PHYSICS

Only study guide for
PHY2601

Classical Mechanics

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Preface

The module

The understanding of the description of the physical world in classical mechanics has contributed greatly to the development of the human society. An individual familiar with the principles and laws of classical mechanics is able to take informed decisions and make mature judgments concerning aspects of application of classical mechanics in everyday life.

The aim of the module, Classical Mechanics (PHY2601), is to

a) equip a student with the principles, laws and methods of classical mechanics,

b) assist a student in learning to recognize and interpret situations related to or requiring application of these principles, laws and methods.

Specific learning outcomes

At the end of the term for the module the qualifying student is expected to be able to :

- discuss and explain the principles, laws and methods of classical mechanics.
- derive equations of motion for a given situation using the principles and laws of classical mechanics.
- integrate the principles, laws and methods of classical mechanics to solve related problems.
- recognize and analyze situations involving the application of the principles and laws of classical mechanics.

The prescribed book

The book prescribed for this module is

Classical Mechanics, 5th Edition,
by T. W. B. Kibble and F. H. Berkshire,
Imperial College Press, 2004.

This book consists of 14 chapters and 4 appendices. The prescribed material from the book **excludes** :
- Chapter 11 – which is the subject of another physics module and
- Chapters 13 and 14 – which cover material beyond the scope of this module.
That is, chapters 11, 13 and 14 should be omitted when studying for PHY2601. Appendices A and B also form part of the prescribed material.

The book gives a *summary* at the end of the presentation of each chapter and also provides *model answers* to the questions/problems listed at the end of every chapter.

The study guide

The objective of this study guide is to assist the student in understanding the presentation of the prescribed concepts of classical mechanics in the book. The study guide is, therefore, based on, but not meant to replace, the prescribed textbook. The guide is organized as follows:

- An overview of the main points in the chapter,
- The learning objectives for the student from the chapter.
- Points for noting and useful equations in understanding the concepts presented in the chapter.
- Additional illustrative examples.

Some of the examples are taken from the references indicated below.

References

1. A. P. Arya, *Introduction to Classical Mechanics*, 2nd Edition, Prentice-Hall, Inc. (1998).
2. P. Smith and R. C. Smith, *Mechanics*, 2nd Edition, Wiley & Sons, (1990).

1 Introduction

The chapter is mainly philosophical. That is, it concentrates on the establishment of laws and definition of concepts in classical mechanics. The validity of such laws and definitions is briefly discussed with reference to the accompanying assumptions and simplifications. Note that the *Principia* is a book written by Isaac Newton.

1.1 Learning Goals

- The process of formulating the laws of classical mechanics.
- Newton's laws of motion.
- Implications of Newton's laws of motion.

1.2 Notes

Take note of the following:

- How the *laws of physics* are established and what constitutes *a universal law*.
- The *relevance of classical mechanics* in understanding other theories in physics.
- General *assumptions* made in formulating the laws of classical mechanics.
- Some *simplifications* adopted in studying dynamical systems.
- Brief *definitions/descriptions* of some concepts involved in the study of classical mechanics
- Newton's laws of motion.

Derive the equations of motion for an object moving with (a) constant speed is and (b) constant acceleration.

(a) The speed v is generally defined as the rate of change in position with respect to the change in time. In the case of a very very small change in position $\mathrm{d}x$ in a very very small change in time $\mathrm{d}t$ the speed is defined by the relation

$$v = \frac{\mathrm{d}x}{\mathrm{d}t} \quad .$$

The distance traveled can be calculated from

$$x = \int_{t_0}^{t} v \, \mathrm{d}t + x_0$$

where t_0 is the initial time and x_0 the initial position. If the speed is constant then the integral can be readily evaluated to obtain

$$x = x_0 + v\,(\,t - t_0\,)$$

which is the distance travelled in the time $t - t_0$. The acceleration of an object moving with changing speed is defined as

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2x}{\mathrm{d}t^2}$$

and when the speed is a function of position

$$a = \frac{\mathrm{d}v}{\mathrm{d}x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} = v\,\frac{\mathrm{d}v}{\mathrm{d}x} \quad .$$

The expression for the speed as a function of the constant acceleration can be determined from the integration

$$v = \int_{t_0}^{t} a\,\mathrm{d}t + v_0$$

and

$$\frac{1}{2}\,v^2 = \int_{x_0}^{x} a\,\mathrm{d}x + \frac{1}{2}\,v_0^2$$

where v_0 is the initial speed. When the acceleration is independent of time elementary integration yields

$$v = v_0 + a\,(\,t - t_0\,) \quad .$$

If this speed is also time-independent then it can be integrated to obtain the distance traveled

$$x = x_0 + v_0\,(\,t - t_0\,) + \frac{1}{2}\,a\,(\,t - t_0\,)^2 \quad .$$

When the acceleration is independent of position then

$$v^2 = v_0^2 + 2\,a\,(\,x - x_0\,) \quad .$$

These equations are valid for constant acceleration and/or speed. Note that the equations are derived from only the definitions of speed and acceleration without reference to the cause of the motion.

2 Linear Motion

Motion along a straight line is motion in one-dimension. Two specific examples of motion along a straight line are being explained, harmonic motion of a single particle and collision of two particles. The two cases are used to introduce the concept of conservation laws for energy and linear momentum.

2.1 Learning Goals

- Definition of a conservative force.
- The equation for harmonic motion.
- Definition of the coefficient of restitution.
- The law of conservation of energy.
- The law of conservation of linear momentum.

2.2 Notes

- Newton's second law of motion

$$m\,\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = F = \frac{\mathrm{d}p}{\mathrm{d}t} \quad .$$

  Note that this law defines the concept of mass.

- The law of conservation of mechanical energy

$$E = T(\dot{x}) + V(x) = \text{constant}$$

  is obtained when the force depends on the position only.

- The speed of a particle with potential energy $V(x)$ and total energy E can be obtained from the law of conservation of energy

$$\dot{x} = \pm\sqrt{\frac{2}{m}\,[\,E - V(x)\,]} \quad .$$

  This speed is classically meaningful/allowed when $E \geq V(x)$ for a confined/trapped particle.

- A conservative force has the property

$$F(x) = -\frac{\mathrm{d}V(x)}{\mathrm{d}x}$$

  where $V(x)$ is the underlying potential.

- At an equilibrium point x_0

$$F(x_0) = -\left.\frac{\mathrm{d}V(x)}{\mathrm{d}x}\right|_{x=x_0} = 0 \ ,$$

  i.e. no motion. This means the potential energy function is differentiated first and then x is set to x_0 in the results.

- Near the equilibrium point $x_0 = 0$, the potential energy function of the system may be expanded as

$$V(x) = V(0) + x \left.\frac{\mathrm{d}V(x)}{\mathrm{d}x}\right|_{x=0} + \frac{x^2}{2} \left.\frac{\mathrm{d}^2V(x)}{\mathrm{d}x^2}\right|_{x=0} + \ldots \ .$$

  Since x is very small, the terms involving the powers x^3 and higher may be neglected.

- General equation for harmonic motion

$$m\,\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \lambda\,\frac{\mathrm{d}x}{\mathrm{d}t} + k\,x = F(t)$$

  where m, λ and k are known constants.

- Conservation of linear momentum

$$P = p_1 + p_2 = \text{constant}$$

  states that the linear momentum in an isolated system is constant.

- The coefficient of restitution is defined as the ratio

$$e = -\frac{v_2 - v_1}{u_2 - u_1}$$

  involving the initial speeds u_1, u_2 and final speeds v_1, v_2. This coefficient is used as a measure of how elastic a collision is. For perfectly elastic collision $e = 1$ and for perfectly inelastic collision $e = 0$.

A sphere sliding along a horizontal plane with speed u_1 collides with a second sphere moving in the same straight line with speed u_2 in the same direction. If the coefficient of restitution is e and both spheres have the same mass m, find the kinetic energy lost in the impact.

Let v_1 and v_2 be the final velocities of the two spheres, respectively. The kinetic energy lost is given by the difference

$$\begin{aligned}\Delta\,E_k &= \frac{1}{2}\,m\,(u_1^2 + u_2^2) - \frac{1}{2}\,m\,(v_1^2 + v_2^2) \\ &= \frac{1}{2}\,m\,\left[u_1^2 + u_2^2) - (v_1^2 + v_2^2)\right] \ .\end{aligned}$$

To determine v_1 and v_2 we use the law of conservation of momentum :

$$m\,v_1 + m\,v_2 = m\,u_1 + m\,u_2$$

or

$$v_1 + v_2 = u_1 + u_2 \ ,$$

and the law of restitution,

$$v_1 - v_2 = e\,(u_2 - u_1) \ .$$

Solving the last two equations simultaneously yields

$$\begin{aligned} v_1 &= \frac{1}{2}\,[u_1 + u_2 - e\,(u_1 - u_2)] \\ v_2 &= \frac{1}{2}\,[u_1 + u_2 + e\,(u_1 - u_2)] \ . \end{aligned}$$

Substituting these results into the first equation we obtain

$$\begin{aligned} \Delta\,E_k &= \frac{1}{4}\,m\,\left[(u_1 - u_2)^2 - e^2\,(u_1 - u_2)^2\right] \\ &= \frac{1}{4}\,m\,(u_1 - u_2)^2\,(1 - e^2) \ . \end{aligned}$$

3 Energy and Angular Momentum

The material presented in this chapter is easily followed when studied after going through Appendix A. Motion in two and three dimensions can be treated as in the case of motion in one dimension if the equations of motion can be written separately for each coordinate component of the force. The geometry of the motion suggests the type of coordinate system that simplifies the study of such motion. Relevant coordinate systems are treated in Appendix A.

3.1 Learning Goals

- Generalization of the equations of motion to three dimensions.
- Generalization of the equation for the work done to three dimensions.
- Generalization of the expressions for the energy to three dimensions.
- Relationship between a vector associated with translational motion and the corresponding vector associated with rotational motion.
- The law of conservation of angular momentum.
- Derivation of Lagrange's equations of motion.

3.2 Notes

- Kinetic energy for motion in general is given by

$$T = \frac{1}{2}\, m \mid \dot{\boldsymbol{r}} \mid^2$$

  where $\mid \dot{\boldsymbol{r}} \mid$ is the magnitude of the velocity in two or three dimensions. The variation of the kinetic energy with time is given by

$$\dot{T} = \dot{\boldsymbol{r}} \cdot \boldsymbol{F} \quad .$$

- The work done W is obtained from the differential

$$\mathrm{d}W = \boldsymbol{F} \cdot \mathrm{d}^3\boldsymbol{r}$$

  where $\mathrm{d}^3\boldsymbol{r}$ is a volume element.

- A general conservative force has the properties

$$\boldsymbol{F}(\boldsymbol{r}) = -\boldsymbol{\nabla}\, V(\boldsymbol{r}) \qquad \text{and} \qquad \boldsymbol{\nabla} \times \boldsymbol{F}(\boldsymbol{r}) = 0$$

  where the *del* operator $\boldsymbol{\nabla}$ is the three-dimensional version of $\mathrm{d}/\mathrm{d}x$. Note that $\boldsymbol{\nabla}$ is a vector.

- The general equation of motion for projectiles may be written in the form

$$m\,\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} + \lambda\left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)^n = m\,\boldsymbol{g}$$

  where $n \geq 0$, while m, λ and g may be treated as constants. When $n = 0$ the motion is independent of the velocity of the projectile. When $n = 1$ the motion dependents linearly on the velocity. And when $n > 1$ the motion dependents non-linearly on the velocity of the projectile. It can be shown that the corresponding terminal velocity v_t is given by $v_t = (\,m\,g/\lambda\,)^{1/n}$.

- Translational motion and rotational motion can be studied along the same principles by expressing rotational quantities in terms of translational quantities.

- Angular momentum is defined as the moment of linear momentum

$$\boldsymbol{J} = \boldsymbol{r} \times \boldsymbol{p}$$

  where $\boldsymbol{p}$ is the linear momentum.

- The moment of force or torque is defined as

$$\boldsymbol{G} = \boldsymbol{r} \times \boldsymbol{F} = \frac{\mathrm{d}\boldsymbol{J}}{\mathrm{d}t} \quad .$$

- In the case of a central force

$$\boldsymbol{r} \times \boldsymbol{F} = 0 \qquad \text{and} \qquad \boldsymbol{J} = \text{constant}$$

  where $\boldsymbol{J} =$ constant is the statement of the law of conservation of angular momentum.

- Cartesian coordinates

$$\begin{aligned}
&\text{Unit vectors} && : && \hat{\boldsymbol{i}},\ \hat{\boldsymbol{j}},\ \hat{\boldsymbol{k}} \\
&\text{General vector} && : && \boldsymbol{A} = A_x\,\hat{\boldsymbol{i}} + A_y\,\hat{\boldsymbol{j}} + A_z\,\hat{\boldsymbol{k}} \\
&\text{Volume element} && : && \mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&\boldsymbol{\nabla} f && = && \hat{\boldsymbol{i}}\,\frac{\partial f}{\partial x} + \hat{\boldsymbol{j}}\,\frac{\partial f}{\partial y} + \hat{\boldsymbol{k}}\,\frac{\partial f}{\partial z}
\end{aligned}$$

- Cylindrical coordinates

$$\begin{aligned}
&\text{Unit vectors} && : && \hat{\boldsymbol{\rho}},\ \hat{\boldsymbol{\varphi}},\ \hat{\boldsymbol{z}} \\
&\text{General vector} && : && \boldsymbol{A} = A_\rho\,\hat{\boldsymbol{\rho}} + A_\varphi\,\hat{\boldsymbol{\varphi}} + A_z\,\hat{\boldsymbol{z}} \\
&\text{Volume element} && : && \mathrm{d}V = \rho\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}z \\
&\boldsymbol{\nabla} f && = && \hat{\boldsymbol{\rho}}\,\frac{\partial f}{\partial \rho} + \hat{\boldsymbol{\varphi}}\,\frac{1}{\rho}\,\frac{\partial f}{\partial \varphi} + \hat{\boldsymbol{z}}\,\frac{\partial f}{\partial z}
\end{aligned}$$

- Spherical coordinates

Unit vectors	:	$\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\varphi}}$
General vector	:	$\boldsymbol{A} = A_r\,\hat{\boldsymbol{r}} + A_\theta\,\hat{\boldsymbol{\theta}} + A_\varphi\,\hat{\boldsymbol{\varphi}}$
Volume element	:	$\mathrm{d}V = r^2\,\sin\,\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi\,\mathrm{d}r$
$\boldsymbol{\nabla}\,f$	=	$\hat{\boldsymbol{r}}\,\dfrac{\partial f}{\partial r} + \hat{\boldsymbol{\theta}}\,\dfrac{1}{r}\,\dfrac{\partial f}{\partial \theta} + \hat{\boldsymbol{\varphi}}\,\dfrac{1}{r\sin\,\theta}\,\dfrac{\partial f}{\partial \varphi}$

  where f is a scalar function.

- The Lagrangian of a system is defined by

$$L(q,\dot{q}) \;=\; T(\dot{q}) - V(q)$$

  where q is a generalized coordinate and $\dot{q}$ a generalized velocity.

- Lagrange's equations of motion for the system have the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$$

  for each coordinate q_i.

- The generalized momentum and generalized force are defined by the equations

$$\frac{\partial L}{\partial \dot{q}} = p_q \qquad , \qquad \frac{\partial L}{\partial q} = F_q \;\; .$$

  With these definitions the Lagrange's equations of motion can be shown to have the same form as Newton's second law of motion.

A ball of mass m is thrown vertically upward with an initial speed v_0. The air resistance is proportional to the square of the speed. Show that the speed with which the ball returns to the original position is

$$\frac{v_0\,v_t}{\sqrt{v_0^2 + v_t^2}}$$

where v_t is the terminal speed.

The equation of motion for the ball is

$$m\,\frac{\mathrm{d}v}{\mathrm{d}t} = \lambda\,v^2 + m\,g \;\; .$$

This equation can be written in the form

$$m\,v\,\frac{\mathrm{d}v}{\mathrm{d}z} = \lambda\,\left(v^2 + \frac{m\,g}{\lambda}\right) = \lambda\,\left(v^2 + v_t^2\right)$$

where $v_t = (\, m\, g/\lambda\,)^{1/2}$ is the terminal velocity of the ball. Rearranging and integrating this equation leads to

$$\int_{v_0}^{0} \frac{v\,\mathrm{d}v}{v^2 + v_t^2} = \int_0^h \frac{\lambda}{m}\,\mathrm{d}z$$

where h is the maximum height reached by the ball. Elementary integration of the equation yields

$$\frac{1}{2}\,\ln\left(\,v^2 + v_t^2\,\right)\Big|_{v_0}^{0} = \frac{\lambda}{m}\,z\Big|_0^h \quad .$$

Using the properties of logarithms we may write

$$\ln\left(\,\frac{v_t^2}{v_0^2 + v_t^2}\,\right) = \frac{2\,\lambda\,h}{m} \quad .$$

Solving the equation of motion for the downward motion yields

$$\int_0^v \frac{v\,\mathrm{d}v}{v^2 - v_t^2} = \int_h^0 \frac{\lambda}{m}\,\mathrm{d}z$$

which leads to

$$\ln\left(\,\frac{v^2 - v_t^2}{-v_t^2}\,\right) = \frac{2\,\lambda\,h}{m} \quad .$$

These imply that

$$\ln\left(\,\frac{v^2 - v_t^2}{-v_t^2}\,\right) = \ln\left(\,\frac{v_t^2}{v_0^2 + v_t^2}\,\right)$$

or

$$\frac{v^2 - v_t^2}{-v_t^2} = \frac{v_t^2}{v_0^2 + v_t^2}$$

whence

$$v^2 = v_t^2 - \frac{v_t^4}{v_0^2 + v_t^2}$$

which simplifies to

$$v = \frac{v_0\,v_t}{\sqrt{v_0^2 + v_t^2}}$$

as required.

4 Central Conservative Forces

This chapter uses examples to illustrate the application of the energy equations to study the motion of a particle experiencing a central conservative force. The concept of collision of small particles is introduced. This is referred to as scattering. The factors involved like beam, flux, scattering cross-sections are explained. In scattering the interacting particles do not necessarily come into physical contact with one another.

4.1 Learning Goals

- Generalization of the law of conservation of energy to three dimensions.
- The implications of the law of conservation of angular momentum.
- The equation of motion for a particle moving in a closed path.
- Terminology : scattering cross-section, mean-free-path. distance of closest approach and impact parameter.

4.2 Notes

- Examples of central conservative forces :
  - restoring force : $\boldsymbol{F} = -k\,\boldsymbol{r}$
  - inverse square force : $\boldsymbol{F} = k\,\hat{\boldsymbol{r}}/r^2$
- Conservation laws for central conservative forces

$$\begin{aligned} E &= T(\dot{\boldsymbol{r}}) + V(r) = \text{constant} \ , \\ J &= \boldsymbol{r} \times \boldsymbol{p} = \text{constant} \end{aligned}$$

  where all the quantities are considered in three dimensions.
- Conservation of angular momentum indicates that motion is only two-dimensional. In spherical coordinates the conservation laws can be written in the form

$$\begin{aligned} E &= \frac{1}{2}\,m\,\dot{r}^2 + \frac{1}{2}\,m\,r^2\,\dot{\theta}^2 + V(r) \ , \\ J &= m\,r^2\,\dot{\theta} \end{aligned}$$

  for the motion confined to the $r\theta$-plane.
- For confined particles the velocity conditions imply

$$E \geq U(r) = V(r) + \frac{J^2}{2\,m\,r^2} \ .$$

  At the turning points $E = U(r)$ and

$$V(r) - E + \frac{J^2}{2\,m\,r^2} = 0 \ .$$

These are the points at which the particle scatter from the potential barrier and change direction.

- An orbit is a closed path of a particle moving under the action of an external force. The general equation of an orbit for a particle under a central conservative force

$$\frac{J^2}{2\,m}\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 + \frac{J^2}{2\,m}\,u^2 + V(1/u) = E$$

where $1/u = r$.

- Collision and scattering are terms referring to similar processes.
Collision usually implies physical contact between participating particles.
Scattering does not necessarily involve contact between participating particles.

- Differential scattering cross-section in the case of a hard sphere of radius R is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{4}\,R^2$$

where $\mathrm{d}\Omega$ is the element of spherical surface.

- Rutherford differential scattering cross-section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{q_1\,q_2}{4\,\pi\,\epsilon_0\,m\,v^2}\right)^2 \frac{1}{4\,\sin^4(\theta/2)}$$

where q_1 and q_2 are charges.

- The total scattering cross-section

$$\sigma = \int \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\,\mathrm{d}\Omega$$

Integration is over the spherical surface area.

- Mean-free-path : $\lambda = 1/(n\,\sigma)$.

- The flux of a particle beam as a function of the penetration depth

$$f(x) = f_0\,\mathrm{e}^{-\,x/\lambda}$$

where f_0 is the initial flux.

- Rate of collisions in a target of thickness z and area A

$$\omega_c = A\,f_0\,\left(1 - \mathrm{e}^{-\,z/\lambda}\right) \quad .$$

In the book, this quantity is not represented by the symbol ω_c.

A particle moves under an attractive central force k/r^n per unit mass. Show that the orbit of the particle satisfies the differential equation

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u = \frac{k}{J^2}u^{n-2}\,,$$

where $u = 1/r$ and J is the angular momentum per unit mass.

The relevant equations of motion are

$$\begin{aligned} \frac{\mathrm{d}^2r}{\mathrm{d}t^2} - r\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 &= -F(r) \\ \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\frac{\mathrm{d}\theta}{\mathrm{d}t}\right) &= 0\,. \end{aligned}$$

The second equation means

$$r^2\frac{\mathrm{d}\theta}{\mathrm{d}t} = J$$

or

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{J}{r^2}$$

where J is a constant. Using this last expression for the angular speed one derives the relation

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{J}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\theta}\,.$$

Using the transformation $u = 1/r$ it can be seen that

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{1}{u}\right) = -\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}\theta}\,.$$

Therefore

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -J\frac{\mathrm{d}u}{\mathrm{d}\theta}\,.$$

Similarly

$$\frac{\mathrm{d}^2r}{\mathrm{d}t^2} = -J^2u^2\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}\,.$$

Substituting for the time derivatives in the radial equation yields

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u = \frac{F(1/u)}{J^2u^2}\,.$$

For $F(r) = k\,r^{-n}$ one obtains

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u = \frac{k}{J^2}\,u^{n-2}\,.$$

5 Rotating Frames

An example of non-inertial frames, a rotating frame, is discussed. The concept of fictitious forces arising in non-inertial frames is explained. Interpretation of motion in a non-inertial frame by an observer located in an inertial frame is given. Up to this point in the book the representation $\mathrm{d}a/\mathrm{d}t = \dot{a}$ was used. In this chapter the book indicates that $\mathrm{d}a/\mathrm{d}t$ does not represent $\dot{a}$ in the same frame. In the notes below the variation with respect to time in a given frame is explicitly indicated.

5.1 Learning Goals

- A fictitious force.
- An expression relating the "variation with respect to time" in a non-inertial frame and an inertial frame for some vectors.
- Some examples of the effect of the Coriolis force.

5.2 Notes

- An inertial observer is one located in an inertial frame of reference. A non-inertial frame is one that is accelerating relative to a given reference point.
- A frame rotating at an angular speed ω about a fixed axis is an example of a non-inertial frame to an observer not located in it.
- The relation between the "rate of change" (with respect to time) of any vector $\boldsymbol{A}$ to an observer in an inertial frame $\left(\dfrac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t}\right)_i$ and to an observer in a frame rotating with an angular velocity $\boldsymbol{\omega}$, $\left(\dfrac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t}\right)_\omega$ is

$$\left(\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t}\right)_i = \left(\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t}\right)_\omega + \boldsymbol{\omega} \times \boldsymbol{A}$$

  Note that it is not the vector $\boldsymbol{A}$ but its variation with time that is being referred to.

- Accelerations in the two frames are related by

$$\boldsymbol{a}_i = \boldsymbol{a}_\omega + 2\,\boldsymbol{\omega} \times \boldsymbol{v}_\omega + \frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} \times \boldsymbol{r}_\omega + \boldsymbol{\omega} \times (\,\boldsymbol{\omega} \times \boldsymbol{r}_\omega\,) \ ,$$

  Coriolis acceleration $= 2\,\boldsymbol{\omega} \times \boldsymbol{v}_\omega$
  Centripetal acceleration $= \boldsymbol{\omega} \times (\,\boldsymbol{\omega} \times \boldsymbol{r}_\omega\,)$

A merry-go-round is revolving with constant angular speed ω. A small mass m is tied to the lower end of a string of length L with the other end fixed at a distance d from the axis of rotation. The mass oscillates with simple harmonic motion with a small amplitude. Discuss the equations of motion of the mass.

Choose the coordinate system such that the rotation axis is parallel to the z-axis and the merry-go-round is on the xy-plane with the origin located on the rotation axis but at the height equal to the fixed point of the pendulum. Then the most general expression for the force acting on the pendulum bob is

$$\boldsymbol{F} = -T\,\sin\theta\,\sin\phi\,\boldsymbol{i} - T\,\sin\theta\,\cos\phi\,\boldsymbol{j} + (\,T\,\cos\theta - m\,g\,)\,\boldsymbol{k}$$

where T is the tension in the string. The corresponding displacement vector $\boldsymbol{r}_\omega$ of the bob, from the origin, is given by

$$\boldsymbol{r}_\omega = L\,\sin\theta\,\sin\phi\,\boldsymbol{i} + (\,d + L\,\sin\theta\,\cos\phi\,)\,\boldsymbol{j} - L\,\cos\theta\,\boldsymbol{k}\ .$$

To simplify the explanation we consider the motion of the pendulum in cylindrical polar coordinates. In that case we have

$$\boldsymbol{F} = -T\,\sin\theta\,\hat{\boldsymbol{\rho}} + (\,T\,\cos\theta - m\,g\,)\,\hat{\boldsymbol{z}}$$

and

$$\boldsymbol{r}_\omega = (\,d + L\,\sin\theta\,)\,\hat{\boldsymbol{\rho}} - L\,\cos\theta\,\hat{\boldsymbol{z}}\ .$$

The

$$\begin{aligned}
\boldsymbol{v}_\omega = \frac{\mathrm{d}\,\boldsymbol{r}}{\mathrm{d}\,t} &= L\,\cos\theta\,\frac{\mathrm{d}\,\theta}{\mathrm{d}\,t}\,\hat{\boldsymbol{\rho}} + L\,\sin\theta\,\frac{\mathrm{d}\,\theta}{\mathrm{d}\,t}\hat{\boldsymbol{z}} \\
\boldsymbol{a}_\omega = \frac{\mathrm{d}^2\,\boldsymbol{r}}{\mathrm{d}\,t^2} &= \left[\,L\,\cos\theta\,\frac{\mathrm{d}^2\,\theta}{\mathrm{d}\,t^2} - L\,\sin\theta\,\left(\frac{\mathrm{d}\,\theta}{\mathrm{d}\,t}\right)^2\,\right]\,\hat{\boldsymbol{\rho}} + \\
&\qquad \left[\,L\,\sin\theta\,\frac{\mathrm{d}^2\,\theta}{\mathrm{d}\,t^2} + L\,\cos\theta\,\left(\frac{\mathrm{d}\,\theta}{\mathrm{d}\,t}\right)^2\,\right]\,\hat{\boldsymbol{z}}\ .
\end{aligned}$$

Since $\boldsymbol{\omega} = \omega\,\hat{\boldsymbol{z}}$, then

$$\begin{aligned}
\boldsymbol{\omega}\times\boldsymbol{\omega}\times\boldsymbol{r}_\omega &= -\omega^2\,(\,d + L\,\sin\theta\,)\,\hat{\boldsymbol{\rho}} \\
2\,\boldsymbol{\omega}\times\boldsymbol{v}_\omega &= -2\,\omega\,L\,\cos\theta\,\frac{\mathrm{d}\,\theta}{\mathrm{d}\,t}\,\boldsymbol{i} \\
\boldsymbol{r}_\omega\times\frac{\mathrm{d}\,\boldsymbol{\omega}}{\mathrm{d}\,t} &= 0\ .
\end{aligned}$$

From the equation

$$\boldsymbol{a}_i = \boldsymbol{a}_\omega + 2\,\boldsymbol{\omega}\times\boldsymbol{v}_\omega + \frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}\times\boldsymbol{r}_\omega + \boldsymbol{\omega}\times(\,\boldsymbol{\omega}\times\boldsymbol{r}_\omega\,)$$

we obtain

$$\begin{aligned} -T \sin\theta &= m \left[L \cos\theta \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} - L \sin\theta \left(\frac{\mathrm{d}\theta}{\mathrm{d}t} \right)^2 - \omega^2 \left(d + L \sin\theta \right) \right] \\ T \cos\theta - m g &= m \left[L \sin\theta \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + L \cos\theta \left(\frac{\mathrm{d}\theta}{\mathrm{d}t} \right)^2 \right] \quad . \end{aligned}$$

Now multiply the second equation by $\cos\theta$, the last equation by $\sin\theta$ and add the resulting equations to obtain

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \left(\frac{g}{L} - \omega^2 \cos\theta \right) \sin\theta - \frac{\omega^2 d}{L} \cos\theta = 0$$

which, for small values of θ becomes

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \left(\frac{g}{L} - \omega^2 \right) \theta - \frac{\omega^2 d}{L} = 0$$

whence the period of the pendulum can be extracted.

6 Potential Theory

The chapter further explains some more properties of inverse-square forces with specific reference to the gravitational and electrostatic forces. Evaluating the potential for approximate values of the fraction $1/|\,\boldsymbol{r}-\boldsymbol{r}'\,|$ is discussed. In this study guide expressions concerning only the gravitational force are given. The corresponding expressions for the electrostatic force are similar.

6.1 Learning Goals

- The equation defining the an "inverse-square" force and the associated potential energy for various source distributions.
- The equation defining the force field generated by an "inverse-square" force, and the underlying potential, for various source distributions.
- Solution of the equations defining the force field and the underlying potential for various source distributions.
- The equations expressing the conservation of mass and charge.

6.2 Notes

- The gravitational force $\boldsymbol{F}$ on and the potential energy V of a point mass m located at $\boldsymbol{r}$ due to
  - another point mass m_1 located at $\boldsymbol{r}_1$

$$\boldsymbol{F}(\boldsymbol{r}) = -G\,\frac{m\,m_1}{|\,\boldsymbol{r}-\boldsymbol{r}_1\,|^2}\,\hat{\boldsymbol{e}}_1 \qquad , \qquad V(\boldsymbol{r}) = -G\,\frac{m\,m_1}{|\,\boldsymbol{r}-\boldsymbol{r}_1\,|}$$

  - a number of point masses m_i located at $\boldsymbol{r}_i$

$$\boldsymbol{F}(\boldsymbol{r}) = -G\,m\sum_i \frac{m_i}{|\,\boldsymbol{r}-\boldsymbol{r}_i\,|^2}\,\hat{\boldsymbol{e}}_i \qquad , \qquad V(\boldsymbol{r}) = -G\,m\sum_i \frac{m_i}{|\,\boldsymbol{r}-\boldsymbol{r}_i\,|}$$

  - a continuous mass distribution ρ centered at $\boldsymbol{r}'$

$$\boldsymbol{F}(\boldsymbol{r}) = -G\,m\int \frac{\rho(\boldsymbol{r}')\mathrm{d}^3\boldsymbol{r}'}{|\,\boldsymbol{r}-\boldsymbol{r}'\,|^2}\,\hat{\boldsymbol{e}}' \qquad , \qquad V(\boldsymbol{r}) = -G\,m\int \frac{\rho(\boldsymbol{r}')\,\mathrm{d}^3\boldsymbol{r}'}{|\,\boldsymbol{r}-\boldsymbol{r}'\,|}\ .$$

  Note that the force is central and conservative.

- The gravitational field $\boldsymbol{g}(\boldsymbol{r})$ on and the gravitational potential $\Phi(\boldsymbol{r})$ due to
  - a point mass m_1 located at $\boldsymbol{r}_1$ is

$$\boldsymbol{g}(\boldsymbol{r}) = -G\,\frac{m_1}{|\,\boldsymbol{r}-\boldsymbol{r}_1\,|^2}\,\hat{\boldsymbol{e}}_1 \qquad , \qquad \Phi(\boldsymbol{r}) = -G\,\frac{m_1}{|\,\boldsymbol{r}-\boldsymbol{r}_1\,|}$$

  - a number of point masses m_i located at $\boldsymbol{r}_i$

$$\boldsymbol{g}(\boldsymbol{r}) = -G\sum_i \frac{m_i}{|\,\boldsymbol{r}-\boldsymbol{r}_i\,|^2}\,\hat{\boldsymbol{e}}_i \qquad , \qquad \Phi(\boldsymbol{r}) = -G\sum_i \frac{m_i}{|\,\boldsymbol{r}-\boldsymbol{r}_i\,|}$$

- a continuous mass distribution ρ centered at $\boldsymbol{r}'$

$$\boldsymbol{g}(\boldsymbol{r}) = -G \int \frac{\rho(\boldsymbol{r}')\mathrm{d}^3\boldsymbol{r}'}{|\,\boldsymbol{r}-\boldsymbol{r}'\,|^2}\,\hat{\boldsymbol{e}}' \qquad , \qquad \Phi(\boldsymbol{r}) = -G \int \frac{\rho(\boldsymbol{r}')\,\mathrm{d}^3\boldsymbol{r}'}{|\,\boldsymbol{r}-\boldsymbol{r}'\,|} \quad .$$

- The unit vector $\hat{\boldsymbol{e}}'$ is defined as

$$\hat{\boldsymbol{e}}' = \frac{(\,\boldsymbol{r}-\boldsymbol{r}'\,)}{|\,\boldsymbol{r}-\boldsymbol{r}'\,|}$$

  with similar expressions for $\hat{\boldsymbol{e}}_1$ and $\hat{\boldsymbol{e}}_i$.

- The gravitational field may be expressed in terms of the gravitational potential in the form

$$\boldsymbol{g}(\boldsymbol{r}) = -\boldsymbol{\nabla}\,\Phi(\boldsymbol{r}) \quad .$$

- The gravitational field equations are

$$\boldsymbol{\nabla}\cdot\boldsymbol{g} + 4\,\pi\,G\,\rho = 0 \qquad \text{and} \qquad \boldsymbol{\nabla}\times\boldsymbol{g} = 0$$

  where ρ is the mass density of the source of the field. These equations express the law of conservation of mass.

Determine the gravitational field strength at a distance r from the center of a solid sphere of radius R, mass M and uniform density ρ.

Apply Gauss' law

$$\oint_S \boldsymbol{g}(\boldsymbol{r})\cdot\mathrm{d}\boldsymbol{S} = 4\,\pi\,G\int_V \rho(\boldsymbol{r})\,\mathrm{d}^3\boldsymbol{r} \quad .$$

Consider a spherical Gaussian surface of radius r concentric with the solid sphere. The gravitational field is directed radially outward, hence, at any given radius the field is constant. The volume of the Gaussian surface is $S_G = 4\,\pi\,r^2$.

(a) $r < R$:
The mass enclosed by the Gaussian surface is

$$m_G = V_G\,\rho = \frac{V_G}{V}\,M = \left(\frac{r}{R}\right)^3 M \quad .$$

where V_G is the volume of the Gaussian surface and V the volume of the solid sphere. Therefore from Gauss' law one obtains

$$g(r)\,S_G = 4\,\pi\,G\,m_G$$

whence the gravitational field strength is

$$g(r) = 4\,\pi\,G\,\frac{m_G}{S_G} = -\frac{G\,M}{R^3}\,r \quad .$$

(b) $r > R$:
In this case the mass enclosed by the Gaussian surface is

$$m_G = V\,\rho = M$$

and the gravitational field strength is

$$g(r) = 4\,\pi\,G\,\frac{m_G}{S_G} = -\frac{G\,M}{r^2} \quad .$$

7 The Two-body Problem

The concept of center-of-mass is introduced and illustrated with a system of two interacting particles. An example of interacting particles is when two particles collide. The transformation relations between the laboratory frame and the center-of-mass frame quantities are given. In the center-of-mass frame the quantities involving only the relative coordinates tend to give more information about the internal structure of the system. Whereas in the book the symbol (*) is used in this guide the symbol ($_{int}$) is used to label quantities involving only the relative coordinates.

7.1 Learning Goals

- Definition of the center-of-mass and reduced mass of two particles.
- Transformation relations between the center-of-mass frame and the laboratory frame quantities.
- The equations of motion of two particles in the center-of-mass frame.

7.2 Notes

- Total mass and reduced mass of the two particles

$$M = m_1 + m_2 \quad , \quad \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \quad .$$

- Relations between position vectors in the laboratory frame $\boldsymbol{r}_1$,$\boldsymbol{r}_2$ and those in the center-of-mass frame $\boldsymbol{r}$,$\boldsymbol{R}$

$$\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2 \quad , \quad \boldsymbol{R} = \frac{m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2}{m_1 + m_2}$$

$$\boldsymbol{r}_1 = \boldsymbol{R} + \frac{m_2}{M}\,\boldsymbol{r} \quad , \quad \boldsymbol{r}_2 = \boldsymbol{R} - \frac{m_1}{M}\,\boldsymbol{r} \quad .$$

  Similar relations can be obtained for the linear momentum of the particles. For more than two particle the relations are become more involved.

- The equations of motion in the two frames, when the particles are exerting a net force on each other, are

$$m_1\,\ddot{\boldsymbol{r}}_1 = \boldsymbol{F} \quad , \quad m_2\,\ddot{\boldsymbol{r}}_2 = -\boldsymbol{F}$$

$$M\,\ddot{\boldsymbol{R}} = 0 \quad , \quad \mu\,\ddot{\boldsymbol{r}} = \boldsymbol{F} \quad .$$

  The equations are different when the particles experience an additional external force.

- The angular momentum

$$\begin{aligned} \boldsymbol{J} &= m_1\,\boldsymbol{r}_1 \times \dot{\boldsymbol{r}}_1 + m_2\,\boldsymbol{r}_2 \times \dot{\boldsymbol{r}}_2 \\ &= M\,\boldsymbol{R} \times \dot{\boldsymbol{R}} + \mu\,\boldsymbol{r} \times \dot{\boldsymbol{r}} \\ &= \boldsymbol{J}_{cm}(\boldsymbol{R}) + \boldsymbol{J}_{int}(\boldsymbol{r}) \end{aligned}$$

  and the kinetic energy

$$\begin{aligned} T &= \frac{1}{2}\,m_1\,\dot{\boldsymbol{r}}_1^2 + \frac{1}{2}\,m_2\,\dot{\boldsymbol{r}}_2^2 \\ &= \frac{1}{2}\,M\,\dot{\boldsymbol{R}}^2 + \frac{1}{2}\,\mu\,\dot{\boldsymbol{r}}^2 \\ &= T_{cm}(\boldsymbol{R}) + T_{int}(\boldsymbol{r}) \end{aligned}$$

  of the system can be separated into a term depending only on the center-of-mass $\boldsymbol{J}_{cm}, T_{cm}$ and a term depending only on the relative coordinate $\boldsymbol{J}_{int}, T_{int}$. In the book the terms $\boldsymbol{J}_{int}, T_{int}$ are indicated as $\boldsymbol{J}^*, T^*$.

- For elastic collisions the scattering angles in the Lab frame θ_L and in the center-of-mass frame θ_C are related by

$$\tan\,\theta_L = \frac{\sin\,\theta_C}{(m_1/m_2) + \cos\,\theta_C} \quad .$$

  This expression can be rearranged to give

$$\cos^2\,\theta_L = \frac{[\,(m_1/m_2) + \cos\,\theta_C\,]^2}{1 + 2\,(m_1/m_2)\,\cos\,\theta_C + (m_1/m_2)^2} \quad .$$

  In the two expressions $m_1 < m_2$.

- The differential scattering cross-sections in the two frames are related by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_L} = \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_C}\,\frac{\sin\,\theta_C\,\mathrm{d}\theta_C}{\sin\,\theta_L\,\mathrm{d}\theta_L} \quad .$$

  Note that $\mathrm{d}(\,\cos\,\theta\,) = \sin\,\theta \mathrm{d}\theta$.

Determine the differential scattering cross-section of a mass m from a fixed force center $\boldsymbol{F} = k/r^3\,\hat{\boldsymbol{r}}$.

The corresponding potential is

$$V(r) = \frac{k}{2\,r^2}\,.$$

The distance of closest approach r_0 is determined from the conservation of the total mechanical energy

$$E = \frac{1}{2}\, m\, v_0^2 = \frac{1}{2}\, m\, v^2 + \frac{k}{2\, r_0^2}$$

and of the total angular momentum

$$J = m\, v_0\, b = m\, v\, r_0\,,$$

where v is the speed of the projectile particle at r_0 and b the impact parameter. Eliminating v from the two conservation laws we obtain

$$r_0 = \sqrt{\frac{J^2 + m\, k}{2\, m\, E}}\,.$$

The scattering angle θ is then given by

$$\theta = \pi - 2\,\alpha$$

where

$$\alpha = \int_{r_0}^{\infty} \frac{J\, \mathrm{d}r}{m\, r^2 \sqrt{\frac{2}{m}\left[E - V(r) - \frac{J^2}{2\, m\, r^2}\right]}}\,.$$

Since the effective potential can be simplified as

$$V(r) + \frac{J^2}{2\, m\, r^2} = \frac{J^2 + m\, k}{2\, m\, r^2} = \frac{E\, r_0^2}{r^2} \quad ,$$

then the integral is evaluated to obtain

$$\alpha = \frac{L}{\sqrt{2\, m\, E}} \int_{r_0}^{\infty} \frac{\mathrm{d}r}{r\sqrt{r^2 - r_0^2}} = -\frac{L}{r_0\sqrt{2\, m\, E}} \arcsin\left(\frac{r_0}{r}\right)\Big|_{r_0}^{\infty} \quad .$$

Note that

$$\arcsin\left(\frac{r_0}{r}\right) = \begin{cases} \pi/2 & \text{for} \quad r = r_0 \\ 0 & \text{for} \quad r \to \infty \end{cases} \quad .$$

Therefore

$$\theta = \pi - \frac{\pi\, J}{\sqrt{J^2 + m\, k}} = \pi\left[1 - \frac{J}{\sqrt{J^2 + m\, k}}\right] \quad .$$

It can be shown from the two conservation laws that $J = \sqrt{2\, m\, E}\, b$. Substituting for J we obtain

$$\theta = \pi\left[1 - \frac{b}{\sqrt{b^2 + k/2\, E}}\right] \quad .$$

Solving for b we get

$$b = \sqrt{\frac{k}{2\, E}}\, \frac{1 - x}{\sqrt{1 - (1 - x)^2}} = \sqrt{\frac{k}{2\, E}}\, \frac{1 - x}{\sqrt{2\, x - x^2}}$$

where $x = \theta/\pi$. Differentiating b with respect to θ yields

$$\frac{\mathrm{d}b}{\mathrm{d}x} \;=\; -\sqrt{\frac{k}{2\,E}}\,\frac{1}{(\,2\,x - x^2\,)^{3/2}}\,.$$

Therefore

$$\sigma(\theta) = -\,\frac{b}{\sin\,\theta}\,\frac{\mathrm{d}b}{\mathrm{d}\theta} = \frac{k}{2\,E}\,\frac{\pi^2}{\sin\,\theta}\,\frac{\pi - \theta}{\theta^2\,(\,2\,\pi - \theta\,)^2}\,.$$

8 Many-body Systems

This chapter is more about the generalization of the two-body problem. Mostly, the generalizations of the two-body equations and expressions of quantities are just stated. However, a variable-mass system is introduced.

8.1 Learning Goals

- The expression for the center-of-mass for a many-particle system.
- The equations of motion for a many-particle system in laboratory coordinates.
- The equation of motion for a variable-mass system.

8.2 Notes

- For a many-particle system there are more relative position vectors involved and only one center-of-mass position vector. The center-of-mass is computed as

$$\boldsymbol{R} = \frac{\sum_i m_i \boldsymbol{r}_i}{\sum_i m_i} \quad , \quad \boldsymbol{R} = \frac{\int \boldsymbol{r} \mathrm{d}m}{\int \mathrm{d}m} \ .$$

  The summation $\sum_i m_i$ is applicable in the case of a system of discrete masses m_i while the integral $\int \mathrm{d}m$ is used in the case of a continuous mass distribution. For a mass distribution with density ρ we have $\mathrm{d}m = \rho(\boldsymbol{r})\, \mathrm{d}^3\boldsymbol{r}$.

- Many quantities like the momentum and kinetic energy of the system can, in many cases, be expressed in the form

$$\begin{aligned} \boldsymbol{P} &= \boldsymbol{P}_{cm} + \boldsymbol{P}_{int} \\ \boldsymbol{J} &= \boldsymbol{J}_{cm} + \boldsymbol{J}_{int} \\ T &= T_{cm} + T_{int} \ . \end{aligned}$$

  However, in this case there are more than one relative coordinate and the resulting expressions for the quantities like $\boldsymbol{P}_{int}$, $\boldsymbol{J}_{int}$, T_{int} involve all the relative coordinate.

- The total linear momentum of the system

$$\boldsymbol{P} = \sum_i m_i \dot{\boldsymbol{r}}_i = M \dot{\boldsymbol{R}} = \boldsymbol{P}_{cm}$$

  is also the momentum of the center-of-mass. The internal linear momentum $\boldsymbol{P}_{int}$ depends only on the relative coordinates and the reduced masses of the particles.

- For a variable mass problem the equation of motion has the form

$$\boldsymbol{F} = \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} + \boldsymbol{v}\frac{\mathrm{d}m}{\mathrm{d}t} \ .$$

A rocket is an example of a variable mass problem.

When some mass m is being lost at a *constant rate* , with respect to time, then this event can be described by the expression

$$\frac{\mathrm{d}m}{\mathrm{d}t} = -\,k$$

where k is the *constant* and the minus sign indicates the loss. Integrating this equation yields

$$m = -\,k\,t + c$$

where c is the integration constant. If

$$m = m_0 \quad \text{when} \quad t = 0\,, \quad \text{then} \quad c = m_0\ .$$

And if

$$m = 0 \quad \text{when} \quad t = t_0 > 0\,, \quad \text{then} \quad k = \frac{m_0}{t_0}\ ,$$

and m as a function of time is given by

$$m = -\,\frac{m_0}{t_0}\,t + m_0 = m_0\,\left(1 - \frac{t}{t_0}\right)\ .$$

- The energy and momentum conservation laws still apply for the internal motions of the system.
- The equations of motion for the system can also be obtained in the laboratory frame and the center-of-mass frame as in the two-particle system.

Find the center-of-mass of a paraboloid $z = a\,(\,x^2 + y^2\,) = a\,r^2$ between $z = 0$ and $z = b$ as shown in the figure. The density is uniform.

The center-of-mass is defined by the relation

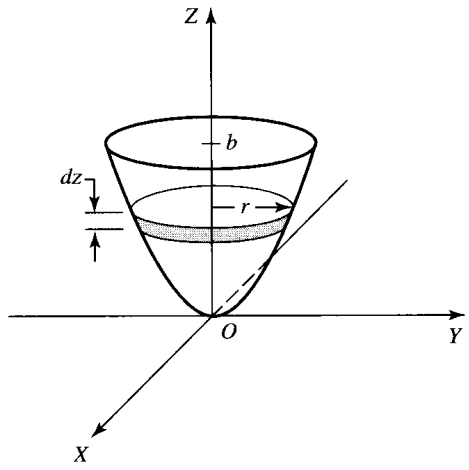
$$\boldsymbol{R} = \frac{\int_0^b \boldsymbol{r}\,\mathrm{d}m}{\int_0^b \mathrm{d}m}\ .$$

Define an element of mass

$$\mathrm{d}m = \rho\,\pi\,r^2\,\mathrm{d}z = \frac{\rho\,\pi}{a}\,z\,\mathrm{d}V$$

using the substitution $z = a\,(\,x^2 + y^2\,) = a\,r^2$. Then the z-coordinate of the center-of-mass is given by

$$z_{cm} = \frac{\int_0^b z^2\,\mathrm{d}z}{\int_0^b z\,\mathrm{d}z} = \frac{\frac{1}{3}\,z^3\,\Big|_0^b}{\frac{1}{2}\,z^2\,\Big|_0^b} = \frac{2}{3}\,b\ .$$



Since the paraboloid is symmetrical on the xy-plane and is centered on the origin, integration with respect to x and y lead to

$$x_{cm} = y_{cm} = 0 \; .$$

Therefore the center-of-mass of the paraboloid is

$$\boldsymbol{R} = \frac{2}{3}\, b\, \boldsymbol{k} \quad .$$

9 Rigid Bodies

The rotational motion of a body with continuous mass distribution and discrete separate masses at fixed positions is studied. The concept of moment of inertia is introduced as the rotational equivalent of mass in translational motion. Expressions for the moment of inertia for bodies of symmetric geometrical shapes about the principal axes are highlighted. The equations of rotational motion are stated.

9.1 Learning Goals

- Definition of the moment of inertia.
- Determination of the principal axes of inertia for bodies of symmetric geometrical shapes.
- The rotational equations of motion.

9.2 Notes

- Moment of inertia of a rigid body is defined by

$$I = \sum_i m_i \, r_i^2 \qquad : \qquad I = \int r^2 \, \mathrm{d}m$$

  where the summation $\sum_i$ is applicable in the case of a system of discrete separate masses m_i while the integral $\int \mathrm{d}m$ is used in the case of continuous mass distribution.

- Moment of inertia about an axis shifted by a from the center-of-mass

$$I = I_c + M \, a^2$$

  where I_c is the moment of inertia about an axis passing through the center-of-mass. The moment of inertia about the three mutually perpendicular axes x, y and z are related by

$$I_z = I_x + I_y \ .$$

- Angular momentum and rotational kinetic energy for a body rotating with angular velocity $\boldsymbol{\omega}$

$$\boldsymbol{J} = \mathrm{I} \, \boldsymbol{\omega} \qquad , \qquad \boldsymbol{T} = \frac{1}{2} \, \boldsymbol{\omega} \cdot \boldsymbol{J}$$

  where I is the moment of inertia.

- The inertia tensor in Cartesian coordinates is given by

$$\mathrm{I} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

where

$$
\begin{aligned}
I_{xx} &= \sum_i (y_i^2 + z_i^2)\, m_i & \qquad I_{xx} &= \iiint (y^2 + z^2)\, \rho(\boldsymbol{r})\, \mathrm{d}V \\
I_{yy} &= \sum_i (x_i^2 + z_i^2)\, m_i & \qquad I_{yy} &= \iiint (x^2 + z^2)\, \rho(\boldsymbol{r})\, \mathrm{d}V \\
I_{zz} &= \sum_i (x_i^2 + y_i^2)\, m_i & \qquad I_{zz} &= \iiint (x^2 + y^2)\, \rho(\boldsymbol{r})\, \mathrm{d}V
\end{aligned}
$$

and

$$
\begin{aligned}
I_{xy} &= -\sum_i x_i\, y_i\, m_i & \qquad I_{xy} &= -\iiint x\, y\, \rho(\boldsymbol{r})\, \mathrm{d}V \\
I_{yz} &= -\sum_i y_i\, z_i\, m_i & \qquad I_{yz} &= -\iiint y\, z\, \rho(\boldsymbol{r})\, \mathrm{d}V \\
I_{xz} &= -\sum_i x_i\, z_i\, m_i & \qquad I_{xz} &= -\iiint x\, z\, \rho(\boldsymbol{r})\, \mathrm{d}V
\end{aligned}
$$

where $\mathrm{d}V = \mathrm{d}^3\boldsymbol{r}$ is the element of the volume and $\rho(\boldsymbol{r})$ the mass density of the body.

- The moment of inertia about the principal axes are determined from solving the equation

$$
\begin{vmatrix}
I_{xx} - I & I_{xy} & I_{xz} \\
I_{yx} & I_{yy} - I & I_{yz} \\
I_{zx} & I_{zy} & I_{zz} - I
\end{vmatrix} = 0 \ .
$$

  The three roots of the resulting polynomial in I are the principal axes I_1, I_2 and I_3.

- Along the principal axes the inertia tensor is

$$
\mathrm{I} = \begin{bmatrix}
I_1 & 0 & 0 \\
0 & I_2 & 0 \\
0 & 0 & I_3
\end{bmatrix}
$$

  that is, $I_{xy} = I_{yz} = I_{xz} = 0$.

- The general rotational equations of motion

$$
\mathrm{I}\, \frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} + \boldsymbol{\omega} \times (\mathrm{I}\, \boldsymbol{\omega}) = \boldsymbol{G} \ .
$$

  In principal axes these equations can be written as

$$
\begin{aligned}
I_1\, \frac{\mathrm{d}\omega_1}{\mathrm{d}t} + (I_3 - I_2)\, \omega_3\, \omega_2 &= G_1 \\
I_2\, \frac{\mathrm{d}\omega_2}{\mathrm{d}t} + (I_1 - I_3)\, \omega_1\, \omega_3 &= G_2 \\
I_3\, \frac{\mathrm{d}\omega_3}{\mathrm{d}t} + (I_2 - I_1)\, \omega_2\, \omega_1 &= G_3
\end{aligned}
$$

where the subscripts 1,2 and 3 represent the directions $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$, respectively.

Determine the principal moments of inertia for a cube of uniform density ρ, mass M and edge length L.

Place the origin of the coordinate system at one corner of the cube. Then the moment of inertia of the cube about the x-axis is

$$\begin{aligned} I_{xx} &= \int_0^L\int_0^L\int_0^L (y^2+z^2)\,\rho\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &= \frac{2}{3}\,\rho\,L^5 = \frac{2}{3}\,M\,L^2 = I_{yy} = I_{zz} \ . \end{aligned}$$

The products of inertia for the cube are

$$\begin{aligned} I_{xy} &= -\int_0^L\int_0^L\int_0^L x\,y\,\rho\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &= -\frac{1}{4}\,\rho\,L^5 = -\frac{1}{4}\,M\,L^2 = I_{yz} = I_{zx} \ . \end{aligned}$$

Because of the symmetry of the cube all the products of inertia are equal. Therefore the inertia tensor for the cube is

$$\mathrm{I} = M\,L^2 \begin{bmatrix} 2/3 & -1/4 & -1/4 \\ -1/4 & 2/3 & -1/4 \\ -1/4 & -1/4 & 2/3 \end{bmatrix} \ .$$

The moment of inertia about the principal axes are determined from solving the equation

$$\begin{vmatrix} 2\,M\,L^2/3 - I & -M\,L^2/4 & -M\,L^2/4 \\ -M\,L^2/4 & 2\,M\,L^2/3 - I & -M\,L^2/4 \\ -M\,L^2/4 & -M\,L^2/4 & 2\,M\,L^2/3 - I \end{vmatrix} = 0$$

or

$$\begin{aligned} & \left(\frac{2\,\gamma}{3} - I\right)\left[\left(\frac{2\,\gamma}{3} - I\right)^2 - \left(\frac{\gamma}{4}\right)^2\right] \\ & -\frac{\gamma}{2}\left[\frac{\gamma}{4}\left(\frac{2\,\gamma}{3} - I\right) + \left(\frac{\gamma}{4}\right)^2\right] \\ = \ & \left(\frac{2\,\gamma}{3} - I\right)\left(\frac{2\,\gamma}{3} - I - \frac{\gamma}{4}\right)\left(\frac{2\,\gamma}{3} - I + \frac{\gamma}{4}\right) \end{aligned}$$

$$
\begin{aligned}
& - \frac{\gamma^2}{8}\left(\frac{2\,\gamma}{3} - I + \frac{\gamma}{4}\right) \\
= \; & \left(\frac{2\,\gamma}{3} - I\right)\left(\frac{5\,\gamma}{12} - I\right)\left(\frac{11\,\gamma}{12} - I\right) - \frac{\gamma^2}{8}\left(\frac{11\,\gamma}{12} - I\right) \\
= \; & \left(\frac{11}{12}\,\gamma - I\right)\left[\left(\frac{2}{3}\,\gamma - I\right)\left(\frac{5}{12}\,\gamma - I\right) - \frac{\gamma^2}{8}\right] \\
= \; & \left(\frac{11}{12}\,\gamma - I\right)\left(\frac{11}{12}\,\gamma - I\right)\left(\frac{1}{6}\,\gamma - I\right) = 0
\end{aligned}
$$

where $\gamma = M\,L^2$. The three roots of this equation are

$$
I_1 = I_2 = \frac{11}{12}\,M\,L^2 \qquad \text{and} \qquad I_3 = \frac{1}{6}\,M\,L^2
$$

which are the required principal moments of inertia for the cube.

10 Lagrangian Mechanics

The derivation of the Lagrange's equations of motion in chapter 3 and their being mentioned again in chapter 8 makes the content of this chapter a bit more familiar. Only the first two sections are instructive, the rest are illustrative examples of the applications of the Lagrange's equations of motion. The emphasis is mainly on the concept of generalized quantities.

10.1 Learning Goals

- The Lagrangian of a system.
- The Lagrange's equations of motion for a given system.
- Generalized momenta and generalized forces.

10.2 Notes

- A general Lagrangian of a system has the form

$$L(q_k, \dot{q}_k) = T(q_k, \dot{q}_k) - V(q_k, \dot{q}_k)$$

  where T and V both depend on the coordinates q_k and velocities $\dot{q}_k$.

- Lagrange's equations of motion for a conservative system are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0$$

  for each coordinate q_k.

- Generalized momenta and generalized forces can be expressed in the form

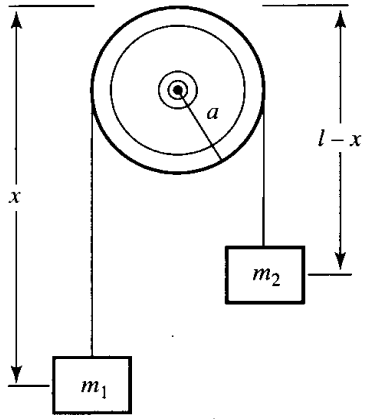
$$p_k = \frac{\partial L}{\partial \dot{q}_k} \qquad \text{and} \qquad \dot{p}_k = \frac{\partial L}{\partial q_k} \ .$$

- For conservative systems

$$\frac{\partial L}{\partial q_k} = -\frac{\partial V}{\partial q_k} \ .$$

  With these definitions the Lagrange's equations of motion can be shown to have the same form as Newton's second law of motion.

Consider a system of two masses m_1 and m_2, and a pulley of radius a and moment of inertia I. The length of the inextensible string connecting the two masses and going



over the pulley is l. Derive the Lagrangian of the system and use it to determine the acceleration of the system.

The velocity of the masses and the angular velocity of the disk may be written as

$$v_1 = \frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x}\ , \qquad v_2 = \frac{\mathrm{d}(\,l - x\,)}{\mathrm{d}t} = -\dot{x}$$

and

$$\omega = \frac{v}{a}$$

where $v = |v_1| = |v_2| = \dot{x}$. The kinetic energy of the system is given by

$$T = \frac{1}{2}\,m_1\,v_1^2 + \frac{1}{2}\,m_2\,v_2^2 + \frac{1}{2}\,I\,\omega^2 = \frac{1}{2}\,m_1\,\dot{x}^2 + \frac{1}{2}\,m_2\,\dot{x}^2 + \frac{1}{2}\,I\,\frac{\dot{x}^2}{a^2}$$

while the potential energy of the system is

$$V = -\,m_1\,g\,x - m_2\,g\,(\,l - x\,)\ .$$

The Lagrangian of the system is

$$L = T - V = \frac{1}{2}\left(\,m_1 + m_2 + \frac{I}{a^2}\,\right)\dot{x}^2 + g\,(\,m_1 - m_2\,)\,x + m_2\,g\,l\ .$$

There is only one degree of freedom, which is the generalized coordinate $q = x$. Lagrange's equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\frac{\mathrm{d}L}{\mathrm{d}\dot{x}} - \frac{\mathrm{d}L}{\mathrm{d}x} = 0\ .$$

It can be easily shown that

$$\begin{aligned}
\frac{\mathrm{d}L}{\mathrm{d}\dot{x}} &= \left(\,m_1 + m_2 + \frac{I}{a^2}\,\right)\dot{x} \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}L}{\mathrm{d}\dot{x}}\right) &= \left(\,m_1 + m_2 + \frac{I}{a^2}\,\right)\ddot{x} \\
\frac{\mathrm{d}L}{\mathrm{d}x} &= g\,(\,m_1 - m_2\,)\ .
\end{aligned}$$

Therefore

$$\left(m_1 + m_2 + \frac{I}{a^2} \right) \ddot{x} = g \left(m_1 - m_2 \right)$$

or

$$\ddot{x} = g \, \frac{m_1 - m_2}{m_1 + m_2 + I/a^2} \quad .$$

11 Small Oscillations and Norma Modes

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12 Hamiltonian Mechanics

This chapter can be viewed as an extension of chapter 10. The Lagrangian is used to define a new function called the Hamiltonian of the system. In the definition of the Hamiltonian the generalized momenta are used instead of the generalized velocities. For natural systems the Hamiltonian is shown to be equivalent to the total energy of the system. The relationship between ignorable coordinates and conservation laws is introduced.

12.1 Learning Goals

- The defining expression for the Hamiltonian of a given system.
- Hamilton's equations of motion for a given system.
- Coordinates in a given system that will give rise to conservation laws.

12.2 Notes

- The Hamiltonian of the system is defined as

$$H = \sum_k p_k \frac{\partial L}{\partial p_k} - L$$

  where $L = T - V$ is the Lagrangian of the system and p_k the momenta corresponding to the coordinates q_k.

- Hamilton's equations of motion for the system are

$$\frac{\partial H}{\partial p_k} = \dot{q}_k \qquad \text{and} \qquad \frac{\partial H}{\partial q_k} = -\dot{p}_k$$

  p_k and q_k are said to be canonically conjugate variables.

- For natural systems the kinetic energy satisfies the condition

$$\sum_k \frac{\partial T}{\partial \dot{q}_k} \dot{q}_k = 2\,T$$

  and the Hamiltonian has the form $H = T + V$.

- In a conservative system energy is conserved

$$H = T + V = E = \text{constant} \quad .$$

  This law is the result of the Hamiltonian of the system being independent of time.

- Hamilton's equations of motion for an ignorable coordinate, q_o, have the form

$$\frac{\partial H}{\partial p_o} = \dot{q}_o \qquad \text{and} \qquad \frac{\partial H}{\partial q_o} = -\dot{p}_o = 0$$

  which means p_o is a constant of the motion.

- Ignorable coordinate express certain symmetries in the system. Symmetries in a system generate certain invariance. An invariant quantity is linked to a conservation law.

Ignorable	–	Conserved
spatial translation	–	linear momentum
rotation	–	angular momentum
time	–	energy

A particle of mass m is attracted toward a given point by a force of magnitude k/r^2, where k is a constant. Derive an expression for the Hamiltonian and Hamilton's equations of motion for the particle.

Choose the coordinates r, θ as the generalized coordinates for the Lagrangian of the system. Then the kinetic energy of the particle is

$$T = \frac{1}{2}\, m \left(\dot{r}^2 + r^2\, \dot{\theta}^2 \right)$$

and the potential energy is

$$V = -\int \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = -\frac{k}{r} \ .$$

The Lagrangian of the system is therefore

$$L = T - V = \frac{1}{2}\, m \left(\dot{r}^2 + r^2\, \dot{\theta}^2 \right) + \frac{k}{r} \ .$$

Since the Hamiltonian requires the use of momenta instead of velocities, one uses the Lagrangian to derive

$$\frac{\mathrm{d}L}{\mathrm{d}\dot{r}} = p_r = m\, \dot{r} \qquad \text{and} \qquad \frac{\mathrm{d}L}{\mathrm{d}\dot{\theta}} = p_\theta = m\, r^2\, \dot{\theta} \ .$$

As a result the expression for the kinetic energy employed is

$$T = \frac{p_r^2}{2\, m} + \frac{p_\theta^2}{2\, m\, r^2} \ .$$

The Hamiltonian is then calculated as

$$H = T + V = \frac{p_r^2}{2\, m} + \frac{p_\theta^2}{2\, m\, r^2} - \frac{k}{r} \ .$$

For the Hamiltonian, the generalized coordinates are r, θ, p_r, p_θ. Therefore Hamilton's equations of motion are

$$\frac{\partial H}{\partial p_r} = \dot{r} = \frac{p_r}{m} \qquad \text{and} \qquad \frac{\partial H}{\partial r} = -\dot{p_r} = -\frac{p_\theta^2}{m\, r^3} + \frac{k}{r^2}$$

$$\frac{\partial H}{\partial p_\theta} = \dot{\theta} = \frac{p_\theta}{m\, r^2} \qquad \text{and} \qquad \frac{\partial H}{\partial \theta} = -\dot{p_\theta} = 0 \ .$$

In this case θ is an ignorable coordinate.

13 Dynamical Systems and Their Geometry

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14 Order and Chaos in Hamiltonian Systems

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