\bar{t}) \quad , \quad x^2 = \bar{x}^2 \quad , \quad x^3 = \bar{x}^3 \quad , \quad t = \beta(\bar{t} + V\bar{x}^1/c^2) \qquad \textbf{(A7.9)}$$

It will be seen that the metric of the new coordinate system is the same in form as the original, that is, on substituting (A7.9) into (A7.7),

$$dS^2 = -(d\bar{x}^1)^2 - (d\bar{x}^2)^2 - (d\bar{x}^3)^2 + c^2(d\bar{t})^2$$

Thus there is no difference between stating that the first system is stationary and the second system is moving with a velocity V and stating that the second system is stationary and the first is moving with a velocity $-V$.

Note that the origin of the $\bar{x}^i$ coordinate system corresponds to the point

$$x^1 = Vt \quad , \quad x^2 = x^3 = 0$$

so that the new coordinate system is moving with a velocity V as perceived from the original coordinate system, as shown in Fig. A7.2.

Suppose that a rod AB lies parallel to the $\bar{x}^1$ axis and moves with this coordinate system, so that the $\bar{x}^1$ ordinates of its ends are the constants $\bar{x}^1_A$ and $\bar{x}^1_B$. At time $t = 0$, its end points are given in the first coordinate system by (A7.8). These are

$$\bar{x}_A^1 = \beta x_A^1 \quad , \quad \bar{x}_B^1 = \beta x_B^1$$

With regard to the 'moving' coordinates, its length is

$$\bar{l} = \bar{x}_B^1 - \bar{x}_A^1$$

but with regard to the 'fixed' coordinates, its length is

$$l = x_B^1 - x_A^1 = \frac{1}{\beta}(\bar{x}_B^1 - \bar{x}_A^1) = (1 - V^2/c^2)^{1/2}\bar{l}$$

so that it appears to have undergone a contraction with respect to the first coordinate system. This is known as the **Lorentz-Fitzgerald contraction**. Similarly, it can be shown that exactly the same contraction is seen taking place in the 'fixed' space by an observer in the 'moving' space.

From (A7.9), a time interval of $\Delta\bar{t}$ at the origin of the second coordinate system corresponds to a longer time interval $\beta\Delta\bar{t}$ in the first. That is, the clocks in the second system appear to have slowed down as observed from the first. This is known as **time dilatation**. As before, from (A7.8), the same phenomenon is seen to occur in the first system by an observer in the second.

If a particle which is at the origin of the second coordinate system at time $\bar{t} = 0$ moves at a constant rate to $\Delta\bar{x}^1$ in time $\Delta\bar{t}$, its velocity is given by $\bar{v}^1 = \Delta\bar{x}^1/\Delta\bar{t}$. In the first system,

$$\Delta x^1 = \beta(\Delta\bar{x}^1 + V\Delta\bar{t}) \quad , \quad \Delta t = (\Delta\bar{t} + V\Delta\bar{x}^1/c^2)$$

Thus the velocity as measured in the first system is

$$v^1 = \frac{\beta(\Delta\bar{x}^1 + V\Delta\bar{t})}{\beta(\Delta\bar{t} + V\Delta\bar{x}^1/c^2)} = \frac{\bar{v}^1 + V}{1 + \bar{v}^1 V/c^2}$$

Thus the velocity of the particle relative to the first coordinate system is less than the simple sum of the two velocities. If $\bar{v}^1$ is the velocity of light, c, it follows that v^1 is also c, in accordance with the second condition mentioned at the outset.

A7.3.2 General Relativity

Riemann tried unsuccessfully to use his formulation of intrinsically curved space to explain the effects of gravity. This was achieved by Einstein's general theory of relativity in which gravitational effects are accounted for by the curved geometry of space-time. Newton's first law is replaced by a statement that the paths or 'world lines' of particles in space-time are geodesics, in the absence of any influence other than gravity. This affects all particles, including photons, so that light paths are curved by gravitating bodies. If a gravitating field is also absent, space-time becomes a flat Minkowski space. The geodesics are then straight lines and Newton's first law again applies.

The **Schwarzschild metric**

$$dS^2 = -\left(1 - \frac{2MG}{c^2 r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2 + c^2\left(1 - \frac{2MG}{c^2 r}\right) dt^2 \qquad \textbf{(A7.10)}$$

describes space-time around a non-rotating spherically-symmetrical body of mass M where G is the universal gravitational constant (6.670×10^{-11} m^3/kg s^2). At radii r which are large

in comparison with MG/c^2, the form tends to that given by special relativity, expressed in terms of spherical polar coordinates.

In the last section, the local coordinates for an observer travelling at a velocity V were $\bar{x}^i$ and $\bar{t}$. Increments in these in these ordinates gave the same increment dS^2 as that given by the corresponding increments in what might be regarded as the global system, x^i and t. Here again, a local coordinate system will be considered, giving the metric

$$dS^2 = -d\bar{l}^2 + c^2 d\bar{t}^2 \qquad (d\bar{l}^2 = d\bar{x}^i d\bar{x}^i) \qquad \textbf{(A7.11)}$$

If we compare time increments at fixed positions in the global and local systems, then from (A7.10) and (A7.11),

$$dt^2 = (1 - 2MG/c^2 r)^{-1} d\bar{t}^2$$

In particular, when

$$r = r_S = 2MG/c^2$$

where r_S is known as the **Schwarzchild radius**, an event in the local coordinate system corresponds to an infinite time period in the global coordinate system.

The Schwarzchild radius defines an 'event horizon' around an uncharged non-rotating black hole. This is a spherical boundary on which no change is discernable to an outside observer. Light cannot escape from within this radius, and it may only escape outwards at less than a certain angle to the radial direction for greater radii up to that of the **photon sphere**. At this radius, light can have a circular orbit around a black hole.

Determining the Christoffel symbols from the metric equation (A7.10), (A7.3) takes the form

$$\ddot{r} - \frac{r_S}{2r^2}\left(1 - \frac{r_S}{r}\right)^{-1}\dot{r}^2 - \left(1 - \frac{r_S}{r}\right)\left(r\dot{\theta}^2 + r\dot{\phi}^2\sin\theta - \frac{c^2 r_S}{2r^2}\dot{t}^2\right) = 0$$

For a null geodesic, from the metric equation,

$$0 = -\left(1 - \frac{r_S}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\theta}^2 - r^2\dot{\phi}^2\sin^2\theta + c^2\left(1 - \frac{r_S}{r}\right)\dot{t}^2$$

Taking there to be a circular light orbit at $r = r_p$ and $\theta = \pi/2$, the above two equations give

$$2r_p^3\dot{\phi}^2 = c^2 r_S \dot{t}^2$$

and

$$r_p^3\dot{\phi}^2 = c^2(r_p - r_S)\dot{t}^2$$

so that the radius of the photon sphere is given by

$$r_p = \tfrac{3}{2}r_S$$

For a Schwarzchild black hole of one solar mass, r_S is 2.96km and r_p is 4.44km; the values for a black hole with the mass of the earth would be 9mm and 13.5mm respectively.

Answers to problems

(1.3) $a = 1/\sqrt{2}$, $-c = d = 1/\sqrt{10}$, $e = -f = 2/\sqrt{10}$, $76.39°$.

(1.4) $\boldsymbol{\theta} = \{\partial u_z/\partial y - \partial u_y/\partial z,\ \partial u_x/\partial z - \partial u_z/\partial x,\ \partial u_y/\partial x - \partial u_x/\partial y\}$.

(1.5) $\omega_{11} = \omega_{22} = \omega_{33} = 0$, $\omega_{12} = -\omega_{21} = d$, $\omega_{13} = -\omega_{31} = e$, $\omega_{23} = -\omega_{32} = f$.
The relative motion is $\boldsymbol{\omega}\mathbf{dr}$. The rotations are $\{-f, e, -d\}$.

(1.6) $\varepsilon'_{xx} = (a-1)\cos^2\theta + (b+c)\sin\theta\cos\theta + (d-1)\sin^2\theta$, $\varepsilon'_{yy} = (a-1)\sin^2\theta$.
If small deflection theory is to apply, $(a - 1)$, b, c and $(d - 1)$ must be taken as small and their products as negligible.

(1.7) $\sigma_{xx} = 110/3$, $\tau_{xy} = 30/\sqrt{6}$, $\tau_{xz} = -35\sqrt{2}/3$, $\sigma_{yy} = 25$, $\tau_{yz} = 5\sqrt{3}$, $\sigma_{zz} = 145/3$.

(1.8) $\varsigma_{xx} = 40/3$, $\varsigma_{yy} = \varsigma_{zz} = -20/3$, $\varsigma_{xy} = \varsigma_{xz} = -10$, $\varsigma_{yz} = 10$.
0, 30 and -10MPa. 20MPa, $-300(\text{MPa})^2$, $0(\text{MPa})^3$.

(1.11) $I_1^4 - 4I_1^2I_2 + 2I_2^2 + 4I_1I_3$.

(1.12) $K_1 = (1 - 2\nu)I_1/E$, $K_2 = [(\nu^2 - 2\nu)I_1^2 + (1+\nu)^2I_2]/E$.
(i) $I_1^2/2E - I_2/2G$ (ii) $G[(4G - E)K_1^2/(6G - 2E) - 2K_2]$.

(1.14) $m^2 = 1/c^2 - \rho/(\lambda + 2\mu)$, $n^2 = 1/c^2 - \rho/\mu$.
$[(\lambda + 2\mu)c^2m^2 - \lambda]A - 2\mu ncB = 0$ and $2mcA - (n^2c^2 + 1)B = 0$,
so that $(2 - c^2/c_2^2)^4 = 16(1 - c^2/c_1^2)(1 - c^2/c_2^2)$ for non-zero A and B.

(1.15) $c_{33} > 0$, $c_{66} > 0$, $c_{33}(c_{12} + c_{66}) > c_{13}^2$, $c_{44}c_{66} > c_{14}^2 + c_{15}^2$.

(1.16) The number of independent constants describing the monoclinic system appears to reduce to ten. However, a material orientation (the z axis) is already defined.

(1.17) No, but a weaker condition defining x and y axes, would reduce the independent elastic constants to twelve.

(1.18) $s_{11} = \frac{1}{4}(1/c_{66} + c_{33}/C)$, $s_{12} = \frac{1}{4}(1/c_{66} - c_{33}/C)$, $s_{13} = -c_{13}/2C$,
$s_{33} = (1 + c_{13}^2/C)/c_{33}$, $s_{44} = 1/c_{44}$, $s_{66} = 1/c_{66}$ where $C = c_{33}(c_{12} + c_{66}) - c_{13}^2$.

(1.19) No, no.

(1.20) $D_{11} = h^3(c_{11}c_{33} - c_{13}^2)/12c_{33}$, $D_{12} = h^3(c_{13}c_{23} - c_{12}c_{33})/12c_{33}$,
$D_{16} = D_{26} = 0$, $D_{22} = h^3(c_{22}c_{33} - c_{23}^2)/12c_{33}$, $D_{66} = h^3c_{66}/12$.

(1.21) 6.7mm.

(1.23) Yes.

(1.24) Yes.

(1.25)
$$\begin{vmatrix} (2-\gamma)\cosh m_1 c & 2\cosh m_2 c \\ (2-2\gamma)\dfrac{1}{m_1 c}\sinh m_1 c & (2-\gamma)\dfrac{1}{m_2 c}\sinh m_2 c \end{vmatrix} = 0$$

where $\gamma = \rho\omega^2/G\lambda^2$, $m_1^2 = \lambda^2(1-\gamma)$ and $m_2^2 = \lambda^2[1 - \frac{1}{2}(1-\nu)\gamma]$.

(2.2) $\bar{A}_{11},\ldots\bar{A}_{33} = (1575, 2626, -107; -377, 2028, -26; 1414, 1469, -223)/845.$

(2.3) $A_{ii} = A_{kk} = 4,\ A_{ij}A_{ij} = A_{kl}A_{kl} = 25.$

(2.5) 6.

(2.7) $e_{11} = Ax_2x_3 + 2Bx_1x_2,\ e_{12} = \tfrac{1}{2}Ax_3(x_1 + x_2) + Bx_1^2$. The rest follow from cyclic permutations of the subscripts. $A(\lambda + \mu) + 2B(\lambda + 2\mu) = 0.$

(2.8) $\sigma_{11} = \lambda(A + 2B)(x_1x_2 + x_2x_3 + x_3x_1) + 2\mu(Ax_2x_3 + 2Bx_1x_2),$
$\sigma_{12} = \mu Ax_3(x_1 + x_2) + 2\mu Bx_1^2$. The rest follow from cyclic permutations of the subscripts.

(2.9) $\boldsymbol{n} = (6\boldsymbol{i}_1 + 2\boldsymbol{i}_2 + 3\boldsymbol{i}_3)/7,\quad \boldsymbol{p} = 2136\boldsymbol{i}_1 + 2328\boldsymbol{i}_2 + 3192\boldsymbol{i}_3.$

(2.11) $e_{ij}e_{jk}e_{kl}e_{li} = [(e_{ii})^4 - 6(e_{ii})^2(e_{jk}e_{kj}) + 8(e_{ii}e_{jk}e_{kl}e_{lj}) + 3(e_{ij}e_{ji})^2]/6.$

(2.13) $s_{ij} = 2\mu e_{ij} + d_3e_{jk}e_{ki}\ldots - \tfrac{1}{3}\delta_{ij}(2\mu e_{kk} + d_3e_{lk}e_{kl}\ldots).$

(2.15) No.

(2.16) $\lambda' = 37.8$ GPa, $\mu' = 32.0$ GPa.

(2.17) $\sigma_{11} = \chi_{3,22} + \chi_{2,33},\ \sigma_{12} = -\chi_{3,12}$ and cyclic permutations of the subscripts give the other stresses.

(2.18) $(\chi_{2,33} + \chi_{3,22})_{,ii} + \Theta_{,11} = 0,\ (\Theta + \chi_{1,ii})_{,23} = 0$ and cyclic permutations of the subscripts give six conditions, where
$\Theta = 2(\lambda + \mu)[(\chi_1 + \chi_2 + \chi_3)_{,ii} - \chi_{11,1} - \chi_{22,2} - \chi_{33,3}]/(3\lambda + 2\mu).$

(2.20) $(\lambda + 2\mu)\phi_{,IIJJ} + 2\alpha\mu(3\lambda + 2\mu)\Delta T_{,II} = 0.$

(3.1) $v^r = (a - c\cot\alpha)\cos\theta + b\sin\theta,\ rv^\theta = (c\cot\alpha - a)\sin\theta + b\cos\theta,\ v^z = c\operatorname{cosec}\alpha;$
$v_r = a\cos\theta + b\sin\theta, v_\theta = -ar\sin\theta + br\cos\theta, v_z = c\sin\alpha + a\cos\alpha; v^iv_i = a^2 + b^2 + c^2.$

(3.2) $A_{11}, A_{12}\ldots A_{33} = 3, -8, 2;\ 4, 0, 16;\ 4, -8, 4.$
$A^1_1, A^1_2\ldots A^3_3 = 3, -8, 2;\ 1, 0, 4;\ 2, -4, 2,$ or $3, -2, 1;\ 4, 0, 8;\ 4, -2, 2.$

(3.3) $A^k_k = 5$ (both forms), $A^{ij}A_{ij} = 83,\ A^{ij}A_{ji} = A^i_jA^j_i$ (both forms) $= -27.$

(3.4) $ds^2 = (1 + 4a^2r^2)dr^2 + r^2d\theta^2;\ g_{rr} = 1/g^{rr} = 1 + 4a^2r^2,\quad g_{\theta\theta} = 1/g^{\theta\theta} = r^2.$

(3.5) $\Gamma^r_{rr} = 4a^2r/(1 + 4a^2r^2),\ \Gamma^r_{\theta\theta} = -r/(1 + 4a^2r^2),\ \Gamma^\theta_{r\theta}\ (= \Gamma^\theta_{\theta r}) = 1/r;$
$R^\theta_{r\theta r} = 4a^2/(1 + 4a^2r^2), R^r_{\theta r\theta} = 4a^2r^2/(1 + 4a^2r^2)^2, R_{\theta r\theta r} = 4a^2r^2/(1 + 4a^2r^2).$

(3.6) $ds^2 = (z\tan\alpha)^2d\theta^2 + \sec^2\alpha\,dz^2;\ \Gamma^\theta_{\theta z} = \Gamma^\theta_{z\theta} = 1/z,\ \Gamma^z_{\theta\theta} = -z\sin^2\alpha.$

(3.7) $g_{\theta,\theta} = -z\sin^2\alpha\,g_z,\ g_{\theta,z} = z^{-1}g_\theta = g_{z,\theta},\ g_{z,z} = 0.$

(3.8) $ds^2 = (z\tan\alpha + y\cos\alpha)^2d\theta^2 + \sec^2\alpha\,dz^2 + dy^2,\ \Gamma^y_{\theta\theta} = -(z\sin\alpha + y\cos^2\alpha).$

(3.9) $-\Gamma^r_{rr} = \Gamma^t_{rt} = \rho/2r(r - \rho),\ \Gamma^r_{\theta\theta} = \rho - r,\ \Gamma^r_{\phi\phi} = (\rho - r)\sin^2\theta,$
$\Gamma^r_{tt} = c^2\rho(r - \rho)/2r^3,\ R_{rttr} = -c^2\rho/r^3$ where $\rho = 2MG/c^2.$

(3.14) $\sigma^{rr} = -2p\sin\theta/\pi r,\quad \sigma^{r\theta} = \sigma^{\theta\theta} = 0.$

(3.15) $\sigma^P_{rr} = -\rho g(5\lambda + 6\mu)(a^2 - r^2)/10a(\lambda + 2\mu),$
$\sigma^P_{\theta\theta} = \sigma^P_{\phi\phi} = -\rho g[5\lambda(a^2 - r^2) + 2\mu(3a^2 - r^2)]/10a(\lambda + 2\mu).$

(3.16) $(\lambda + 2\mu)(u_{r,r} + 2u_r/r)_{,r} = 0,\ u_r = Ar + B/r^2.$

(3.17) $\sigma^{rr} = pa^3(1 - b^3/r^3)/(b^3 - a^3),\ \sigma^{\theta\theta} = \sigma^{\phi\phi}\sin^2\theta = pa^3(1 + b^3/2r^3)/r^2(b^3 - a^3).$

(3.18) $\mu u_r = P[\mu(z - R)/Rr(\lambda + \mu) + zr/R^3]/4\pi,\ \mu u_z = P[z^2 + R^2(\lambda + 2\mu)/(\lambda + \mu)]/R^3.$

(3.19) $\sigma^{rr} = -P[\mu(z - R)/Rr^2(\lambda + \mu) + 3zr^2/R^5]/2\pi,\quad \sigma^{zz} = -3Pz^3/2\pi R^5.$
$\sigma^{\theta\theta} = P\mu(zR^2 + zr^2 - R^3)/2\pi R^3r^2(\lambda + \mu),\quad R\sigma^{rz} = rp^z = -3Pz^2r/2\pi R^4.$

(3.20) $N_\theta = \rho gz(h - z)\tan\alpha\sec\alpha,\quad N_z = \rho gz(\tfrac{1}{2}h - \tfrac{1}{3}z)\tan\alpha\sec\alpha.$

(3.21) $\ddot{\boldsymbol{r}} = (\ddot{r} - r\dot{\theta}^2)\boldsymbol{g}_r + (\ddot{\theta} + 2\dot{r}\dot{\theta}/r)\boldsymbol{g}_\theta + \ddot{z}\boldsymbol{g}_z.$

(4.1) $\gamma_{11} = \tfrac{1}{2}(a_2^2 + a_3^2)$, $\gamma_{12} = \tfrac{1}{2}(a_1 + a_2 + a_3^2)$ etc.

(4.2) $\boldsymbol{G}_1 = \boldsymbol{i}_1 + a_2\boldsymbol{i}_2 + a_3\boldsymbol{i}_3$, $\boldsymbol{G}^1 = [(1 - a_2a_3)\boldsymbol{i}_1 + a_1(a_3 - 1)\boldsymbol{i}_2 + a_1(a_2 - 1)\boldsymbol{i}_3]/G$,
$U_1 = [a_1(x_2 + x_3) + a_2^2(x_3 + x_1) + a_3^2(x_1 + x_2)]$,
$G = 1 + 2a_1a_2a_3 - a_1a_2 - a_2a_3 - a_3a_1$.

(4.4) $\eta_{11} = \tfrac{1}{2}\{1 + [a_2a_3(2a_2 + 2a_3 + 2 - 3a_2a_3) - 1 - a_2^2 - a_3^2]/G^2\}$,
$\eta_{12} = [(a_1 + a_2)(a_3^2 - a_3 + 1) + a_1a_2a_3(a_3 - 2)]/2G^2$ etc.

(4.5) $\gamma_1^1, \gamma_2^2, \gamma_3^3 = 6\tfrac{1}{2}, 5, 2\tfrac{1}{2}$; $\gamma_i^i = 14$;
$\eta_1^1, \eta_2^2, \eta_3^3 = -3\tfrac{1}{4}, -2, -\tfrac{3}{4}$ or $-5\tfrac{3}{4}, -\tfrac{1}{2}, \tfrac{1}{4}$; $\eta_i^i = -6$.

(4.6) $J_1, J_2, J_3 = 28, 1, -26$; $K_1, K_2, K_3 = 14, \tfrac{1}{4}, -3\tfrac{1}{4}$; $I_1, I_2, I_3 = 31, 60, 4$.

(4.7) $\lambda_1^2 = 28.931$, $\lambda_2^2 = 0.069$, $\lambda_3^2 = 2$.

(4.8) $[G_{jk}][g^{ki}] = [\delta_j^i + 2\gamma_j^i]$.

(4.9) $I_3 = 1 + 2\gamma_i^i(1 + \gamma_k^k + \tfrac{2}{3}\gamma_j^j\gamma_k^k - 2\gamma_k^j\gamma_j^k) - 2\gamma_k^j\gamma_j^k + \tfrac{8}{3}\gamma_j^i\gamma_k^j\gamma_i^k$.

(4.10) $U = \tfrac{1}{2}(\lambda + 2\mu)K_1^2 - 2\mu K_2 + \tfrac{1}{3}(l + 2m)K_1^3 - 2mK_1K_2 + nK_3 + O(\gamma^4)$.

(4.11) $s_{ij} = \delta_{ij}(\lambda\gamma_{kk} + d_1\gamma_{kk}\gamma_{ll} + d_2\gamma_{kl}\gamma_{lk}) + 2(\mu + d_2\gamma_{kk})\gamma_{ij} + d_3\gamma_{jk}\gamma_{ki} + O(\gamma^3)$.

(4.12) s^{11}, s^{12}, s^{13}, s^{22}, s^{23}, s^{33} = -187, -147, -98, -149, -75, -87 TPa. At such high strains, elasticity and convergence of the series cannot be assumed.

(4.13) t^{ij} = -4582, -3972, -2487; -3930, -3487, -2078; -2559, -2166, -1509 TPa.

(4.14) $\boldsymbol{p} = -(9994\,\boldsymbol{i}_1 + 8569\,\boldsymbol{i}_2 + 5709\,\boldsymbol{i}_3)$TPa.

(4.15) $2a_1a_2a_3 = a_1a_2 + a_2a_3 + a_3a_1$.

(4.16) $$\gamma_{11}^N = \sqrt{[(1 + u_{1,1})^2 + u_{2,1}^2 + u_{3,1}^2]} - 1,$$

$$\gamma_{12}^N = \sin^{-1}\left(\frac{(1 + u_{1,1})u_{1,2} + (1 + u_{2,2})u_{2,1} + u_{3,1}u_{3,2}}{\sqrt{\{[(1 + u_{1,1})^2 + u_{2,1}^2 + u_{3,1}^2][(1 + u_{2,2})^2 + u_{1,2}^2 + u_{3,2}^2]\}}}\right).$$

No.

(4.18) $B = (b^3 - a^3 + A^3)^{1/3}$; $\lambda_r = r^2/R^2$, $\lambda_\theta = \lambda_\phi = R/r$ $[R = (r^3 - a^3 + A^3)^{1/3}]$.

(4.19) $\hat{t}_{ji} = \lambda'\delta_{ij}v_{k,k} + \mu'(v_{i,j} + v_{j,i}) - 3e(\tfrac{1}{2}\lambda + \mu)(v_{i,j} - v_{j,i})$
where $\lambda' = \lambda - e(6d_1 + 4d_2 + 2\lambda)$, $\mu' = \mu - 3e(d_2 + \tfrac{1}{2}\lambda + \mu) - ed_3$.

(4.20) The stiffness can be expected to increase with compression.

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