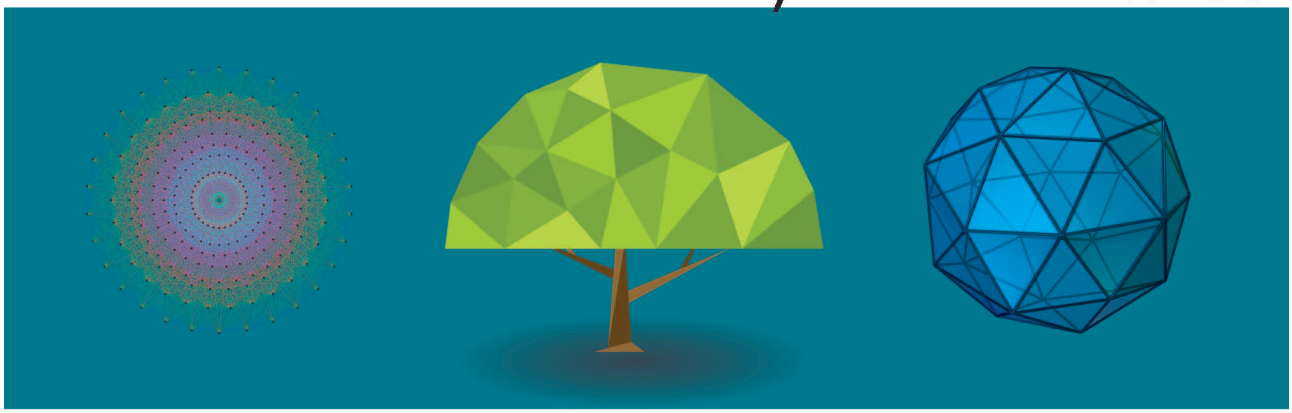
Basic Numeracy

Only study guide for
BNU1501



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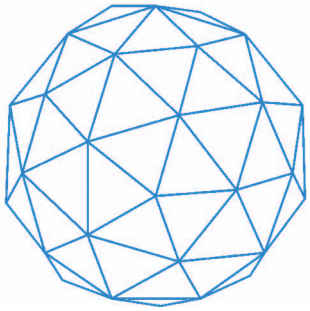

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Introduction

We must all make important decisions in life-, and especially, financial decisions. Some of these decisions require the basic ability to use numbers. In this course, we want to guide you to acquire some basic numeracy skills to use when making decisions. We will guide you to apply these numeracy skills to solve problems encountered in daily life and to make rational and responsible decisions.

Firstly, we will introduce you to the numeracy tools. Then we will let you practise until you are proficient with these tools. Lastly, we will show you examples of where these tools can be applied in everyday decision-making.

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Chapter 1

Numbers and variables

On completion of this chapter, you will be able to

- identify the different kinds of numbers
- draw a number line
- round off numbers
- explain what a variable is
- write down an elementary expression or equation by using variables

In this first chapter, we will explain some of the basic terminology and notations you will need in this course.

1.1 Numbers

We use numbers to describe and measure quantities. A number is an abstract entity, an idea we use to express a quantity. Various symbols can be used to represent the same number, for instance, “4”, “3+1”, “four” and “IV”. Let us tell you about the different kinds of numbers we will use in this course.

The most fundamental system of numbers is the set of ***natural numbers*** that we use to count things, starting with the number one. We can write it as a set, which is a grouping or collection of similar things. Curly brackets are used to indicate sets.

The set of natural numbers is

$$\{1;2;3;4;5;\ldots\}.$$

(The three dots ... at the end indicate that the set continues indefinitely.)

The elements of the set are 1, 2, 3, 4, 5 and so on.

Natural numbers can be used for counting ("there are two (2) books on the table") and for ordering or arranging in sequence ("this is the fourth (4th) largest apple").

Natural numbers are also called ***positive integers***.

To indicate the absence of certain things (such as when your team fails to score), the number 0 was added. This gives us the set of ***counting numbers***:

$$\{0;1;2;3;4;5;\ldots\}.$$

Counting numbers are also called ***non-negative integers***.

Situations may arise in which positive numbers alone are not sufficient. For example, we can subtract a larger number from a smaller one. The negative numbers are usually represented by a minus sign in front of the number concerned. The set of ***negative integers*** is

$$\{\ldots;-5;-4;-3;-2;-1\}.$$

When working with negative numbers, it is important to distinguish between the sign of an operation (addition and subtraction) and the sign of a number (positive or negative). The $\boxed{+/-}$, $\boxed{(-)}$ or $\boxed{\text{NEG}}$ key on different calculators is used to enter a negative number or to change its sign, while the $\boxed{-}$ is used for subtraction.

Negative numbers are used to describe values below zero, such as temperature and debt. Negative numbers provide a convenient way of indicating opposite direction with a minimum of words, for example, steps inside a building (see figure 1.1).

The following example illustrates the positive and negative integers. Let us look at something we are all familiar with – steps inside a building.

We indicate ground level, where there is no step, as 0 (zero). The steps above ground level are numbered $+1, +2, +3,$ and so on. The plus sign does not indicate the operation of addition; it is a sign of direction, in this case meaning ***above*** ground level. In the basement, the steps are numbered $-1, -2, -3,$ and so on, indicating ***below*** ground level (the opposite direction).

Combining the negative integers and the non-negative integers (or zero plus the positive integers) yields the set of integers:

$$\{\cdots -3;-2;-1;\ 0;\ 1;\ 2;\ 3;\ \cdots\}.$$

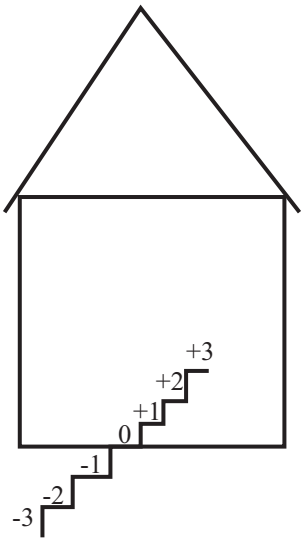


Figure 1.1: Steps inside a building.

One property of this set is that for every positive integer, there is a corresponding negative integer such that their sum is zero. In other words, if we add -3 and $+3$, the answer is zero (0).

The number line is one way to represent a set of numbers graphically. To illustrate the construction of a number line, we place the elements of the integer set in one-on-one correspondence with points on a line. Draw a straight horizontal line and mark a point near the middle as 0 (zero) on it. We call this point the origin. Since integers are equally spaced, we select points around 0 (zero) with equal distances between them. This distance between any two points is called the scale, a familiar example being the scale on a thermometer. Certain conventions are usually followed when using number lines; positive integers are associated with points to the right of 0 (zero) and negative integers are associated with points to the left of 0 (zero). See figure 1.2 below.

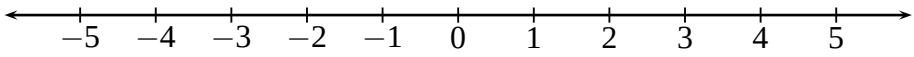


Figure 1.2: The number line.

The arrows at each end of the number line mean that the numbers continue infinitely.

Thus far, we have not mentioned any numbers other than integers. The number line also gives us a picture of the relationship between integers and other numbers. The integers are marked on the line, but each point in between the integer markings also represents a number. Examples are the points representing the numbers $\frac{1}{2}$ (located halfway between 0 and 1) and $-2,5$ (located halfway between -2 and -3) on the line. There are in fact an infinite num-

ber of points between two integer markings. All these points on the number line (that is, the solid line) represent the set of ***real numbers***. Note that integers are real numbers. They are special real numbers.

Exercise 1.1

† (See page 139 for the solutions.)

1. What kind of number is the following?
   (a) -1
   (b) $\frac{3}{4}$
2. Indicate the two numbers in question 1 above on a number line.

1.2 Rounding of numbers

Rounding is the process of reducing the number of digits in a number while trying to keep the value similar. Rounded numbers are less accurate, but easier to use.

In this course, we will be using the common method to do rounding. The rule for applying this method is as follows:

- Decide which is the last digit that you want to keep.
- Increase this digit by one if the ***next*** digit to the right is "5" or more, and discard the rest.
- Leave this digit the same if the ***next*** digit to the right is less than "5", and discard the rest.

Example 1.1

1. Round 4,36923 to three decimal digits.

   You want to keep three digits, so the "9" will be the last digit. The next digit to the right of the "q" is "2", which is less than 5. Drop the "23" part to get 4,369.

2. Round 4,36923 to two decimal digits.

   You want to keep two digits, so the "6" will be the last digit. The next digit is "9", which is more than 5. Add 1 to the "6" and drop the "923" part to get 4,37.

3. Round 4,36923 to an integer.

   You want to keep only integers, so the “4” is the last digit. The next digit is “3”, which is less than 5. Drop the “36923” part to get 4.

Before rounding a number, you need to think carefully. Keep the following in mind:

- If a interest rate is 16,25%, then you should not round it to 16%. There is a big difference if you calculate interest on a loan using a 16% interest rate instead of a 16,25% interest rate!
- However, what does it mean if you calculate the price of an article as R27,5394? When working with money, it makes more sense if we round to two decimal digits.
- Do not round to integers unless it is obvious, or asked for.
- Do not round in the middle of a problem. Use all the digits, and **only round your final answer**.

† (See page 139 for the solutions.)

Exercise 1.2

1. Round the following numbers to three decimal digits:
   (a) 4,57849
   (b) 0,39887
   (c) 10,0004
2. Round the following numbers to two decimal digits:
   (a) 7,9747
   (b) 10,495
   (c) 8,998
3. Round the following numbers to integers:
   (a) 12,034
   (b) 9,8746
   (c) 0,0342

1.3 Variables

A variable is a symbol for a number we do not know. It can be seen as a place holder for an unknown number. Variables are useful because they allow us to write down unknown values in expressions[1] and equations[2]. In the expression $x+7$, the symbol x indicates a variable because we do not know its value. We call an expression with variables in it an algebraic expression. Algebra is basically a shorthand system.

For example, an expression for the area of a rectangular plot of land of unknown size can be constructed. Call the length of the rectangle ℓ and its width w. Then you can say that the area of the rectangle is $\ell \times w$. This will be true for any value of ℓ and w. Once the values of ℓ and w are specified we can calculate the value of the expression. For example, if the length is 4 metres and the width is 3 metres, we substitute ℓ with 4 and w with 3 to get the area of the plot: $\ell \times w = 4 \times 3 = 12$. The area is 12 square metres.

This is the great value of using variables — we can write down expressions which are completely general. To find the answer in a particular case, all we need to do is to substitute our particular values, as explained above.

Multiplication of numbers by variables can be represented in more than one way, for example:

- with a multiplication sign: $4 \times a$
- with a dot: $4 \cdot a$
- with brackets: $4(a)$
- or straight: $4a$

Example 1.2

Make use of variables and write the following sentences as algebraic expressions or equations:

(a) Add 7 to an unknown value.

$x+7$ (or $a+7$ or $p+7$, and so on)

(b) Subtract a from b.

$$b-a$$

[1] An expression is a combination of numbers, operators, brackets and variables. Two examples of expressions are $3x-6$ and 2×5.

[2] An equation is a statement that says that what is on the left of the equal sign ("=") is equal to what is on the right of it. Two examples of equations are $7x-4=1$ and $2+4=6$.

(c) The sum of x and y is equal to z.

$$x + y = z$$

To summarise

- We can represent any unknown value by a symbol, and we call this symbol a variable.
- A variable can represent or assume any numerical value.
- The symbols most often used to represent variables are lower case letters of the alphabet, such as a, b, x and y.
- Acronyms are used as variables, for example, PV for present value.
- Choose a symbol that is short and that makes sense.

(Rosser 2003)

Example 1.3

A biscuit manufacturer uses the following ingredients for each packet of biscuits produced: 0,2 kg of flour, 0,05 kg of sugar and 0,1 kg of butter.

One way to specify the total amount of flour used is: 0,2 kg times the number of packets of biscuits produced. However, it is easier and simpler if we let the letter q (for quantity) represent the number of packets of biscuits produced.

We can then say that

- the total amount of flour required is $0{,}2q$ kilograms
- the total amount of sugar required is $0{,}05q$ kilograms
- the total amount of butter required is $0{,}1q$ kilograms

Exercise 1.3

† (See page 141 for the solutions.)

Make use of variables and write the following sentences as algebraic expressions or equations:

(a) Write an expression to subtract 4 from an unknown value and multiply this difference by 2.

(b) If q is the number of loaves of bread you buy and p is the price per loaf, write an equation for the total cost, c, when buying bread.

(c) A cookery book recommends a cooking time of 30 minutes for every kilogram of turkey plus another quarter of an hour. Write an equation for the total cooking time for turkey in terms of its weight using your own variables

i. in minutes

ii. in hours

(d) The petrol consumption of your car is 10 kilometres per litre. Let x be the distance you travel in kilometres and p the price per litre of petrol in rand. Write an expression for

i. the amount of petrol you will use to travel x kilometres

ii. your total petrol cost

(e) You want to make a booking at a restaurant for a group of people. You are told that there is a set menu that costs R95 per adult and R50 per child. There is also a fixed service charge of R10 per adult. Write an equation for the total cost of the meal, in rand, if there are x adults and y children.

Chapter 2

Basic operations on numbers and variables

On completion of this chapter, you will be able to

- perform basic operations on numbers and variables
- use basic powers and roots in calculations
- perform multiple basic operations in the correct order

2.1 Basic operations on numbers

You are already familiar with the basic rules for working with numbers. We are going to review them and let you practise them on your calculator.

If you have not yet taken your SHARP EL-738FB calculator out of its box, now is the time to do so! You will find notes on how to use the calculator, on the BNU1501 module site, under **Additional Resources**. Work through it up to *"Financial calculator mode"*.

The four basic operations are addition, subtraction, multiplication and division.

2.1.1 Addition and subtraction

When numbers are *added*, the answer is called the *sum* or the *total* of the numbers.

The sum of two real numbers is unique, that is, it does not matter which number is placed first and which second — the result is the same. For example

$$6+2=2+6 \text{ (both are equal to 8).}$$

The sum of three numbers also does not depend on which two are added first; the result is unique. For example

$$2+(5+4)=(2+5)+4 \text{ (both are equal to 11).}$$

Try these two examples on your calculator and see if you come to the same conclusion.

When numbers are *subtracted*, the result is called the *difference* of the two numbers.

Let us investigate if the difference of two real numbers is unique. Again, take the two numbers 6 and 2:

$$6-2=4$$

and

$$2-6=-4$$

We see that $6-2 \neq 2-6$. (The sign $\neq$ is read as "does not equal".) Changing the order when subtracting leads to the negative of the initial result, therefore the difference is not unique. This result means that we need to make very sure that the order is correct before we do any subtraction.

Exercise 2.1

1. Which of the following statements are true? Show your calculations.
   (a) $7+(3+2)=(7+3)+2$
   (b) $6+(-5+3)=(6+-5)+3$
2. Write the following as an equation:
   (a) Subtract 5 from 8 to get a difference of 3.
   (b) Subtract -5 from -8 to get a difference of -3.

† (See page 142 for the solutions.)

Consider the following problem (Rosser 2003) involving multiple operations of addition and subtraction.

Example 2.1

1. A train leaves its terminus with 19 passengers on board. At the first station, 4 passengers get off and 9 get on. At the second station, 15 get off and 1 gets on. How many passengers remain on the train?

   Most of you would probably answer this by saying $19-4=15$ and $15+9=24$. Then $24-15=9$ and $9+1=10$ passengers remaining, which is the correct answer. If you were faced with the problem

   $$19-4+9-15+1=?$$

   you should simplify it in the same way, that is, **from the left to the right**. That means

   $$19-4+9-15+1=10$$

   When an expression contains only the two operations, of **addition and subtraction**, without any brackets, the rule is to perform them **from left to right**.

2.1.2 Multiplication and division

Multiplication

When two numbers are *multiplied*, the answer is called the *product* of the two numbers. The numbers we multiply are called *factors*.

Again, the product of two real numbers is unique, that is, it does not matter which number is placed first and which second — the result is the same. For example

$$6\times 2=2\times 6 \quad \text{(both are equal to 12).}$$

The product of three numbers also does not depend on which two are multiplied first; the result is unique. For example

$$2\times(5\times 4)=(2\times 5)\times 4 \quad \text{(both are equal to 40).}$$

Try these two examples on your calculator and see if you come to the same conclusion.

When multiplying with plus and minus signs, the rule is as follows:

- If we multiply two numbers with the same signs, the answer is a positive number, that is

$$+ \times + = +$$

and

$$- \times - = +$$

- If we multiply two numbers with different signs, the answer is a negative number, that is

$$+ \times - = -$$

and

$$- \times + = -$$

Note that we usually do not write the positive sign, +. A number without a sign always indicates a positive number.

Example 2.2

1. $-2 \times 3 = -6$
2. $-2 \times -3 = +6 = 6$

Division

When one number is *divided* by another number, the answer is called the *quotient*.

The specific terminology we use in *division* is

$$\text{dividend} \div \text{divisor} = \text{quotient}.$$

If we look at $6 \div 2 = 3$, 6 is the *dividend*, 2 the *divisor* and 3 the *quotient*.

Let us investigate if the quotient of two real numbers is also unique. Again, take the two numbers 6 and 2:

$$6 \div 2 = 3$$

and

$$2 \div 6 = \tfrac{1}{3}$$

We see that $6 \div 2 \neq 2 \div 6$. Changing the order when dividing leads to the inverse of the initial result (that is, $\frac{1}{3}$ instead of 3); therefore, the quotient is not unique.

When dividing with plus and minus signs, the rule is as follows:

- If we divide two numbers with the same signs, the answer is a positive number, that is

$$+ \div + = +$$

and

$$- \div - = +$$

- If we divide two numbers with different signs, the answer is a negative number, that is

$$+ \div - = -$$

and

$$- \div + = -$$

Example 2.3

1. $12 \div -3 = -4$
2. $-12 \div -3 = +4 = 4$

When adding long rows of numbers, it does not matter which numbers we add first and which last. A similar remark applies to multiplication, but when we come to subtraction and division, we must be very careful, as the following examples show:

$$10 - (5 - 2) \neq (10 - 5) - 2$$

and

$$(8 \div 4) \div 2 \neq 8 \div (4 \div 2)$$

This means that an expression like $8 \div 4 \div 2$ is highly ambiguous because the result depends on the order in which we perform the division. The SHARP EL-738FB interprets $8 \div 4 \div 2$ as $(8 \div 4) \div 2$, which is equal to $(2) \div 2 = 1$, since it works from left to right. When an expression contains only the two operations of **multiplication** and **division**, without any brackets, the rule is to perform the operation from left to right. Therefore

$$8 \div 2 \times 4 = 16.$$

You can enforce a specific order by using brackets, for example

$$8 \div (2 \times 4) = 8 \div 8 = 1.$$

We can conclude that, although we can change the order of the numbers for addition and multiplication, we cannot do it for subtraction and division.

Exercise 2.2

1. Which of the following statements are true? Show your calculations.
   (a) $-7 \times (2 \times -3) = (-7 \times 2) \times -3$
   (b) $12 \div (-4 \div 2) = (12 \div -4) \div 2$
2. Write the following as an equation:
   Divide 8 by -2 to get a quotient of -4.

† (See page 142 for the solutions.)

Consider the following problem (Rosser 2003) involving multiple operations of multiplication and addition.

Example 2.4

1. A restaurant catering for a large party sits 6 people to a table. Each table requires two dishes of vegetables. How many dishes of vegetables are required for a party of 60?

   Most people would answer this by saying $60 \div 6 = 10$ tables and $10 \times 2 = 20$ dishes, which is correct.

   If this problem is set as $60 \div 6 \times 2 =$? then the left to right rule must be used. If you did not use this rule, you might get

   $$60 \div 6 \times 2 = 60 \div 12 = 5$$

   which is *incorrect*.

Exercise 2.3

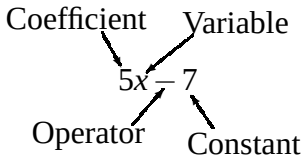
1. $43 - 13 + 1 =$?
2. $6 + 16 - 7 =$?
3. $66 + 3 - 30 - 4 + 12 =$?
4. $22 \times 8 \div 4 =$?
5. $460 \div 5 \times 4 =$?
6. $200 \div 25 \times 8 \times 3 \div 4 =$?

† (See page 142 for the solutions.)

2.2 Basic operations on variables

Before we proceed with the basic operations on variables, we need to discuss some more terminology.

Let us look at the following expression:



You already know that a ***variable*** is a symbol for an unknown number. In this expression the variable is x. An ***operator*** is a symbol representing an operation, in this case subtraction. A ***constant*** always has the same value; it is a fixed number, in this case 7. A ***coefficient*** is the number in front of a variable. In this case the coefficient of x is 5.

If we look at the expression $3x + 7y + 2$, we see that the constant is 2, the coefficient of y is 7 and the coefficient of x is 3.

Some basic rules must be followed when working with variables. In language, we put letters or sounds together to form words, and words put together make sentences. If we think of an expression as a mathematical sentence, the words could be seen as *terms* made up of numbers and variables. For example, in the expression

$$5x + 4ab - 6y$$

there are three terms, namely, $5x$, $4ab$ and $-6y$. The numbers and variables we multiply together to produce a term are called *factors*. In the term $4ab$, the following are all factors of $4ab$:

$$4,\ a,\ b,\ 4a,\ 4b \text{ and } ab$$

Take note of two more general notes concerning notation:

- Note that when we have more than one variable in a factor, we arrange the variables alphabetically. For example, we write $4ab$, rather than $4ba$.
- If the coefficient is 1, we usually do not write it. For example, $1y$ is written as y, or $1xy$ is written as xy, and $-1x$ is written as $-x$.

A special notation is also used when a factor is repeated in a term, for example, $4 \times 4 \times 4$. We first write down the factor that is repeated, in this case 4. Then we count how many times the factor is repeated, 3, and write it down to the right of and higher than the factor (superscript) to get: 4^3. We call the "3" the exponent of the number 4, but we say "four to the power three" or "the third power of four". Similarly, $5 \times 5 = 5^2$ and we say "five to the power two" or "the second power of five".

The repeated factor can also be a variable, so that $x \times x \times x \times x \times x = x^5$ and we say "x to the power five" or "the fifth power of x".

We will discuss powers and exponents later again.

2.2.1 Addition and subtraction

The expression $4xy + 2x + 5y - 3z$ has 4 terms, namely $4xy$; $2x$; $5y$ and $-3z$. We can say that terms are separated by $+$ and $-$ signs. (Factors are separated by $\times$ signs.)

When adding or subtracting, we only add or subtract *like terms*. Terms are alike when they have exactly the same combination of letter variables. For example, we can add $4a$, $8a$ and $3a$, but we cannot add $4a$, $2b$ and $8ab$.

We cannot simplify the expression $2x + 5y - 3z$ because $2x$, $5y$ and $-3z$ are not like terms. We can, however, simplify the expression $2x + 5x - 3z$ by adding the two like terms to get $7x - 3z$.

When adding or subtracting like terms, we add or subtract the coefficients and keep the letter variable part as it is. For example, when we add $6a$ to $3a$, we add the 6 to the 3 to get 9 and keep the a to get the sum of $9a$:

$$6 + 3 = 9$$
$$6a + 3a = 9a$$

Example 2.5

Simplify the following expressions:

(a) $x + x$

$2x$ (Remember that there is actually a coefficient of 1 in front of x.)

(b) $-7a - 3a$

$-10a$

(c) $3x^2 + 4x$

$3x^2 + 4x$ (The two terms are not like terms.)

Take note:

- xy and yx are like terms when ordered (and should be ordered) and they can thus be added. That means

$$xy + yx = xy + xy = 2xy$$

- x^2y and xy^2 are not like terms. (The exponents must also be exactly the same.) That means

$$x^2y + xy^2 = x^2y + xy^2$$

  which is the simplest form of the expression.

- x^2y^3 and x^3y^2 are not like terms, therefore

$$x^2y^3 + x^3y^2 = x^2y^3 + x^3y^2$$

  in its simplest form.

† (See page 143 for the solutions.)

Exercise 2.4

Simplify the following expressions:

(a) $4b + b$

(b) $8y - 5y$

(c) $2x^2 + 4x^2$

(d) $7b^2 - 4b$

(e) $20ab - 5ab$

(f) $5x^2 + 2x + 3x^2 + 5x$

(g) $7x + 2y - 4x$

(h) $x^2 + y^2 - 3x^2 - 5y^2 + xy$

(i) $x^3y^3 - 3xy^3 + 5x^3y^3 - y^3x - y^3x^3$

(j) $12wx + 5 - 3wy + 4yw + xw + wx + 3$

2.2.2 Multiplication and division

We are familiar with the multiplication of two numbers, but not necessarily with the multiplication of two variables or the multiplication of a number and a variable.

In section 1.3, we saw that $6c$ means that 6 is multiplied by c. In the term $6c$ we say that both 6 and c are factors of $6c$.

When we want to multiply $6c$ with $8a$, we first multiply the numbers (the coefficients): $6 \times 8 = 48$. (Always keep the sign rule in mind.) Then multiply

the variables: $c \times a = ca = ac$. (Always arrange the variables in the product alphabetically.) Thus

$$6c \times 8a = 48ac$$

Now what about $6c \times 6c$? We follow the same method as above and first multiply the coefficients: $6 \times 6 = 36$. Then multiply the variables and remember the special notation for repeated factors: $c \times c = c^2$. Thus

$$6c \times 6c = 36c^2$$

Let us look at a few more examples.

Example 2.6

1. Simplify $6a \times -2b$

   $6 \times -2 = -12$
   $a \times b = ab$

   Thus

   $6a \times -2b = -12ab$

2. Simplify $-5x \times -4y$

   $-5x \times -4y = 20xy$

3. Simplify $-3xy \times 4xy$

   $-3xy \times 4xy = -12x^2y^2$

Now it is your turn to do a few multiplications!

Exercise 2.5

† (See page 144 for the solutions.)

Simplify

(a) $6x \times 2x$

(b) $3xy \times 2xy$

(c) $3x^2 \times 3x$

(d) $2ab \times 4ac$

(e) $xy \times y$

(f) $3lp \times p$

(g) $17ab \times 10b$

(h) $-4kl \times -5kp$

(i) $14x - 2cd$

(j) $4ac(-5ad)$

Let us look at examples for division of terms containing variables and numbers.

Example 2.7

Simplify:

(a) $\frac{-6a}{2a}$

We divide the numbers $-6 \div 2 = -3$.

The a's cancel out, that is, $a \div a = 1$.

Thus

$$\frac{-6a}{2a} = \frac{-6 \times 1}{2 \times 1} = -\frac{6}{2} = -3$$

(b) $\frac{20xy}{-5x}$

Divide the coefficients first, keeping the sign rule in mind:

$$20 \div -5 = -4$$

The x's cancel out, leaving only y.

Thus

$$\frac{20xy}{-5x} = \frac{20 \times x \times y}{-5 \times x} = \frac{20 \times 1 \times y}{-5 \times 1} = -4y$$

(c) $\frac{-5x}{20xy}$

We get

$$\begin{aligned} \frac{-5x}{20xy} &= \frac{-\cancel{5}x}{4 \cdot \cancel{5}xy} \\ &= -\frac{1}{4y} \end{aligned}$$

Note: The forward strokes used in

$$\frac{-\cancel{5}x}{4 \cdot \cancel{5}xy}$$

over the numbers and variables mean that we cancel out factors.

(d) $\frac{18x^2y}{3xy^2}$

We get

$$\begin{aligned} \frac{18x^2y}{3xy^2} &= \frac{\cancel{3} \cdot 6 \cdot x \cdot x \cdot \cancel{y}}{\cancel{3} \cdot x \cdot \cancel{y} \cdot y} \\ &= \frac{6x}{y} \end{aligned}$$

When you multiply or divide expressions, always use the following order:

- Establish the sign of the answer by using the applicable sign rule.
- Multiply or divide the numbers (coefficients).
- Multiply or divide the variables.

It is time for some more exercises.

Exercise 2.6

† (See page 145 for the solutions.)

Simplify:

(a) $56x^3y^3 \div 8x^2y^2$

(b) $-81ac \div 3ad$

(c) $32p^2 \div -64p$

(d) $-10cd \div -d$

(e) $12xy \div 12xy$

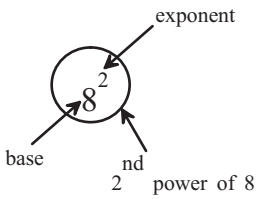
2.3 Powers

In section 2.2, you were briefly introduced to powers. Here we revisit the special notation and introduce you to the rules (laws) of exponents.

A power consists of a base and an exponent for example, 8^2 is a power, the second power of 8.

2.3.1 Exponents

In the expression 8^2, the base is 8, the exponent is 2, and the power is the whole expression, 8^2.



The exponent tells you how many times the base is multiplied by itself. In this example: $8^2 = 8 \cdot 8 = 64$. In words, we would say “eight to the power two” or “eight squared”. We can also say “the second power of eight”.

Note that

$$\begin{array}{lrcl} & 2+2+2 & = & 3\times 2=6 \\ \text{and} & 2\times 2\times 2 & = & 2^3=8 \\ \text{Also:} & x+x+x & = & 3x \\ \text{and} & x\times x\times x & = & x^3 \end{array}$$

In general, we say that if a number a is multiplied by itself n times, it is written as a^n.

$$\underbrace{a\times a\times a\times\cdots\times a}_{n \text{ times}}=a^n$$

In this general case, a is the base and n the exponent.

Note that $a^1=a$, for example $4^1=4$.

Remember that the value of any base **to the power one** is always just the base. For example,

$$5^1=5 \text{ and } x^1=x.$$

What about a number or variable **to the power zero**?

By definition, any number or variable to the power zero is one, so that

$$10^0=1, 4^0=1 \text{ and } a^0=1.$$

See the notes on the module site, under **Additional Resources**, on using the calculator, to obtain the value of an exponential expression by using the power key $\boxed{y^x}$.

Exercise 2.7

† (See page 146 for the solutions.)

1. Identify the base and the exponent in the following expressions/powers.
   (a) 6^4
   (b) 3^2
   (c) 10^8
   (d) 206^0
2. Calculate the value of the following expressions/powers:
   (a) 5^4
   (b) 10^5
   (c) $3,1^3$
   (d) 206^0

Now what about **negative exponents**? If $10^2 = 100$, what does 10^{-2} mean?

By definition,

$$10^{-2} = \frac{1}{10^2} = \frac{1}{100}$$

Note that 10^{-2} is *not* a negative number.

To deal with this, we use the concept of the inverse (or reciprocal) of a number. The inverse or reciprocal of a number is the result we obtain when we divide 1 with the number. For example, the inverse of 10 is $\frac{1}{10}$. Now we can write $\frac{1}{10}$ as 10^{-1}, or 10^{-1} as $\frac{1}{10}$.

Thus, for any base with an exponent, the *inverse* is found by using the same base, but changing the *sign* of the exponent.

Therefore, by definition

$$a^{-1} = \frac{1}{a}$$

and in general

$$a^{-n} = \frac{1}{a^n}$$

We can write: $10^{-1} = \frac{1}{10}$; $5^{-2} = \frac{1}{5^2}$; $x^{-5} = \frac{1}{x^5}$ and so on.

Vice versa, we can write $\frac{1}{3}$ as 3^{-1}, $\frac{1}{x^2}$ as x^{-2}, etc.

The key on the calculator that we can use to calculate the inverse is $\boxed{1/x}$. (On other calculators, it may be $\boxed{x^{-1}}$.) See the notes on using the calculator which show you how to obtain the value of an exponential expression with a negative exponent.

Exercise 2.8

Express the following powers with positive exponents and simplify as far as possible:

(a) 3^{-1}

(b) 5^{-3}

(c) $5^{-1} \times 5^2$

† (See page 147 for the solutions.)

2.3.2 Laws of exponents

In this section, we will deduce the laws of exponents by using examples.

1. (a)

$$\begin{aligned} a^2 \times a^3 &= (a \times a) \quad \times \quad (a \times a \times a) \\ &= a \times a \times a \times a \times a \\ &= a^5 \end{aligned}$$

(b)

$$\begin{aligned} 2^3 \times 2^2 &= (2 \times 2 \times 2) \quad \times \quad (2 \times 2) \\ &= 2 \times 2 \times 2 \times 2 \times 2 \\ &= 2^5 \end{aligned}$$

(c)

$$\begin{aligned} b^4 \times b^2 &= (b \times b \times b \times b) \quad \times \quad (b \times b) \\ &= b \times b \times b \times b \times b \times b \\ &= b^6 \end{aligned}$$

Thus, we can deduce from the above that when we multiply two or more terms with **similar bases**, we **add** the exponents.

$$a^m \times a^n = a^{m+n}$$

Note: $a^m \times b^n = a^m b^n$ (It cannot be simplified further because the bases are **not** the same.)

2. (a)

$$\begin{aligned} \frac{a^4}{a^2} &= \frac{a \times a \times \cancel{a} \times \cancel{a}}{\cancel{a} \times \cancel{a}} \\ &= a^2 \end{aligned}$$

(b)

$$\begin{aligned} \frac{b^7}{b^5} &= \frac{b \times b \times \cancel{b} \times \cancel{b} \times \cancel{b} \times \cancel{b} \times \cancel{b}}{\cancel{b} \times \cancel{b} \times \cancel{b} \times \cancel{b} \times \cancel{b}} \\ &= b^2 \end{aligned}$$

(c)

$$\begin{aligned}\frac{8^3}{8^2} &= \frac{8 \times \cancel{8} \times \cancel{8}}{\cancel{8} \times \cancel{8}} \\ &= 8\end{aligned}$$

Thus, when we divide two powers with **similar bases**, we **subtract** the exponents from each other.

$$\frac{a^m}{a^n} = a^{m-n}$$

Note: $\frac{a^m}{b^m}$ cannot be simplified because the bases differ.

3. (a)

$$\begin{aligned}(a^2)^3 &= (a^2) \times (a^2) \times (a^2) \\ &= a \times a \quad \times \quad a \times a \quad \times \quad a \times a \\ &= a^6\end{aligned}$$

(b)

$$\begin{aligned}(3^3)^2 &= (3^3) \times (3^3) \\ &= (3 \times 3 \times 3) \times (3 \times 3 \times 3) \\ &= 3^6\end{aligned}$$

Thus, when we raise a power (base with exponent) to a power, we multiply the exponents with each other.

$$(a^m)^n = a^{m \times n} = a^{mn}$$

4. (a)

$$\begin{aligned}(ab)^3 &= (ab) \times (ab) \times (ab) \\ &= a \times b \times a \times b \times a \times b \\ &= a \times a \times a \times b \times b \times b \\ &= a^3 b^3\end{aligned}$$

(b)

$$\begin{aligned}(a^3 b^2)^3 &= (a^3 b^2) \times (a^3 b^2) \times (a^3 b^2) \\ &= aaabbaaabbaaabb \\ &= a^9 b^6\end{aligned}$$

(c)

$$\begin{aligned}(2^3 3^2)^3 &= (2^3 3^2) \times (2^3 3^2) \\ &= 2^6 3^4\end{aligned}$$

Thus, when we raise a product to a power, it is equal to the product of each factor raised to the same power.

$$(ab)^m = a^m b^m$$

Note: $(a+b)^m = (a+b)^m$ (cannot be simplified further)

† (See page 147 for the solutions.)

Exercise 2.9

Simplify the following by using the exponent laws:

(a) $b^4 \times b^2$

(b) $2x^5 \times 3x^4$

(c) $a^2b^3 \times a^4b^5$

(d) $2xy \times 4x^2y^3$

(e) $2ab \times 4a^2 \times 3ab^2$

(f) $\dfrac{x^6}{x^3}$

(g) $\dfrac{15a^7}{5a^2}$

(h) $6cd^2e^2 \div 3cde^2$

(i) $(x^4)^2$

(j) $(3c^4)^2$

(k) $(4a^2b^3)^3$

(l) $(3xy^2z^3)^4$

(m) $\frac{3xy^3}{9x^2y^2}$

2.4 Roots

2.4.1 Roots of numbers

In the previous paragraph, when we multiplied 8 by itself, we wrote

$$8 \times 8 = 8^2$$

and in words, we said “eight squared”. More examples of squares are

4 squared $= 4^2 = 4 \times 4 = 16$

5 squared $= 5^2 = 5 \times 5 = 25$

6 squared $= 6^2 = 6 \times 6 = 36$

A *square root* goes in the opposite direction:

3 squared is 9, so the square root of 9 is 3.

The square root of a number is the value that, when multiplied by itself, gives the number.

We not only have square roots, but also third roots, fourth roots and so on.

The symbol we use for roots in general is $\sqrt[n]{\ }$. Examples of specific roots are as follows:

square root: $\sqrt[2]{\ }$

third root: $\sqrt[3]{\ }$

fourth root: $\sqrt[4]{\ }$

Note: By convention, we do not write the small 2 in front of the symbol for the **square root**, but only $\sqrt{\ }$. If you come across a root symbol without a small number in front, you know that it is a *square* root. Most calculators have a square root key $\sqrt{\ }$. (See the notes on using the calculator.)

Another example:

$$7^2 = 49 \text{ and } \sqrt{49} = 7$$

(We say “the square root of 49 is 7".)

By using the SHARP EL738FB calculator:

Example 2.8

1. $$\sqrt{49} = ?$$

   Calculator steps:

   2ndF $\sqrt{\ }$ 49 =

   The answer is 7.

2. $$\sqrt[3]{216} = ?$$

Calculator steps:

3 [2ndF] [$\sqrt[x]{}$] 216 [=]

The answer is 6.

† (You can use the calculator to get to the answer directly, but you should also be able to do it without a calculator. See page 148 for the solutions.)

Exercise 2.10

Determine the following:

(a) $\sqrt{16}$

(b) $\sqrt{25}$

(c) $\sqrt[4]{16}$

(d) $\sqrt[3]{125}$

1. Calculate:

   (a) $\sqrt{196}+\sqrt{144}$

   (b) $(\sqrt{5})^2$

   (c) $(\sqrt{64})^2$

   (d) $(\sqrt{1})^2$

2.4.2 Roots of variables

When we multiply x by itself, we have

$$x \times x = x^2$$

In words, we say "x squared".

More examples of squares are

$$\begin{aligned} y \text{ squared} &= y \times y = y^2 \\ p \text{ squared} &= p \times p = p^2 \\ (ab) \text{ squared} &= (ab) \times (ab) = (ab)^2 = a^2b^2 \\ \left(a^2b^3\right) \text{ squared} &= \left(a^2b^3\right)^2 = \left(a^2b^3\right) \times \left(a^2b^3\right) = a^4b^6 \end{aligned}$$

To find the **square roots** of variables, we work in the opposite direction. Therefore, in words

- the square root of x^2 is x, because $x \times x = x^2$

- the square root of $16y^2$ is $4y$, because $4y \times 4y = 16y^2$
- the square root of $25x^8$ is $5x^4$, because $5x^4 \times 5x^4 = 25x^8$

If we write these in mathematics symbols, we write

- $\sqrt{x^2} = x$
- $\sqrt{16y^2} = 4y$
- $\sqrt{25x^8} = 5x^4$

Example 2.9

1. $$\begin{aligned}\sqrt{16x^{16}y^8} &= \sqrt{(4x^8y^4) \times (4x^8y^4)} \\ &= 4x^8y^4\end{aligned}$$

2. $$\begin{aligned}\sqrt{100a^6b^{10}} &= \sqrt{(10a^3b^5) \times (10a^3b^5)} \\ &= 10a^3b^3\end{aligned}$$

Exercise 2.11

Simplify the following expressions:

(a) $\sqrt{16b^8c^{16}}$

(b) $\sqrt{36x^2y^{64}}$

(c) $\sqrt{25x^{36}y^{100}}$

(d) $\sqrt{36x^{36}}$

(e) $\sqrt{16a^{36}b^{16}}$

(f) $\sqrt{100p^{14}q^{100}}$

(g) $\sqrt{81m^8n^4}$

† (See page 149 for the solutions.)

2.5 Order of operations

When you see something like $3 + 2 \times 5$, which part should you calculate first? Your instinct might be to start the operations from the left and go to the right, as if you were reading. That might seem natural, but you will get a wrong answer in the case of $3 + 2 \times 5$. In problems involving more than one operation (such as addition and multiplication in this case) certain rules, called **priority rules**, must be followed.

It is essential that when we see an expression, we all understand it to mean the same thing; then we will all obtain the same result. Mathematicians agreed to perform the operations in a specific order. Therefore, the **commonly accepted convention or rule for operations is the following:**

Highest priority

- Brackets
- Powers and roots
- Multiplication and division
- Addition and subtraction

Lowest priority

Example 2.10

Calculate the value of the expression $2\times 3-4\times 5+6\div 3$.

There are no brackets, powers or roots in the expression, so we start with the multiplication and division:

$$\underbrace{2\times 3}_{6}-\underbrace{4\times 5}_{20}+\underbrace{6\div 3}_{2}$$

Now we have $6-20+2$, which is equal to $-14+2=-12$.

Therefore, $2\times 3-4\times 5+6\div 3=-12$.

Exercise 2.12

† (See page 150 for the solutions.)

Calculate the value of the following two expressions. Decide beforehand on the order in which the operations will have to be performed.

(a) $4\times 3\div 2+5\times 6-20\div 4\times 2$

(b) $3^2\times 7-\sqrt{9}\times 8+2^3\div 4\times 3$

Example 2.11

$$2\times(3+4)\times 5$$

Brackets in an expression indicate that we must calculate the expression in brackets, $(3+4)$, first, and then do the other operations:

$$2\times\underbrace{(3+4)}_{7}\times 5$$

Now we have $2\times 7\times 5=70$ as the value of the expression.

We call "(" the left or open bracket and ")" the right or closed bracket. The number of open brackets and closed brackets must always be the same, otherwise the expression is not uniquely specified.

Exercise 2.13

† (See page 151 for the solutions.)

Calculate the value of the following expressions:

(a) $7 \times 8 + 8 \times 9$

(b) $7 \times (8 + 8) \times 9$

(c) $(7 \times 8 + 8) \times 9$

(d) $7 \times (8 + 8 \times 9)$

(e) $200 + 16^3 \div 4$

We now know that brackets within an expression have the highest priority and should always be calculated first. But what about brackets within brackets? Here we always work from the innermost bracket outwards. Note that inside a set of brackets, the usual priority rules apply. An expression underneath a root sign, should be seen as if in brackets.

Example 2.12

Calculate the value of the following expression:

$$\sqrt{3 \times \sqrt{4^2 + 3^2} + (4 + 3)^2}$$

Step by step, we will do the calculation as follows:

Step	**Reduced expression**
1	$\sqrt{\left(3 \times \sqrt{(4^2 + 3^2)} + (4 + 3)^2\right)}$
2	$\sqrt{(3 \times \sqrt{(16 + 9)} + (7)^2)}$
3	$\sqrt{(3 \times \sqrt{25} + 7^2)}$
4	$\sqrt{(3 \times 5 + 49)}$
5	$\sqrt{(15 + 49)}$
6	$\sqrt{64}$
7	8

Let us look at the order of operations in the following example:

- **Do the brackets first:**

$$\begin{aligned} 3\times(5+6)=3\times 11 &= \mathbf{33} \quad \text{(right)} \\ 3\times(5+6)=15+6 &= 21 \quad \text{(wrong)} \end{aligned}$$

- **Powers and roots before multiplication, division, addition and subtraction:**

$$\begin{aligned} 5\times 2^2=5\times 4 &= \mathbf{20} \quad \text{(right)} \\ 5\times 2^2=10^2 &= 100 \quad \text{(wrong)} \end{aligned}$$

- **Multiplication or division before addition and subtraction:**

$$\begin{aligned} 2+5\times 3=2+15 &= \mathbf{17} \quad \text{(right)} \\ 2+5\times 3=7\times 3 &= 21 \quad \text{(wrong)} \end{aligned}$$

- **Otherwise, just go from left to right:**

$$\begin{aligned} 30\div 5\times 3=6\times 3 &= 18 \quad \text{(right)} \\ 30\div 5\times 3=30\div 15 &= 2 \quad \text{(wrong)} \end{aligned}$$

Exercise 2.14

Determine the value of

$$\left(4^2-(6\times 3-80\div 5)^2\right)\div\left(9^2-77\right)$$

† (See page 151 for the solutions.)

Chapter 3

More operations on numbers

On completion of this chapter, you should be able to

- list all the factors of a certain number
- determine the lowest common multiple (LCM)
- do basic operations on fractions and decimals
- do basic operations involving percentages and ratios

3.1 Factors

3.1.1 What is a factor?

All integers have factors. Some numbers have more factors than others.

A factor of a number is a number by which it can be divided without leaving a remainder.

1 and the number itself are always factors of a certain number. Let us look at the following example.

Example 3.1

1. The factors of 6.

   If 6 is divided by 1, the answer is 6.

If 6 is divided by 2, the answer is 3.

If 6 is divided by 3, the answer is 2.

If 6 is divided by 6, the answer is 1.

Therefore, the factors of 6 are: 1, 2, 3 and 6, because all these numbers divide exactly into 6, that is, without a remainder.

If we write $6 = 2 \times 3$, we have factorised 6.

2. The factors of 8.

   If 8 is divided by 1, the answer is 8.

   If 8 is divided by 2, the answer is 4.

   If 8 is divided by 4, the answer is 2.

   If 8 is divided by 8, the answer is 1.

   Therefore, the factors of 8 are 1, 2, 4 and 8.

   If we write $8 = 2 \times 4$, we have factorised it.

   If we write $8 = 2 \times 2 \times 2$, we have factorised 8 in prime factors.

   Note: Prime numbers all have only two factors, namely 1 and the number itself, for example, 7 has no other factors than 1 and 7. Another such a number is 11. We can try to list the prime numbers:

   1, 2, 3, 5, 7, 11, 13, 17, 19, 23, etc.

3.1.2 The lowest common multiple (LCM)

Firstly, we must know what a multiple of a whole number is. A multiple of a number is found by taking the *product* of that number and any counting number. For example, to find the multiples of 3, multiply 3 by 1, or 3 by 2, or 3 by 3, and so on. The multiples of 3 are thus 3, 6, 9, 12, 15, etc. To find the multiples of 5 multiply 5 by 1, 5 by 2, 5 by 3, and so on. The multiples of 5 are thus 5, 10, 15, 20, etc.

If we have two numbers and wish to determine the lowest common multiple (LCM) of the two numbers, we want the smallest number which is a multiple of both these numbers.

Example 3.2

1. Determine the LCM of 6 and 8.

   Method 1.

   The multiples of 6 are

   6; 12; 18; (24); ....

   The multiples of 8 are

   8; 16; (24); ....

   It is clear from the above, that the **lowest** common multiple of 6 and 8 is 24.

   Method 2.

   First we factorise 6 and 8 in **prime** factors:

$$\begin{aligned} 6 &= 2 \times 3 \\ 8 &= 2 \times 2 \times 2 \end{aligned}$$

   Now to determine the LCM, we write down the prime factors of the first number (6), in factorised form, that is

$$2 \times 3$$

   Now see which prime factors of the second number have NOT been written down yet. We see that the prime factors of the second number (that is the 8) are $2 \times 2 \times 2$. Only one 2 has been written down, so we must add another two 2s, as factors of the LCM.

$$\begin{aligned} 6 &= 2 \times 3 \\ 8 &= 2 \times 2 \times 2 \\ \text{Therefore, LCM} &= 2 \times 3 \times 2 \times 2 \\ &= 24 \end{aligned}$$

2. Determine the LCM of

   (a) 30 and 45

   Method 1.

$$\begin{aligned} 30 &= 2 \times 15 = 2 \times 3 \times 5 \\ 45 &= 5 \times 9 = 5 \times 3 \times 3 \\ \text{Therefore, LCM} &= 2 \times 3 \times 5 \times 3 \\ &= 90 \end{aligned}$$

Method 2.

The multiples of 30 are 30; 60; (90); 120; 150; ... etc.

The multiples of 45 are 45; (90); ... etc.

Therefore, the LCM of 30 and 45 is 90.

(b) 3 and 8 and 24

Method 1.

$$\begin{aligned} 3 &= 3 \\ 8 &= 2 \times 2 \times 2 \\ 24 &= 3 \times 2 \times 2 \times 2 \\ \text{LCM} &= 3 \times 2 \times 2 \times 2 \\ &= 24 \end{aligned}$$

Method 2.

The multiples of 3 are 3; 6; 9; 12; 15; 18; 21; (24); 27; ... etc.

The multiples of 8 are 8; 16; (24); 32; 40; ... etc.

The multiples of 24 are (24); 48; ... etc.

Therefore, the LCM of 3, 8 and 24 is 24.

Exercise 3.1

† (See page 152 for the solutions.)

Determine the LCM of

(a) 4 and 6

(b) 6 and 16

(c) 5 and 30

(d) 14 and 20

3.2 Fractions

3.2.1 What is a fraction?

A fraction is a numerical quantity that is not a whole number.

A fraction consists of a numerator and a denominator:

$$\frac{3}{4} : \frac{\text{numerator}}{\text{denominator}}$$

The numerator tells us how many parts of the denominator there are. Thus, there are 3 quarters in a $\frac{3}{4}$.

3.2.2 Types of fractions

- Proper fractions:

  A proper fraction is a fraction with a numerator that is smaller than the denominator.

  $$\frac{3}{7}; \frac{2}{5}; \frac{13}{17}; \text{ etc}$$

- Improper fractions

  An improper fraction is a fraction with a numerator that is larger (bigger) than the denominator:

  $$\frac{4}{3}; \frac{7}{5}; \frac{13}{12}; \text{ etc}$$

- Mixed fractions

Consider the improper fraction $\frac{3}{2}$, meaning “three halves”. We know that three halves are equal to $1\frac{1}{2}$, which we call a mixed fraction. How did we get the $1\frac{1}{2}$? We did the following:

We divided the numerator by the denominator. This gave us the integer, with the remainder being the numerator over the existing denominator.

Example 3.3

1. Change the following improper fractions into mixed fractions:

   (a) $\frac{7}{5}$

   We divide 5 into 7, giving us 1 with a remainder of 2.

   $$\frac{7}{5} = 1\frac{2}{5}$$

   (b) $\frac{17}{6}$

   We divide 6 into 17, giving us 2 with a remainder of $(17-12)=5$.

   $$\frac{17}{6} = 2\frac{5}{6}$$

Can one change mixed fractions into improper fractions? The answer is YES.

This is how we do it: Multiply the whole number (integer) by the denominator and add this answer to the numerator, with the denominator staying the same:

$$\begin{aligned} & 2\tfrac{5}{6} \\ = & \ \tfrac{2\times 6+5}{6} \\ = & \ \tfrac{12+5}{6} \\ = & \ \tfrac{17}{6} \end{aligned}$$

Alternative explanation:

$$\begin{aligned} 2\tfrac{5}{6} &= 2+\tfrac{5}{6} \\ &= \tfrac{12}{6}+\tfrac{5}{6} \\ &= \tfrac{17}{6} \end{aligned}$$

3.2.3 Simplification of fractions

If both the numerator and the denominator of a fraction are multiplied or divided by the same number, the fraction does not change, as it still represents the same quantity.

To have a fraction in its simplest form, we want the smallest possible numbers in the numerator and denominator and still have a fraction with the same value.

Example 3.4

(a)

$$\begin{aligned} \frac{6}{18} &= \frac{2}{6} && \text{Divide both the denominator and the numerator by 3,} \\ \text{and } \frac{2}{6} &= \frac{1}{3} && \text{Divide both the denominator and the numerator by 2.} \end{aligned}$$

(b)

$$\begin{aligned} \frac{12}{16} &= \frac{6}{8} && \text{Divide both the denominator and the numerator by 2,} \\ \text{and } \frac{6}{8} &= \frac{3}{4} && \text{Divide both the denominator and the numerator by 2.} \end{aligned}$$

(c)

$$\begin{aligned} \frac{18}{24} &= \frac{9}{12} && \text{Divide both the denominator and the numerator by 2,} \\ &= \frac{3}{4} && \text{and then by 3.} \end{aligned}$$

The fractions in the examples above have been simplified.

† (See page 152 for the solutions.)

Exercise 3.2

1. Change the following improper fractions into mixed fractions:
   (a) $\frac{8}{3}$
   (b) $\frac{9}{4}$
   (c) $\frac{14}{5}$
   (d) $\frac{41}{12}$
2. Change the following mixed fractions into improper fractions:
   (a) $3\frac{1}{3}$
   (b) $8\frac{3}{7}$
   (c) $7\frac{4}{5}$
   (d) $2\frac{3}{8}$
3. Simplify the following fractions:

(a) $\frac{7}{14}$

(b) $\frac{20}{15}$

(c) $\frac{8}{12}$

(d) $\frac{14}{16}$

3.2.4 Multiplication of fractions

Follow these **three simple steps** to multiply fractions:

1. Multiply the numerators.
2. Multiply the denominators.
3. Simplify the fraction, if possible.

Example 3.5

1.
$$\begin{aligned} & \tfrac{2}{3} \times \tfrac{5}{7} \\ = \; & \tfrac{2\times 5}{3\times 7} \\ = \; & \tfrac{10}{21} \end{aligned}$$

2.
$$\begin{aligned} & \tfrac{3}{4} \times \tfrac{2}{3} \\ = \; & \tfrac{6}{12} \\ = \; & \tfrac{1}{2} \end{aligned}$$

OR:

We could first simplify the fractions by cross cancellation, which is allowed.

$$\begin{aligned} & \tfrac{3}{4} \times \tfrac{2}{3} && \text{2 divides into 4; and 3 divides into 3.} \\ = \; & \tfrac{1}{2} \end{aligned}$$

3.
$$\begin{aligned} & \tfrac{6}{15} \times \tfrac{5}{4} \\ = \; & \tfrac{3}{3} \times \tfrac{1}{2} && \text{Divide 2 into 4 and 6; and 5 into 15.} \\ = \; & \tfrac{1\times 1}{1\times 2} && \text{We divided 3 into 3.} \\ = \; & \tfrac{1}{2} \end{aligned}$$

4.

$$\begin{aligned} & \tfrac{2}{3} \times \tfrac{4}{5} \times \tfrac{15}{20} \\ = \; & \tfrac{1\times4\times5}{1\times5\times10} && \text{Divide 2 into 20; and divide 3 into 15.} \\ = \; & \tfrac{1\times2\times1}{1\times1\times5} && \text{Divide 5 into 5; and 2 into 4 and 10.} \\ = \; & \tfrac{2}{5} \end{aligned}$$

5.

$$\begin{aligned} & \tfrac{5a}{7} \times \tfrac{21}{15a^2} \\ = \; & \tfrac{5a}{7} \times \tfrac{21}{15\times a\times a} && \text{Divide } 5a \text{ into } 15a^2\text{; and divide 7 into 21.} \\ = \; & \tfrac{3}{3a} && \text{Divide 3 into 3.} \\ = \; & \tfrac{1}{a} \end{aligned}$$

6.

$$5 \times \frac{3}{5}$$

Let us first change the 5 (an integer) into a fraction, with the numerator being 5 and the denominator 1. As we know that 5 divided by 1 is 5, we will normally write "5" and not "$\frac{5}{1}$". Now,

$$\begin{aligned} & \tfrac{5}{1} \times \tfrac{3}{5} \\ = \; & 3 \end{aligned}$$

3.2.5 Division of fractions

Follow these **three simple steps** to divide fractions:

1. Invert (turn around) the second fraction, and make the $\div$ sign a $\times$ sign.
2. Multiply the first fraction with the inverted second fraction.
3. Simplify the resulting fraction, if possible.

$$\begin{aligned} & 5 \div 3 \\ = \; & \tfrac{5}{1} \div \tfrac{3}{1} \\ = \; & \tfrac{5}{1} \times \tfrac{1}{3} \\ = \; & \tfrac{5}{3} \end{aligned}$$

Example 3.6

1. (a)

$$\begin{aligned} & \tfrac{4}{7} \div \tfrac{6}{14} \\ = \ & \tfrac{4}{7} \times \tfrac{14}{6} \\ = \ & \tfrac{2}{1} \times \tfrac{2}{3} \\ = \ & \tfrac{4}{3} \\ = \ & 1\tfrac{1}{3} \end{aligned}$$

(b)

$$\begin{aligned} & \tfrac{3}{4} \times \tfrac{8}{9} \\ = \ & \tfrac{1}{1} \times \tfrac{2}{3} \\ = \ & \tfrac{2}{3} \end{aligned}$$

(c)

$$\begin{aligned} & \tfrac{3}{8} \times \tfrac{16}{21} \times \tfrac{7}{10} \\ = \ & \tfrac{1}{1} \times \tfrac{2}{7} \times \tfrac{7}{10} \\ = \ & \tfrac{1}{1} \times \tfrac{1}{1} \times \tfrac{1}{5} \\ = \ & \tfrac{1}{5} \end{aligned}$$

(d)

$$\begin{aligned} & \tfrac{3a}{4b} \div \tfrac{12a}{14b} \\ = \ & \tfrac{3a}{4b} \times \tfrac{14b}{12a} \\ = \ & \tfrac{3}{4} \times \tfrac{14}{12} \\ = \ & \tfrac{1}{2} \times \tfrac{7}{4} \\ = \ & \tfrac{7}{8} \end{aligned}$$

(e)

$$\begin{aligned} & \tfrac{2}{3} \times \tfrac{4}{9} \div \tfrac{12}{15} \\ = \; & \tfrac{2}{3} \times \tfrac{4}{9} \times \tfrac{15}{12} && \text{Only invert the term directly to the right-hand side of the division sign.} \\ = \; & \tfrac{1}{1} \times \tfrac{4}{9} \times \tfrac{5}{6} \\ = \; & \tfrac{1}{1} \times \tfrac{2}{9} \times \tfrac{5}{3} \\ = \; & \tfrac{2 \times 5}{9 \times 3} \\ = \; & \tfrac{10}{27} \end{aligned}$$

(f)

$$\begin{aligned} & \tfrac{5}{6} \div \tfrac{1}{12} \times \tfrac{3}{15} \\ = \; & \tfrac{5}{6} \times \tfrac{12}{1} \times \tfrac{3}{15} && \text{Only invert the term directly to the right-hand side of the division sign.} \\ = \; & 2 \end{aligned}$$

(g)

$$\begin{aligned} & \tfrac{13b^2}{4a} \div \left(\tfrac{3b}{4a} \times \tfrac{8}{9}\right) \\ = \; & \tfrac{13bb}{4a} \times \left(\tfrac{3a}{2b}\right) && \text{First do the brackets, and then the division.} \\ = \; & \tfrac{39b}{8} \end{aligned}$$

† (See page 153 for the solutions.)

Exercise 3.3

1. Simplify
   (a) $\frac{3}{4} \times \frac{1}{3}$
   (b) $\frac{3}{4} \div \frac{1}{3}$
   (c) $\frac{3}{7} \div \frac{9}{21}$
   (d) $\frac{5}{6} \times \frac{1}{4} \div \frac{5}{12}$
   (e) $\frac{5}{6} \div \frac{1}{12} \times \frac{3}{15}$
   (f) $\frac{7}{24} \div \frac{14}{32} \times \frac{3}{16}$

3.2.6 Addition and subtraction of fractions

We know that if we add $\frac{1}{4}$ to $\frac{1}{2}$, the answer is $\frac{3}{4}$. How did we get to this answer?

Follow these **three simple steps** to add fractions:

1. Make sure the **denominators** are the same (LCM).
2. Add the numerators of the fractions and put the answer over the denominator.
3. Simplify the fraction, if possible.

So, how do we add $\frac{1}{4}$ to $\frac{1}{2}$?

We get the denominator to be the same by determining the LCM of the denominators:

$$\begin{aligned} 2 &= 2 \\ 4 &= 2 \times 2 \\ \text{LCM} &= 2 \times 2 \\ &= 4 \end{aligned}$$

Therefore

$$\begin{aligned} & \tfrac{1}{4} + \tfrac{1}{2} \\ = \ & \tfrac{1}{4} + \tfrac{2}{4} \\ = \ & \tfrac{2+1}{4} \\ = \ & \tfrac{3}{4} \end{aligned}$$

Example 3.7

1. $\frac{1}{4} + \frac{5}{6}$

   First determine the LCM of the denominators:

   $$\begin{aligned} 4 &= 2 \times 2 \\ 6 &= 2 \times 3 \\ \text{LCM} &= 2 \times 3 \times 2 \\ &= 12 \end{aligned}$$

   $$\begin{aligned} & \tfrac{1}{4} + \tfrac{5}{6} \\ = \ & \tfrac{1 \times 3 + 5 \times 2}{12} \\ = \ & \tfrac{3+10}{12} \\ = \ & \tfrac{13}{12} \\ = \ & 1\tfrac{1}{12} \end{aligned}$$

Alternative method:

$$\begin{aligned} & \tfrac{1}{4}+\tfrac{5}{6} \\ = \ & \tfrac{1\times3}{4\times3}+\tfrac{5\times2}{6\times2} \\ = \ & \tfrac{3}{12}+\tfrac{10}{12} \\ = \ & \tfrac{3+10}{12} \\ = \ & \tfrac{13}{12} \\ = \ & 1\tfrac{1}{12} \end{aligned}$$

2. $\frac{3}{5}+\frac{4}{7}$

First determine the LCM of the denominators:

$$\begin{aligned} 5 &= 5 \\ 7 &= 7 \\ \text{LCM} &= 5\times7 \\ &= 35 \end{aligned}$$

$$\begin{aligned} & \tfrac{3}{5}+\tfrac{4}{7} \\ = \ & \tfrac{3\times7+4\times5}{35} \\ = \ & \tfrac{21+20}{35} \\ = \ & \tfrac{41}{35} \\ = \ & 1\tfrac{6}{35} \end{aligned}$$

Alternative method:

$$\begin{aligned} & \tfrac{3}{5}+\tfrac{4}{7} \\ = \ & \tfrac{3\times7}{5\times7}+\tfrac{4\times5}{7\times5} \\ = \ & \tfrac{21}{35}+\tfrac{20}{35} \\ = \ & \tfrac{41}{35} \\ = \ & 1\tfrac{6}{35} \end{aligned}$$

3. $\frac{7}{9}+\frac{5}{6}$

Determine the LCM:

$$\begin{aligned} 9 &= 3\times3 \\ 6 &= 2\times3 \\ \text{LCM} &= 3\times3\times2 \\ &= 18 \end{aligned}$$

$$
\begin{aligned}
& \frac{7}{9}+\frac{5}{6} \\
= \ & \frac{7\times 2+5\times 3}{18} \\
= \ & \frac{14+15}{18} \\
= \ & \frac{29}{18} \\
= \ & 1\frac{11}{18}
\end{aligned}
\qquad \text{OR} \qquad
\begin{aligned}
& \frac{7}{9}+\frac{5}{6} \\
= \ & \frac{7\times 2}{9\times 2}+\frac{5\times 3}{6\times 3} \\
= \ & \frac{14}{18}+\frac{15}{18} \\
= \ & \frac{29}{18} \\
= \ & 1\frac{11}{18}
\end{aligned}
$$

To subtract one fraction from another, we basically take the same steps as for addition.

4. $\frac{13}{15}-\frac{2}{6}$

Determine the LCM:

$$
\begin{aligned}
15 &= 3\times 5 \\
6 &= 2\times 3 \\
\text{LCM} &= 3\times 5\times 2 \\
&= 30
\end{aligned}
$$

$$
\begin{aligned}
& \frac{13}{15}-\frac{2}{6} \\
= \ & \frac{13\times 2-2\times 5}{30} \\
= \ & \frac{26-10}{30} \\
= \ & \frac{16}{30} \\
= \ & \frac{8}{15}
\end{aligned}
\qquad \text{OR} \qquad
\begin{aligned}
& \frac{13}{15}-\frac{2}{6} \\
= \ & \frac{13\times 2}{15\times 2}-\frac{2\times 5}{6\times 5} \\
= \ & \frac{26}{30}-\frac{10}{30} \\
= \ & \frac{26-10}{30} \\
= \ & \frac{16}{30} \\
= \ & \frac{8}{15}
\end{aligned}
$$

5. $\frac{16a}{b}+\frac{9b}{c}$

Determine the LCM:

$\text{LCM} = b\times c = bc$

$$
\begin{aligned}
& \frac{16a}{b}+\frac{9b}{c} \\
= \ & \frac{16a\times c+9b\times b}{bc} \\
= \ & \frac{16ac+9b^2}{bc}
\end{aligned}
\qquad \text{OR} \qquad
\begin{aligned}
& \frac{16a}{b}+\frac{9b}{c} \\
= \ & \frac{16ac}{bc}+\frac{9bb}{bc} \\
= \ & \frac{16ac+9b^2}{bc}
\end{aligned}
$$

6. $\frac{1}{6}-\frac{3}{5}$

Determine the LCM:

$$
\begin{aligned}
5 &= 5 \\
6 &= 2\times 3 \\
&= 30
\end{aligned}
$$

$$\begin{aligned} & \tfrac{1}{6}-\tfrac{3}{5} \\ = \ & \tfrac{1\times5}{6\times5}-\tfrac{3\times6}{5\times6} \\ = \ & \tfrac{1\times5-3\times6}{30} \\ = \ & \tfrac{5-18}{30} \\ = \ & \tfrac{-13}{30} \\ = \ & -\tfrac{13}{30} \end{aligned}$$

OR

$$\begin{aligned} & \tfrac{1}{6}-\tfrac{3}{5} \\ = \ & \tfrac{1\times5}{6\times5}-\tfrac{3\times6}{5\times6} \\ = \ & \tfrac{5}{30}-\tfrac{18}{30} \\ = \ & \tfrac{5-18}{30} \\ = \ & \tfrac{-13}{30} \\ = \ & -\tfrac{13}{30} \end{aligned}$$

7. $\frac{2}{5}+\frac{3}{6}-\frac{1}{4}$

Determine the LCM:

$$\begin{aligned} 5 &= 5 \\ 6 &= 2\times3 \\ 4 &= 2\times2 \\ \text{LCM} &= 5\times2\times3\times2 \\ &= 60 \end{aligned}$$

$$\begin{aligned} & \tfrac{2}{5}+\tfrac{3}{6}-\tfrac{1}{4} \\ = \ & \tfrac{2\times12}{5\times12}+\tfrac{3\times10}{6\times10}-\tfrac{1\times15}{4\times15} \\ = \ & \tfrac{24+30-15}{60} \\ = \ & \tfrac{39}{60} \\ = \ & \tfrac{13}{20} \end{aligned}$$

OR

$$\begin{aligned} & \tfrac{2}{5}+\tfrac{3}{6}-\tfrac{1}{4} \\ = \ & \tfrac{2\times12}{5\times12}+\tfrac{3\times10}{6\times10}-\tfrac{1\times15}{4\times15} \\ = \ & \tfrac{24+30-15}{60} \\ = \ & \tfrac{39}{60} \\ = \ & \tfrac{13}{20} \end{aligned}$$

8.

$$\begin{aligned} & 2\tfrac{1}{2}+1\tfrac{1}{2} \\ = \ & 2+1+\tfrac{1}{2}+\tfrac{1}{2} && \text{Separate the integers from the fractions.} \\ = \ & 3+\tfrac{1+1}{2} && \text{Add the integers and then add the} \\ & 3+\tfrac{2}{2} && \text{two fractions.} \\ = \ & 3+1 \\ = \ & 4 \end{aligned}$$

NB: We know that

$$\begin{aligned} & 2-\tfrac{1}{2} \\ = \ & \tfrac{2\times2-1\times1}{2} \\ = \ & \tfrac{4-1}{2} \\ = \ & \tfrac{3}{2} \\ = \ & 1\tfrac{1}{2} \end{aligned}$$

Note: We may only cross cancel over a **multiplication** sign, not over a $-$ or a $+$ sign.

It is **TOTALLY INCORRECT** to do the following:

$$\begin{aligned} & \frac{2}{1} - \frac{1}{2} \\ = \; & 1 - 1 \\ = \; & 0 \end{aligned}$$

Cancel the 2 in the numerator by the 2 in the denominator.

You may only cancel out factors!

Exercise 3.4

† (See page 154 for the solutions.)

1. Simplify the following:
   (a) $\frac{1}{4} + \frac{3}{4}$
   (b) $\frac{1}{5} + \frac{2}{3}$
   (c) $\frac{2}{3} - \frac{1}{6}$
   (d) $\frac{3}{7} - \frac{13}{28}$
   (e) $\frac{5}{6} - \frac{3}{4} + \frac{7}{24}$
   (f) $\frac{6}{11} + \frac{3}{2} - \frac{5}{3}$
   (g) $1\frac{1}{3} + 4\frac{3}{4}$

3.3 Decimals

A decimal system involves counting in units of 10. A decimal is a number consisting of a comma, followed by one or more numbers which represent tenths, hundredths, and so on.

To change a fraction into a decimal, we must divide the denominator into the numerator.

Example 3.8

1. Write $\frac{1}{2}$ as a decimal.

   $2\overline{)1}$ We cannot divide 2 into 1; therefore, we add a zero to the 1.

   $2\overline{)1,0}$ Because we have added the zero, we must write a comma above the division line.

   $\begin{array}{r} 0,5 \\ 2\overline{)1,0} \end{array}$ Now we can divide 2 into 10 to get 5. The answer is therefore 0,5. We usually write a zero in front of the comma.

   Thus, $\frac{1}{2} = 0,5$.

2. Write $\frac{3}{4}$ as a decimal.

   We must divide 4 into 3,000 ...

   $$\begin{array}{r} 0,75 \\ 4\overline{)3,00} \\ \underline{28} \\ 20 \\ \underline{20} \\ 0 \end{array}$$

   $4 \times 7 = 28$
   Subtract 28 from 30 and add zero to the answer of 2.
   $4 \times 5 = 20$
   Subtract 20 from 20 to get 0.

   Therefore, $\frac{3}{4} = 0,75$.

3. Write the following as decimals:

   (a) $\frac{1}{10}$

   If we must divide the numerator by 10, we just shift the comma in the numerator **one** unit to the left. Note that 1 = 1,0.

   Therefore, $\frac{1}{10} = 0,1$.

   We usually write a zero in front of the comma.

   (b) $\frac{1}{100} = 0,01$

   If we must divide the numerator by 100, we shift the comma in the numerator **two** units to the left.

   Therefore, $\frac{1}{100} = 0,01$.

   100 has 2 zeros; therefore, there, are two numbers after the comma.

   (c) $\frac{3}{100} = 0,03$.

   Here again we shifted the comma in 3,0 two units to the left to get 0,03.

   (d) $\frac{17}{1000} = 0,017$.

   There are **three zeros** in the denominator 1 000; therefore, we shift the comma in the numerator 17 **three** units to the left to get 0,017.

   (e) $\frac{4}{100000} = 0,00004$

There are **five zeros** in 100 000; therefore, we shift the comma **five units** to the left to get 0,00004.

You can use your calculator to check the answers.

4. Write $2\frac{1}{4}$ as a decimal.

   To change a mixed fraction into a decimal, we do the following:

   Write the $\frac{1}{4}$ as a decimal and add it to the integer (whole number).

$$2\frac{1}{4} = 2,25$$

Exercise 3.5

† (See page 156 for the solutions.)

1. Write the following as decimals:
   (a) $\frac{2}{3}$
   (b) $\frac{3}{5}$
   (c) $\frac{5}{6}$
   (d) $3\frac{3}{4}$
2. Write the following as fractions:
   (a) $0,25$
   (b) $0,3$
   (c) $1,45$
   (d) $2,6$

3.4 Percentages

Percentage means per hundred. Five percent or 5% means 5 per hundred, which can be written as 5% of $\frac{5}{100}$ or 0,05.

33% means 33 out of a hundred or 0,33.

115% means $\frac{115}{100}$ or 1,15, etc.

Example 3.9

1. Tim would like to buy a bicycle that costs R1 200. His grandmother is prepared to give him 40% of the price of the bicycle as a birthday present. How much money will she give him?

$$\begin{aligned} \text{The amount received} &= 40\% \text{ of } 1\,200 \\ &= \frac{40}{100} \times \frac{1\,200}{1} \\ &= 480 \end{aligned}$$

   His grandmother will give him R480.

2. John receives a commission of 2,5% on his sales. His monthly sales for December amounted to R175 000. Determine his commission.

$$\begin{aligned} \text{The amount of commission} &= \frac{2{,}5}{100} \times \frac{175\,000}{1} \\ &= 4\,375{,}00 \end{aligned}$$

   John's commission for December was R4 375,00.

3. Kate bought a dress, priced at R1 800, at Elegant Boutique. She was given a discount of 30%. How much did Kate pay for the dress?

$$\begin{aligned} \text{Discount} &= \frac{30}{100} \times \frac{1\,800}{1} \\ &= 540 \end{aligned}$$

   The amount paid for the dress is the original price of the dress, minus the discount:

$$\begin{aligned} \text{Price paid} &= 1\,800 - 540 \\ &= 1\,260 \end{aligned}$$

   Therefore, Kate paid R1 260 for the dress.

4. James received an increase of R1 200 per month on his monthly salary of R9 600. Determine the percentage increase in his salary.

$$\begin{aligned} \text{Increase} &= \frac{1\,200}{9\,600} \times \frac{100}{1} \\ &= 12{,}5\% \end{aligned}$$

   The percentage increase was 12,5%.

5. You obtained 12 out of 15 for your first assignment. What is your mark as a percentage for this assignment?

$$\begin{aligned} \text{Percentage} &= \frac{12}{15} \times \frac{100}{1} \\ &= 80\% \end{aligned}$$

   You obtained 80% for the first assignment.

6. Joseph bought a television set that normally costs R2 500, for R2 000. Determine the percentage discount that Joseph received.

   First determine the amount of discount that Joseph received:

   That is $\mathrm{R}\,(2\,500 - 2\,000) = \mathrm{R}500$.

   Remember that percentage is always calculated on the original amount.

$$\begin{aligned} \text{Percentage discount} &= \frac{500}{2\,500} \times \frac{100}{1} \\ &= 20,00\% \end{aligned}$$

   Joseph received 20% discount.

Exercise 3.6

† (See page 157 for the solutions.)

1. Sera bought a box of apples. There were 125 apples in the box and she discovered that 4% of them were rotten. How many apples could she sell?
2. Peter's monthly salary is R9 000. An amount of R3 150 is deducted for income tax. Determine the percentage income tax that Peter paid.
3. A survey reveals that in the small town of Duncanville 4 150 families out of a total of 11 275 families own DVD players. What percentage of families own DVD players?
4. Mary bought a pair of shoes on which she had to pay 14% VAT, amounting to R216,60. Determine the price of the shoes.
5. The price of a loaf of bread, currently costing R5,12, is increased by 9,5%. Determine the new price of a loaf of bread.

3.5 Ratios

A ratio is a way of comparing two or more quantities.

Suppose we have a pupil : teacher ratio of 40 to 1. Then we can symbolically write that the pupil **to** teacher ratio is 40:1, or alternatively $\frac{40}{1}$. We read 40:1 as 40 to 1.

The order in which the numbers is written is important. The 40 refers to the first object mentioned, namely the pupils, and the 1 refers to the second, namely, teachers.

Example 3.10

1. Albert has R10 000 which he wishes to divide between his two sisters, Alice and April, in the ratio of 6:4. How much will each of the sisters receive?

   First, we determine the portion that Alice will receive, and then the portion that April will receive. If the money must be divided in the ratio of 6:4, we divide the money into 10 equal parts $(10 = 6 + 4)$.

   Then

   $$\begin{aligned} \text{Alice will receive} &= \frac{6}{10} \times \frac{\text{R10\,000}}{1} \\ &= 6\,000 \end{aligned}$$

   $$\begin{aligned} \text{April will receive} &= \frac{4}{10} \times \frac{\text{R10\,000}}{1} \\ &= 4\,000 \end{aligned}$$

   Alice will receive R6 000 and April R4 000.

   Note: If you add the two amounts together, you get R10 000, which is the original amount that had to be divided between the two sisters.

2. Jerome picked 77 apples, and he wishes to divide them between Sipho and Simon in the ratio of 3:8. Determine how many apples Sipho will receive.

   The total number of parts in the ratio is 11.

   $$\begin{aligned} \text{Sipho will receive} &= \frac{3}{11} \times \frac{77}{1} \\ &= 21 \end{aligned}$$

   Sipho will receive 21 apples.

Exercise 3.7

1. There are 72 Smarties in a packet containing red and green Smarties in the ratio of 5:3. Determine the number of red Smarties in the packet.
2. The ratio of stone, sand and cement in a concrete mixture is 6:4:1. Determine
   (a) the percentage of stone
   (b) the percentage of sand
   (c) the percentage of cement in the mixture

† (See page 158 for the solutions.)

Chapter 4

Measurements

On completion of this chapter, you will be able to

- convert length units from one unit to another
- determine the perimeter of different figures
- determine the area of different figures
- determine the volume of different objects
- convert litres to cubic units
- convert cubic units to litres
- apply measurements to assist in decision-making in daily life problems

Have you ever renovated the tiling in the bathroom or kitchen of your home? If yes, then you will know that in order to do that, you had to decide how many tiles to buy. Were you able to determine the number of tiles needed or did you ask someone else?

In this chapter, we will start off by showing you the basics of measurement. This will enable you to tackle problems similar to the ones described above (and others) with confidence. It will place you in a position where you can do the calculations and make the decisions yourself.

4.1 The international system

It is difficult to imagine a world without a system for measuring things. You would not know the distance to another town, the capacity in litres of a fuel tank, whether your body mass is within limits, and so on.

The earliest measuring system originated in the barter system, and units corresponded with things like the length or the size of a hand or foot. Many countries each had their own measuring system. In the modern world, which is characterised by international trade and extended industrial and technological development, it became necessary to have a common measuring system. A modern international system, the *Système international d'unités*, also known as SI, was developed. South Africa was one of the first countries to accept this system.

The following table gives us a good indication of the SI system:

Factor	**Power**	**Common name**	**SI name**	**SI symbol**
10	10^1	ten	deca-	D
100	10^2	hundred	hecto-	h
1 000	10^3	thousand	kilo-	k
1 000 000	10^6	million	mega-	M
0,1	10^{-1}	tenth	deci-	d
0,01	10^{-2}	hundredth	centi-	c
0,001	10^{-3}	thousandth	milli-	m
0,000001	10^{-6}	millionth	micro-	μ

4.2 Length

The SI base unit for length is **metre**. The symbol used for this base unit is **m**. Therefore, we write 10 metres as 10 m. Note the space between the number and the unit.

Decimal multiples and submultiples in the SI system are as follows:

10	millimetres	(mm)	=	1	centimetre	(cm)
10	centimetres	(cm)	=	1	decimetre	(dm)
10	decimetres	(dm)	=	1	metre	(m)
10	meters	(m)	=	1	decametre	(Dm)
10	decametres	(Dm)	=	1	hectometre	(hm)
10	hectometres	(hm)	=	1	kilometre	(km)

In the SI system, it is very easy to convert from one unit to another because we work in multiples of ten.

The rule is as follows:

- If you must convert to a greater unit, say from m to km, *divide* by the appropriate multiple of 10.
- If you must convert to a smaller unit, say from m to mm, *multiply* by the appropriate multiple of 10.

We normally use km, m, cm and mm.

1 km = 1 000 m and 1 m = 100 cm = 1 000 mm

"Distance" is a length. Examples of units used for distances are the following:

- kilometres to measure long distances, like between cities, say 63 km
- metres to measure medium-sized distances, like your height, say 1,6 m
- centimetres to measure small lengths, like your body measurements, say your waist is 80 cm
- millimetres to measure very small lengths or thicknesses, like the width of a paper clip, say 6 mm.

Example 4.1 Convert the length of a 2,5 km tunnel into centimetres.

The first step is to change 2,5 km into metres by multiplying the number of kilometres by 1 000:

$$2{,}5\text{ km} = 2{,}5 \times 1\,000\text{ m} = 2\,500\text{ m}$$

Therefore,

$$2{,}5\text{ km} = 2\,500\text{ m}.$$

The second step is to change 2 500 m into centimetres by multiplying the number of metres by 100:

$$2\,500\text{ m} = 2\,500 \times 100\text{ cm} = 250\,000\text{ cm}$$

The length of the tunnel is thus 250 000 cm.

† (See page 159 for the solutions.)

Exercise 4.1

Convert the following:

(a) 400 m to km
(b) 18 cm to mm
(c) 4 450 m to km
(d) 5 km to m
(e) 12,4 m to cm
(f) 5 246,3 m to km

(g) a km to cm

(h) 246 cm to m

(i) 246 cm to km

(j) (50 m + 0,5 km) to m

4.3 Perimeter

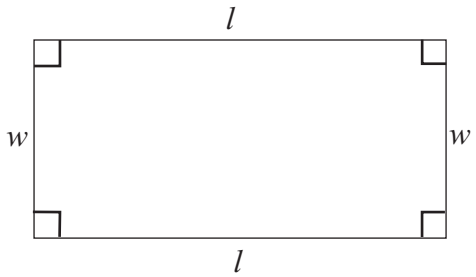
The perimeter C of a figure is the distance around the edge or boundary of a figure (the length of the boundary). We know that distance is measured as a length, therefore, we would use the same SI units for length to measure perimeter.

A homeowner who wants to fence his or her property would, for example, measure the perimeter of the property to know how much fencing he or she needs.

To measure the perimeter of a regular figure we usually make use of the properties of the specific regular figure to simplify the calculations. Let us look at a rectangle, a square, a triangle and a circle.

4.3.1 Rectangles

A rectangle is a four-sided figure with all 4 angles equal to 90° and the opposite sides equal in length.



The little squares in the corners of the figure above indicate that the angles are 90°. Suppose the long sides of the rectangle are of length l and the short sides are of length w. We call them the length and width of the rectangle.

The perimeter of the rectangle is the sum of all 4 sides:

$$\begin{aligned} C &= l+w+l+w \\ &= 2l+2w \\ &= 2(l+w) \end{aligned}$$

The formula for the **perimeter of a rectangle** is

$$C = 2(l + w)$$

Therefore, to calculate the perimeter of the rectangle, we add the long side of the rectangle to the short side of the rectangle and multiply this sum by two.

Example 4.2

Calculate the perimeter of a rectangle with a length of 10 m and a width of 4 m.

The first step is to add the long side of 10 m to the short side of 4 m to get the sum of 14 m. The second step is to multiply this sum by two to get the perimeter of 28 m.

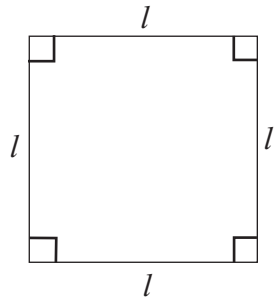
Mathematically, we write it as follows:

$$\begin{aligned} C &= 2(l + w) \\ &= 2(10 + 4) \\ &= 2(14) \\ &= 28 \end{aligned}$$

The perimeter is 28 m.

4.3.2 Squares

A square is a rectangle with all 4 sides equal in length, l.



The perimeter is the sum of all 4 sides:

$$\begin{aligned} C &= l + l + l + l \\ &= 4l \end{aligned}$$

The formula for the **perimeter of a square** is

$$C = 4l$$

Example 4.3

Calculate the perimeter of a square with sides of 7 cm.

Multiply the side length by 4 to get the perimeter of 28 cm.

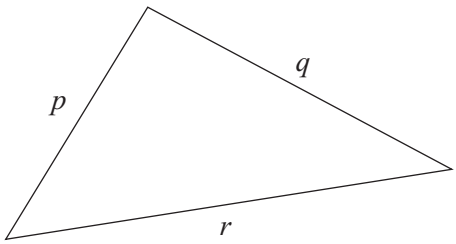
Mathematically, we write it as

$$\begin{aligned} C &= 4l \\ &= 4 \times 7 \\ &= 28 \end{aligned}$$

The perimeter is 28 cm.

4.3.3 Triangles

A triangle is a three-sided figure.



Again, the perimeter is the sum of the lengths of the sides.

$$C = p + q + r$$

Example 4.4

Calculate the perimeter of a triangle with sides of 3 cm, 5 cm and 7 cm.

Add the lengths of the three sides together to get the perimeter of 15 cm.

Mathematically, we write it as

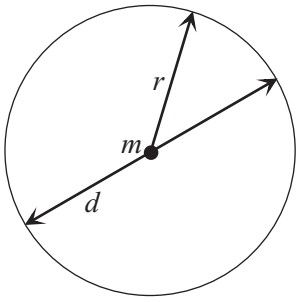
$$\begin{aligned} C &= 3 + 5 + 7 \\ &= 15 \end{aligned}$$

The perimeter is 15 cm.

4.3.4 Circles

Note: We refer to the perimeter of a circle as the **circumference** of the circle.

A circle is a perfectly round plane figure in which every point on its circumference is equidistant from its centre. ("Equidistant" means the same distance from the centre of the circle.)



We call the distance from the centre (m) to the circumference the **radius** (r) of the circle. The **diameter** (d) is the straight line from side to side passing through the centre (m). The diameter (d) is two times as long as the radius (r), therefore $d = 2r$.

The irrational number π (pronounced "pie") is the ratio of the circumference of a circle to its diameter.

The formula for the **circumference (C) of a circle** is

$$C = 2\pi r$$

Note: We use the π key on the calculator in calculations because it is more accurate than $\frac{22}{7}$.

Example 4.5

Calculate the circumference of a circle with a radius of 2 cm.

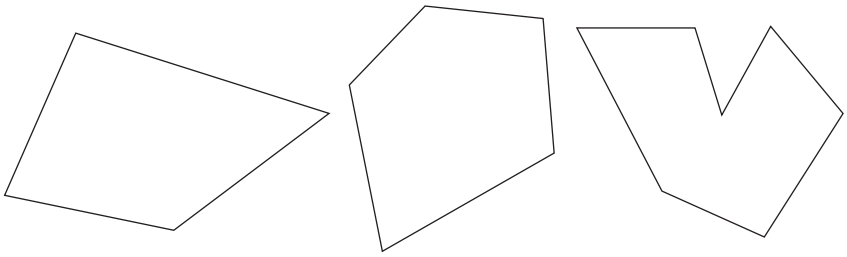
Using the given formula, the circumference is calculated as

$$\begin{aligned} C &= 2\pi r \\ &= 2 \times \pi \times 2 \\ &\approx 12,57 \end{aligned}$$

Therefore, the circumference of the circle is 12,57 cm.

4.3.5 Irregular figures

Here are some examples of irregular figures:



The perimeter is simply the sum of the lengths of all the sides.

Example 4.6

Calculate the perimeter of an irregular figure with five sides of length 2 m, 4 m, 5 m, 6 m and 3 m.

Add the lengths of the five sides together to get the perimeter of 20 m.

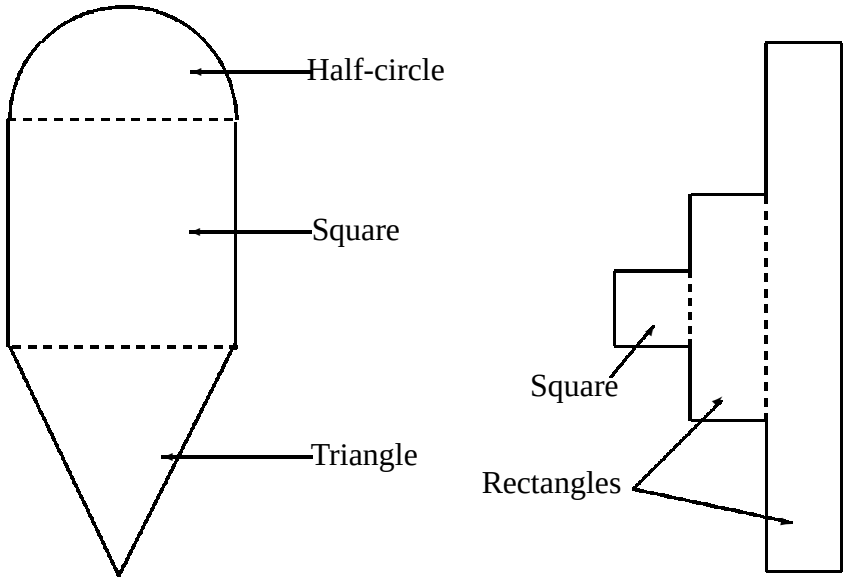
Mathematically, we write it as

$$\begin{aligned} C &= 2+4+5+6+3 \\ &= 20 \end{aligned}$$

The perimeter is 20 m.

4.3.6 Composite figures

A composite figure is a combination of regular figures. Here are two examples:



Again, the perimeter is the sum of the lengths of all the sides.

Exercise 4.2

† (See page 161 for the solutions.)

1. Calculate the perimeter of each of the following figures:

   (a) a rectangle with a length of 1,2 km and a width of 375 m.

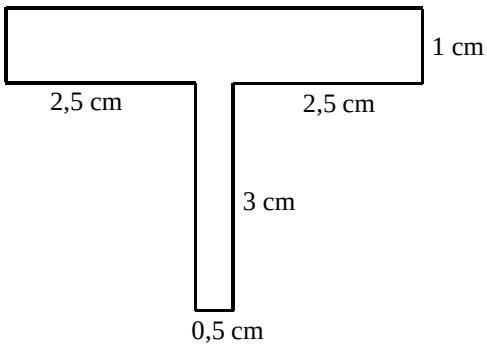
   (b) a square with sides of 3 cm.

   (c) a triangle with sides of 30 mm, 25 mm and 15 mm.

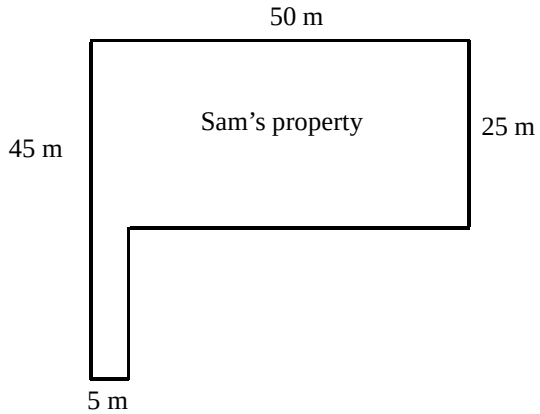
   (d) a circle with a radius of 4,5 m.

   (e) a figure with 4 sides of lengths:
   12 m, 35 m, 7 m and 13 m.

2. Calculate the perimeter of the figure below:



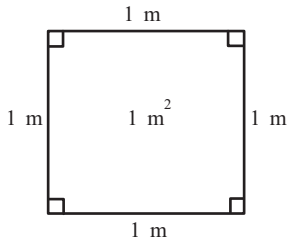
3. Sam wants to obtain a quotation to fence his entire property. Below is a plan of Sam's property. Sam asks you to help him determine the total metres of fencing he needs.



4.4 Area

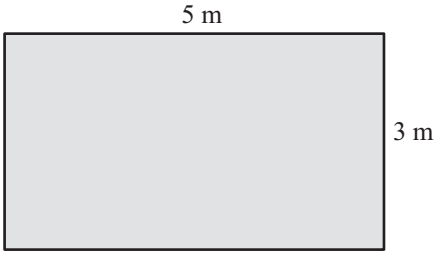
Area (*A*) is the word for the size of a surface. To specify the size of the surface we calculate the area of the surface.

The basic SI unit which is used for area is square metre (m^2). One square metre is the size of a square, where each side of a square is one metre in length.



4.4.1 Rectangles

Suppose we want to tile a rectangular surface area that is 5 m long and 3 m wide.

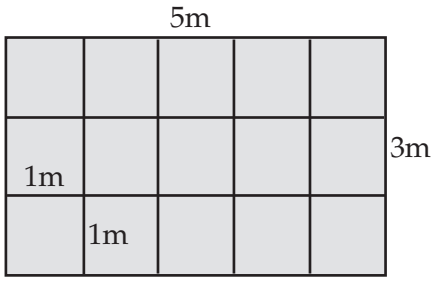


Furthermore, suppose we want to use big square tiles with side lengths 1 m.

The surface area of one tile is $1\,m \times 1\,m = 1\,m^2$.

(Remember our rules of multiplication: $1 \times 1 = 1$ and $m \times m = m^2$.)

Now, how many of these tiles do we need to cover the surface area of $5\,m \times 3\,m$? (We say 5 m *by* 3 m.)

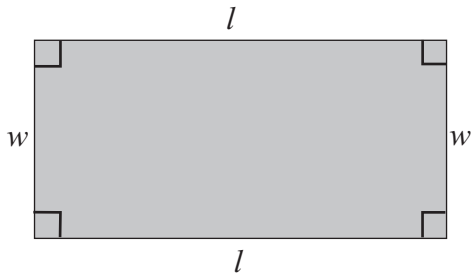


It is clear that we need 5 tiles in the length and 3 tiles in the width.

The number of tiles needed is $5 \times 3 = 15$.

We can now say that the area is 15 square metres.
(Also, $5\,\text{m} \times 3\,\text{m} = 15\,\text{m}^2$.)

Thus, to calculate the area of a rectangle, we multiply the length by the width and give the answer as a "square $\cdots$". The unit following the "square" will depend on the unit we are working with.



The formula for the **area (A) of a rectangle** is

$$\begin{aligned} A &= length \times width \\ &= l \times w \end{aligned}$$

Instead of saying the length is x and the width is y, we talk about a rectangle of size x **by** y and write $x \times y$.

When the numbers become large, it is convenient to convert to a larger unit as the next example shows.

Example 4.7

To calculate the area of a rectangle of size $1\,000\,\text{m} \times 1\,000\,\text{m}$ we do the following:

$$\begin{aligned} A &= 1\,000\,\text{m} \times 1\,000\,\text{m} \\ &= 1\,000\,000\,\text{m}^2. \end{aligned}$$

The area is $1\,000\,000\,\text{m}^2$.

Instead of working with $1\,000\,000\,\text{m}^2$, we convert the $1\,000\,\text{m}$ to km. We know that $1\,000\,\text{m}$ is equal to $1\,\text{km}$ therefore:

$$\begin{aligned} A &= 1\,000\,\text{m} \times 1\,000\,\text{m} \\ &= \frac{1\,000}{1\,000}\,\text{km} \times \frac{1\,000}{1\,000}\,\text{km} \\ &= 1\,\text{km} \times 1\,\text{km} \\ &= 1\,\text{km}^2 \end{aligned}$$

So we have $1\,000\,000\,\text{m}^2 = 1\,\text{km}^2$.

Other units we use for area are square metres, square centimetres, etc.

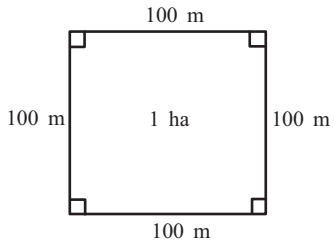
$$\begin{aligned} 1\,\text{m}^2 &= 1\text{m} \times 1\text{m} \\ &= 100\,\text{cm} \times 100\,\text{cm} \\ &= 10\,000\,\text{cm}^2 \end{aligned}$$

and

$$\begin{aligned} 1\,\text{cm}^2 &= 1\text{cm} \times 1\text{cm} \\ &= 10\,\text{mm} \times 10\text{mm} \\ &= 100\,\text{mm}^2 \end{aligned}$$

When we are talking about land and its size, even km^2 will result in large numbers. For this purpose, we use a hectare (ha).

A **hectare** is the area of a square, with each side of length 100 m.



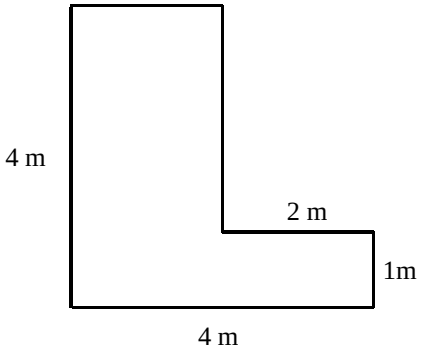
Thus

$$\begin{aligned} 1\,\text{ha} &= 100\,\text{m} \times 100\,\text{m} \\ &= 10\,000\,\text{m}^2 \end{aligned}$$

Exercise 4.3

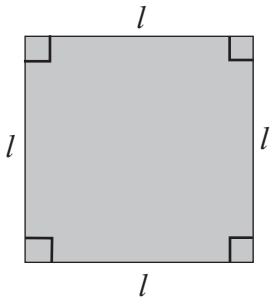
† (See page 163 for the solutions.)

1. How many hectares are there in one square kilometre?
2. Calculate the area of a rectangle with size $24\,\text{cm} \times 18\,\text{cm}$.
3. Calculate the area of the following figure:



4.4.2 Squares

A square is a special type of rectangle with sides of equal length l.



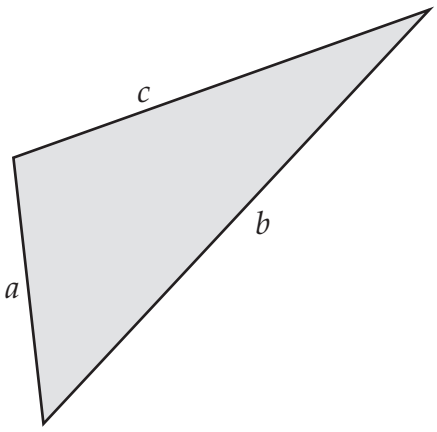
Therefore, the area of a square can easily be derived from the area formula for rectangles by substituting the width with length l.

The formula for the **area (A) of a square** is

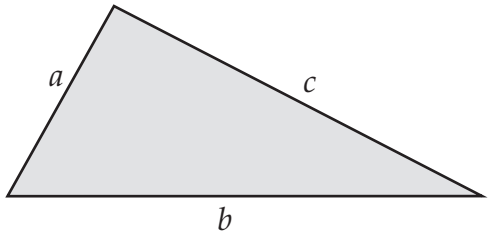
$$\begin{aligned} A &= \textit{length} \times \textit{length} \\ &= l \times l \\ &= l^2 \end{aligned}$$

4.4.3 Triangles

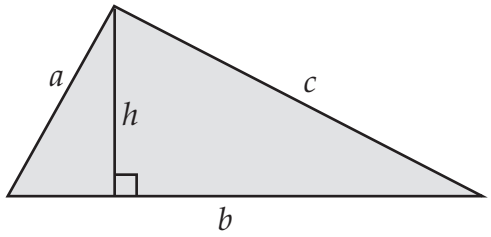
Calculating the area of a triangle is a little more complicated, but we can make it easier by using a sketch. Use any triangle and call the sides a, b and c.



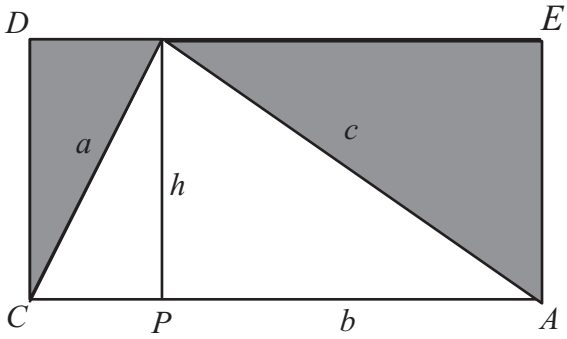
Now turn the triangle so that we can use any side as the base. Let us use side b.



Draw a line from the top vertex (angular point) perpendicular to the base *b*, and call it *h* (for height).



Now draw a rectangle over this triangle by drawing a line parallel to the base *b* through the top of the triangle and two more lines parallel to *h* so that we have the following:



Call this rectangle *ACDE*. The original triangle is called *ACB* and is made up of two triangles, *PCB* and *APB*.

It is clear that the triangle *PCB* is exactly one half of the rectangle *DBPC*, and the triangle *APB* is one half of rectangle *BEAP*. This means that the area of the triangle *ACB* is exactly one half that of the rectangle *ACDE*.

The area of the rectangle $ACDE = AC \times CD$.

But $CD = EA = h$, therefore the area of the rectangle is $b \times h$.

Therefore, the area of $ACDE = b \times h$.

We have already found that the area of the triangle is half that of the rectangle *ACDE*.

Therefore, the formula for **the area of a triangle** is

$$\begin{aligned} A &= \frac{1}{2}\text{base} \times \text{height} \\ &= \frac{1}{2}b \times h \end{aligned}$$

Example 4.8

Calculate the area of a triangle with base 9,3 mm and height 7,6 mm.

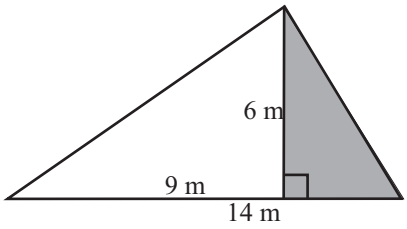
The area of a triangle is

$$\begin{aligned} A &= \frac{1}{2}b \times h \\ &= \frac{1}{2} \times 9,3 \times 7,6 \\ &\approx 35,34 \end{aligned}$$

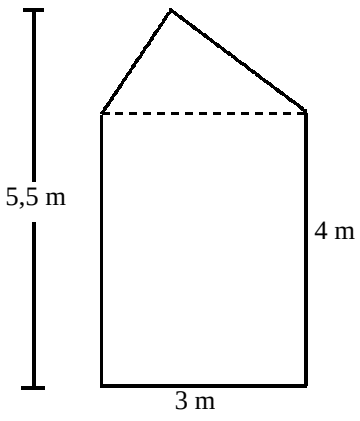
The area of the triangle is $35,34\,\text{mm}^2$.

Exercise 4.4

1. Calculate the area of the following shaded figure:



2. Calculate the area of the following figure:

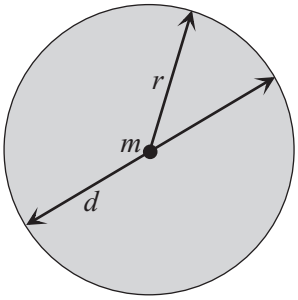


† (See page 165 for the solutions.)

4.4.4 Circles

The formula for the **area of a circle** is

$$A = \pi r^2$$



How we have arrived at this formula is not important in this module. Let us look at an example.

Example 4.9

Calculate the area of a circle with radius 3 m.

The area of the circle is:

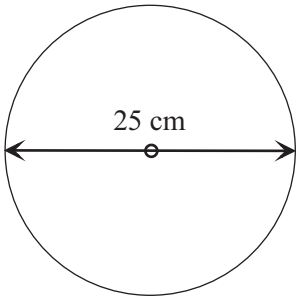
$$\begin{aligned} \text{A} &= \pi r^2 \\ &= \pi(3)^2 \\ &= \pi \cdot 9 \\ &\approx 28,27 \end{aligned}$$

The area is $28,27\,\text{m}^2$.

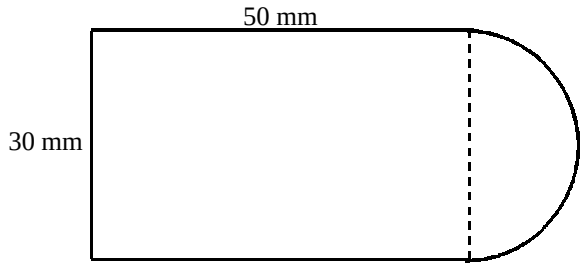
Exercise 4.5

1. Calculate the area of the following circle:

† (See page 167 for the solutions.)



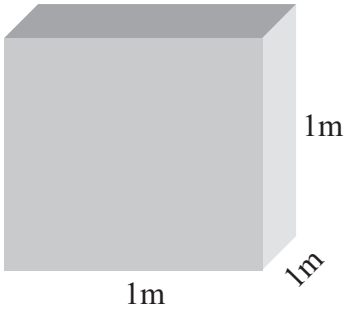
2. Calculate the area of the following figure:



4.5 Volume

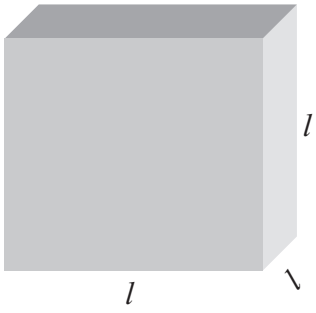
Volume (V) is the word used for the capacity of a three-dimensional object.

The basic SI unit for volume is **cubic metre (m^3)**. The volume of a cube, with length of each side 1 metre, is **one cubic metre**.



4.5.1 Cubes

A cube is a block with sides all equal in length, that is, its length, width and height are all equal in length.

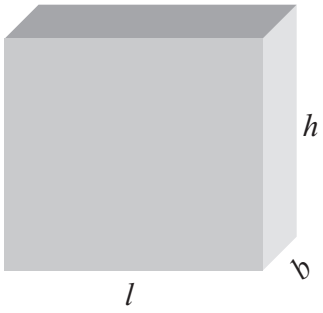


The formula for the **volume (V) of a cube** is

$$\begin{aligned} V &= l \times l \times l \\ &= l^3 \end{aligned}$$

4.5.2 Prisms

In a rectangular prism, the length, width and height can be of different lengths.

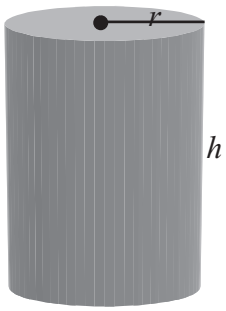


The formula for the **volume (V) of a prism** is

$$\begin{aligned} V &= \textit{area of the base} \times \textit{perpendicular height} \\ &= l \times b \times h \end{aligned}$$

4.5.3 Cylinders

A cylinder is any object that has circles at its base and top and has a perpendicular height that joins the top and base.



The formula for the **volume of a cylinder** is

$$\begin{aligned} V &= \textit{area of base} \times \textit{perpendicular height} \\ &= \pi r^2 \times h \\ &= \pi r^2 h \end{aligned}$$

4.5.4 Conversions and litres

Volume units other than m^3 (cubic metres) are also used. It is easy to convert from one unit to another in the SI system, as we work in multiples of ten.

To convert from cm^3 to mm^3:

$$\begin{aligned} 1\,cm^3 &= 1\,cm \times 1\,cm \times 1\,cm \\ &= 10\,mm \times 10\,mm \times 10\,mm \\ &= 1\,000\,mm^3 \end{aligned}$$

In words: One cubic centimetre is equal to a thousand cubic millimetres.

Therefore, to convert from cm^3 to mm^3 we must **multiply** by 1 000, and to convert from mm^3 to cm^3 we must **divide** by 1 000.

To convert from m^3 to cm^3:

$$\begin{aligned} 1\,m^3 &= 1\,m \times 1\,m \times 1\,m \\ &= 100\,cm \times 100\,cm \times 100\,cm \\ &= 1\,000\,000\,cm^3 \end{aligned}$$

In words: One cubic metre is equal to a million cubic centimetres.

Therefore, to convert from **m^3 to cm^3** we must **multiply by 1 000 000**, and to convert from **cm^3 to m^3** we must **divide by 1 000 000**.

Another unit often used to measure volume is **litre**. The abbreviation for litre is ℓ.

$$1\,\ell = 10\,cm \times 10\,cm \times 10\,cm = 1\,000\,cm^3$$

Therefore, to change from **litre to cm^3** we must **multiply by 1 000**. To change from **cm^3 to ℓ** we must **divide by 1 000**.

$$\begin{aligned} 1\,\text{m}^3 &= 1\,\text{m} \times 1\,\text{m} \times 1\,\text{m} \\ &= 100\,\text{cm} \times 100\,\text{cm} \times 100\,\text{cm} \\ &= 1\,000\,000\,\text{cm}^3 \\ &= \frac{1\,000\,000}{1\,000}\,\ell \\ &= 1\,000\,\ell \end{aligned}$$

Therefore,

$$1\,\text{m}^3 = 1\,000\,\ell$$

To change from **m^3 to litre**, we **multiply** by 1 000.

Example 4.10

1. Determine the volume of a cube with sides 50 mm long.
   Give the answer in cm^3.

   The volume of the cube is:

$$\begin{aligned} V &= 50 \times 50 \times 50 \\ &= 125\,000 \end{aligned}$$

   The volume is 125 000 mm^3.

   We know that

$$\begin{aligned} 1\,000\,\text{mm}^3 &= 10\,\text{mm} \times 10\,\text{mm} \times 10\,\text{mm} \\ &= 1\,\text{cm} \times 1\,\text{cm} \times 1\,\text{cm} \\ &= 1\,\text{cm}^3 \end{aligned}$$

   Therefore, to change the volume to cm^3, divide by 1 000.

   Thus

$$\begin{aligned} V &= 125\,000 \div 1\,000 \\ &= 125 \end{aligned}$$

   The volume is 125 cm^3.

   Alternatively, first convert the side lengths from mm to cm to get

$$\begin{aligned} V &= 50\,\text{mm} \times 50\,\text{mm} \times 50\,\text{mm} \\ &= 5\,\text{cm} \times 5\,\text{cm} \times 5\,\text{cm} \quad (10\,\text{mm} = 1\,\text{cm}) \\ &= 125\,\text{cm}^3 \end{aligned}$$

2. Determine the volume of a prism with $l = 2\,\text{m}$, $b = 3\,\text{m}$ and $h = 3\,\text{m}$. Give the answer in litres.

   The volume of the prism is:

   $$\begin{aligned} V &= l \times b \times h \\ &= 2 \times 3 \times 3 \\ &= 18 \end{aligned}$$

   The volume is $18\,\text{m}^3$.

   To change the volume to litres, multiply by 1 000 because

   $1\,\text{m}^3 = 1\,000\,\ell$. Thus

   $$\begin{aligned} V &= (18 \times 1\,000) \\ &= 18\,000 \end{aligned}$$

   The volume is 18 000 litres.

3. Determine the volume of a water pipe that is 2 km long with a radius of 25 cm. Give the answer in litres.

   First convert all the measurements to the same units:

   $$\begin{aligned} 2\,\text{km} &= 2\,000\,\text{m} \\ &= (2\,000 \times 100)\,\text{cm} \\ &= 200\,000\,\text{cm} \end{aligned}$$

   The volume of the water pipe, which is in the form of a cylinder, is

   $$\begin{aligned} V &= \pi \text{r}^2 h \\ &= \pi (25)^2 \times 200\,000 \\ &= \pi \cdot 125\,000\,000 \\ &\approx 392\,699\,081,7 \end{aligned}$$

   The volume of the water pipe is thus $392\,699\,081,7\,\text{cm}^3$.

   To convert this volume to litres, we divide by 1 000. The volume is thus 392 699 litres.

Exercise 4.6

† (See page 168 for the solutions.)

1. How many cm^3 do you get in half a litre?
2. The perpendicular height of a circular dam is 1,5 m and the diameter is 5 m. What is the volume of the dam in litres?
3. Calculate the volume of a prism of size 12 cm by 9 cm by 4 cm.

4.6 Summary of formulas

Perimeter	
Rectangle:	$C = 2(l + w)$
Square:	$C = 4l$
Triangle:	$C = a + b + c$
Circle:	$C = 2\pi r$
Area	
Rectangle:	$A = l \times w$
Square:	$A = l^2$
Triangle:	$A = \frac{1}{2}bh$
Circle:	$A = \pi r^2$
Volume	
Cube:	$V = l^3$
Prism:	$V = l \times b \times h$
Cylinder:	$V = \pi r^2 h$

4.7 Applications

Now that you have mastered the basics of measurement, we will apply it to problems encountered in our daily lives. These problems usually demand some type of decision from us, that is, how many tiles to buy, how often to phone the oil recycling company, and so on. From what you have learned, you will be in a position to make such decisions confidently. The following examples will illustrate different applications.

Example 4.11

1. You want to tile your lounge floor of size 5,0 m by 2,5 m. How many tiles are needed if the tiles are 250 mm by 250 mm? Add 10% for breakage.

   The area of the floor is a rectangle, thus we can calculate it as:

$$\begin{aligned} A &= l \times w \\ &= 2,5 \times 5,0 \\ &= 12,5 \end{aligned}$$

   The area of the lounge floor is 12,5 m^2.

   Since we must use the same unit right through, we need to calculate the area of one square tile in m^2. To do that we convert the side length of the tile from **mm to m**.

$$\begin{aligned} \text{Remember, } 1\,000\,\text{mm} &= 1\,\text{m} \\ \text{Therefore, } \quad 250\,\text{mm} &= \frac{250}{1\,000}\,\text{m} \\ &= 0,25\,\text{m} \end{aligned}$$

The area of one tile is

$$\begin{aligned} A &= l^2 \\ &= (0,25)^2 \\ &= 0,0625 \end{aligned}$$

The area of one tile is thus $0,0625\,\text{m}^2$.

To get the number of tiles needed we divide the floor area by the area of one tile. (Remember, the area of the floor and tile must be in the same units!)

$$\begin{aligned} \text{Number of tiles} &= \frac{12,5}{0,0625} \\ &= 200 \end{aligned}$$

Thus, 200 tiles will cover the $5\,\text{m} \times 2,5\,\text{m}$ floor.

To provide extra tiles for possible breakage, we add 10%. Therefore,

$$\begin{aligned} \text{the total number of tiles} &= 200 + 10\% \text{ of } 200 \\ &= 200 + \frac{10}{100} \times 200 \\ &= 200 + 20 \\ &= 220 \end{aligned}$$

The number of tiles we need to purchase is 220.

2. How much will it cost to fill your swimming pool to the brim if the pool is in the form of a prism of size 6 m by 4 m by 2 m? You can buy water at R5 per kilolitre.

   First, we calculate the volume of the swimming pool:

$$\begin{aligned} V &= l \times b \times h \\ &= 6 \times 4 \times 2 \\ &= 48 \end{aligned}$$

   The volume of the swimming pool is $48\,\text{m}^3$, which is 48 kilolitre.

As discussed in the section on volume, we know that

$$1\,\text{m}^3 = 1\,000\,\ell = 1 \text{ kilolitre}$$

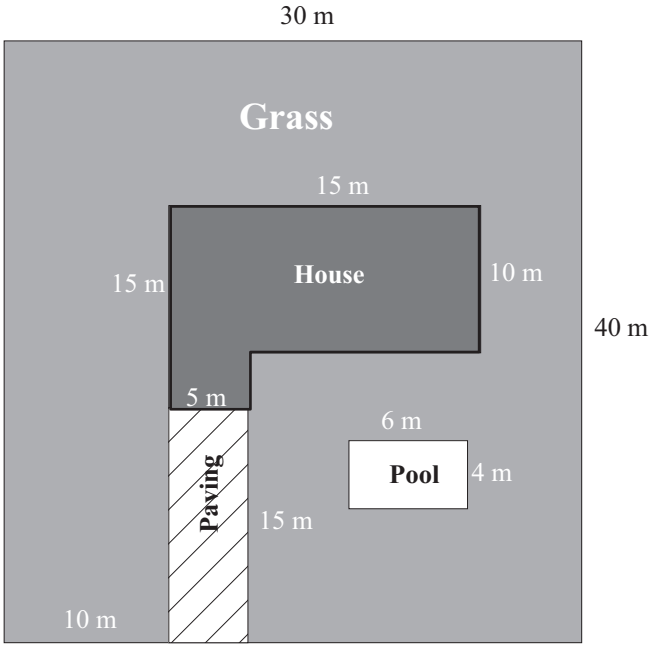
1 kilolitre costs R5, therefore the cost to fill the pool is as follows:

$$\begin{aligned} \text{cost} &= \text{price per kilolitre} \times \text{number of kilolitre used} \\ &= 5 \times 48 \\ &= 240 \end{aligned}$$

It will cost R240 to fill your swimming pool to the brim. ⌟

Exercise 4.7

1. You want to cover the walls of your bedroom with wall paper. The bedroom floor is 4,3 m by 3,4 m and the walls are 2,5 m high. One roll of paper covers 4,5 m^2. How many rolls of wall paper will you need to purchase? Ignore any doors and windows in your calculation.
2. You have moved into your newly built house and want to cover the whole garden with grass initially. Your garden budget is R15 000. The figure below shows the area you want to cover with grass. A garden contractor has quoted you R15 per m^2 to plant kikuyu grass and R20 per m^2 to plant LM grass. Will you be able to use this contractor and stay within your budget?



† (See page 169 for the solutions.)

Chapter 5

Expressions, equations and formulas

On completion of this chapter, you will be able to

- manipulate equations
- solve equations
- change the subject of a formula in an equation

5.1 The difference between expressions, equations and formulas

An **expression** is a combination of numbers, operators, brackets and variables, for example:

$$3x-6,\ 2\times 5,\ \text{ and } 2\,(3x-5)$$

An **equation** states that two expressions are equal to each other.

For example: $2x+3=9$

An equation always contains an equal sign.

A **formula** is a specific expression/equation which represents the *relationship between different variables*. It is, in fact, a recipe for calculating the value of some desired variable, called the *dependent* variable, from the values of the relevant *independent* variables.

You should be familiar with the use of formulas by now, because we have used formulas in chapter 4.

In this chapter, you will learn how to manipulate given equations, to solve them, and to change the subject of a formula.

5.2 Manipulation and solving of equations

What does it mean to solve an equation?

For example, what does it mean to solve for x in the following equation?

$$2x+1=5$$

It simply means that we must **find the value that x should have to make the equation true.**

In other words, what number should x be so that 2 times the number, plus 1, is equal to 5?

By inspection, it seems that x should be the number 2, because

$$2x+1=2\times x+1=2\times 2+1=5$$

It means that if the number 2 is substituted for x, it makes the equation

$$2x+1=5 \text{ true.}$$

In mathematics, we say

"the **solution** of the equation $2x+1=5$, is 2,"

or

$$x=2.$$

Unfortunately, it is not always easy or even possible to solve equations by inspection.

In these cases, we manipulate an equation step-by-step mathematically correct, to find the unknown variable alone on the left-hand side (LHS) of the manipulated equation. In other words, we end with

$$x=? \quad \text{(a number)}$$

To get the unknown variable alone on the left, we must manipulate the LHS with mathematics operations like addition, subtraction, multiplication or division, etc.

Example 5.1

(a) To get rid of the 2 in $2x$ (2 times x), we divide $2x$ by 2, that is

$$\frac{2x}{2} = \frac{\cancel{2}^{1} \times x}{\cancel{2}^{1}} = x$$

(b) To get rid of the 2 in $\frac{x}{2}$, we multiply $\frac{x}{2}$ by 2, that is

$$\frac{x}{\cancel{2}} \times \frac{\cancel{2}}{1} = x$$

(c) To get rid of the 2 in $x-2$, we add 2 to $x-2$, that is

$$x-2+2=x$$

(d) To get rid of the 2 in $x+2$, we subtract 2 from $x+2$, that is

$$x+2-2=x$$

Important note: The LHS of a given equation is equal to its RHS. Therefore, to keep the equation true, the sides should stay equal to each other after manipulation.

Therefore:

> Whatever you do on the left-hand side (LHS) of an equation you **must** also do on the right-hand side (RHS) to keep the equation true.

Let us go back to our original equation, and solve it with a step-by-step manipulation to have x alone on the LHS.

Solve

	$2x+1$	$=$	5	
that is	$2x+1-1$	$=$	$5-1$	It is easiest to get rid of the 1 on the LHS first by subtracting 1 on both sides of the equation.
that is	$2x$	$=$	4	
that is	$\frac{2x}{2}$	$=$	$\frac{4}{2}$	The only way to get rid of the 2 that is multiplied by x is to divide $2x$ by 2.
that is	x	$=$	2	Solution!!

Example 5.2

Solve the following equations:

1. $$b+4=9$$

   To solve for b we must subtract 4 on both sides of the equation. Thus

   $$\begin{aligned} b+4-4 &= 9-4 \\ b &= 5 \end{aligned}$$

   The value of b is 5. That means the equation

   $$b+4=9$$

   is true if $b=5$.

2. $$p-4=9$$

   To solve for p we must add 4 on both sides of the equation. Thus

   $$\begin{aligned} p-4+4 &= 9+4 \\ p &= 13 \end{aligned}$$

   The value of p is 13.

3. $$4k=24$$

   To solve for k we must divide by 4 on both sides. Thus

   $$\begin{aligned} \frac{4k}{4} &= \frac{24}{4} \\ k &= 6 \end{aligned}$$

   The value of k is 6.

4. $$\frac{k}{4}=24$$

   To solve for k we must multiply by 4 on both sides. Thus

   $$\begin{aligned} \frac{k}{4}\times 4 &= 24\times 4 \\ k &= 96 \end{aligned}$$

   The value of k is 96.

Check all 4 of these answers by substituting the answer in the original equation. Do you get the same answer on the LHS and RHS?

To summarise:

Apply the inverse basic operation on **both sides**.

The inverse operation for

- addition ($+$) is subtraction ($-$)
- subtraction ($-$) is addition ($+$)
- multiplication ($\times$) is division ($\div$)
- division ($\div$) is multiplication ($\times$).

Example 5.3

Solve the variable in each of the following equations:

1. $a+3=5$

   The operation on the left-hand side of the equation is addition. To solve the equation we must apply the inverse operation of addition, namely subtraction, on both sides and simplify:

   $$\begin{aligned} a+3 &= 5 \\ a+3-3 &= 5-3 \\ a &= 2 \end{aligned}$$

   The value of a is 2.

2. $b-7=16$

   To solve the equation we must apply the inverse operation of addition, on both sides:

   $$\begin{aligned} b-7 &= 16 \\ b-7+7 &= 16+7 \\ b &= 23 \end{aligned}$$

   The value of b is 23.

3. $x-2=6$

   To solve the equation add 2 on both sides:

$$\begin{aligned} x-2 &= 6 \\ x-2+2 &= 6+2 \\ x &= 8 \end{aligned}$$

   The value of x is 8.

4. $3p=14$

   To solve the equation, divide by 3 on both sides:

$$\begin{aligned} 3p &= 14 \\ \tfrac{3p}{3} &= \tfrac{14}{3} \\ p &= 4\tfrac{2}{3} \end{aligned}$$

   The value of p is $4\frac{2}{3}$.

5. $\dfrac{r}{5}=4$

   To solve the equation, multiply by 5 on both sides:

$$\begin{aligned} \frac{r}{5} &= 4 \\ \frac{r}{5}\times\frac{5}{1} &= 4\times 5 \\ r &= 20 \end{aligned}$$

   The value of r is 20.

Exercise 5.1

† (See page 171 for the solutions.)

Solve for x in the following equations:

(a) $x+4=12$

(b) $6+x=2$

(c) $x-7=-3$

(d) $5x=23$

(e) $\dfrac{x}{4}=10$

Our next step will be to look at a combination of the basic operations. Let us look at some other examples.

Example 5.4

Solve for x.

$$2x-3=5$$

Add 3 on both sides and simplify.

$$\begin{aligned} 2x-3+3 &= 5+3 \\ 2x &= 8 \end{aligned}$$

Divide both sides by 2.

$$\begin{aligned} \frac{2x}{2} &= \frac{8}{2} \\ x &= 4 \end{aligned}$$

Thus, x is 4.

Check:

We look at the LHS and RHS separately:

$$\begin{aligned} \text{LHS} &= 2x-3 \quad \text{(given)} \\ &= 2(4)-3 \\ &= 8-3 \\ &= 5 \\ \text{RHS} &= 5 \quad \text{(given)} \end{aligned}$$

Therefore, the LHS = RHS for $x=4$.

This indicates that the given equation is true for $x=4$.

Example 5.5

Solve for x.

$$\frac{x}{3}+2=11$$

Subtract 2 on both sides.

$$\begin{aligned} \frac{x}{3}+2-2 &= 11-2 \\ \frac{x}{3} &= 9 \end{aligned}$$

Multiply both sides by 3.

$$\begin{aligned} \frac{x}{3} \times \frac{3}{1} &= \frac{9}{1} \times \frac{3}{1} \\ x &= 27 \end{aligned}$$

Thus, x is 27.

Check:

$$\begin{aligned} \text{LHS} &= \tfrac{x}{3} + 2 \quad \text{(given)} \\ &= \tfrac{27}{3} + 2 \\ &= 9 + 2 \\ &= 11 \\ \text{RHS} &= 11 \quad \text{(given)} \end{aligned}$$

Thus, LHS = RHS and the given equation is true for $x = 27$.

Example 5.6

Solve for p.

$$7p + 5 = 33$$

Subtract 5 on both sides and simplify.

$$\begin{aligned} 7p + 5 - 5 &= 33 - 5 \\ 7p &= 28 \end{aligned}$$

Divide by 7 on both sides and simplify.

$$\begin{aligned} \frac{7p}{7} &= \frac{28}{7} \\ p &= 4 \end{aligned}$$

Check: LHS = $7p + 5 = 7(4) + 5 = 28 + 5 = 33 =$ RHS.

Always remove the number *furthest* from the variable *first*.

Exercise 5.2

† (See page 172 for the solutions.)

Solve the variables in the following equations. Always check your answer.

(a) $2x+3=6$

(b) $6x-4=26$

(c) $-5x+4=19$

(d) $\frac{x}{7}-2=9$

Let us look at more complicated examples:

Example 5.7

Solve the following equations:

1. $4+4x-2 = 7$

   Add the constants on the left:

   $4x+2 = 7$

   Subtract 2 on both sides:

   $4x = 5$

   Divide by 4 on both sides and simplify:

   $x = \frac{5}{4}$

   The value of x is $\frac{5}{4}$ or 1,25.

2. $8x+6 = 3x+26$

   Subtracting $3x$ on both sides and simplify:

   $8x-3x+6 = 3x-3x+26.$

   $5x+6 = 26$

   Subtract 6 on both sides and simplify:

   $5x = 20$

   Divide by 5 on both sides and simplify:

   $x = 4$

3. $2(x+3)=18$

   Let us look at the LHS separately:

LHS: $2(x+3)$

According to the order of operations (see 2.4 in section 2.5), we must remove the brackets first. We cannot simplify what is in the brackets: The x and the 3 are not like terms. What we do is to multiply what is in front of the brackets by each term inside the brackets:

We get

$$\begin{aligned} & 2\times x+2\times 3 \\ &= 2x+6. \end{aligned}$$

Remember this simplification of a bracket, as you will use it often in solving equations. In general, we have

$$a(b+c) = ab+ac$$

Now back to our example:

$$\begin{aligned} 2(x+3) &= 18 \\ 2x+6 &= 18 \\ 2x &= 18-6 \\ 2x &= 12 \\ x &= 6 \end{aligned}$$

Exercise 5.3

Solve for the variable in the following equations:

† (See page 174 for the solutions.)

(a) $5(3-x) = 4(5+5x)$

(b) $6a+3+1 = 4a-8-3a$

(c) $5x+1-2x = 9-x-4$

(d) $5x-3(x-(3x-4)) = 2(5x-3(1-2x))+5$

5.3 Changing the subject of a formula

In formulas, we usually have more than one variable. We can manipulate the formula to have another variable on the left. We call this manipulation "changing the subject".

Consider the following formula:

$$v = u + at$$

(a) To change the subject of the formula, $v = u + at$, from v to u, we want to have u alone on the left-hand side of the equation. Firstly, we swop the LHS and the RHS of the equation around to get the side, containing the u, on the LHS. After this step, the formula is still true.

That is, if

$$v = u + at$$

then

$$u + at = v$$

Now, to get rid of the term at on the left we must subtract at on the left – and on the right-hand side.

Thus, from

$$\begin{array}{rrcl} & u + at & = & v \\ \text{we get} & u + at - at & = & v - at \\ \text{that is} & u & = & v - at \end{array}$$

Now we have changed the formula to have u as the subject.

(b) To change the subject of $v = u + at$ to a, we want a alone on the left of the equation.

We may switch the left-hand side and the right-hand side around at any time, because they are equal to each other. We do that to get a on the left.

Thus

$$v = u + at$$

becomes

$$u + at = v$$

Now, to get rid of the u on the LHS of the equation we must subtract u on the left. However, to keep the sides of the equation equal to each other we must subtract u on the right-hand side as well.

Thus

$$\begin{array}{rrcl} & u + at - u & = & v - u \\ \text{that is} & at & = & v - u \end{array}$$

Next, we must get rid of the t on the LHS. We must divide both sides of the equation by t.

Thus,

$$at = v - u$$

becomes

$$\frac{at}{t} = \frac{v-u}{t}$$

that is

$$a = \frac{v-u}{t}$$

The variable on the left is called the **dependent** variable and those on the right are the **independent** variable(s).

In the formula $v = u + at$, v is the dependent variable and u, a and t are the independent variables.

Example 5.8

Change the subject of the following formulas to the variable indicated:

1. Change the subject to y in
$$2y - 3x = 4$$
Add $3x$ on both sides and simplify.
$$\begin{aligned} 2y - 3x + 3x &= 4 + 3x \\ 2y &= 4 + 3x \end{aligned}$$
Divide by 2 on both sides and simplify.
$$\begin{aligned} y &= \tfrac{4+3x}{2} \\ &= 2 + \tfrac{3}{2}x \\ \text{Therefore, } y &= \tfrac{3}{2}x + 2 \end{aligned}$$

2. Change the subject to y in
$$y - mx - c = 0$$
Add c on both sides and simplify.
$$\begin{aligned} y - mx - c + c &= 0 + c \\ y - mx &= c \end{aligned}$$
Add mx on both sides and simplify.
$$\begin{aligned} y - mx + mx &= c + mx \\ y &= mx + c \end{aligned}$$

3. Change the subject to a in
$$\sqrt{a} = b$$

Square on both sides.
$$(\sqrt{a})^2 = (b)^2$$
Simplify using the fact that $(\sqrt{a})^2 = a$
$$a = b^2$$

4. Change the subject to a in
$$ab - c = d$$

Add c on both sides and simplify.
$$ab - c + c = d + c$$
$$ab = c + d$$
Divide by b on both sides and simplify.
$$\frac{ab}{b} = \frac{c+d}{b}$$
$$a = \frac{c+d}{b}$$

5. Change the subject to y in
$$\frac{x+y}{z} = t$$

Multiply by z on both sides and simplify.
$$\frac{x+y}{z} \cdot \frac{z}{1} = \frac{t}{1} \cdot \frac{z}{1}$$
$$x + y = tz$$
Subtract x on both sides and simplify:
$$x - x + y = tz - x$$
$$y = tz - x$$

6. Change the subject to b in
$$\frac{\sqrt{a+b}}{t} = s$$

Multiply by t on both sides and simplify.
$$\frac{\sqrt{a+b}}{t} \times \frac{t}{1} = s \times t$$
$$\sqrt{a+b} = st$$
Square on both sides and simplify.
$$(\sqrt{a+b})^2 = (st)^2$$
$$a + b = (st)^2$$
Subtract a on both sides and simplify.
$$a - a + b = (st)^2 - a$$
$$b = s^2t^2 - a$$

7. Change the subject to h in

$$t = \frac{\sqrt{2h}}{a}$$

Multiply by a on both sides and simplify.

$$\frac{t}{1} \times \frac{a}{1} = \frac{\sqrt{2h}}{a} \times \frac{a}{1}$$

$$at = \sqrt{2h}$$

Switch the sides around.

$$\sqrt{2h} = at$$

Square on both sides and simplify.

$$(\sqrt{2h})^2 = (at)^2$$

$$2h = (at)^2$$

Divide by 2 on both sides and simplify.

$$\frac{2h}{2} = \frac{(at)^2}{2}$$

$$h = \frac{a^2t^2}{2}$$

8. Change the subject to x in

$$x^2 = y$$

Take the square root on both sides and use the fact that $\sqrt{x^2} = x$.

$$\sqrt{x^2} = \sqrt{y}$$

$$x = \sqrt{y}$$

9. Change the subject to r in

$$A = \frac{\pi r^2}{4}$$

Multiply by 4 on both sides and simplify.

$$4A = \pi r^2$$

Switch the two sides around.

$$\pi r^2 = 4A$$

Divide by π on both sides and simplify.

$$r^2 = \frac{4A}{\pi}$$

Take the square root on both sides and simplify.

$$\sqrt{r^2} = \sqrt{\frac{4A}{\pi}}$$

$$r = \sqrt{\frac{4A}{\pi}}$$

† (See page 176 for the solutions.)

Exercise 5.4

Change the subject of the following formulas to the variable indicated in brackets:

(a) $\sqrt{xy}-z=w$ $\ldots(y)$

(b) $q=\dfrac{4r}{t^2}$ $\ldots(t)$

(c) $l=(mk)^2$ $\ldots(k)$

(d) $r^2=x^2+y^2$ $\ldots(r)$

(e) $r^2=x^2+y^2$ $\ldots(x)$

(f) $I=Prt$ $\ldots(r)$

(g) $S=P(1+rt)$ $\ldots(t)$

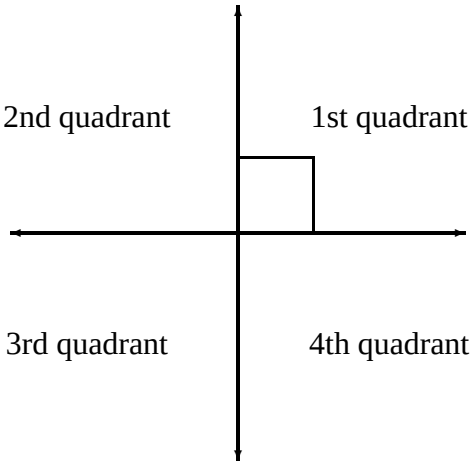
Chapter 6

Straight lines

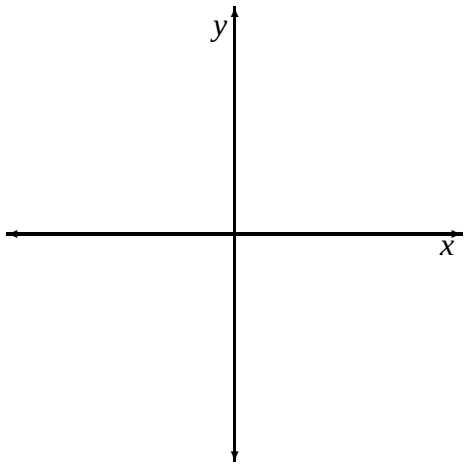
On completion of this chapter, you should be able to

- draw a straight line on a system of axes
- determine the slope of a straight line
- determine the equation of a straight line

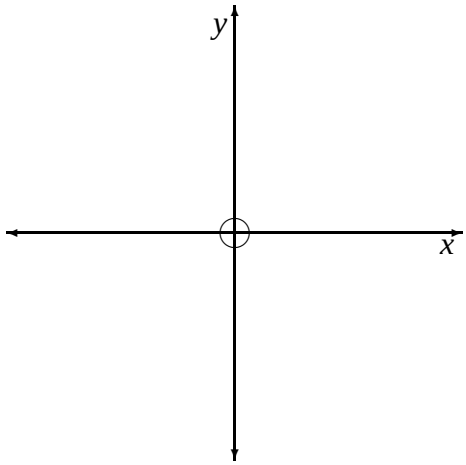
If a vertical and horizontal number line intersect perpendicularly, they form right angles (of 90°) with each other and a **co-ordinate system of axes** is formed. These two lines intersect to form 4 sections, called quadrants.



The horizontal line is called the *x*-axis, and the vertical line is called the *y*-axis. These two axes MUST be clearly indicated with an *x* and a *y*.

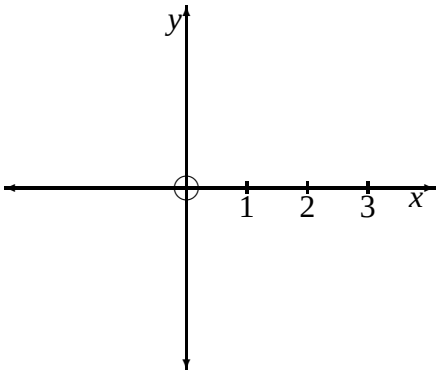


The point at which the two axes intersect (cross each other) is called the *origin* and is indicated by a zero.

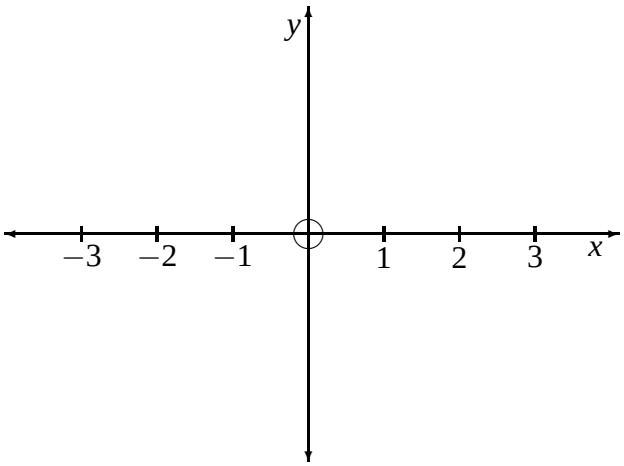


Both the *x*-axis (that is, the horizontal line) and the *y*-axis (that is, the vertical line) must be divided into sections of equal length to form a scale.

Starting at the origin and moving to the right, we divide the axis into equal parts. Note that the distance between the units (numbers) MUST always be the same. The numbers on the right-hand side of the origin are positive numbers.

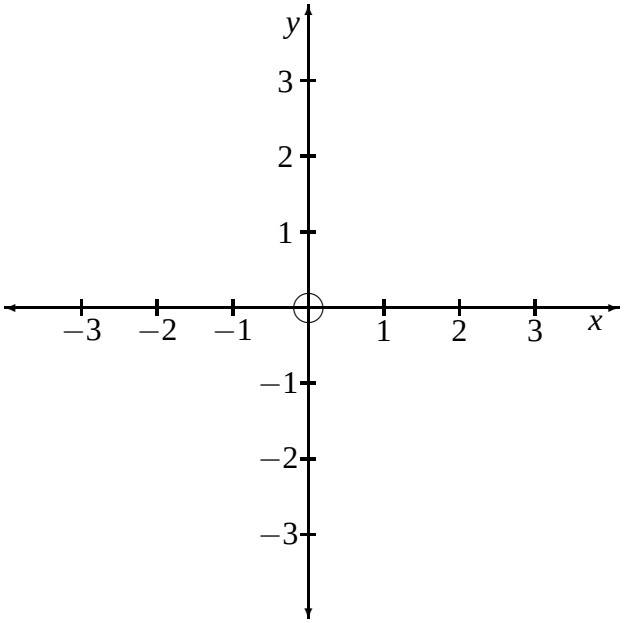


Now moving to the left of the origin, we divide the axis into units (numbers) of exactly the same length as those on the positive side. The numbers on the left-hand side of the origin are negative numbers (as is the case with the number line).



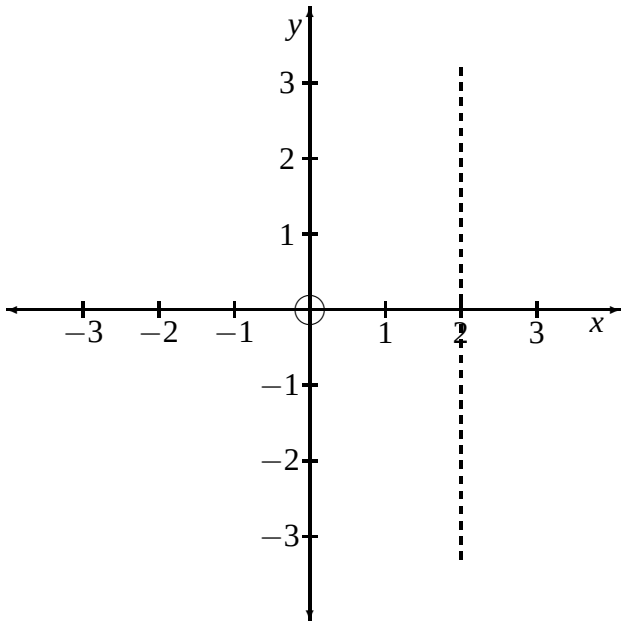
We follow the same procedure on the vertical *y*-axis. The numbers above the origin are positive numbers and the numbers below the origin are negative numbers.

A neat system (set) of axes is given in the sketch below.



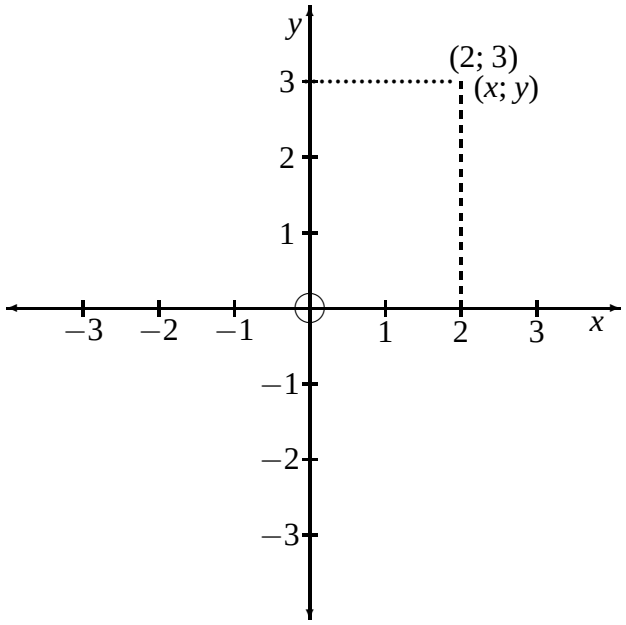
How do we use the numbers on the set of axes?

If we look at the set of axes below, we see that in any point on the broken vertical line, *x* is equal to 2.



A specific point is indicated by two coordinates, written as $(x; y)$.

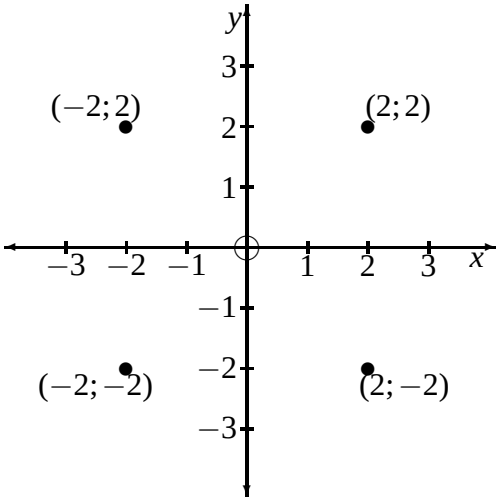
For example, to find the exact point at which $(2; 3)$ lies on the system of axes, we do the following: All the points on the dotted line above have an x-coordinate of 2 and all the points on the horizontal dotted line in the following sketch have a y-coordinate of 3.



The broken line and the dotted line intersect at a point at which $x = 2$ **and** $y = 3$, and we write it down as $(2; 3)$ – first the x-value and then the y-value. The order in which the numbers are written is important.

Note: **On** the horizontal x-**axis** the value of y is always **zero**. **On** the vertical y-**axis** the value of x **is always zero**.

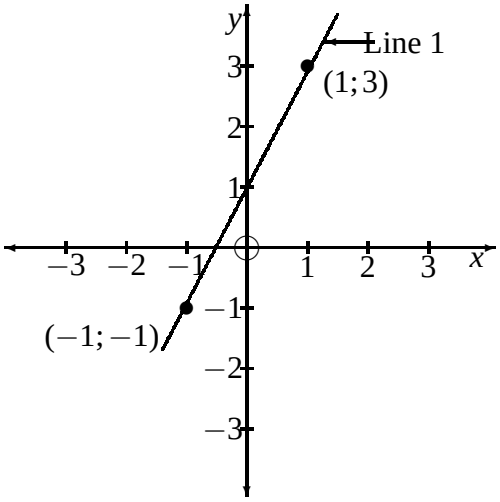
Looking at all 4 the quadrants, we see that in the first quadrant, both x and y are positive. In the second quadrant, x is negative and y is positive. In the third quadrant, the values of both x and y are negative. In the fourth quadrant, y is negative and x is positive.



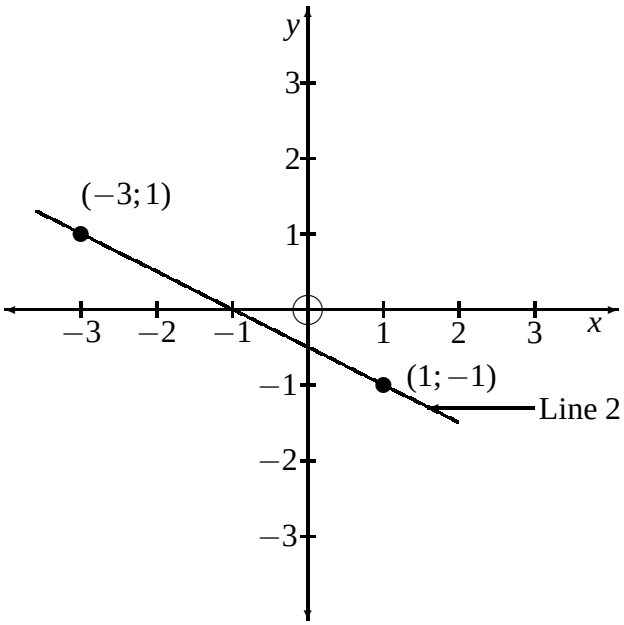
If we wish to draw a straight line (also called a graph) on a set of axes, we need two points.

Let us draw a straight line (line 1) through points $(x_1; y_1) = (1; 3)$ and $(x_2; y_2) = (-1; -1)$. We do it as follows:

1. Draw a neat set of axes and name the axes.
2. Divide the axes into equal suitable length units.
3. Plot the given points on the system of axes.
4. Join the points.
5. Name the line.

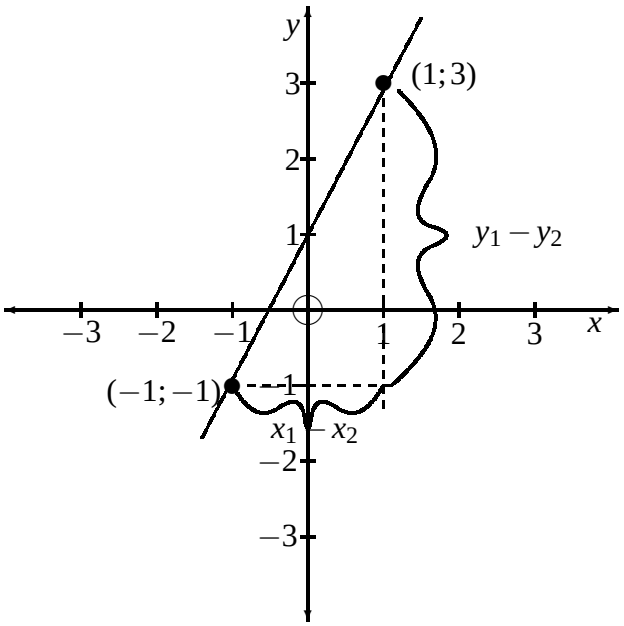


Let us draw another straight line through the points $(x_1; y_1) = (-3; 1)$ and $(x_2; y_2) = (1; -1)$ (line 2).



If we look carefully at the two graphs above, we see that the first straight line (line 1) runs from the third quadrant upwards (that is, ascends from left to right) to the first quadrant. The second line (line 2) starts in the second quadrant and runs downwards (that is, descends) to the fourth quadrant. We also see that the second line is "flatter" than the fist line. This "steepness" with which straight lines ascend or descend is called the *slope* or *gradient* of the line. The slope is the ratio of change in the y-value from one point to the other to the change in the corresponding x-value.

Let us look again to the line passing through the points $(1; 3)$ and $(-1; -1)$ (that is, line 1).

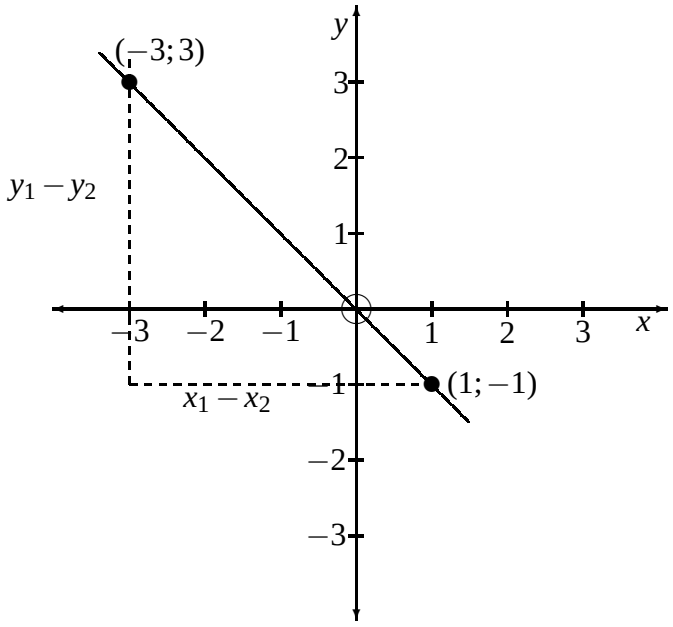


The slope of this line $=$ $\dfrac{\textbf{change in the } y\textbf{-value}}{\textbf{change in the } x\textbf{-value}}$

$$= \frac{y_1 - y_2}{x_1 - x_2}$$
$$= \frac{3-(-1)}{1-(-1)}$$
$$= \frac{4}{2}$$
$$= 2$$

The slope of this line is positive 2. Any straight line that runs from the third quadrant upwards (that is, ascends) from left to right has a positive slope.

Let us draw a straight line going through the points $(-3;\ 3)$ and $(1;\ -1)$.

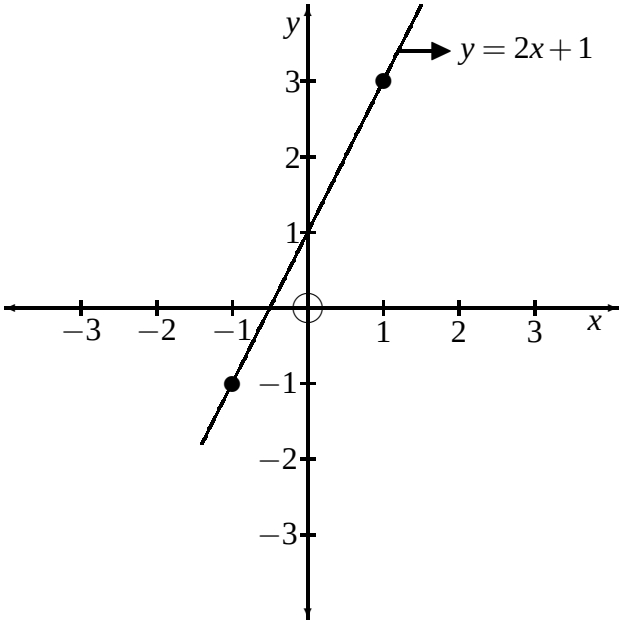


$$\text{The slope of this line} = \frac{\text{change in the } y\text{-value}}{\text{change in the } x\text{-value}}$$
$$= \frac{y_1 - y_2}{x_1 - x_2}$$
$$= \frac{3-(-1)}{-3-(1)}$$
$$= \frac{4}{-4}$$
$$= -1$$

The slope of this line is negative. Any line that has a descending (that is downwards) trend from left to right has a negative slope.

Note: Almost all straight lines intersect the *y*-axis.

Refer to the straight line passing through the points $(1; 3)$ and $(-1; -1)$ (line 1) again.



We have calculated that the slope of this line is 2. We see from the graph that the line intersects the vertical axis, which is the *y*-axis, at 1. Can this information be represented in an equation?

Yes, there is a general functional equation that we can use to represent this information.

The equation for a straight line is

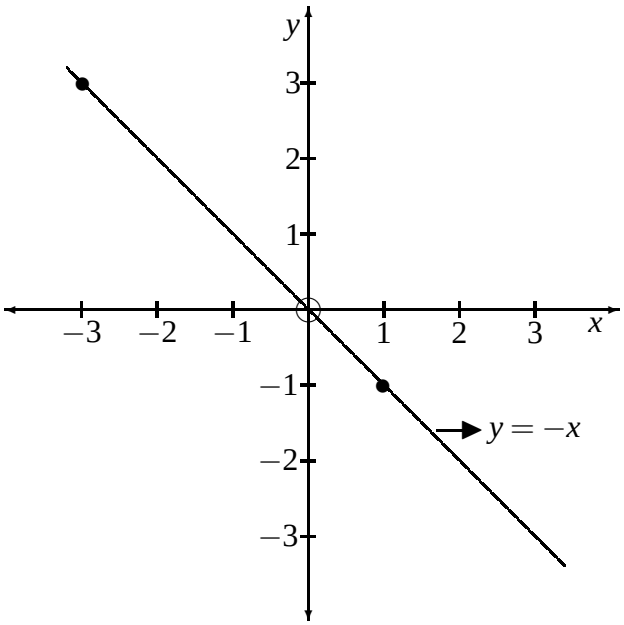
$$y = bx + a$$

where b is the slope of the line and a the *y*-intercept.

The equation for a straight line passing through the points $(1; 3)$ and $(-1; -1)$ is given by

$$y = 2x + 1$$

Let us determine the equation for the straight line passing through points $(-3; 3)$ and $(1; -1)$ (line 3).



We have calculated that the slope of the line is -1 and, looking at the graph, we see that the y-intercept is zero.

The equation for this line is thus

$$\begin{aligned} y &= bx+c \\ &= -1x+0 \\ &= -x \end{aligned}$$

Can we determine the **equation for a straight line passing through two points** without drawing a graph? Yes, we can by using the following equation:

$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$$

Example 6.1

1. Determine the equation for the line through the two points

   $(x_1; y_1) = (1; 3)$ and $(x_2; y_2) = (-1; -1)$ and draw this line.

$$\begin{aligned}
\frac{y-y_1}{x-x_1} &= \frac{y_2-y_1}{x_2-x_1} \\
\frac{y-3}{x-1} &= \frac{-1-3}{-1-1} \\
\frac{y-3}{x-1} &= \frac{-4}{-2} \\
\frac{y-3}{x-1} &= 2 \\
y-3 &= 2(x-1) \\
y-3 &= 2x-2 \\
y &= 2x-2+3 \\
y &= 2x+1
\end{aligned}$$

If one has the equation for a straight line, can one then draw that line? Yes! This is how we do it:

Draw up a table, choosing three values for x – the third point is to check if the calculations are correct:

x	0	1	2
y			

Now substitute the x-values into the equation:

$$\begin{aligned}
y &= 2x+1 \\
\text{If } x=0, \text{ then } y &= 2(0)+1 \\
&= 0+1 \\
&= 1
\end{aligned}$$

$$\begin{aligned}
\text{If } x=1, \text{ then } y &= 2(1)+1 \\
&= 2+1 \\
&= 3
\end{aligned}$$

$$\begin{aligned}
\text{If } x=2, \text{ then } y &= 2(2)+1 \\
&= 4+1 \\
&= 5
\end{aligned}$$

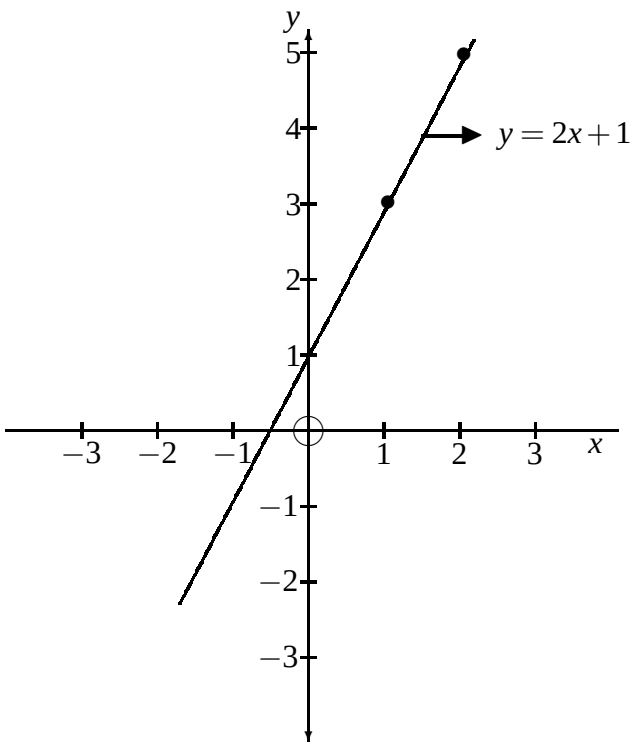
Complete the table by filling in the y values, obtained above:

x	0	1	2
y	1	3	5

This means that the points $(0;1)$; $(1;3)$ and $(2;5)$ lie on the line for which the equation is

$$y=2x+1$$

Let us draw the graph by plotting the x- and the corresponding y-values (points) on the graph and join the points. We then name the graph.



2. Draw the graph of

$$y = -x + 2$$

If $x = 0$, then $y = -(0) + 2 = 2$

If $x = 1$, then $y = -(1) + 2 = 1$

If $x = 2$, then $y = -(2) + 2 = 0$.

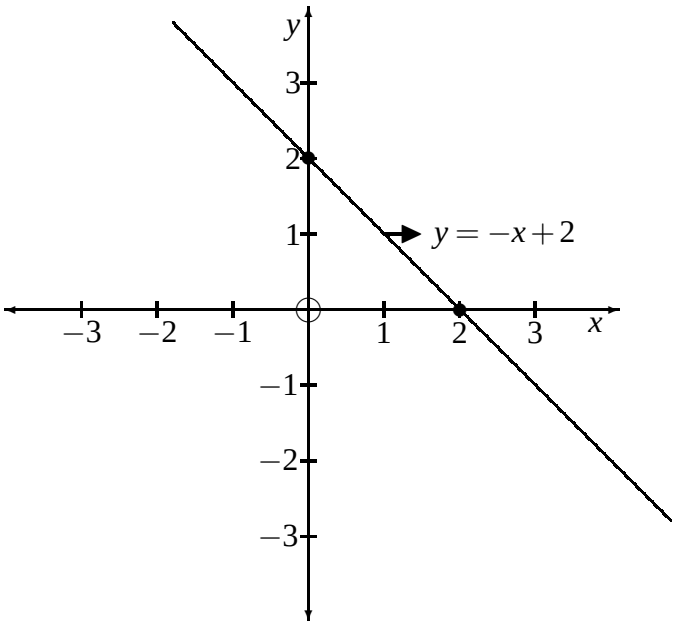
Draw up a table and fill in the y values.

x	0	1	2
y	2	1	0

The points $(0; 2)$, $(1; 1)$ and $(2; 0)$ lie on the line of

$$y = -x + 2$$

Draw a neat set of axes, name the axes and divide them up according to a suitable scale, plot the points, join the points, and name the graph.



We can draw more than one graph on the same set of axes.

3. Draw the graphs for

   (a) $y = 2x - 1$

   (b) $y = -3x + 2$

   on the same set of axes.

   (a) $y = 2x - 1$

   If $x = 0$, then $y = 2(0) - 1 = -1$

   If $x = 1$, then $y = 2(1) - 1 = 1$

   If $x = 2$, then $y = 2(2) - 1 = 3$

x	0	1	2
y	-1	1	3

   The points $(0; -1)$, $(1; 1)$ and $(2; 0)$ lie on the line of

$$y = 2x - 1$$

   (b) $y = -3x + 2$

   If $x = 0$, then $y = -3(0) + 2 = 2$

   If $x = 1$, then $y = -3(1) + 2 = -1$

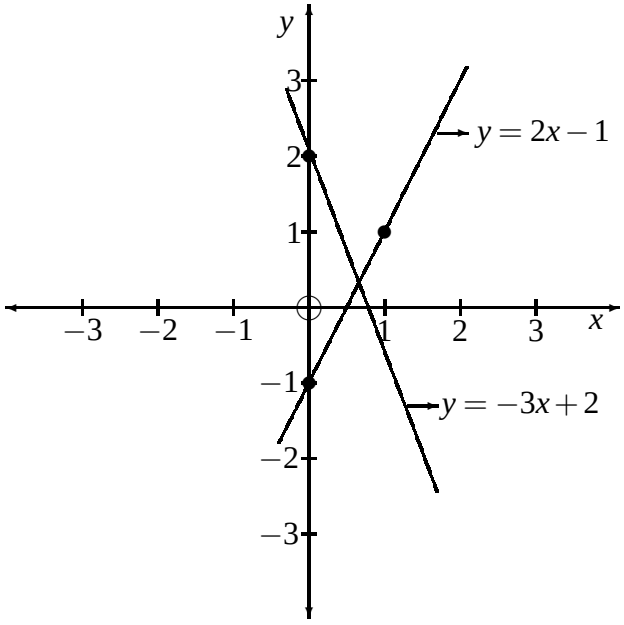
   If $x = 2$, then $y = -3(2) + 2 = -4$

x	0	1	2
y	2	-1	-4

The points $(0;2),(1;-1)$ and $(2;-4)$ lie on the line of

$$y=-3x+2$$

Draw a neat set of axes, name the axes and divide them into equal lengths according to a suitable scale, plot the points for the first graph, join the points and name the line. Do exactly the same for the second equation.



† (See page 178 for the solutions.)

Exercise 6.1

1. Draw a straight line through the points $(-2;4)$ and $(3;-1)$.
2. Draw a straight line through the points $(-1;-2)$ and $(2;3)$.
3. Determine the equation for the straight line going through the points $(2;3)$ and $(-2;-2)$.
4. Determine the equation for the straight line going through the points $(-1;-2)$ and $(2;2)$.
5. Draw the following two straight lines on the same set of axes:

$$\begin{aligned} y &= -x+2 \\ y &= 2x-1 \end{aligned}$$

Chapter 7

Basic financial calculations

On completion of this chapter, you should be able to

- do basic simple and compound interest calculations
- do basic annuity calculations
- calculate the payments on a mortgage loan
- set up a basic amortisation schedule

Note:
- **Appendix A** contains the solutions to all exercises by substituting into the relevant formulas and using ordinary calculator keys, when necessary.
- **Appendix B** contains the solutions to the financial exercises by using the ordinary or **financial keys** on the calculator.

7.1 Interest

Interest is money that you receive if you have invested a sum of money, or money that you pay if you have borrowed money. The amount of interest is calculated as a fraction of the amount borrowed or saved over a period of time. The fraction, also known as the interest rate, is usually expressed as a percentage per year. The period of time should be expressed in years.

7.1.1 Simple interest

If you borrow money from the bank and the amount that you owe the bank for the loan is calculated at the end of the period for which you borrowed the money, you pay simple interest – only one calculation is necessary.

Example 7.1

1. How much simple interest is payable on a loan of R20 000, borrowed for a period of one year at a simple interest rate of 10% per year?

   The interest payable per year is 10% of R20 000:

   $$\begin{aligned} \text{Interest payable} &= 10\% \times 20\,000 \\ &= \tfrac{10}{100} \times 20\,000 \\ &= 0,10 \times 20\,000 \\ &= 2\,000 \end{aligned}$$

   The above means that at the end of the year (that is, after 12 months), R2 000 interest is owed to the bank.

   Can we formulate an equation for calculating simple interest?

   Yes! Let us try to do so:

   The interest (I) payable equals the original amount borrowed, also called the principal (P) or **present value**, multiplied by the interest rate (r) per year, multiplied by the time in years (t) for which we borrowed the money.

   The formula to calculate the **simple interest** is

   $$I = Prt$$

   where P is the principal amount borrowed or the present value of the loan, r is the simple interest rate per year, and t is the time in years for which the principal is borrowed. Let us use this equation and see if the above calculation was correct.

   $$\begin{aligned} I &= Prt \\ &= 20\,000 \times 0,10 \times 1 \\ &= 2\,000 \end{aligned}$$

   Now, let us create a formula for the **future value** or accumulated amount (S) – that is, the total amount payable at the end of the period for which the money has been borrowed.

   $$\begin{aligned} S &= P\,(\text{principal}) + I\,(\text{interest payable}) \\ &= P + Prt \qquad (\text{Replace } I \text{ with } Prt.) \\ &= P\,(1 + rt) \quad (\text{Take out the common factor } P.) \end{aligned}$$

Thus, the **future value** formula for **simple interest** is

$$S = P(1+rt)$$

Let us test our new formula:

$$\begin{aligned} S &= P(1+rt) \\ &= 20\,000\left(1+\tfrac{10}{100}\times 1\right) \\ &= 20\,000(1+0,10\times 1) \\ &= 22\,000 \end{aligned}$$

See Appendix B (7.1) for the method to enter this calculation on your calculator.

An amount of R22 000 must be paid back at the end of year 1.

We could also use our calculator to do this calculation.

The answer is 22 000.

If we wish to borrow the money for a period of 3 years, the total amount payable is

$$\begin{aligned} S &= P(1+rt) \\ &= 20\,000\left(1+\tfrac{10}{100}\times 3\right) \\ &= 20\,000(1+0,10\times 3) \\ &= 26\,000 \end{aligned}$$

See Appendix B (7.2).

2. Martha invests R1 500 for 21 months. The applicable simple interest rate is 9% per year. Determine the amount of interest that Martha will receive in 21 months' time.

   In the formula, time has been defined in years, therefore the 21 months should be written in years and must be divided by 12:

$$\begin{aligned} I &= Prt \\ &= 1\,500\times\tfrac{9}{100}\times\tfrac{21}{12} \\ &= 236,25 \end{aligned}$$

See Appendix B (7.3).

   Martha will receive R236,25 interest after 21 months.

3. Sera must pay Stephan an amount of R3 500 in 6 years' time. A 12,5% simple interest rate is applicable. Determine the amount that Sera borrowed from Stephan 6 years ago.

$$\begin{aligned} S &= P(1+rt) \\ 3\,500 &= P\left(1+\tfrac{12,5}{100}\times 6\right) \\ P &= \frac{3\,500}{\left(1+\frac{12,5}{100}\times 6\right)} \\ &= 2\,000 \end{aligned}$$

See Appendix B (7.4).

   Sera borrowed R2 000 from Stephan.

4. Jane borrowed R5 000 from John at a simple interest rate of 7,5%. The amount that she needs to repay is R7 500. Determine the time under consideration.

   First manipulate the formula to get t on the left-hand side of the equal sign and the rest of the variables on the right-hand side of the equal sign.

$$\begin{aligned} S &= P(1+rt) \\ \frac{S}{P} &= 1+rt && \text{(Get rid of the } P.\text{)} \\ \frac{S}{P}-1 &= rt && \text{(Get rid of the 1.)} \\ rt &= \frac{S}{P}-1 && \text{(Switch around the terms.)} \\ t &= \left(\frac{S}{P}-1\right) \div r && \text{(Get rid of the } r.\text{)} \end{aligned}$$

   Now fill in the values of the variables:

$$\begin{aligned} t &= \left(\frac{7\,500}{5\,000}-1\right) \div \frac{7,5}{100} \\ &= (1,5-1) \times \frac{100}{7,5} \\ &= 0,5 \times \frac{100}{7,5} \\ &= 6,6667 \end{aligned}$$

See Appendix B (7.5).

   Jane needs to pay Johan R7 500 in 6,67 years' time.

5. Temba borrowed R12 500 from Thabo. He must pay him R20 000 back 6 years from now. Determine the applicable simple interest rate at which Temba borrowed the money.

   From the question, we see that in 6 years' time Temba must pay Thabo R(20 000 − 12 500) = R7 500 in interest. The following formula may be used to find the interest rate, r:

$$\begin{aligned} & I = Prt \\ \text{i.e.} \quad & Prt = I \\ \text{i.e.} \quad & \frac{Prt}{Pt} = \frac{I}{Pt} \\ \text{i.e.} \quad & r = \frac{I}{Pt} \\ \text{i.e.} \quad & r = \frac{7\,500}{12\,500 \times 6} \\ \text{i.e.} \quad & r = 0,1 \\ \text{i.e.} \quad & r = \frac{10}{100} \\ \text{i.e.} \quad & r = 10\% \end{aligned}$$

See Appendix B (7.6).

   Themba borrowed the money at a simple interest rate of 10% per year.

† (See page 181 for the solutions.)

Exercise 7.1

1. James borrowed R2 000 from Tim at a simple interest rate of 8% per year. Determine the amount that James must pay Tim after 7 years.
2. Debbie must pay Douglas an amount of R6 400 in 4 years' time. Determine the amount that she borrowed from Douglas if a simple interest rate of 15% per year is applicable.

7.1.2 Compound interest

In the case of simple interest, only one calculation of interest is done, namely at the end of the contract or investment period.

In the case of compound interest, interest is calculated at regular intervals, without paying or withdrawing the interest, which means that interest is also calculated on interest during the following compounding periods.

Example 7.2

1. **Simple interest**

   Suppose you invest R1 000 for 3 years at a simple interest rate of 10% per year. At the end of the 3-year period, you will have earned interest, I, where

   $$\begin{aligned} I &= Prt \\ &= 1\,000 \times 0,10 \times 3 \\ &= \text{R}300 \end{aligned}$$

   and the accumulated amount will be $\text{R}(1\,000 + 300) = \text{R}1\,300$.

2. **Compound interest**

   Suppose the interest is calculated at the end of each year and instead of withdrawing the interest, you leave the interest in the account. Then, **after one year**, you will have earned

   $$\begin{aligned} I &= Prt \\ &= 1\,000 \times 0,10 \times 1 \\ &= 100 \end{aligned}$$

   This R100 is added to the principal and at the end of year 2, you will receive interest on the original principal of R1 000 plus the interest of R100 − thus on R1 100.

The interest earned in the second year is

$$\begin{aligned} I &= Prt \\ &= 1\,100 \times 0,10 \times 1 \\ &= 110 \end{aligned}$$

At the **end of year 2**, you will have a balance of
$\mathrm{R}(1\,100 + 110) = \mathrm{R}1\,210$ in your account.

The interest earned in the third year is

$$\begin{aligned} I &= Prt \\ &= 1\,210 \times 0,1 \times 1 \\ &= 121 \end{aligned}$$

and at the **end of the third year**, you will have a balance of
R(1 210 + 121) = R1 331,00 in your account.

As you see, if interest is calculated more frequently and not withdrawn, you earn $\mathrm{R}(1\,331 - 1\,300)$ = R31 more in interest than when the calculation is done at the end of the investment time. Interest that is calculated more frequently and not withdrawn is called compound interest.

The above implies an interest-on-interest situation. Compound interest is just the repeated application of simple interest to the principal, which at each stage is increased by the simple interest earned in the previous period.

Instead of doing a separate calculation for every compound period, we do only one calculation by using the **compound interest formula**, namely as follows:

COMPOUND INTEREST

$$S = P(1+i)^n$$

where

S = the accumulated amount or future value
P = the principal or present value
i = the given interest rate per year divided by the number of compounding periods per year
n = the time in years multiplied by the number of compounding periods per year.

Let us use the formula to check our calculations above:

$$\begin{aligned} S &= P(1+i)^n \\ &= 1\,000\left(1+\tfrac{10\%}{1}\right)^{3\times 1} \\ &= 1\,000\,(1+0,10)^3 \qquad \text{because } 10\% = \tfrac{10}{100} = 0,10 \\ &= 1\,331,0000 \end{aligned}$$

Ordinary calculator steps:

See Appendix B (7.7).

1 000 [×] [(] 1 [+] 0,10 [)] [2ndF] [y^x] 3 [=]

The accumulated amount after 3 years, if R1 000 is invested at 10% compound interest, compounded annually, is R1 331,00.

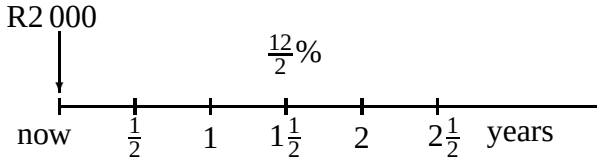
For using the financial keys on the calculator, see Appendix B (7.7).

Note: The prescribed financial calculator has been pre-programmed to do these types of compound interest calculations.

See **Additional Resources** on the module site for an explanation of the use of the financial keys on the calculator. It is very important to study the calculator examples in Appendix B, in this study guide.

3. Cheryl invested R2 000 at an interest rate of 12% per year, compounded every 6 months for $2\frac{1}{2}$ years. What will this investment amount to?

The above may be represented on a time line as



By using the formula

$$\begin{aligned} S &= P(1+i)^n \\ \text{where } P &= 2000 \\ i &= 12\% \div 2 \qquad \text{The interest is compounded every 6 months and that is twice a year } \left(\tfrac{12}{6}=2\right) \\ n &= 2,5\times 2 \end{aligned}$$

$$\begin{aligned} S &= 2\,000\left(1+\tfrac{12\%}{2}\right)^{2,5\times 2} \\ S &= 2\,000\left(1+\tfrac{0,12}{2}\right)^{2,5\times 2} \qquad \text{because } 12\% = \tfrac{12}{100} = 0,12 \\ &= 2\,676,4512 \end{aligned}$$

Ordinary calculator steps used for the calculation above:

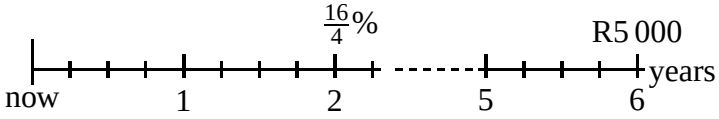
2 000 × (1 + 0,12 ÷ 2) 2ndF y^x (2,5 × 2) =

Cheryl will receive R2 676,45 after $2\frac{1}{2}$ years.

For using the financial keys on the calculator, see Appendix B (7.8).

See Appendix B (7.8).

4. Carl borrowed money from Daphne at an interest rate of 16% per year, compounded quarterly. The total amount to be paid back in 6 years' time is R5 000. Determine the amount of money that Carl borrowed from Daphne.



By using the formula

$$\begin{aligned} S &= P(1+i)^n \\ \text{with } S &= 5\,000 \\ i &= 16\% \div 4 \\ n &= 6 \times 4 \end{aligned}$$

$$\begin{aligned} 5\,000 &= P\left(1+\tfrac{16\%}{4}\right)^{6\times 4} \\ 5\,000 &= P\left(1+\tfrac{0{,}16}{4}\right)^{6\times 4} \\ P &= 5\,000 \div \left(1+\tfrac{0{,}16}{4}\right)^{24} \\ P &= 1\,950{,}6074 \end{aligned}$$

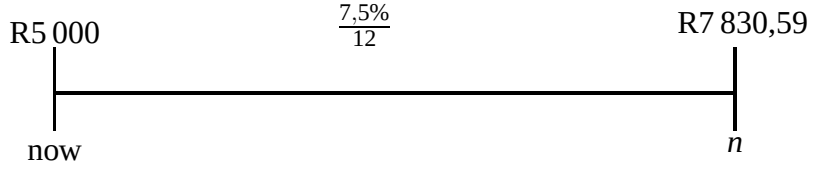
Ordinary calculator steps used for the calculation above:
5 000 ÷ (1 + 0,16 ÷ 4) 2ndF y^x 24 =

Carl borrowed R1 950,61 from Daphne.

For using the financial keys on the calculator, see Appendix B (7.9).

See Appendix B (7.9).

5. Debbie invested R5 000 in an account earning 7,5% interest, compounded monthly. She received R7 830,59. Determine the number of periods under consideration.



By using the formula

$$\begin{aligned} S &= P(1+i)^n \\ S &= 7\,830,59 \\ P &= 5\,000 \\ i &= 7,5\% \div 12 \\ 7\,830,59 &= 5\,000\left(1+\tfrac{7,5\%}{12}\right)^n \\ 7\,830,59 &= 5\,000\left(1+\tfrac{0,075}{12}\right)^n \quad \text{because } 7,5\% = \tfrac{7,5}{100} = 0,075 \\ n &= 72 \quad \text{(Logarithms must be used.)} \end{aligned}$$

See Appendix B (7.10).

Debbie invested her money for 72 months, which is 6 years.

Note: One needs to know **logarithms** if the unknown is in the exponent. Logarithms are not part of the syllabus of this module, therefore it is advised that students use the **financial keys** on the calculator in such a case. See Appendix B (7.10).

Exercise 7.2

1. Sam borrowed R10 000 at 7,5% per year, compounded monthly. Determine the amount that Sam owes after 3 years.
2. Marcus must pay Mika R12 000 in 6 years' time from now. Determine the amount of money that Marcus borrowed 4 years ago if an interest rate of 11,5% per year, compounded weekly, was applicable.
3. Jill invested R10 000 in an account earning 9,2% interest, compounded quarterly. If she received R24 832,78, determine the periods under consideration.
4. Abel invested R15 000. After 10 years, he received R45 000. Determine the yearly interest rate (compounded yearly).

† (See page 182 for solutions.)

† (See A(3), A(4), A(5) and A(6) in Appendix B for the solutions by using the financial keys on the calculator.)

7.2 Annuities

In each of the above instances, only one amount was either lent or borrowed.

In the case of simple interest, only one calculation was done and the interest was added to the principal at the end of the whole period.

In the case of compound interest, only one amount was either lent or borrowed and the interest was calculated according to the number of compounding periods per year, which can differ from one investment to another. After each calculation, the interest was added to the principal and the interest for the following compounding period was calculated on the new principal, and so forth.

There are also situations in which the same amount of money is paid into an account at certain fixed equal time intervals. For example:

Every Monday morning, you deposit R100 into a bank account without withdrawing any money from this account. You could also deposit, say, R1 000 into an account every 15th of a month, without withdrawing any money from this account. You could further deposit the same amount every year as long as the intervals between the payments are exactly the same. This type of transaction is called an annuity.

An annuity is a sequence of equal (same amount) payments at equal time intervals.

There are two types of formulas to determine different values of annuities.

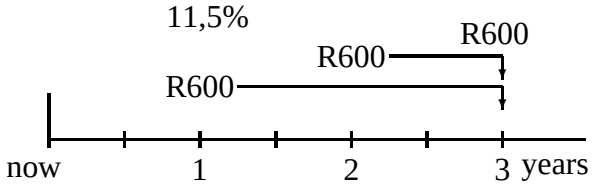
7.2.1 Future value of an annuity

The **future value** of an annuity is the balance in an account after a series of payments have been made at equal time intervals.

Example 7.3

1. Suppose Bert deposited an amount of R600 at the end of every year into an account earning 11,5% interest per year for 3 years. Determine the accumulated amount at the end of year 3.

   Let us do the calculations step-by-step first:



   The first deposit will earn interest for **2 years**, and will amount to

$$\begin{aligned} S &= P(1+i)^n \\ &= 600(1+11,5\%)^2 \\ &= 600(1+0,115)^2 \qquad \text{because } 11,5\% = \tfrac{11,5}{100} = 0,115 \\ &= 745,94 \end{aligned}$$

The second deposit will earn interest for **one year**, and will amount to

$$\begin{aligned} S &= P(1+i)^n \\ &= 600(1+11,5\%)^1 \\ &= 600(1+0,115)^1 \\ &= 669,00 \end{aligned}$$

The third R600 will earn no interest.

At the end of year 3, Bert will have a balance of

R(754,94 + 669,00 + 600) = R2 014,94 in the account.

Just think what it would be like if you had to do, say, 30 of the above calculations!

Instead, the following formula may be used for determining the future value of an annuity.

The formula for the **future value** of an annuity is as follows:

FUTURE VALUE

$$S = R\left(\frac{(1+i)^n - 1}{i}\right)$$

where S = the future value or accumulated amount
R = equal payments made at equal time intervals
i = the applicable interest rate divided by the number of compounding periods per year
n = time in years multiplied by the number of payments per year

Let us check our previous example 7.3.

The values of the variables are as follows:

$$\begin{aligned} R &= 600 \\ i &= 11,5\% \div 1 = 0,115 \\ n &= 3 \text{ (years)} \times 1 \text{ (compounded yearly).} \\ S &= 2014,94 \end{aligned}$$

By substituting into the formula

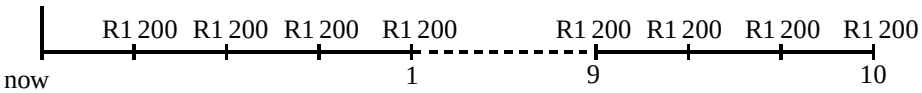
$$\begin{aligned} S &= R\left(\frac{(1+i)^n - 1}{i}\right) \\ &= 600\left(\frac{(1+0,115)^{3\times 1} - 1}{0,115}\right) \\ &= 2014,9350 \end{aligned}$$

Ordinary calculator steps used above: 600 × (((1 + 0,115) 2ndF y^x 3 − 1) ÷ 0,115 =

What does this mean? It means that if you deposit an amount of R600 at the end of every year into an account earning 11,5% interest per year, there will be a balance of R2 014,94 in the account after 3 years.

For using the financial keys on the calculator, see appendix B (7.11). See Appendix B (7.11).

2. Determine the accumulated amount of an annuity with quarterly payments of R1 200, at an interest rate of 16% per year, compounded quarterly, made over a period of 10 years.



The values of the variables are as follows:

$R = 1\,200$

$n = 10\,(\text{years}) \times 4\,(\text{number of compounding periods per year})$

$i = \frac{16\%}{4} = \frac{0{,}16}{4}$

By using the formula

$$\begin{aligned} S &= R\left(\frac{(1+i)^n-1}{i}\right) \\ &= 1\,200\left(\frac{\left(1+\frac{0{,}16}{4}\right)^{10\times 4}-1}{\frac{0{,}16}{4}}\right) \\ &= 114\,030,6188 \end{aligned}$$

Ordinary calculator steps: 1 200 × (((1 + 0,16 ÷ 4) 2ndF y^x 40 − 1) ÷ (0,16 ÷ 4 =

The accumulated amount of R1 200 made every quarter for 10 years amounts to R114 030,62.

For using the financial keys on the calculator, see Appendix B (7.12). See Appendix B (7.12).

7.2.2 Present value of an annuity

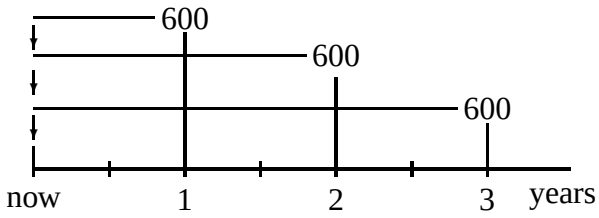
The **present value** of an annuity is the amount of money (balance) in an account *now*, which must be paid off in equal payments at equal time intervals at a set interest rate to ensure that the balance will be zero at the end of the time period.

Example 7.4

1. Determine the **present value** of an annuity with yearly payments of R600 made at the end of each year, at an interest rate of 11,5% per year, for a term of 3 years.

   If we do a step-by-step calculation, without a formula, we do it as follows:

   The relevant time line is given below.



   We wish to determine the value of the annuity *now*. We must find the present value of each payment of R600 now.

   The present value of the first R600 is P in the formula

$$\begin{aligned} S &= P(1+i)^n \\ 600 &= P(1+11,5\%)^1 \quad \text{(1 year discounted)} \\ P &= \frac{600}{(1+11,5\%)^1} \\ P &= \frac{600}{(1+0,115)^1} \\ &= 538,1166 \\ &\approx 538,12 \end{aligned}$$

   The present value of the second R600 is P in the formula

$$\begin{aligned} S &= P(1+i)^n \\ 600 &= P(1+11,5\%)^2 \quad \text{(2 years discounted)} \\ P &= \frac{600}{(1+11,5\%)^2} \\ P &= \frac{600}{(1+0,115)^2} \\ &= 482,6158 \\ &\approx 482,62 \end{aligned}$$

   The present value of the third R600 is P in the formula

$$\begin{aligned} S &= P(1+i)^n \\ 600 &= P(1+11,5\%)^3 \quad \text{(3 years discounted)} \\ P &= \frac{600}{(1+11,5\%)^3} \\ P &= \frac{600}{(1+0,115)^3} \\ &= 432,8393 \\ &\approx 432,84 \end{aligned}$$

The present value of the annuity is thus

R(538,12 + 482,62 + 432,84) = R1 453,58.

We can use the following formula, however, to do **present value** annuities calculations:

PRESENT VALUE

$$P = R\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right]$$

where P = the present value or principal
R = equal payments made at equal time intervals
i = the applicable interest rate divided by the number of compounded periods per year
n = time in years multiplied by the number of payments per year

Let us use the formula to check our example above, Example 7.4.

The values of the variables are as follows:

$$\begin{aligned} R &= 600 \\ n &= 3\text{ (years)} \times 1\text{(compounded yearly)} = 3 \\ i &= 11,5\% = 0,115 \\ P &= 1\,453,5716 \end{aligned}$$

Note: There is a one cent difference because we rounded three times in the step-by-step calculation and only once by using the formula.

By substituting into the formula

$$\begin{aligned} P &= R\left(\frac{(1+i)^n-1}{i(1+i)^n}\right) \\ &= 600\left(\frac{(1+0{,}115)^3-1}{0{,}115(1+0{,}115)^3}\right) \\ &= 1\,453{,}5716 \end{aligned}$$

Ordinary calculator steps used above: 600 [×] [(] [(] [(] 1 [+] 0,115 [)] [2ndF] [y^x] 3 [−] 1 [)] [÷] [(] 0,115 [×] [(] 1 [+] 0,115 [)] [2ndF] [y^x] 3 [=]

This means that if you were to pay R600 every year for 3 years at an interest rate of 11,5% per year, you would be able to pay off a debt of R1 453,57.

For using the financial keys on the calculator, see Appendix B (7.13).

See Appendix B (7.13).

2. Determine the **present value** of R1 500 quarterly payments made into an account earning 12,6% interest, compounded quarterly. The time under consideration is 6 years.

   First determine the value of the variables:

   $R = 1\,500$
   $n = 6 \times 4$, the number of years multiplied by the number of compounding periods per year
   $i = 12{,}6\%$
   $P = 24\,997{,}6658$

   By using the formula

$$\begin{aligned} P &= R\left(\frac{(1+i)^n-1}{(1+i)^n \cdot i}\right) \\ &= 1\,500\left(\frac{\left(1+\frac{0{,}126}{4}\right)^{24}-1}{\left(1+\frac{0{,}126}{4}\right)^{24}\cdot\frac{0{,}126}{4}}\right) \\ &= 24\,997{,}6658 \end{aligned}$$

   Ordinary calculator steps: 1 500 [×] [(] [(] [(] 1 [+] 0,126 [÷] 4 [)] [2ndF] [y^x] 24 [−] 1 [)] [÷] [(] [(] 1 [+] 0,126 [÷] 4 [)] [2ndF] [y^x] 24 [×] 0,126 [÷] 4 [=]

   This answer means that you were able to pay off a debt (loan) of R24 997,67 in 3 years' time.

For using the financial keys on the calculator, see Appendix B (7.14).

See Appendix B (7.14).

3. Every Friday, Sophie deposits R160 into an account earning 7,8% interest per year, compounded weekly. Determine the balance (accumulated amount) in the account after 2 years.

   The values of the variables are as follows:

$$\begin{aligned} R &= 160 \\ i &= 7,8\% \div 52 \text{ (compounded weekly) that is, 52 times per year} \\ &= (0,078 \div 52) \\ n &= 2\,(\text{years}) \times 52 \\ S &= 17\,993,5568 \end{aligned}$$

   By substituting into the formula

$$\begin{aligned} S &= R\left[\frac{(1+i)^n-1}{i}\right] \\ &= 160\left[\frac{(1+\frac{0.078}{52})^{52\times 2}-1}{\frac{0.078}{52}}\right] \\ &= 17\,993,5568 \end{aligned}$$

   Ordinary calculator steps used above: 160 [×] [(] [(] [(] 1 [+] 0,078 [÷] 52 [)] [2ndF] [y^x] [(] 52 [×] 2 [)] [−] 1 [)] [÷] [(] 0,078 [÷] 52 [)] [=]

   Sophie's account had a balance of R17 993,56 after 2 years.

For using the financial keys on the calculator, see Appendix B (7.15).

See Appendix B (7.15).

4. Neville wishes to buy a motor cycle that will cost him R48 000. Determine how much money he must deposit every month into an account earning 8,4% per year, compounded monthly, to be able to buy the motor cycle in 3 years' time.

   The values of the variables are as follows:

$$\begin{aligned} S &= 48\,000 \text{ (future value)} \\ i &= 48\% \div 12 \text{ (compounded monthly)} \\ &= (0,48 \div 12) \\ n &= 3 \text{ (years)} \times 12 = 36 \\ R &= 1\,177,0186 \end{aligned}$$

By substituting into the formula

$$\begin{aligned} S &= R\left[\frac{(1+i)^n-1}{i}\right] \\ R &= S \div \left[\frac{(1+i)^n-1}{i}\right] \\ &= 48\,000 \div \left[\frac{(1+\frac{0.084}{12})^{12\times 3}-1}{\frac{0.084}{12}}\right] \\ &= 1\,177,0186 \end{aligned}$$

Ordinary calculator steps used above: 48 000 ÷ (((1 + 0,084 ÷ 12) 2ndF y^x 36 − 1) ÷ (0,084 ÷ 12 =

Neville must deposit R1 177,02 every month in order to have R48 000 available in 3 years' time to buy his motor cycle.

For using the financial keys on the calculator, see Appendix B (7.16).

See Appendix B (7.16).

5. How long will it take Eva to save R55 000 if she deposits R9 500 every year into an account earning 7,34% interest per year?

   The values of the variables are as follows:

$$\begin{aligned} S &= 55\,000 \text{ (future value)} \\ R &= 9\,500 \text{ (payment)} \\ i &= 7,34\% \\ &= 0,0734 \text{ (compounded yearly)} \\ n &= 5. \end{aligned}$$

By substituting into the formula

$$\begin{aligned} S &= R\left[\frac{(1+i)^n-1}{i}\right] \\ \text{that is } Si &= R\left[(1+i)^n-1\right] \\ \text{that is } \frac{Si}{R} &= (1+i)^n-1 \\ \text{that is } (1+i)^n &= \frac{Si}{R}+1 \\ \text{that is } n &= \frac{\log\left(\frac{Si}{R}+1\right)}{\log(1+i)} \\ &= \frac{\log\left(\frac{55\,000\times 0.0734}{9\,500}+1\right)}{\log(1+0,0734)} \\ &= 4,9997 \text{ years} \end{aligned}$$

Eva will have to deposit R9 500 every year for **5 years** to save R55 000.

For using the financial keys on the calculator, see Appendix B (7.17).

Note: Logarithms are not part of the syllabus of this module. Therefore, it is advised that students use the financial calculator keys in exercises where the time (in the exponent) must be calculated. See Appendix B (7.17).

See Appendix B (7.17).

6. Harold must pay Rodney R12 500 every month for 5 years to pay for the gym equipment he bought. The interest rate is 12,3% per year, compounded monthly. Determine the price (present value) of the gym equipment.

   The values of the variables are as follows:

$$\begin{aligned} R &= 125\,000 \\ n &= 5 \text{ (years)} \times 12 \text{ (compounded monthly)} \\ &= 60 \\ i &= 12,3\% \div 12 = (0,123 \div 12) \\ p &= 558\,126,7996 \end{aligned}$$

   By substituting into the formula

$$\begin{aligned} P &= R\left[\frac{(1+i)^n-1}{(1+i)^n i}\right] \\ &= 12\,500\left[\frac{\left(1+\frac{0,123}{12}\right)^{12\times 5}-1}{\left(1+\frac{0,123}{12}\right)^{12\times 5}\cdot\frac{0,123}{12}}\right] \\ &= 558\,126,7996 \end{aligned}$$

   The price of the gym equipment is R558 126,80.

   For using the financial keys on the calculator, see Appendix B (7.18).

See Appendix B (7.18).

7. Ronel owes Ilze an amount of R120 000. This debt must be paid off in 6 years' time. An interest rate of 13,6% per year, compounded quarterly, is applicable. Determine Ronel's quarterly payments.

   The values of the variables are as follows:

$$\begin{aligned} P &= 120\,000 \text{ (present value)} \\ n &= 6 \text{ (years)} \times 4 \text{ (compounded quarterly)} \\ i &= 13,6\% \div 4 = (0,136 \div 4) \\ R &= 7\,394,4704 \end{aligned}$$

By substituting into the formula

$$\begin{aligned} P &= R\left[\frac{(1+i)^n-1}{(1+i)^n i}\right] \\ \text{i.e} \quad R &= P \div \left[\frac{(1+i)^n-1}{(1+i)^n i}\right] \\ &= 120\,000 \div \left[\frac{\left(1+\frac{0{,}136}{4}\right)^{24}-1}{\left(1+\frac{0{,}136}{4}\right)^{24}.\frac{0{,}136}{4}}\right] \\ &= 7\,394{,}4704 \end{aligned}$$

Ronel must pay Ilze R7 394,47 every quarter.

For using the financial keys on the calculator, see Appendix B (7.19).

See Appendix B (7.19).

Exercise 7.3

† (See page 183 for the solutions.)

† (See Appendix B, A(7), A(8), A(9) and A(10) for the solutions by using the financial keys on the calculator.)

1. Determine the future value of an annuity after 5 payments of R6 000 each paid annually at an interest rate of 10% per year.
2. Determine the present value of an annuity with semi-annual payments of R800 each at 16% per year, compounded half-yearly and with a term of 10 years.
3. Determine the present and the future value of an annuity, with payments of R200 per month for 5 years at 8% per year, compounded monthly.
4. Mapule needs R25 000 in 5 years' time to pay for a new lounge suite. She invests an amount of money monthly into an account earning 9,95% interest per year, compounded monthly. Determine her monthly deposits into this account.

7.3 Amortisation

When we buy a house, we usually borrow money from a bank to pay for the house. The bank will then issue a mortgage loan that must be paid back over a period of time. A mortgage loan is said to be amortised if all the liabilities (that is, the principal and the interest) are paid in equal payments at equal time intervals.

Note: We use the **present value formula** to determine the monthly payment and the outstanding principal.

Example 7.5

1. A loan of R50 000 is to be amortised at an interest rate of 15% per year, compounded monthly, by means of equal monthly payments over a period of 10 years. The first payment is due at the end of the first month. Determine the monthly payment.

   The values of the variables are as follows:

$$\begin{aligned} P &= 50\,000 \text{ (present value)} \\ i &= 15\% \div 12 \text{ (compounded monthly)} \\ &= (0,15 \div 12) \\ n &= 10 \text{ (years)} \times 12 = 120 \end{aligned}$$

   By using the present value formula

$$\begin{aligned} P &= R\left[\frac{(1+i)^n-1}{(1+i)^n.i}\right] \\ \text{that is } R &= P \div \left[\frac{(1+i)^n-1}{(1+i)^n i}\right] \\ &= 50\,000 \div \left[\frac{\left(1+\frac{0,15}{12}\right)^{120}-1}{\left(1+\frac{0,15}{12}\right)^{120}.\frac{0,15}{12}}\right] \\ &= 806,6748 \end{aligned}$$

   The monthly payment is R806,67.

   For using the financial keys on the calculator, see Appendix B (7.20).

See Appendix B (7.20).

   Note: An extra amount must be paid when a loan is amortised. The monthly payment includes the interest on the outstanding balance in the account. This may best be understood by drawing up an amortisation schedule (table).

2. A loan of R5 000 is to be repaid at an interest rate of 15% per year over a 5-year period.

   If a step-by-step calculation is done, it will have to be done as follows:

   The **yearly payments** are as follows:

The values of the variables are:

$$\begin{aligned} P &= 5\,000 \text{ (present value)} \\ i &= 15\% \div 1 \text{ (compounded yearly)} = 0,15 \\ n &= = 5 \text{ (years)} \times 1 = 5 \end{aligned}$$

By using the formula

$$\begin{aligned} R &= P \div \left[\frac{(1+i)^n - 1}{(1+i)^n . i}\right] \\ &= 5\,000 \div \left[\frac{\left(1+\frac{0,15}{1}\right)^{5\times 1} - 1}{\left(1+\frac{0,15}{1}\right)^{5} . \frac{0,15}{1}}\right] \\ &= 1\,491,5778 \end{aligned}$$

For using the financial keys on the calculator, see Appendix B (7.21).

See Appendix B (7.21).

At the end of year 1, the interest due is

$$\begin{aligned} I &= Prt \\ &= 5\,000 \times 15\% \times 1 \\ &= 5\,000 \times 0,15 \times 1 \\ &= 750,00 \end{aligned}$$

This amount must be subtracted from the payment of R1 491,58 to have the amount of capital paid off with this payment. Thus, an amount of only R(1 491,58 – 750) = R741,58 was paid off.

The outstanding balance at the beginning of year 2 is thus R(5 000 – 741,58) = 4 258,42.

Note: Before the total yearly payment can be deducted from the outstanding balance, 15% interest is to be paid on the above amount:
Interest due = Prt = 4 258,42 × 15% = 638,76

Therefore, the actual amount that can be deducted from the loan is R(1 491,58 – 638,76) = R852,82. The outstanding principal at the beginning of year 3 is thus R(4 258,42 – 852,82) = R3 405,60.

The interest due is 15% × 3 405,60 = R510,84. The loan is therefore reduced by an amount of R(1 491,58 – 510,84) = R980,74. The outstanding amount at the beginning of year 4 is
R(3 405,60 – 980,74) = R2 424,86.

The interest due is 15% of R2 424,86 = R363,73 while the amount by which the loan is reduced is R(1 491,58 – 363,73) = R1 127,85. The outstanding balance at the beginning of year 5 is

R(2 424,86 – 1 127,85) = R1 297,01.

The interest of 15% of R1 297,01 amounts to R194,55. If we deduct this amount from the yearly payment of R1 491,58, it becomes R1 297,03, which is 2 cent more than the outstanding principal at the beginning of year 5. This is due to rounding errors.

The information above may be represented in a **amortisation schedule** as follows:

Payments made (year)	**Outstanding principal (balance in rand)**	**Interest (rand)**	**Payment (rand)**	**Principal repaid (rand)**
	5 000,00			
1	4 258,42	750,00	1 491,58	741,58
2	3 405,60	638,76	1 491,58	852,82
3	2 424,86	510,84	1 491,58	980,74
4	1 297,01	363,73	1 491,58	1 127,85
5	–0,02	194,55	1 491,58	1 297,03

Note: Drawing up an amortisation table can be a tedious process, therefore we would rather use our financial calculator to assist us with these calculations.

For example, to determine the outstanding balance at the beginning of year 4, we will do the following:

For using the financial keys on the calculator, see Appendix B (7.22).

First, we determine the yearly payment by using the financial calculator keys. For the whole procedure, see Appendix B (7.22).

See Appendix B (7.22).

Exercise 7.4

1. Mr and Mrs Smart bought a house for R750 000. They obtained a mortgage loan at 11,6% per year, compounded monthly, with a term of 20 years. Determine their monthly payments.
2. Draw up an amortisation schedule for a loan of R12 000 with interest at 10% per year, compounded yearly, and a term of 5 years.

† (See page 185 for the solutions.)

† (See Appendix B, A(12) and A(13) for the solutions by using the financial keys on the calculator.)

Appendix A

Solutions to exercises

Chapter 1. Numbers and variables

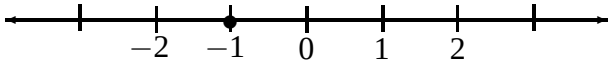
Exercise 1.1:1(a)

-1 is a negative integer (or a negative real number).

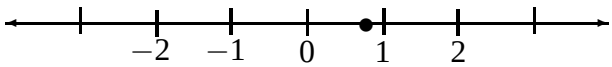
Exercise 1.1:1(b)

$\frac{3}{4}$ is a positive real number.

Exercise 1.1:2

-1 : 

number line

$\frac{3}{4}$: 

number line

Exercise 1.2:1(a)

4,578

Exercise 1.2:1(b)

0,399

Exercise 1.2:1(c)

10,000

Exercise 1.2:2(a)

7,97

Exercise 1.2:2(b)

10,50

Exercise 1.2:2(c)

9,00

Exercise 1.2:3(a)

12

Exercise 1.2:3(b)

10

Exercise 1.2:3(c)

0

Exercise 1.3(a)

Take the unknown value to be x (or any other variable).

Subtract 4 from an unknown value: $x-4$

Multiply this difference by 2: $2\times(x-4)$ or $2(x-4)$

Exercise 1.3(b)

$$\begin{aligned}\text{Total cost} &= \text{Number of loaves} \times \text{price per loaf}\\ c &= q\times p\\ &= pq\end{aligned}$$

Exercise 1.3(c)

Let

$$\begin{aligned}TCT_m &= \text{Total cooking time in minutes}\\ TCT_h &= \text{Total cooking time in hours}\\ W &= \text{Weight in kilogram}\end{aligned}$$

Exercise 1.3(c)

(i)

$$\begin{aligned} TCT_m &= 30 \times W + 15 \quad \left(\tfrac{1}{4} \times 60 = 15\right) \\ &= 30W + 15 \end{aligned}$$

Exercise 1.3(c)

(ii)

$$\begin{aligned} TCT_h &= \tfrac{30}{60} \times W + \tfrac{1}{4} \\ &= \tfrac{1}{2}W + \tfrac{1}{4} \end{aligned}$$

Exercise 1.3(d)

It is given that

$$\begin{aligned} x &= \text{distance in kilometre} \\ p &= \text{price per litre in rand} \end{aligned}$$

(i)

For 10 km, your car uses 1 litre.
Thus, for 1 km, your car uses $\frac{1}{10}$ litre,
and for x km, your car uses $\frac{1}{10} \times x$ litre
$= \frac{x}{10}$ litre.
The amount of petrol used to travel x km is $\frac{x}{10}$ litres.

Exercise 1.3(d)

(ii)
Let $c =$ total petrol cost

$$\begin{aligned} \text{Petrol cost} &= \text{Amount of petrol} \times \text{price} \\ c &= \tfrac{x}{10} \times p \\ &= \tfrac{xp}{10} \end{aligned}$$

Your total petrol cost will be $\frac{xp}{10}$ rand.

Exercise 1.3(e)

$$\begin{aligned} \text{Total cost of meal} &= \text{Number of adults} \times (\text{menu price per adult} + \\ &\quad \text{fixed charge}) + \text{Number of children} \times \\ &\quad (\text{menu price per child}) \\ &= x \times (95 + 10) + y \times 50 \\ &= 105x + 50y \end{aligned}$$

Chapter 2. Basic operations on numbers and variables

Exercise 2.1:1(a)

$$\begin{array}{rcrcl} 7+(3+2) & = & 7+(5) & = & 12 \\ (7+3)+2 & = & (10)+2 & = & 12 \end{array}$$

The statement is true.

Exercise 2.1:1(b)

$$\begin{array}{rcrcl} 6+(-5+3) & = & 6+(-2) & = & 4 \\ (6+-5)+3 & = & (1)+3 & = & 4 \end{array}$$

The statement is true.

Exercise 2.1:2(a)

$8-5=3$

Exercise 2.1:2(b)

$-8-(-5)=-3$

Exercise 2.2:1(a)

$$\begin{array}{rcrcrcl} -7\times(2\times-3) & = & -7\times(-6) & = & +42 & = & 42 \\ (-7\times2)\times-3 & = & (-14)\times-3 & = & +42 & = & 42 \end{array}$$

The statement is true.

Exercise 2.2:1(b)

$$\begin{array}{rcrcl} 12\div(-4\div2) & = & 12\div(-2) & = & -6 \\ (12\div-4)\div2 & = & -3\div2 & = & -\frac{3}{2} \end{array}$$

The statement is not true.

Exercise 2.2:2

$8\div-2=-4$.

Exercise 2.3:1

$43-13+1=30+1=31$

Exercise 2.3:2

$6+16-7=22-7=15$

Exercise 2.3:3

$$\begin{aligned} & 66+3-30-4+12 \\ = \; & 69-30-4+12 \\ = \; & 39-4+12 \\ = \; & 35+12 \\ = \; & 47 \end{aligned}$$

Exercise 2.3:4

$22 \times 8 \div 4 = 176 \div 4 = 44$

Exercise 2.3:5

$460 \div 5 \times 4 = 92 \times 4 = 368$

Exercise 2.3:6

$$\begin{aligned} & 200 \div 25 \times 8 \times 3 \div 4 \\ = \; & 8 \times 8 \times 3 \div 4 \\ = \; & 64 \times 3 \div 4 \\ = \; & 192 \div 4 \\ = \; & 48 \end{aligned}$$

Exercise 2.4(a)

$4b + b = (4+1)b = 5b$

Exercise 2.4(b)

$8y - 5y = (8-5)y = 3y$

Exercise 2.4(c)

$2x^2 + 4x^2 = (2+4)x^2 = 6x^2$

Exercise 2.4(d)

$7b^2 - 4b = 7b^2 - 4b$ (b^2 and b are not like terms.)

Exercise 2.4(e)

$20ab - 5ab = (20-5)ab = 15ab$

Exercise 2.4(f)

$$\begin{aligned} & 5x^2 + 2x + 3x^2 + 5x \\ = \; & 5x^2 + 3x^2 + 2x + 5x \quad \text{(Group like terms together.)} \\ = \; & 8x^2 + 7x \end{aligned}$$

Exercise 2.4(g)

$$\begin{aligned} & 7x+2y-4x \\ = \; & 7x-4x+2y \quad \text{(Group like terms together.)} \\ = \; & 3x+2y \end{aligned}$$

Exercise 2.4(h)

$$\begin{aligned} & x^2+y^2-3x^2-5y^2+xy \\ = \; & x^2-3x^2+y^2-5y^2+xy \\ = \; & (1-3)x^2+(1-5)y^2+xy \\ = \; & -2x^2-4y^2+xy \end{aligned}$$

Exercise 2.4(i)

$$\begin{aligned} & x^3y^3-3xy^3+5x^3y^3-y^3x-y^3x^3 \\ = \; & x^3y^3-3xy^3+5x^3y^3-xy^3-x^3y^3 \quad \text{(Rearrange factors alphabetically inside terms.)} \\ = \; & x^3y^3+5x^3y^3-x^3y^3-3xy^3-xy^3 \quad \text{(Group like terms together.)} \\ = \; & 5x^3y^3-4xy^3 \end{aligned}$$

Exercise 2.4(j)

$$\begin{aligned} & 12wx+5-3wy+4yw+xw+wx+3 \\ = \; & 12wx+5-3wy+4wy+wx+wx+3 \quad \text{(Rearrange factors alphabetically inside terms.)} \\ = \; & 12wx+wx+wx-3wy+4wy+5+3 \quad \text{(Group like terms together.)} \\ = \; & 14wx+wy+8 \end{aligned}$$

Exercise 2.5(a)

$6x \times 2x = (6 \times 2) \times (x \times x) = 12 \times x^2 = 12x^2$

Exercise 2.5(b)

$$\begin{aligned} 3xy \times 2xy = (3 \times 2) \times (xy \times xy) &= 6 \times (x \times x \times y \times y) \\ &= 6 \times (x^2 \times y^2) \\ &= 6x^2y^2 \end{aligned}$$

Exercise 2.5(c)

$$\begin{aligned} 3x^2 \times 3x = (3 \times 3) \times (x^2 \times x) &= 9 \times (x \times x \times x) \\ &= 9x^3 \end{aligned}$$

Exercise 2.5(d)

$$\begin{aligned} 2ab \times 4ac = 8 \times (ab \times ac) &= 8(a \times a \times b \times c) \\ &= 8a^2bc \end{aligned}$$

Exercise 2.5(e)

$xy \times y = (1 \times 1) \times (x \times y \times y) = 1xy^2 = xy^2$

Exercise 2.5(f)

$3lp \times p = (3 \times 1) \times (l \times p \times p) = 3lp^2$

Exercise 2.5(g)

$17ab \times 10b = (17 \times 10) \times (a \times b \times b) = 170ab^2$

Exercise 2.5(h)

$$\begin{aligned} -4kl \times -5kp &= (-4 \times -5) \times (k \times l \times k \times p) \\ &= +20 \times (k^2lp) \\ &= 20k^2lp \end{aligned}$$

Exercise 2.5(i)

$14x - 2cd$ is already in its simplest form.

Exercise 2.5(j)

$4ac(-5ad) = -20a^2cd$

Exercise 2.6(a)

$$\begin{aligned} & 56x^3y^3 \div 8x^2y^2 \\ = & \frac{\overset{7}{\cancel{56}} \times \cancel{x} \times x \times x \times \cancel{y} \times y \times \cancel{y} \times y}{\underset{1}{\cancel{8}} \times \cancel{x} \times x \times \cancel{y} \times \cancel{y}} \\ = & \frac{7 \times x \times y}{1} \\ = & 7xy \end{aligned}$$

Exercise 2.6(b)

$$\begin{aligned} & -81ac \div 3ad \\ = & \frac{\overset{-27}{\cancel{-81}}\cancel{a}c}{\underset{1}{\cancel{3}}\cancel{a}d} \\ = & -\frac{27c}{d} \end{aligned}$$

Exercise 2.6(c)

$$
\begin{aligned}
& 32p^2 \div -64p \\
= & \frac{\overset{1}{\cancel{32}}\cancel{p} \times p}{\underset{-2}{\cancel{-64}}\cancel{p}} \\
= & -\frac{p}{2}
\end{aligned}
$$

Exercise 2.6(d)

$$
\begin{aligned}
& -10cd \div -d \\
= & \frac{\cancel{-10c}\cancel{d}}{\cancel{-d}} \\
= & \ 10c
\end{aligned}
$$

Exercise 2.6(e)

$$
\begin{aligned}
& 12xy \div 12xy \\
= & \frac{\overset{1}{\cancel{12}}\cancel{xy}}{\underset{1}{\cancel{12}}\cancel{xy}} \\
= & \ 1 \qquad \text{Note: The answer is 1 and not 0.}
\end{aligned}
$$

Exercise 2.7:1(a)

Base: 6
Exponent: 4

Exercise 2.7:1(b)

Base: 3
Exponent: 2

Exercise 2.7:1(c)

Base: 10
Exponent: 8

Exercise 2.7:1(d)

Base: 206
Exponent: 0

Exercise 2.7:2(a)

$5^4 = 5 \times 5 \times 5 \times 5 = 625$

Exercise 2.7:2(b)

$10^5 = 10 \times 10 \times 10 \times 10 \times 10 = 100\,000$

Exercise 2.7:2(c)

$3,1^3 = 3,1 \times 3,1 \times 3,1 = 29,791$

Exercise 2.7:2(d)

$206^0 = 1$

Exercise 2.8(a)

$3^{-1} = \frac{1}{3}$

Exercise 2.8(b)

$5^{-3} = \frac{1}{5^3} = \frac{1}{125}$

Exercise 2.8(c)

$5^{-1} \times 5^2 = \frac{1}{5} \times (5 \times 5) = \frac{25}{5} = 5$
(Check your answer with a calculator.)

Exercise 2.9(a)

$b^4 \times b^2 = b^{4+2} = b^6$

Exercise 2.9(b)

$2x^5 \times 3x^4 = 6x^{5+4} = 6x^9$

Exercise 2.9(c)

$a^2b^3 \times a^4b^5 = a^{2+4} \times b^{3+5} = a^6 \times b^8 = a^6b^8$

Exercise 2.9(d)

$2xy \times 4x^2y^3 = 8x^{1+2}y^{1+3} = 8x^3y^4$

Exercise 2.9(e)

$2ab \times 4a^2 \times 3ab^2 = 24a^{1+2+1}b^{1+2} = 24a^4b^3$

Exercise 2.9(f)

$\frac{x^6}{x^3} = x^{6-3} = x^3$

Exercise 2.9(g)

$\frac{15a^7}{5a^2} = 3a^{7-2} = 3a^5$

Exercise 2.9(h)

$6cd^2e^2 \div 3cde^2 = 2c^{1-1}d^{2-1}e^{2-2} = 2c^0d^1e^0 = 2d$

Exercise 2.9(i)

$\left(x^4\right)^2 = 1^{1\times2}x^{4\times2} = 1^2x^8 = x^8$

Exercise 2.9(j)

$\left(3c^4\right)^2 = 3^2\left(c^4\right)^2 = 9c^8$

Exercise 2.9(k)

$\left(4a^2b^3\right)^3 = 4^{1\times3}a^{2\times3}b^{3\times3} = 4^3a^6b^9 = 64a^6b^9$

Exercise 2.9(l)

$\left(3xy^2z^3\right)^4 = 3^{1\times4}x^{1\times4}y^{2\times4}z^{3\times4} = 81x^4y^8z^{12}$

Exercise 2.9(m)

$\frac{3xy^3}{9x^2y^2} = \frac{1y^{3-2}}{3x^{2-1}} = \frac{y}{3x}$

Exercise 2.10

You can use the calculator to get to the answer directly, but we will also show you how to do it without a calculator.

Exercise 2.10(a)

$\sqrt{16} = \sqrt{4\times4} = 4$

Calculator steps: [2ndF] [$\sqrt{\ }$] 16 [=]

Exercise 2.10(b)

$\sqrt{25} = \sqrt{5\times5} = 5$

Calculator steps: [2ndF] [$\sqrt{\ }$] 25 [=]

Exercise 2.10(c)

$\sqrt[4]{16} = \sqrt[4]{2\times2\times2\times2} = 2$

Calculator steps: 4 [2ndF] [$\sqrt[x]{\ }$] 16 [=]

Exercise 2.10(d)

$\sqrt[3]{125} = \sqrt[3]{5\times5\times5} = 5$

Calculator steps: 3 [2ndF] [$\sqrt[x]{\ }$] 125 [=]

Exercise 2.10:1(a)

$\sqrt{196}+\sqrt{144}=\sqrt{14\times14}+\sqrt{12\times12}=14+12=26$

Calculator steps: 2ndF $\sqrt{\ }$ 196 $+$ 2ndF $\sqrt{\ }$ 144 $=$

Exercise 2.10:1(b)

$\left(\sqrt{5}\right)^2=\sqrt{5}\times\sqrt{5}=\sqrt{5\times5}=5$

Exercise 2.10:1(c)

$\left(\sqrt{64}\right)^2=\sqrt{64}\times\sqrt{64}=8\times8=64$

Exercise 2.10:1(d)

$\left(\sqrt{1}\right)^2=\sqrt{1}\times\sqrt{1}=1\times1=1$

Exercise 2.11

Exercise 2.11(a)

$$\begin{aligned} & \sqrt{16b^8c^{16}} \\ = \ & \sqrt{(4b^4c^8)\times(4b^4c^8)} \text{ Note that: } b^4\times b^4=b^{4+4}=b^8 \text{ and} \\ = \ & 4b^4c^8 \qquad\qquad\qquad\qquad\qquad\qquad c^8\times c^8=c^{8+8}=c^{16} \end{aligned}$$

Exercise 2.11(b)

$$\begin{aligned} & \sqrt{36x^2y^{64}} \\ = \ & \sqrt{(6xy^{32})\times(6xy^{32})} \\ = \ & 6xy^{32} \end{aligned}$$

Exercise 2.11(c)

$$\begin{aligned} & \sqrt{25x^{36}y^{100}} \\ = \ & \sqrt{(5x^{18}y^{50})\times(5x^{18}y^{50})} \\ = \ & 5x^{18}y^{50} \end{aligned}$$

Exercise 2.11(d)

$$\begin{aligned} & \sqrt{36x^{36}} \\ = \ & \sqrt{(6x^{18})\times(6x^{18})} \\ = \ & 6x^{18} \end{aligned}$$

Exercise 2.11(e)

$$\begin{aligned} & \sqrt{16a^{36}b^{16}} \\ = \ & \sqrt{(4a^{18}b^8)\times(4a^{18}b^8)} \\ = \ & 4a^{18}b^8 \end{aligned}$$

Exercise 2.11(f)

$$\begin{aligned} & \sqrt{100p^{14}q^{100}} \\ = \ & \sqrt{(10p^7q^{50})\times(10p^7q^{50})} \\ = \ & 10p^7q^{50} \end{aligned}$$

Exercise 2.11(g)

$$\begin{aligned} & \sqrt{81m^8n^4} \\ = \ & \sqrt{(9m^4n^2)\times(9m^4n^2)} \\ = \ & 9m^4n^2 \end{aligned}$$

Exercise 2.12(a)

There are no brackets, powers or roots, therefore we will start with $\times$ and $\div$ from left to right:

$$\begin{aligned} & 4\times3\div2+5\times6-20\div4\times2 \\ = \ & 12\div2+30-5\times2 \\ = \ & 6+30-10 \quad (\text{Now } + \text{ and } -) \\ = \ & 26. \end{aligned}$$

Therefore, $4\times3\div2+5\times6-20\div4\times2=26$.

Exercise 2.12(b)

$3^2\times7-\sqrt{9}\times8+2^3\div4\times3$

First, do the powers and roots:

$9\times7-3\times8+8\div4\times3$

Then do $\times$ and $\div$:

$$\begin{aligned} & 63-24+2\times3 \\ = \ & 63-24+6. \end{aligned}$$

Lastly, do $+$ and $-$:

$$\begin{aligned} & 39+6 \\ = \ & 45 \end{aligned}$$

Therefore, $3^2\times7-\sqrt{9}\times8+2^3\div4\times3=45$.

Exercise 2.13(a)

$$\begin{aligned} & 7\times 8+8\times 9 \\ = \; & 56+72 \\ = \; & 128 \end{aligned}$$

Exercise 2.13(b)

$$\begin{aligned} & 7\times (8+8)\times 9 \\ = \; & 7\times (16)\times 9 \\ = \; & 1\,008 \end{aligned}$$

Exercise 2.13(c)

$$\begin{aligned} & (7\times 8+8)\times 9 \\ = \; & (56+8)\times 9 \\ = \; & 64\times 9 \\ = \; & 576 \end{aligned}$$

Exercise 2.13(d)

$$\begin{aligned} & 7\times (8+8\times 9) \\ = \; & 7\times (8+72) \\ = \; & 7\times 80 \\ = \; & 560 \end{aligned}$$

Exercise 2.13(e)

$$\begin{aligned} & 200+16^3\div 4 \\ = \; & 200+4\,096\div 4 \\ = \; & 200+1\,024 \\ = \; & 1\,224 \end{aligned}$$

Exercise 2.14

$$\begin{aligned} & \left(4^2-(6\times 3-80\div 5)^2\right)\div\left(9^2-77\right) \\ = \; & \left(4^2-(18-16)^2\right)\div(81-77) \\ = \; & \left(4^2-(2)^2\right)\div(4) \\ = \; & (16-4)\div 4 \\ = \; & 12\div 4 \\ = \; & 3 \end{aligned}$$

Chapter 3. More operations on numbers

Exercise 3.1(a)

Method 1.

$$\begin{aligned} 4 &= 2\times 2 \\ 6 &= 2\times 3 \\ \text{LCM} &= 2\times 2\times 3 \\ &= 12 \end{aligned}$$

Method 2.
Multiples of 4 are
4; 8; (12); 16; 20; ... etc.
Multiples of 6 are
6; (12); 18; 24; ... etc.
Therefore, the LCM of 4 and 6 is 12.

Exercise 3.1(b)

Method 1.

$$\begin{aligned} 6 &= 2\times 3 \\ 16 &= 2\times 2\times 2\times 2 \\ \text{LCM} &= 2\times 3\times 2\times 2\times 2 \\ &= 48 \end{aligned}$$

Method 2.
Multiples of 6 are
6; 12; 18; 24; 30; 36; 42; (48); ... etc.
Multiples of 16 are
16; 32; (48); 64; ... etc.
Therefore, the LCM of 6 and 16 is 48.

Exercise 3.1(c)

Method 1.

$$\begin{aligned} 5 &= 5 \\ 30 &= 5\times 2\times 3 \\ \text{LCM} &= 2\times 3\times 5 \\ &= 30 \end{aligned}$$

Method 2.
Multiples of 5 are
5; 10; 15; 20; 25; (30); 35; ... etc.
Multiples of 30 are
(30); 60; 90; ... etc.
Therefore, the LCM of 5 and 30 is 30.

Exercise 3.1(d)

Method 1.

$$\begin{aligned} 14 &= 2\times 7 \\ 20 &= 2\times 2\times 5 \\ \text{LCM} &= 2\times 7\times 2\times 5 \\ &= 140 \end{aligned}$$

Method 2.
Multiples of 14 are
14; 28; 42; 56; 70; 84; 98; 112; 126; (140); 154; 168; ... etc.
Multiples of 20 are
20; 40; 60; 80; 100; 120; (140); 160; 180; ... etc.
Therefore, the LCM of 14 and 20 is 140.

Exercise 3.2:1(a)

$\frac{8}{3} = 2\frac{2}{3}$

Exercise 3.2:1(b)

$\frac{9}{4} = 2\frac{1}{4}$

Exercise 3.2:1(c)

$\frac{14}{5} = 2\frac{4}{5}$

Exercise 3.2:1(d)

$\frac{41}{12} = 3\frac{5}{12}$

Exercise 3.2:2(a)

$3\frac{1}{3} = \frac{3(3)+1}{3} = \frac{10}{3}$ OR $3\frac{1}{3} = 3 + \frac{1}{3} = \frac{9}{3} + \frac{1}{3} = \frac{10}{3}$

Exercise 3.2:2(b)

$8\frac{3}{7} = \frac{8(7)+3}{7} = \frac{59}{7}$ OR $8\frac{3}{7} = 8 + \frac{3}{7} = \frac{56}{7} + \frac{3}{7} = \frac{59}{7}$

Exercise 3.2:2(c)

$7\frac{4}{5} = \frac{7(5)+4}{5} = \frac{39}{5}$ OR $7\frac{4}{5} = 7 + \frac{4}{5} = \frac{35}{5} + \frac{4}{5} = \frac{39}{5}$

Exercise 3.2:2(d)

$2\frac{3}{8} = \frac{2(8)+3}{8} = \frac{19}{8}$ OR $2\frac{3}{8} = 2 + \frac{3}{8} = \frac{16}{8} + \frac{3}{8} = \frac{19}{8}$

Exercise 3.2:3(a)

$\frac{7}{14} = \frac{1\times7}{2\times7} = \frac{1}{2}$

Exercise 3.2:3(b)

$\frac{20}{15} = \frac{2\times2\times5}{3\times5} = \frac{4}{3}$

Exercise 3.2:3(c)

$\frac{8}{12} = \frac{2\cdot2\cdot2}{2\cdot2\cdot3} = \frac{2}{3}$

Exercise 3.2:3(d)

$\frac{14}{16} = \frac{2\cdot7}{2\cdot2\cdot2\cdot2} = \frac{7}{8}$

Exercise 3.3:1(a)

$\frac{3}{4} \times \frac{1}{3} = \frac{1}{4}$

Exercise 3.3:1(b)

$\frac{3}{4} \div \frac{1}{3} = \frac{3}{4} \times \frac{3}{1} = \frac{9}{4}$

Exercise 3.3:1(c)

$$\begin{aligned}\tfrac{3}{7} \div \tfrac{9}{21} &= \tfrac{3}{7} \times \tfrac{21}{9} \\ &= \tfrac{3}{7} \times \tfrac{3\times 7}{3\times 3} \\ &= 1\end{aligned}$$

Exercise 3.3:1(d)

$$\begin{aligned}&\tfrac{5}{6} \times \tfrac{1}{4} \div \tfrac{5}{12} \\ =\ &\tfrac{5}{6} \times \tfrac{1}{4} \times \tfrac{12}{5} \\ =\ &\tfrac{5}{2\times 3} \times \tfrac{1}{2\times 2} \times \tfrac{2\times 2\times 3}{5} \\ =\ &\tfrac{1}{2}\end{aligned}$$

Exercise 3.3:1(e)

$$\begin{aligned}&\tfrac{5}{6} \div \tfrac{1}{12} \times \tfrac{3}{15} \\ =\ &\tfrac{5}{2\times 3} \times \tfrac{2\times 2\times 3}{1} \times \tfrac{3}{3\times 5} \\ =\ &2\end{aligned}$$

Exercise 3.3:1(f)

$$\begin{aligned}&\tfrac{7}{24} \div \tfrac{14}{32} \times \tfrac{3}{16} \\ =\ &\tfrac{7}{2\times 2\times 2\times 3} \times \tfrac{2\times 2\times 2\times 2\times 2}{2\times 7} \times \tfrac{3}{2\times 2\times 2\times 2} \\ =\ &\tfrac{1}{8}\end{aligned}$$

Exercise 3.4:1(a)

$$\begin{aligned}&\tfrac{1}{4} + \tfrac{3}{4} \\ =\ &\tfrac{1+3}{4} \\ =\ &\tfrac{4}{4} \\ =\ &1\end{aligned}$$

Exercise 3.4:1(b)

$$\begin{aligned}&\tfrac{1}{5} + \tfrac{2}{3} \\ =\ &\tfrac{1(3)+2(5)}{15} \qquad \text{LCM} = 5\times 3 = 15 \\ =\ &\tfrac{3+10}{15} \\ =\ &\tfrac{13}{15}\end{aligned}$$

Exercise 3.4:1(c)

$$\begin{aligned} & \tfrac{2}{3} - \tfrac{1}{6} \\ = \ & \tfrac{2(2)-1(1)}{6} \qquad \text{LCM} = 3 \times 2 = 6 \\ = \ & \tfrac{4-1}{6} \\ = \ & \tfrac{1}{2} \end{aligned}$$

Exercise 3.4:1(d)

$$\begin{aligned} & \tfrac{3}{7} - \tfrac{13}{28} \\ = \ & \tfrac{3(4)-13}{28} \qquad \text{LCM} = 7 \times 4 = 28 \\ = \ & \tfrac{12-13}{28} \\ = \ & -\tfrac{1}{28} \end{aligned}$$

Exercise 3.4:1(e)

$$\begin{aligned} & \tfrac{5}{6} - \tfrac{3}{4} + \tfrac{7}{24} \\ = \ & \tfrac{5(4)-3(6)+7}{24} \qquad \text{LCM} = 2 \times 3 \times 2 \times 2 = 24 \\ = \ & \tfrac{20-18+7}{24} \\ = \ & \tfrac{2+7}{24} \\ = \ & \tfrac{9}{24} \\ = \ & \tfrac{3 \times 3}{2 \times 2 \times 2 \times 3} \\ = \ & \tfrac{3}{8} \end{aligned}$$

Exercise 3.4:1(f)

$$\begin{aligned} & \tfrac{6}{11} + \tfrac{3}{2} - \tfrac{5}{3} \\ = \ & \tfrac{6(6)+3(33)-5(22)}{66} \qquad \text{LCM} = 11 \times 2 \times 3 = 66 \\ = \ & \tfrac{36+99-110}{66} \\ = \ & \tfrac{135-110}{66} \\ = \ & \tfrac{25}{66} \end{aligned}$$

Exercise 3.4:1(g)

$$
\begin{aligned}
& 1\tfrac{1}{3}+4\tfrac{3}{4} \\
= \; & 1+4+\tfrac{1(4)+3(3)}{12} \qquad \text{LCM} = 3\times 2\times 2 = 12 \\
= \; & 1+4+\tfrac{13}{12} \\
= \; & 1+4+1\tfrac{1}{12} \\
= \; & 6\tfrac{1}{12}
\end{aligned}
$$

Exercise 3.5:1(a)

$\frac{2}{3} = 0,66\dot{}$

$$
\begin{array}{r|l}
 & 0,66\dot{} \\
\hline
3 & 2,00 \\
 & \underline{1\,8} \qquad (3\times 6 = 18) \\
 & 20 \\
 & \underline{18} \\
 & 2
\end{array}
$$

Exercise 3.5:1(b)

$\frac{3}{5} = 0,6$

$$
\begin{array}{r|l}
 & 0,6 \\
\hline
5 & 3,0 \\
 & \underline{3\,0} \qquad (6\times 5 = 30) \\
 & 0
\end{array}
$$

Exercise 3.5:1(c)

$\frac{5}{6} = 0,83\dot{3}$

$$
\begin{array}{r|l}
 & 0,83\dot{3} \\
\hline
6 & 5,000 \\
 & \underline{48} \qquad\qquad (6\times 8 = 48) \\
 & 20 \qquad\quad \text{and} \quad (3\times 6 = 18) \\
 & \underline{18} \\
 & 20 \\
 & \ddots
\end{array}
$$

Exercise 3.5:1(d)

$3\frac{3}{4} = 3,75$

$$
\begin{array}{r|l}
 & 0,75 \\
\hline
4 & 3,00 \\
 & \underline{2\,8} \qquad (7\times 4 = 28) \\
 & 20 \\
 & \underline{20} \qquad (5\times 4 = 20) \\
 & 0
\end{array}
$$

Exercise 3.5:2(a)

$$\begin{aligned} 0,25 &= \frac{25}{100} \\ &= \frac{5\times5}{2\times2\times5\times5} \\ &= \frac{1}{4} \end{aligned}$$

Exercise 3.5:2(b)

$$0,3 = \frac{3}{10}$$

Exercise 3.5:2(c)

$$\begin{aligned} 1,45 &= 1\frac{45}{100} \\ &= 1\frac{3\times3\times5}{2\times2\times5\times5} \\ &= 1\frac{9}{20} \end{aligned}$$

Exercise 3.5:2(d)

$$\begin{aligned} 2,6 &= 2\frac{6}{10} \\ &= 2\frac{2\times3}{2\times5} \\ &= 2\frac{3}{5} \end{aligned}$$

Exercise 3.6:1

$$\begin{aligned} \text{Rotten apples} &= \frac{4}{100}\times\frac{125}{1} \\ &= \frac{4}{2\times2\times5\times5}\times\frac{5\times5\times5}{1} \\ &= 5 \end{aligned}$$

Sera can sell 120 apples $(125-5)$.

Exercise 3.6:2

Percentage income tax paid

$$\begin{aligned} &= \frac{3150}{9000}\times100 \\ &= 35\% \end{aligned}$$

Peter pays 35% of his salary to income tax.

Exercise 3.6:3

$$\begin{aligned} \text{Percentage DVD player owners} &= \frac{4150}{11275}\times\frac{100}{1} \\ &= 36,81\% \end{aligned}$$

The percentage of families who own a DVD player are 36,81.

Exercise 3.6:4

$$\begin{aligned} & 114\% \text{ of the price is R216,60} \\ \text{Thus,} \quad & 1\% \text{ of the price is } \frac{\text{R}216,60}{114} \\ \text{and} \quad & 100\% \text{ of the price is } \frac{\text{R}216,60}{114} \times \frac{100}{1} = \text{R}190,00 \end{aligned}$$

The price of the pair of shoes was R190,00.

Exercise 3.6:5

The price of a loaf of bread is R5,12.

$$\begin{aligned} \text{Increase in price} &= \frac{9,5}{100} \times \frac{5,12}{1} \\ &= 0,4864 \end{aligned}$$

Therefore the price of a loaf of bread increased by 49 cents.

The new price of a loaf of bread is $\text{R}(5,12+0,49) = \text{R}5,61$.

Exercise 3.7:1

The total number of parts in this ratio is $5+3=8$.

$$\begin{aligned} \text{Therefore, the number of red Smarties} &= \frac{5}{8} \text{ of } 72 \\ &= \frac{5}{8} \times \frac{72}{1} \\ &= 45 \end{aligned}$$

The number of red Smarties is 45.

Exercise 3.7:2(a)

The total number of parts in this ratio is $6+4+1=11$.

The percentage stone is $\frac{6}{11} \times \frac{100}{1} = 54,55\%$

Exercise 3.7:2(b)

$$\begin{aligned} \text{The percentage sand} &= \frac{4}{11} \times \frac{100}{1} \\ &= 36,36\% \end{aligned}$$

Exercise 3.7:2(c)

$$\begin{aligned} \text{The percentage cement} &= \frac{1}{11} \times \frac{100}{1} \\ &= 9,09\% \end{aligned}$$

Chapter 4. Measurements

Exercise 4.1(a)

400 m to km:

We know that 1 000 m = 1 km.
Therefore

$$400\,\text{m} = \frac{400}{1\,000}\,\text{km} = 0,4\,\text{km}.$$

400 m is equal to 0,4 km.

Exercise 4.1(b)

18 cm to mm:

We know that $1\,\text{cm} = 10\,\text{mm}$.
Therefore

$$18\,\text{cm} = 18 \times 10\,\text{mm} = 180\,\text{mm}$$

18 cm is equal to 180 mm.

Exercise 4.1(c)

4 450 m to km:

We know that $1\,000\,\text{m} = 1\,\text{km}$.
Therefore

$$4\,450\,\text{m} = \frac{4\,450}{1\,000}\,\text{km} = 4,45\,\text{km}$$

4 450 m is equal to 4,45 km.

Exercise 4.1(d)

5 km to m:
We know that $1\,\text{km} = 1\,000\,\text{m}$.
Therefore

$$5\,\text{km} = 5 \times 1\,000\,\text{m} = 5\,000\,\text{m}$$

5 km is equal to 5 000 m.

Exercise 4.1(e)

12,4 m to cm:
We know that $1\,\text{m} = 100\,\text{cm}$.
Therefore

$$12,4\,\text{m} = 12,4 \times 100\,\text{cm} = 1\,240\,\text{cm}$$

12,4 m is equal to 1 240 cm

Exercise 4.1(f)

5 246,3 m to km:

We know that $1\,000\,\text{m} = 1\,\text{km}$.
Therefore

$$5\,246,3\,\text{m} = \frac{5\,246,3}{1\,000}\,\text{km} = 5,2463\,\text{km}$$

5 246,3 m is equal to 5,2463 km.

Exercise 4.1(g)

a km to cm:
We know that $1\,\text{km} = 1\,000\,\text{m} = (1\,000 \times 100)\,\text{cm} = 100\,000\,\text{cm}$.
Therefore

$$a \times 1\,\text{km} = a \times 1\,000\,\text{m} = 1\,000a\,\text{m} = 1\,000a \times 100\,\text{cm} = 100\,000a\,\text{cm}.$$

a km is equal to $100\,000a$ cm.

Exercise 4.1(h)

246 cm to m:
We know that $100\,\text{cm} = 1\,\text{m}$.
Therefore

$$246\,\text{cm} = 246 \div 100\,\text{m} = 2,46\,\text{m}.$$

246 cm is equal to 2,46 m.

Exercise 4.1(i)

246 cm to km:
We know that $1\,\text{km} = 1\,000\,\text{m} = (1\,000 \times 100)\,\text{cm} = 100\,000\,\text{cm}$.
Therefore

$$246\,\text{cm} = 246 \div 100\,000\,\text{km} = 0,002\,46\,\text{km}.$$

246 cm is equal to 0,00246 km

Exercise 4.1(j)

(50 m + 0,5 km) to m:
We know that $1\,\text{km} = 1\,000\,\text{m}$.
Therefore

$$(50\,\text{m} + 0,5\,\text{km}) = 50\,\text{m} + (0,5 \times 1\,000)\,\text{m} = 50\,\text{m} + 500\,\text{m} = 550\,\text{m}$$

$(50\,\text{m} + 0,5\,\text{km})$ is equal to 550 m.

Exercise 4.2:1(a)

Given: A rectangle with $l = 1,2$ km and $w = 375$ m.
Convert the width also to kilometre by dividing it by 1 000:
$375 \div 1000 = 0,375$

The perimeter is calculated as

$$\begin{aligned} C &= 2(l+w) \\ &= 2(1,2+0,375) \\ &= 2(1,575) \\ &= 3,15 \end{aligned}$$

The perimeter of the rectangle is 3,15 km.

Exercise 4.2:1(b)

Given: A square with $l = 3$ cm.
The perimeter is calculated as

$$\begin{aligned} C &= 4l \\ &= 4 \times 3 \\ &= 12 \end{aligned}$$

The perimeter of the square is 12 cm.

Exercise 4.2:1(c)

Given: A triangle with sides of 30 mm, 25 mm and 15 mm.
The perimeter is calculated as

$$\begin{aligned} C &= 30+25+15 \\ &= 70 \end{aligned}$$

The perimeter of the triangle is 70 mm.

Exercise 4.2:1(d)

Given: A circle with $r = 4,5$ m.
The circumference is calculated as

$$\begin{aligned} C &= 2\pi r \\ &= 2 \times \pi \times 4,5 \\ &\approx 28,27 \end{aligned}$$

The **circumference of the circle** is 28,27 m.

Exercise 4.2:1(e)

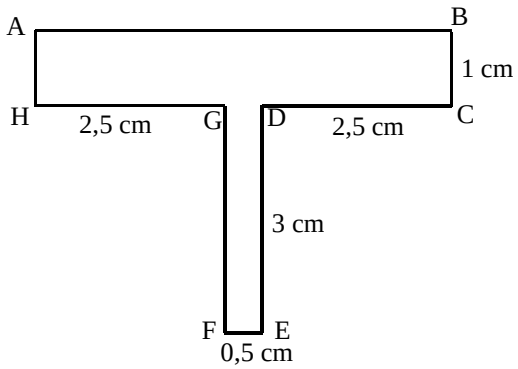
Given: A figure with 4 sides of length 12 m, 35 m, 7 m, and 13 m.
The perimeter is calculated as

$$\begin{aligned} C &= 12+35+7+13 \\ &= 67 \end{aligned}$$

The perimeter of the figure is 67 m.

Exercise 4.2:2

Given: The following figure:



We see that the length of AB is equal to the sum of the lengths of CD, EF and GH. Therefore,

$$\begin{aligned} \text{Length of AB} &= \text{Length of CD} + \text{Length of EF} + \text{Length of GH} \\ &= 2,5+0,5+2,5 \\ &= 5,5 \end{aligned}$$

The length of AB is thus 5,5 cm.

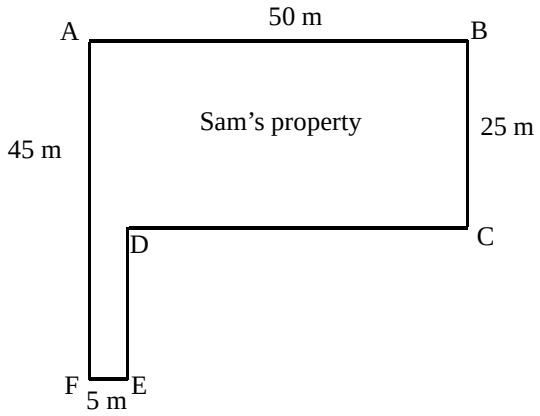
The perimeter is calculated as

$$\begin{aligned} C &= AB+BC+CD+DE+EF+FG+GH+HA \\ &= 5,5+1+2,5+3+0,5+3+2,5+1 \\ &= 19 \end{aligned}$$

The perimeter of the figure is 19 cm.

Exercise 4.2:3

Given: The following figure:



First calculate the lengths of CD and DE:

$$\begin{aligned} \text{Length of CD} &= \text{Length of AB} - \text{Length of EF} \\ &= 50 - 5 \\ &= 45 \end{aligned}$$

The length of CD is 45 m.

$$\begin{aligned} \text{Length of DE} &= \text{Length of AF} - \text{Length of BC} \\ &= 45 - 25 \\ &= 20 \end{aligned}$$

The length of DE is 20 m.

The perimeter is calculated as

$$\begin{aligned} C &= AB + BC + CD + DE + EF + FA \\ &= 50 + 25 + 45 + 20 + 5 + 45 \\ &= 190. \end{aligned}$$

Sam needs 190 m of fencing.

Exercise 4.3:1

We know that $1\,000\,\text{m} = 1\,\text{km}$ and $1\,\text{ha} = 100\,\text{m} \times 100\,\text{m} = 10\,000\,\text{m}^2$. Therefore

$$\begin{aligned} 1\,\text{km}^2 &= 1\,000\,\text{m} \times 1\,000\,\text{m} \\ &= 1\,000\,000\,\text{m}^2 \\ &= \frac{1\,000\,000}{10\,000}\ \text{hectare} \\ &= 100\ \text{hectare}. \end{aligned}$$

There are 100 hectare in one km^2.

Exercise 4.3:2

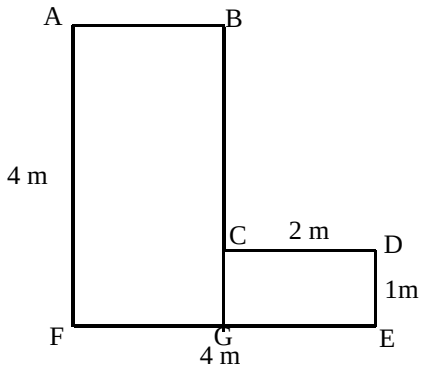
Given: A rectangle with $l = 24\,\text{cm}$ and $w = 18\,\text{cm}$.
The area is calculated as

$$\begin{aligned} A &= l \times w \\ &= 24 \times 18 \\ &= 432 \end{aligned}$$

The area of the rectangle is $432\,\text{cm}^2$.

Exercise 4.3:3

Given: The following figure:



We can subdivide the figure into two rectangles ABGF and CDEG to make calculations easier.

First calculate the length of AB:

$$\begin{aligned} \text{Length of AB} &= \text{Length of EF} - \text{Length of CD} \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The length of AB is 2 m.

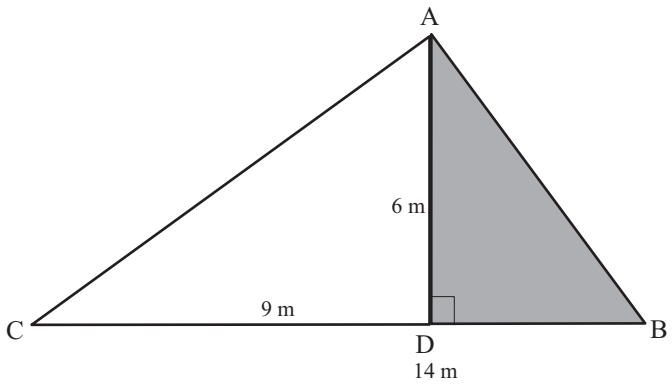
The area is a composite area of two rectangles and it is calculated as

$$\begin{aligned} A &= \text{Area ABGF} + \text{Area CDEG} \\ &= (\text{AB} \times \text{BG}) + (\text{CD} \times \text{DE}) \\ &= (2 \times 4) + (2 \times 1) \\ &= 8 + 2 \\ &= 10 \end{aligned}$$

The area of the figure is $10\,\text{m}^2$.

Exercise 4.4:1

Given: The following figure:



Calculate the length of BD.

$$\begin{aligned} \text{Length of BD} &= \text{Length of BC} - \text{Length of CD} \\ &= 14 - 9 \\ &= 5 \end{aligned}$$

The length of BD is 5 m.

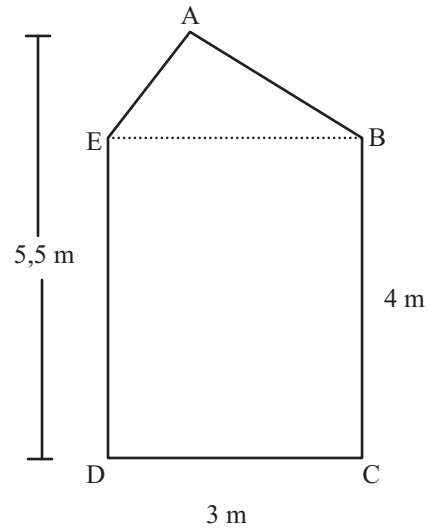
The area is calculated as

$$\begin{aligned} A &= \text{Area ABD} \\ &= \frac{1}{2}bh \\ &= \frac{1}{2}\text{BD} \times \text{AD} \\ &= \frac{1}{2} \times 5 \times 6 \\ &= 15. \end{aligned}$$

The area of the shaded figure is $15\,\text{m}^2$.

Exercise 4.4:2

Given: The following figure:



The area is a composite area of one rectangle and one triangle.

First calculate the perpendicular height of the triangle ABE:

$$\begin{aligned} h &= \text{Total height} - \text{Length of BC} \\ &= 5,5 - 4 \\ &= 1,5 \end{aligned}$$

The perpendicular height of the triangle is 1,5 m.

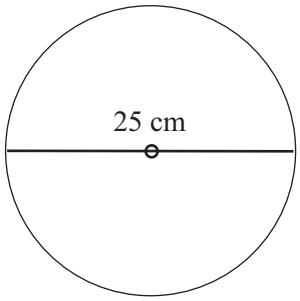
The area is calculated as

$$\begin{aligned} A &= \text{Area ABE} + \text{Area BCDE} \\ &= \frac{1}{2}bh + (l \times w) \\ &= \frac{1}{2}\text{BE} \times h + (\text{BC} \times \text{CD}) \\ &= (\frac{1}{2} \times 3 \times 1,5) + (4 \times 3) \\ &= 2,25 + 12 \\ &= 14,25. \end{aligned}$$

The total area of the figure is 14,25 m^2.

Exercise 4.5:1

Given: The following figure:



The diameter of the circle is given as 25 cm. Therefore, the radius is

$$\begin{aligned} r &= d \div 2 \\ &= 25 \div 2 \\ &= 12,5. \end{aligned}$$

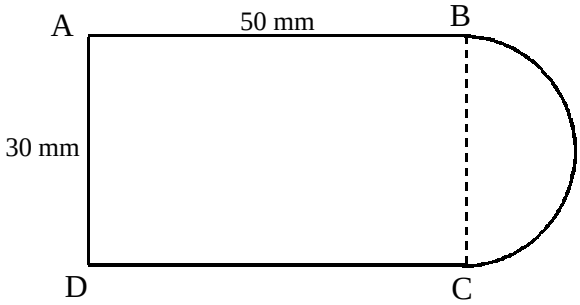
The area is calculated as

$$\begin{aligned} A &= \pi r^2 \\ &= \pi(12,5)^2 \\ &= \pi \times 156,25 \\ &\approx 490,87. \qquad \text{(using the calculator)} \end{aligned}$$

The area of the circle is 490,87 cm^2.

Exercise 4.5:2

Given: The following figure:



The area is a composite area of one rectangle and one half-circle, also called a semi-circle.

First, calculate the radius of the half-circle:

$$\begin{aligned} r &= d \div 2 \\ &= 30 \div 2 \\ &= 15 \end{aligned}$$

The radius of the half-circle is thus 15 mm.

The area of the figure is calculated as

$$\begin{aligned} A &= \text{Area ABCD} + \text{Area of half-circle} \\ &= (l \times w) + \frac{1}{2}\pi r^2 \\ &= (\text{AB} \times \text{AD}) + \frac{1}{2}(\pi r^2) \\ &= (50 \times 30) + \frac{1}{2}(\pi \times (15)^2) \\ &= 1\,500 + 353,43 \\ &= 1\,853,4292 \end{aligned}$$

The total area of the figure is $1\,853,43\,\text{mm}^2$.

Exercise 4.6:1

We know that there are $1\,000\,\text{cm}^3$ in one litre.

Therefore, in half a litre, we have $(1\,000 \div 2) = 500\,\text{cm}^3$.

Exercise 4.6:2

The dam is in the form of a cylinder with diameter 5 m and perpendicular height of 1,5 m. The volume is calculated as

$$\begin{aligned} V &= \pi \text{r}^2 h \\ &= \pi \times (5 \div 2)^2 \times 1,5 \\ &= \pi \times 9,375 \\ &\approx 29,4524 \end{aligned}$$

The volume of the dam is $29,452\,\text{m}^3$.

We know that $1\,\text{m}^3 = 1\,000$ litres.
Therefore, the volume of the dam is $(29.452 \times 1\,000)\ell = 29\,452\,\ell$.

Exercise 4.6:3

Given: A prism of size $12\,\text{cm} \times 9\,\text{cm} \times 4\,\text{cm}$.
The volume is calculated as:

$$\begin{aligned} V &= l \times b \times h \\ &= 12 \times 9 \times 4 \\ &= 432 \end{aligned}$$

The volume of the prism is $432\,\text{cm}^3$.

Exercise 4.7:1

The total area of the walls is a composite of **four** rectangles, two sets of two equal-sized rectangles. We can calculate it as

$$\begin{aligned} A &= 2(l \times h) + 2(b \times h) \\ &= 2(4,3 \times 2,5\,\text{m}) + 2(3,4 \times 2,5) \\ &= 21,5 + 17 \\ &= 38,5 \end{aligned}$$

The total area of the walls is 38,5 m^2.

Each roll of wallpaper covers $4,5\,\text{m}^2$.

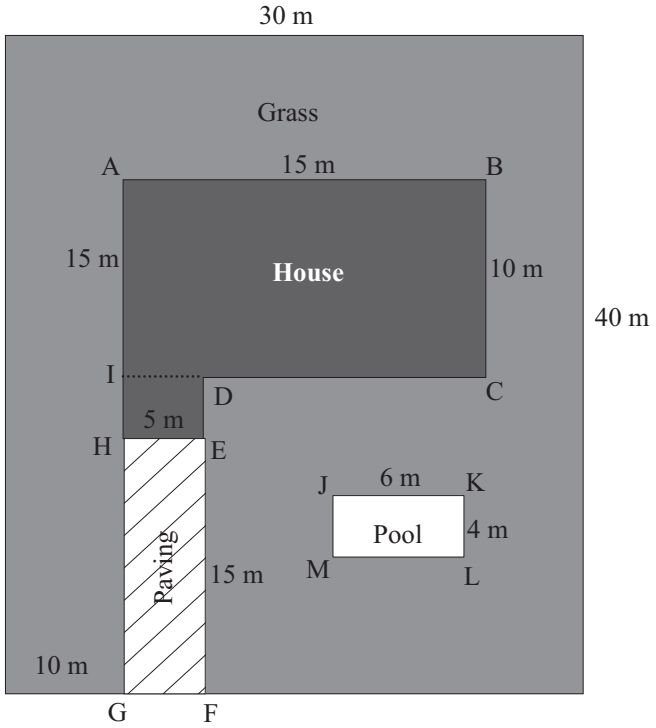
Therefore, the number of rolls needed is as follows:

$$\begin{aligned} \text{Number of rolls} &= \frac{38,5}{4,5} \\ &= 8,56 \end{aligned}$$

The number of wallpaper rolls needed is 8,56, but as we can buy only whole rolls, we round it off to 9 rolls.

Exercise 4.7:2

The area to be covered with grass must be calculated. The easiest is to calculate the area of the stand and then subtract the area of the house, paving and pool.



The area of the stand is:

$$\begin{aligned} A &= l \times w \\ &= 40 \times 30 \\ &= 1\,200 \end{aligned}$$

The area of the stand is $1\,200\,\text{m}^2$.

The area of the house is area ABCI plus area DEHI.

$$\begin{aligned} \text{Area ABCI} &= \text{AB} \times \text{BC} \qquad \text{BC} = \text{AI} = 10 \\ &= 15 \times 10 \\ &= 150 \end{aligned}$$

$$\begin{aligned} \text{Area DEHI} &= \text{DE} \times \text{HE} \qquad \text{DE} = \text{AH} - \text{AI} \\ &= 5 \times 5 \qquad\quad = 15 - 10 \\ &= 25 \qquad\quad = 5 \\ &= 25 \end{aligned}$$

$$\begin{aligned} \text{The area of the house} &= \text{ABCI} + \text{DEHI} \\ &= 150 + 25 \\ &= 175 \end{aligned}$$

The area of the house is $175\,\text{m}^2$.

$$\begin{aligned} \text{The area of the pool} &= \text{Area of JKLM} \\ &= 6 \times 4 \\ &= 24 \end{aligned}$$

The area of the pool is thus $24\,\text{m}^2$.

$$\begin{aligned} \text{The area of the paving} &= \text{Area of EFGH} \\ &= 15 \times 5 \\ &= 75 \end{aligned}$$

The area of the paving is thus $75\,\text{m}^2$.

$$\begin{aligned} &\text{The total area that will have to be planted with grass} \\ &= \text{Area of stand} - \text{Area of house} - \text{Area of pool} - \text{Area of paving} \\ &= 1\,200 - 175 - 24 - 75 \\ &= 926 \end{aligned}$$

The total area that will have to be planted with grass is thus $926\,m^2$.

$$\begin{aligned} \text{The } \textbf{cost} \text{ to plant with kikuyu grass} &= 15 \times 926 \\ &= 13\,890 \end{aligned}$$

It will cost you R13 890 to plant $926\,m^2$ of kikuyu grass.

$$\begin{aligned} \text{The cost to plant with LM grass} &= 20 \times 926 \\ &= 18\,520 \end{aligned}$$

It will cost you R18 520 to plant $926\,m^2$ of LM grass.

Your budget allows you to spend R15 000, therefore kikuyu grass will be your choice.

Chapter 5. Expressions, equations and formulas

Exercise 5.1(a)

To solve for x subtract 4 on both sides and simplify:

$$\begin{aligned} x+4 &= 12 \\ x+4-4 &= 12-4 \\ x &= 8. \end{aligned}$$

The value of x is 8.

Exercise 5.1(b)

To solve for x subtract 6 on both sides and simplify:

$$\begin{aligned} 6+x &= 2 \\ 6+x-6 &= 2-6 \\ x &= -4 \end{aligned}$$

The value of x is -4.

Exercise 5.1(c)

To solve for x add 7 on both sides and simplify:

$$\begin{aligned} x-7 &= -3 \\ x-7+7 &= -3+7 \\ x &= 4 \end{aligned}$$

The value of x is 4.

Exercise 5.1(d)

To solve for x divide by 5 on both sides and simplify:

$$\begin{aligned} 5x &= 23 \\ \frac{5x}{5} &= \frac{23}{5} \\ x &= 4\frac{3}{5} \\ x &= 4,6 \end{aligned}$$

The value of x is 4,6.

Exercise 5.1(e)

To solve for x multiply by 4 on both sides and simplify:

$$\begin{aligned} \frac{x}{4} &= 10 \\ \frac{x}{4} \times 4 &= 10 \times 4 \\ x &= 40 \end{aligned}$$

The value of x is 40.

Exercise 5.2(a)

To solve for x remove the number *furthest* from the x first and simplify:

$$\begin{aligned} 2x+3 &= 6 \\ 2x+3-3 &= 6-3 \\ 2x &= 3 \\ \frac{2x}{2} &= \frac{3}{2} \\ x &= \frac{3}{2} \end{aligned}$$

The value of x is 5.

Check your answer:

$$\text{RHS} = 6$$

$$\begin{aligned} \text{LHS}: \quad 2\left(\frac{3}{2}\right)+3 &= \frac{2}{1} \times \frac{3}{2} + 3 \\ &= 3+3 \\ &= 6 \end{aligned}$$

The RHS is equal to the LHS, therefore the answer of $\frac{3}{2}$ is correct.

Exercise 5.2(b)

Remove -4, which is the furthest from x, by adding 4 on both sides and simplify:

$$\begin{aligned} 6x-4 &= 26 \\ 6x-4+4 &= 26+4 \\ 6x &= 30 \end{aligned}$$

Divide by 6 on both sides and simplify:

$$\begin{aligned} \frac{6x}{6} &= \frac{30}{6} \\ x &= 5 \end{aligned}$$

The value of x is 5.

Check the answer:

$$\begin{aligned} \text{RHS} &= 26 \\ \text{LHS} &= 6x-4 \\ &= 6(5)-4 \\ &= 30-4 \\ &= 26 \end{aligned}$$

The RHS is equal to the LHS, therefore the answer of 5 is correct.

Exercise 5.2(c)

To solve for x: Subtract 4 on both sides, simplify, divide by 5 on both sides and simplify:

$$\begin{aligned} -5x+4 &= 19 \\ -5x+4-4 &= 19-4 \\ -5x &= 15 \\ \frac{-5x}{5} &= \frac{15}{5} \\ -x &= 3 \\ x &= -3 \qquad \text{(Multiply or divide both sides by } -1.) \end{aligned}$$

Check the answer:

$$\begin{aligned} \text{RHS} &= 19 \\ \text{LHS} &= -5x+4 \\ &= -5(-3)+4 \\ &= 15+4 \\ &= 19 \end{aligned}$$

The RHS is equal to the LHS, therefore the answer of -3 is correct.

Exercise 5.2(d)

Add 2 on both sides, simplify, multiply by 7 on both sides and simplify to solve for x:

$$\begin{aligned} \frac{x}{7}-2 &= 9 \\ \frac{x}{7}-2+2 &= 9+2 \\ \frac{x}{7} &= 11 \\ \frac{x}{7}\times\frac{7}{1} &= 11\times 7 \\ x &= 77 \end{aligned}$$

Check the answer:

$$\begin{aligned} \text{RHS} &= 9 \\ \text{LHS} &= \frac{x}{7}-2 \\ &= \frac{77}{7}-2 \\ &= 11-2 \\ &= 9 \end{aligned}$$

The RHS is equal to the LHS, therefore the answer of 77 is correct.

Exercise 5.3(a)

$$5(3-x) = 4(5+5x)$$

Remove the brackets by multiplying 4 by both terms inside the bracket, to get

$$15-5x = 20+20x$$

Subtract 15 on both sides of the equation.

$$15-15-5x = 20-15+20x$$

Simplify to get

$$-5x = 5+20x$$

Subtract $20x$ on both sides of the equation and simplify:

$$\begin{aligned} -5x-20x &= 5+20x-20x \\ -25x &= 5 \end{aligned}$$

Divide by -25 on both sides and simplify:

$$\begin{aligned}\frac{-25x}{-25} &= \frac{5}{-25}\\ x &= -\frac{1}{5}\\ x &= -0,2\end{aligned}$$

The value of $x = -0,2$ or $-\frac{1}{5}$.

Exercise 5.3(b)

$$6a+3+1 = 4a-8-3a$$

Add the like terms on both sides.

$$6a+4 = a-8$$

Subtract 4 on both sides and simplify.

$$\begin{aligned}6a+4-4 &= a-8-4\\ 6a &= a-12\end{aligned}$$

Subtract a on both sides and simplify.

$$\begin{aligned}6a-a &= a-a-12\\ 5a &= -12\end{aligned}$$

Divide by 5 on both sides and simplify.

$$\begin{aligned}\frac{5a}{5} &= -\frac{12}{5}\\ a &= -\frac{12}{5}\end{aligned}$$

The value of a is $-\frac{12}{5}$ or $-2,4$.

Exercise 5.3(c)

$$5x+1-2x = 9-x-4$$

Simplify the like terms on both sides.

$$3x+1 = 5-x$$

Subtract 1 on both sides and simplify.

$$\begin{aligned}3x+1-1 &= 5-1-x\\ 3x &= 4-x\end{aligned}$$

Add x on both sides and simplify.

$$\begin{aligned}3x+x &= 4-x+x\\ 4x &= 4\\ x &= 1\end{aligned}$$

The value of x is 1.

Exercise 5.3(d)

$$5x-3(x-(3x-4)) = 2(5x-3(1-2x))+5$$

Remove the inner brackets first and simplify.

$$\begin{aligned} 5x-3(x-3x+4) &= 2(5x-3+6x)+5 \\ 5x-3x+9x-12 &= 10x-6+12x+5 \\ 11x-12 &= 22x-1 \end{aligned}$$

Add 12 on both sides and simplify.

$$\begin{aligned} 11x-12+12 &= 22x-1+12 \\ 11x &= 22x+11 \end{aligned}$$

Subtract $22x$ on both sides and simplify.

$$\begin{aligned} 11x-22x &= 22x-22x+11 \\ -11x &= 11 \\ x &= -1 \end{aligned}$$

The value of x is -1.

Exercise 5.4

(a) $\sqrt{xy}-z=w$

To change the subject to y, we first add z on both sides of the equation and simplify.

$$\begin{aligned} \sqrt{xy}-z+z &= w+z \\ \sqrt{xy} &= w+z \end{aligned}$$

To eliminate the $\sqrt{\ }$ we square both sides and simplify.

$$\begin{aligned} \left(\sqrt{xy}\right)^2 &= (w+z)^2 \\ xy &= (w+z)^2 \end{aligned}$$

Divide by x on both sides.

$$y = \frac{(w+z)^2}{x} \qquad \text{(Note that } (w+z)^2 \neq w^2+z^2\text{.)}$$

(b) $q = \frac{4r}{t^2}$

To change the subject to t, we start by getting rid of the fraction.

We multiply by t^2 on both sides and simplify.

$$\begin{aligned} q \times t^2 &= \frac{4r}{t^2} \times t^2 \\ qt^2 &= 4r \end{aligned}$$

To get t alone on the left, we divide by q on both sides and simplify.

$$\begin{aligned} \frac{qt^2}{q} &= \frac{4r}{q} \\ t^2 &= \frac{4r}{q} \end{aligned}$$

To eliminate the power, we take the square root on both sides and simplify.

$$\begin{aligned}\sqrt{t^2} &= \sqrt{\tfrac{4r}{q}}\\ t &= \sqrt{\tfrac{4r}{q}}\end{aligned}$$

(c) $\ell = (mk)^2$

To change the subject to k, we first get rid of the square on the right, by taking the square root on both sides of the equation.

$\sqrt{\ell} = mk$

We switch the left-hand and right-hand sides around, to have k on the left-hand side.

$mk = \sqrt{\ell}$

Then we divide both sides by m.

$$\begin{aligned}\tfrac{mk}{m} &= \tfrac{\sqrt{\ell}}{m}\\ k &= \tfrac{\sqrt{\ell}}{m}.\end{aligned}$$

(d) $r^2 = x^2+y^2$

To change the subject to r, we take the square root on both sides.

$r = \sqrt{x^2+y^2}$

(e) $r^2 = x^2+y^2$

To change the subject to x, we switch the left-hand and right-hand sides around.

$x^2+y^2 = r^2$

Then we subtract y^2 on both sides.

$x^2 = r^2-y^2$

Then we take the square root on both sides to make x the subject.

$x = \sqrt{r^2-y^2}$

(f) **Note:** The formulas in (f) and (g) are the formulas used in simple interest calculations, where I is the interest, P the principal amount invested or paid, r the interest rate and t the time period.

$I = Prt$

To change the subject to r, we switch the left-hand and right-hand sides around.

$Prt = I$

Then we divide by Pt on both sides and simplify.

$$\begin{aligned}\tfrac{Prt}{Pt} &= \tfrac{I}{Pt}\\ r &= \tfrac{I}{Pt}\end{aligned}$$

(g) $S = P(1+rt)$

To change the subject to t, we first remove the brackets on the right, by multiplying P by both terms inside the bracket.

$S = P+Prt$

Then we subtract P on both sides, simplify and switch the left-hand and right-hand sides around.

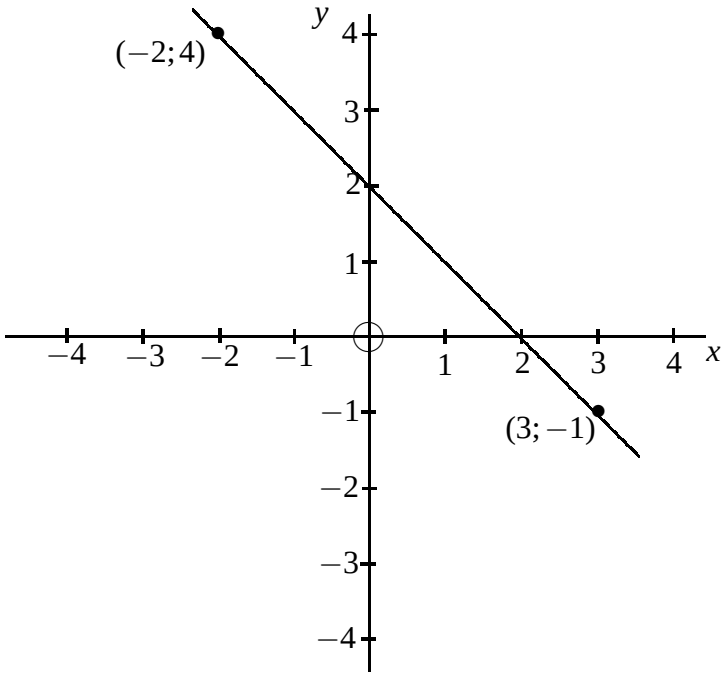
$Prt = S-P$

Divide by Pr on both sides.

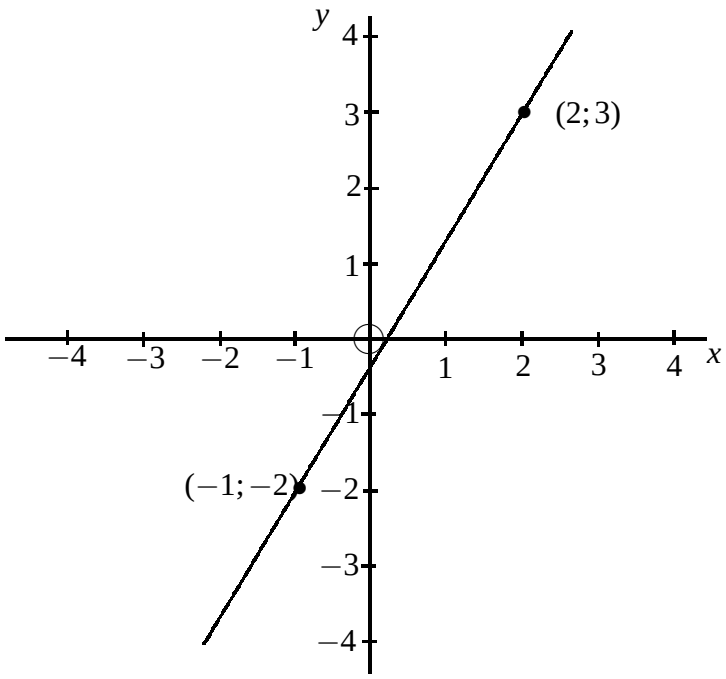
$t = \frac{S-P}{Pr}$

Chapter 6. Straight lines

Exercise 6.1:1



Exercise 6.1:2



Exercise 6.1:3

$$\begin{aligned}
\frac{y-y_1}{x-x_1} &= \frac{y_2-y_1}{x_2-x_1} \\
\frac{y-3}{x-2} &= \frac{-2-3}{-2-2} \\
\frac{y-3}{x-2} &= \frac{-5}{-4} \\
y-3 &= \frac{5}{4}(x-2) \\
y &= \frac{5}{4}x-\frac{5}{4}\cdot\frac{2}{1} \\
y &= \frac{5}{4}x-\frac{10}{4}+3 \\
&= \frac{5}{4}x-\frac{10}{4}+\frac{12}{4} \\
&= \frac{5}{4}x+\frac{2}{4} \\
&= \frac{5}{4}x+\frac{1}{2} \quad \text{(in ordinary fraction form)} \\
&= 1,25x+0,5 \quad \text{(in decimal form)}
\end{aligned}$$

Exercise 6.1:4

$$\begin{aligned}
\frac{y-y_1}{x-x_1} &= \frac{y_2-y_1}{x_2-x_1} \\
\frac{y-(-2)}{x-(-1)} &= \frac{2-(-2)}{2-(-1)} \\
\frac{y+2}{x+1} &= \frac{2+2}{2+1} \\
y+2 &= \frac{4}{3}(x+1) \\
y &= \frac{4}{3}x+\frac{4}{3}-2 \\
&= \frac{4}{3}x+\frac{4-2(3)}{3} \\
&= \frac{4}{3}x+\frac{4-6}{3} \\
&= \frac{4}{3}x-\frac{2}{3} \\
&= 1,333x-0,667 \quad \text{(rounded to three decimal digits)} \\
&= 1,33x-0,67 \quad \text{(rounded to two decimal digits)}
\end{aligned}$$

Exercise 6.1:5

$$\begin{array}{lllllll}
 & & y & = & -x+2 & & \\
 & \text{that is} & y & = & -1x+2 & & \\
\text{If } x = 0, \text{ then} & & y & = & -1(0)+2 & = & 2. \\
\text{If } x = 1, \text{ then} & & y & = & -1(1)+2 & = & 1. \\
\text{If } x = 2, \text{ then} & & y & = & -1(2)+2 & = & 0.
\end{array}$$

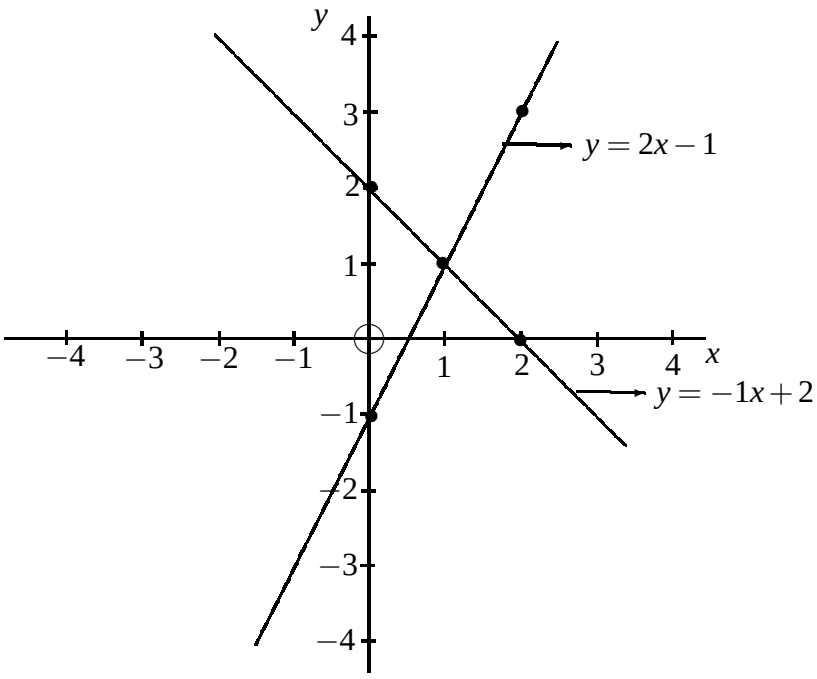
x	0	1	2
y	2	1	0

Points $(0;2)$, $(1;1)$ and $(2;0)$ lie on $y=-x+2$.

$$\begin{array}{lllllll}
 & & y & = & 2x-1 & & \\
\text{If } x = 0, \text{ then} & & y & = & 2(0)-1 & = & -1. \\
\text{If } x = 1, \text{ then} & & y & = & 2(1)-1 & = & 1. \\
\text{If } x = 2, \text{ then} & & y & = & 2(2)-1 & = & 3.
\end{array}$$

x	0	1	2
y	-1	1	3

Points $(0;-1)$, $(1;1)$ and $(2;3)$ lie on $y=2x-1$.



Chapter 7. Basic financial calculations

Exercise 7.1:1

By using the Simple Interest Formula:

$$\begin{aligned} S &= P(1+rt) \\ S &= 2\,000\,(1+8\%\times 7) \\ &= 2\,000\,(1+0,08\times 7) \qquad 8\% = \tfrac{8}{100} = 0,08 \\ &= 3\,120,00 \end{aligned}$$

See Appendix B (A.1) for the method to enter this calculation on your calculator.

James must pay Tim R3 120 after 7 years.

Exercise 7.1:2

By using the Simple Interest Formula:

$$\begin{aligned} S &= P(1+rt) \\ 6\,400 &= P(1+15\%\times 4) \\ 6\,400 &= P(1+0,15\times 4) \qquad 15\% = \tfrac{15}{100} = 0,15 \\ P &= \tfrac{6\,400}{1+0,15\times 4} \\ &= 4\,000,00 \end{aligned}$$

See Appendix B (A.2).

Four years ago, Debbie borrowed R4 000 from Douglas.

Exercise 7.2:1

By substituting into the Compound Interest Formula:

$$\begin{aligned} S &= P(1+i)^n \\ &= 10\,000\left(1+\tfrac{7{,}5\%}{12}\right)^{3\times 12} \\ &= 10\,000\left(1+\tfrac{0{,}075}{12}\right)^{3\times 12} \\ &= 12\,514{,}4614 \end{aligned}$$

For using the financial keys on the calculator, see Appendix B (A.3). See Appendix B (A.3).

Sam owes R12 514,46 after 3 years.

Exercise 7.2:2

By substituting into the Compound Interest Formula:

$$\begin{aligned} S &= P(1+i)^n \\ 12\,000 &= P\left(1+\tfrac{11{,}5\%}{52}\right)^{10\times 52} \\ 12\,000 &= P\left(1+\tfrac{0{,}115}{52}\right)^{10\times 52} \\ P &= \frac{12\,000}{\left(1+\frac{0{,}115}{52}\right)^{10\times 52}} \\ &= 3\,804{,}4689 \end{aligned}$$

For using the financial keys on the calculator, see Appendix B (A.4). See Appendix B (A.4).

Marcus borrowed R3 804,47 from Mika 4 years ago.

Exercise 7.2:3

By using the Compound Interest Formula:

$$\begin{aligned} S &= P(1+i)^n \\ 24\,832{,}78 &= 10\,000\left(1+\tfrac{9{,}2\%}{4}\right)^{n} \\ n &= 40{,}0000 \end{aligned}$$

That is 40 quarters, which are 10 years, because the number of compounding periods per year is 4. Jill will receive R24 832,78 after 10 years.

For using the financial keys on the calculator, see Appendix B (A.5). See Appendix B (A.5).

Exercise 7.2:4

By using the Compound Interest Formula:

$$\begin{aligned} S &= P(1+i)^n \\ 45\,000 &= 15\,000(1+i)^{10} \\ \tfrac{45\,000}{15\,000} &= (1+i)^{10} \\ \sqrt[10]{3} &= 1+i \\ i &= \sqrt[10]{3}-1 \\ i &= 0,1161 \end{aligned}$$

Therefore, $\frac{\text{interest rate}}{100} = 0,1161$, which means the interest rate = 11,61%.

See Appendix B (A.6). **For using the financial keys on the calculator, see Appendix B (A.6).**

Abel invested R15 000 for 10 years at an interest rate of 11,61% per year.

Exercise 7.3:1

By using the present value formula

$$\begin{aligned} S &= R\left[\tfrac{(1+i)^n-1}{i}\right] \\ &= 6\,000\left[\tfrac{(1+0,1)^5-1}{0,1}\right] \\ &= 36\,630,6000 \end{aligned}$$

See Appendix B (A.7). **For using the financial keys on the calculator, see Appendix B (A.7).**

The future value of R6 000 payments for 5 years is R36 630,00.

Exercise 7.3:2

By using the future value formula

$$\begin{aligned} S &= R\left[\frac{(1+i)^n-1}{(i+i)^n \cdot i}\right] \\ &= 800\left[\frac{(1+\frac{0,16}{2})^{2\times 10}-1}{(1+\frac{0,16}{2})^{2\times 10}\,.\,\frac{0,16}{2}}\right] \\ &= 7\,854,5179 \end{aligned}$$

See Appendix B (A.8). **For using the financial keys on the calculator, see Appendix B (A.8).**

The present value of an annuity with R800 payments made very 6 months for 10 years, equals R7 854,52.

Exercise 7.3:3

By using the present value formula

$$\begin{aligned} S &= R\left[\frac{(1+i)^n-1}{(i+i)^n \cdot i}\right] \\ &= 200\left[\frac{(1+\frac{0{,}08}{12})^{12\times 5}-1}{(1+\frac{0{,}08}{12})^{12\times 5}\cdot\frac{0{,}08}{12}}\right] \\ &= 9\,863{,}6867 \end{aligned}$$

For using the financial keys on the calculator, see Appendix B (A.9). See Appendix B (A.9).

The present value of an annuity, with monthly payments of R200 each made for 5 years, is R9 863,69.

By using the future value formula

$$\begin{aligned} S &= R\left[\frac{(1+i)^n-1}{i}\right] \\ &= 200\left[\frac{(1+\frac{0{,}08}{12})^{12\times 5}-1}{\frac{0{,}08}{12}}\right] \\ &= 14\,695{,}3712 \end{aligned}$$

For using the financial keys on the calculator, see Appendix B (A.10). See Appendix B (A.10).

The future value of an annuity of R200 monthly payments made for 5 years is R14 695,37.

Exercise 7.3:4

By using the future value formula

$$\begin{aligned} S &= R\left[\frac{(1+i)^n-1}{i}\right] \\ \text{Thus,}\quad R &= S \div \left[\frac{(1+i)^n-1}{i}\right] \\ &= 25\,000 \div \left[\frac{(1+\frac{0{,}0995}{12})^{5\times 12}-1}{\frac{0{,}0995}{12}}\right] \\ &= 323{,}2696 \end{aligned}$$

For using the financial keys on the calculator, see Appendix B (A.11). See Appendix B (A.11).

Mapule needs to invest R323,27 every month for 5 years to have R25 000 available to pay for her new lounge suite.

Exercise 7.4:1

By using the present value formula

$$\begin{aligned} P &= R\left[\frac{(1+i)^n-1}{(1+i)^n i}\right] \\ R &= P \div \left[\frac{(1+i)^n-1}{(1+i)^n i}\right] \\ &= 750\,000 \div \left[\frac{\left(1+\frac{0{,}116}{12}\right)^{12\times 20}-1}{\left(1+\frac{0{,}116}{12}\right)^{240}\cdot\frac{0{,}116}{12}}\right] \\ &= 8\,049{,}9620 \end{aligned}$$

See Appendix B (A.12). **For using the financial calculator keys, see Appendix B (A:12).**

The monthly payments are R8 049,96.

See Appendix B (A.13). **Exercise 7.4:2**

Note:

Since drawing up an amortisation table step-by-step by using formulas is a very tedious process, we use the financial keys on the calculator for this process. **See Appendix B (A.13).**

Appendix B

Key operations for the calculator: Chapter 7

Calculator example 7:1

Enter into the calculator as 20 000 [(] 1 [+] 0.10 [×] 1 [)] [=]
The answer is 22 000.

Calculator example 7:2

Enter into the calculator as 20 000 [(] 1 [+] 0.10 [×] 3 [)] [=]
The answer is 26 000.

Calculator example 7:3

Enter into the calculator as 1 500 [×] 9 [÷] 100 [×] 21 [÷] 12 [=]
The answer is 236.25.

Calculator example 7:4

Enter into the calculator as

3 500 [÷] [(] 1 [+] [(] 12.5 [÷] 100 [×] 6 [)] [)] [=]

The answer is 2 000.

Calculator example 7:5

Enter into the calculator as (7500 [÷] 5000 [−] 1 [)] [÷] [(] 7.5 [÷] 100 [)] [=]

6.6667 appears on the screen.

Calculator example 7:6

Enter into the calculator as 7 500 [÷] [(] 12 500 [×] 6 [)] [=]

0.1000 appears on the screen, and that is 10%.

Calculator example 7:7

What is given?

(i) The present value, PV is R1 000.

(ii) The interest rate, I/Y, is 10,0% per year.

(iii) Interest is compounded yearly, that is once a (1 time per) year.

(iv) The time, *n*, of the investment is 3 years.

What is required?

The future value, FV, of the investment.

Enter into the calculator as

[2ndF] [CA] to clear the register of the calculator.

[2ndF] [P/Y] 1 [ENT] (Enter the number of compounding periods per year first.)

[ON/C]

[+/−] 1 000 [PV]

10 [I/Y]

3 [2ndF] [× P/Y] [N]

[COMP] [FV]

FV = 1 331.0000

Calculator example 7:8

What is given?

(i) The present value, PV, is R2 000.

(ii) The interest rate, I/Y, is 12% per year.

(iii) Interest is compounded every 6 months, that is twice a (2 times per) year.

(iv) The time, *n*, of the investment is $2\frac{1}{2}$ years.

What is required?

The future value, FV, of the investment.

Enter into the calculator as

[2ndF] [CA]

[2ndF] [P/Y] 2 [ENT]

[ON/C]

[+/−] 2 000 [PV]

12 [I/Y]

2.5 [2ndF] [× P/Y] [N]

[COMP] [FV]

FV = 2 676.4512

Calculator example 7:9

What is given?

(i) The future value, FV, of the loan is R5 000.

(ii) The interest rate, I/Y, is 16% per year.

(iii) Interest is compounded quarterly, that is, 4 times per year.

(iv) The time of the loan is 6 years.

What is required?

The present value, PV, of the loan.

Enter into the calculator as

[2ndF] [CA]

[2ndF] [P/Y] 4 [ENT]

[ON/C]

[+/−] 5 000 [FV]

16 [I/Y]

6 [2ndF] [× P/Y] [N]

[COMP] [PV]

PV = 1 950.6074

Calculator example 7:10

What is given?

(i) The present value of the investment, PV, is R5 000.

(ii) The interest rate, I/Y, is 7,5% per year.

(iii) Interest is compounded monthly, that is, 12 times per year.

(iv) The future value of the investment, FV, is R7 830,59.

What is required?

The time period, *n*, of the investment.

Enter into the calculator as

[2ndF] [CA]

[2ndF] [P/Y] 12 [ENT]

[ON/C]

[+/−] 5 000 [PV]

7 830.59 [FV]

7.5 [I/Y]

[COMP] [N]

N $= 72,0001$ appears on the screen, which is **72 months**, that is **6 years**.

Calculator example 7:11

What is given?

(i) The yearly payment, PMT, is R600.

(ii) The interest rate, I/Y, is 11,5% per year.

(iii) The time period, *n*, of the loan is 3 years.

What is required?

The future value, FV, of the annuity.

Enter into the calculator as

[2ndF] [CA]

[2ndF] [P/Y] 1 [ENT]

[ON/C]

[+/−] 600 [PMT]

11.5 [I/Y]

3 [2ndF] [× P/Y] [N]

[COMP] [FV]

FV = 2 014.9350

Calculator example 7:12

Enter into the calculator as

[2ndF] [CA]

[2ndF] [P/Y] 4 [ENT]

[ON/C]

[+/−] 1 200 [PMT]

10 [2ndF] [× P/Y] [N]

16 [I/Y]

[COMP] [FV]

FV = 114 030.6188

Calculator example 7:13

Enter into the calculator as

[2ndF] [CA]

[2ndF] [P/Y] 1 [ENT]

[ON/C]

[+/−] 600 [PMT]

3 [2ndF] [× P/Y] [N]

11.5 [I/Y]

[COMP] [PV]

PV = 1 453.5716

Calculator example 7:14

Enter into the calculator as

[2ndF] CA

[2ndF] [P/Y] 4 [ENT]

ON/C

+/− 1 500 PMT

12.6 I/Y

6 2ndF × P/Y N

COMP PV

PV = 24 997.6658

Calculator example 7:15

Enter into the calculator as

2ndF CA

2ndF P/Y 52 ENT

ON/C

+/− 160 PMT

2 2ndf × P/Y N

7.8 I/Y

COMP FV

FV = 17 993.5568

Calculator example 7:16

Enter into the calculator as

2ndF CA

2ndF P/Y 12 ENT

ON/C

+/− 48 000 FV

3 2ndF × P/Y N

8.4 I/Y

COMP PMT

PMT = 1 177.0186

Calculator example 7:17

Enter into the calculator as

2ndF CA

2ndF P/Y 1 ENT

ON/C

+/− 55 000 FV

9 500 PMT

7.34 I/Y

COMP N

N = 4,9997 years.

Calculator example 7:18

Enter into the calculator as

2ndF CA

2ndF P/Y 12 ENT

ON/C

+/− 12 500 PMT

12.3 I/Y

5 2ndF × P/Y N

COMP PV

PV = 558 126.7996

Calculator example 7:19

Enter into the calculator as

2ndF CA

2ndF P/Y 4 ENT

ON/C

+/− 120 000 PV

6 2ndF × P/Y N

13.6 I/Y

COMP PMT

PMT = 7 394.4704

Calculator example 7:20

Enter into the calculator as

[2ndF] [CA]

[2ndF] [P/Y] 12 [ENT]

[ON/C]

[+/−] 50 000 [PV]

10 [2ndF] [× P/Y] [N]

15 [I/Y]

[COMP] [PMT]

PMT = 806.6748

Calculator example 7:21

Enter into the calculator as

[2ndF] [CA]

[2ndF] [P/Y] 1 [ENT]

[ON/C]

[+/−] 5 000 [PV]

5 [2ndF] [× P/Y] [N]

15 [I/Y]

[COMP] [PMT]

PMT = 1 491.5776

Calculator example 7:22

Enter into the calculator as

[2ndF] [CA]

[2ndF] [P/Y] 1 [ENT]

[ON/C]

[+/−] 5 000 [PV]

5 [2ndF] [× P/Y] [N]

15 [I/Y]

[COMP] [PMT]

PMT = 1 491.5776

Now press [AMRT] (4th row, 1st key) 4 [ENT].

Press [▼] 4 [ENT]

Press [▼]

BALANCE = −1 297.01 appears on the screen, meaning that an amount of R1 297,01 is still outstanding after 4 payments.

Press [▼] again.

∑ PRINCIPAL = 1 127,85 is the amount paid off.

Press [▼] again.

∑ INTEREST = 363.73 is the amount of interest paid.

To complete the schedule press

[▼] 5 [ENT] twice and continue as above.

Calculator example A:1

Enter into the calculator as 2 000 [×] [(] 1 [+] 0,08 [×] 7 [)] [=].

3 120,0000 appears on the screen.

Calculator example A:2

Enter into the calculator as

6 400 [÷] [(] 1 [+] 0.15 [×] 4 [)] [=]

The answer is 4 000.0000

Calculator example A:3

What is given?

(i) The present value of the loan, PV, is R10 000.

(ii) The interest rate, I/Y, is 7,5% per year.

(iii) Interest is compounded **monthly**, that is, 12 times per year.

(iv) The time period of the loan, *n*, is 3 years.

What is required?

The future value, FV, of the loan is required.

Enter into the calculator as

[2ndF] [CA]

2ndF P/Y 12 ENT
ON/C
+/− 10 000 PV
7.5 I/Y
3 2ndF × P/Y N
COMP FV

FV = 12 514.4614

Calculator example A:4

What is given?

(i) The future value, FV, is R12 000.

(ii) The interest rate, I/Y, is 11,5% per year.

(iii) Interest is compounded **weekly**, that is, 52 times per year.

(iv) The time period of the loan, *n* is 6 years.

What is required?

The present value, FV, of the loan is required.

Enter into the calculator as:

2ndF CA
2ndF P/Y 52 ENT
ON/C
+/− 12 000 FV
11.5 I/Y
10 2ndF × P/Y N
COMP PV

PV = 3 804.4689

Calculator example A:5

What is given?

(i) The present value, PV, of the investment is R10 000.

(ii) The yearly interest rate, I/Y, is 9,2%.

(iii) Interest is compounded **quarterly**, that is, 4 times per year.

(iv) The future value, FV, of the investment is R24 832,78.

What is required?

The time period of the investment is required.

Enter into the calculator as

2ndF CA
2ndF P/Y 4 ENT
ON/C
+/− 10 000 PV
24 832,78 FV
9.2 I/Y
COMP N

N = 40,0000.

That is 40 **quarters**, which is $\frac{40}{4} = 10$ **years**.

Calculator example A:6

What is given?

(i) The present value, PV, of the investment is R15 000.

(ii) The future value, FV, of the investment is R45 000.

(iii) The time period of the investment is 10 years.

(iv) Interest is compounded yearly, that is, once a (1 time per) year.

What is required?

The yearly interest rate is required.

Enter into the calculator as

2ndF CA
2ndF P/Y 1 ENT
ON/C
+/− 45 000 FV
15 000 PV
10 2ndF × P/Y N
COMP I/Y
I/Y = 11.6123

Calculator example A:7

Enter into the calculator as

2ndF CA
2ndF P/Y 1 ENT
ON/C
+/− 6 000 PMT
10 I/Y

5 [2ndF] [× P/Y] [N]
[COMP] [FV]

FV = 36 630.6000

Calculator example A:8

Enter into the calculator as

[2ndF] [CA]
[2ndF] [P/Y] 2 [ENT]
[ON/C]
[+/−] 800 [PMT]
16 [I/Y]
10 [2ndF] [× P/Y] [N]
[COMP] [PV]

PV = 7 854.5179

Calculator example A:9

Enter into the calculator as

[2ndF] [CA]
[2ndF] [P/Y] 12 [ENT]
[ON/C]
[+/−] 200 [PMT]
8 [I/Y]
5 [2ndF] [× P/Y] [N]
[COMP] [PV]

PV = 9 863.6867

Calculator example A:10

Enter into the calculator as

[2ndF] [CA]
[2ndF] [P/Y] 12 [ENT]
[ON/C]
[+/−] 200 [PMT]
8 [I/Y]
5 [2ndF] [× P/Y] [N]
[COMP] [FV]

FV = 14 695.3712

Calculator example A:11

Enter into the calculator as

2ndF CA
2ndF P/Y 12 ENT
ON/C
+/− 25 000 FV
9.95 I/Y
5 2ndF × P/Y N
COMP PMT

PMT = 323.2696

Calculator example A:12

Enter into the calculator as

2ndF CA
2ndF P/Y 12 ENT
ON/C
+/− 750 000 PV
11.6 I/Y
20 2ndF × P/Y N
COMP PMT

PMT = 8 049.9620

Calculator example A:13

Note: We must always find the payment first (yearly, in this case).

Enter into the calculator as

2ndF CA
2ndF P/Y 1 ENT
ON/C
+/− 12 000 PV
10 I/Y
5 2ndF × P/Y N
COMP PMT

PMT = 3 165.5689 ≈ 3 165,57

Enter this amount into the **payment** column and keep it on the calculator.

Consequent steps to follow:

Press AMRT 1 ENT

Press [▼] 1 [ENT]

Press [▼]

BALANCE $= -10\,034.4302$ appears on the screen. The amount of R10 034,43 must be filled in under **Outstanding principal** at the beginning of year 2.

Press [▼] again.

$\sum$ PRINCIPAL $= 1\,965.5698$ appears on the screen. The amount of R1 965,57 must be filled in under **Principal repaid**.

Press [▼] again.

$\sum$ INTEREST $= 1\,200.0000$ appears on the screen. The amount of R1 200,00 must be filled in under **Interest**.

Press [▼] 2 [ENT] twice.

Press [▼]

BALANCE $= -7\,872.3034$ appears.

Press [▼]

$\sum$ PRINCIPAL $= 2\,162.1268$ appears.

Press [▼]

$\sum$ INTEREST $= 1\,003.4430$ appears.

To continue to year 3:

Press [▼] 3 [ENT] twice.

Then press [▼] for the balance.

Press [▼] for the principal.

Press [▼] for the interest.

For year 4:

Press [▼] 4 [ENT] twice.

Press [▼] to find the principal and the interest paid.

For year 5:

press [▼] 5 [ENT] twice.

Continue as above.

The balance is 0.00. The principal repaid is R2 877,79 and the interest due is R287,78.

Fill all of this data into the amortisation schedule.

The amortisation schedule is as follow:

Payments made (year)	Outstanding principal (balance in rand)	Interest (rand)	Payment (rand)	Principal repaid (rand)
	12 000,00			
1	10 034,43	1 200,00	3 165,57	1 965,57
2	7 872,30	1 003,44	3 165,57	2 162,13
3	5 493,96	787,23	3 165,57	2 378,34
4	2 877,79	549,40	3 165,57	2 616,17
5	−0,00	287,78	3 165,57	2 877,79