

# A Collection of Problems in Differential Calculus

Problems Given At the Math 151 - Calculus I and Math 150 - Calculus I With  
Review Final Examinations

Department of Mathematics, Simon Fraser University

2000 - 2010

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*To my sons, my best teachers.* - Veselin Jungic

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# Preface

The purpose of this *Collection of Problems* is to be an additional learning resource for students who are taking a differential calculus course at Simon Fraser University. The Collection contains problems given at Math 151 - Calculus I and Math 150 - Calculus I With Review final exams in the period 2000-2009. The problems are sorted by topic and most of them are accompanied with hints or solutions.

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No project such as this can be free from errors and incompleteness. The authors will be grateful to everyone who points out any typos, incorrect solutions, or sends any other suggestion on how to improve this manuscript.

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# Recommendations for Success in Mathematics

The following is a list of various categories gathered by the Department of Mathematics. This list is a recommendation to all students who are thinking about their well-being, learning, and goals, and who want to be successful academically.

## Tips for Reading these Recommendations:

- Do not be overwhelmed with the size of this list. You may not want to read the whole document at once, but choose some categories that appeal to you.
- You may want to make changes in your habits and study approaches after reading the recommendations. Our advice is to take small steps. Small changes are easier to make, and chances are those changes will stick with you and become part of your habits.
- Take time to reflect on the recommendations. Look at the people in your life you respect and admire for their accomplishments. Do you believe the recommendations are reflected in their accomplishments?

## Habits of a Successful Student:

- **Acts responsibly:** This student
  - reads the documents (such as course outline) that are passed on by the instructor and acts on them.
  - takes an active role in their education.
  - does not cheat and encourages academic integrity in others.

- **Sets goals:** This student
  - sets attainable goals based on specific information such as the academic calendar, academic advisor, etc..
  - is motivated to reach the goals.
  - is committed to becoming successful.
  - understands that their physical, mental, and emotional well-being influences how well they can perform academically.
- **Is reflective:** This student
  - understands that deep learning comes out of reflective activities.
  - reflects on their learning by revisiting assignments, midterm exams, and quizzes and comparing them against posted solutions.
  - reflects why certain concepts and knowledge are more readily or less readily acquired.
  - knows what they need to do by having analyzed their successes and their failures.
- **Is inquisitive:** This student
  - is active in a course and asks questions that aid their learning and build their knowledge base.
  - seeks out their instructor after a lecture and during office hours to clarify concepts and content and to find out more about the subject area.
  - shows an interest in their program of studies that drives them to do well.
- **Can communicate:** This student
  - articulates questions.
  - can speak about the subject matter of their courses, for example by explaining concepts to their friends.
  - takes good notes that pay attention to detail but still give a holistic picture.
  - pays attention to how mathematics is written and attempts to use a similar style in their written work.
  - pays attention to new terminology and uses it in their written and oral work.
- **Enjoys learning:** This student

- is passionate about their program of study.
- is able to cope with a course they don't like because they see the bigger picture.
- is a student because they made a positive choice to be one.
- reviews study notes, textbooks, etc..
- works through assignments individually at first and way before the due date.
- does extra problems.
- reads course related material.

- **Is resourceful:** This student

- uses the resources made available by the course and instructor such as the Math Workshop, the course container on WebCT, course websites, etc..
- researches how to get help in certain areas by visiting the instructor, or academic advisor, or other support structures offered through the university.
- uses the library and internet thoughtfully and purposefully to find additional resources for a certain area of study.

- **Is organized:** This student

- adopts a particular method for organizing class notes and extra material that aids their way of thinking and learning.

- **Manages his/her time effectively:** This student

- is in control of their time.
- makes and follows a schedule that is more than a timetable of course. It includes study time, research time, social time, sports time, etc..

- **Is involved:** This student

- is informed about their program of study and their courses and takes an active role in them.
- researches how to get help in certain areas by visiting the instructor, or academic advisor, or other support structures offered through the university.

- joins a study group or uses the support that is being offered such as a Math Workshop (that accompanies many first and second year math courses in the Department of Mathematics) or the general SFU Student Learning Commons Workshops.
- sees the bigger picture and finds ways to be involved in more than just studies. This student looks for volunteer opportunities, for example as a Teaching Assistant in one of the Mathematics Workshops or with the MSU (Math Student Union).

## How to Prepare for Exams:

- Start preparing for an exam on the FIRST DAY OF LECTURES!
- Come to all lectures and listen for where the instructor stresses material or points to classical mistakes. Make a note about these pointers.
- Treat each chapter with equal importance, but distinguish among items within a chapter.
- Study your lecture notes in conjunction with the textbook because it was chosen for a reason.
- Pay particular attention to technical terms from each lecture. Understand them and use them appropriately yourself. The more you use them, the more fluent you will become.
- Pay particular attention to definitions from each lecture. Know the major ones by heart.
- Pay particular attention to theorems from each lecture. Know the major ones by heart.
- Pay particular attention to formulas from each lecture. Know the major ones by heart.
- Create a cheat sheet that summarizes terminology, definitions, theorems, and formulas. You should think of a cheat sheet as a very condensed form of lecture notes that organizes the material to aid your understanding. (However, you may not take this sheet into an exam unless the instructor specifically says so.)
- Check your assignments against the posted solutions. Be critical and compare how you wrote up a solution versus the instructor/textbook.

- Read through or even work through the paper assignments, online assignments, and quizzes (if any) a second time.
- Study the examples in your lecture notes in detail. Ask yourself, why they were offered by the instructor.
- Work through some of the examples in your textbook, and compare your solution to the detailed solution offered by the textbook.
- Does your textbook come with a review section for each chapter or grouping of chapters? Make use of it. This may be a good starting point for a cheat sheet. There may also be additional practice questions.
- Practice writing exams by doing old midterm and final exams under the same constraints as a real midterm or final exam: strict time limit, no interruptions, no notes and other aides unless specifically allowed.
- Study how old exams are set up! How many questions are there on average? What would be a topic header for each question? Rate the level of difficulty of each question. Now come up with an exam of your own making and have a study partner do the same. Exchange your created exams, write them, and then discuss the solutions.

## Getting and Staying Connected:

- Stay in touch with family and friends:
  - A network of family and friends can provide security, stability, support, encouragement, and wisdom.
  - This network may consist of people that live nearby or far away. Technology in the form of cell phones, email, facebook, etc. is allowing us to stay connected no matter where we are. However, it is up to us at times to reach out and stay connected.
  - Do not be afraid to talk about your accomplishments and difficulties with people that are close to you and you feel safe with, to get different perspectives.
- Create a study group or join one:
  - Both the person being explained to and the person doing the explaining benefit from this learning exchange.

- Study partners are great resources! They can provide you with notes and important information if you miss a class. They may have found a great book, website, or other resource for your studies.
- Go to your faculty or department and find out what student groups there are:
  - The Math Student Union (MSU) seeks and promotes student interests within the Department of Mathematics at Simon Fraser University and the Simon Fraser Student Society. In addition to open meetings, MSU holds several social events throughout the term. This is a great place to find like-minded people and to get connected within mathematics.
  - Student groups or unions may also provide you with connections after you complete your program and are seeking either employment or further areas of study.
- Go to your faculty or department and find out what undergraduate outreach programs there are:
  - There is an organized group in the Department of Mathematics led by Dr. Jonathan Jedwab that prepares for the William Lowell Putnam Mathematical Competition held annually the first Saturday in December: <http://www.math.sfu.ca/ugrad/putnam.shtml>
  - You can apply to become an undergraduate research assistant in the Department of Mathematics, and (subject to eligibility) apply for an NSERC USRA (Undergraduate Student Research Award): <http://www.math.sfu.ca/ugrad/awards/nsercsu.shtml>
  - You can attend the Math: Outside the Box series which is an undergraduate seminar that presents on all sorts of topics concerning mathematics.

## Staying Healthy:

- A healthy mind, body, and soul promote success. Create a healthy lifestyle by taking an active role in this lifelong process.
- Mentally:
  - Feed your intellectual hunger! Choose a program of study that suits your talents and interests. You may want to get help by visiting with an academic advisor: [math\\_advice@sfu.ca](mailto:math_advice@sfu.ca).
  - Take breaks from studying! This clears your mind and energizes you.

- Physically:
  - Eat well! Have regular meals and make them nutritious.
  - Exercise! You may want to get involved in a recreational sport.
  - Get out rain or shine! Your body needs sunshine to produce vitamin D, which is important for healthy bones.
  - Sleep well! Have a bed time routine that will relax you so that you get good sleep. Get enough sleep so that you are energized.
- Socially:
  - Make friends! Friends are good for listening, help you to study, and make you feel connected.
  - Get involved! Join a university club or student union.

## Resources:

- Old exams for courses serviced through a workshop that are maintained by the Department of Mathematics: <http://www.math.sfu.ca/ugradworkshops>
- WolframAlpha Computational Knowledge Engine: <http://www.wolframalpha.com/examples/Math.html>
- Survival Guide to 1st Year Mathematics at SFU: <http://www.math.sfu.ca/ugrad/guide1.shtml>
- Survival Guide to 2nd-4th Year Mathematics at SFU: <http://www.math.sfu.ca/ugrad/guide2.shtml>
- SFU Student Learning Commons: <http://learningcommons.sfu.ca/>
- SFU Student Success Programs: <http://students.sfu.ca/advising/studentsuccess/index.html>
- SFU Writing for University: <http://learningcommons.sfu.ca/strategies/writing>
- SFU Health & Counselling Services: <http://students.sfu.ca/health/>
- How to Ace Calculus: The Streetwise Guide: <http://www.math.ucdavis.edu/hass/Calculus/HTAC/excerpts/excerpts.html>
- 16 Habits of Mind (1 page summary): [http://www.chsvt.org/wdp/Habits\\_of\\_Mind.pdf](http://www.chsvt.org/wdp/Habits_of_Mind.pdf)

## References:

Thien, S. J. Bulleri, A. The Teaching Professor. Vol. 10, No. 9, November 1996. Magna Publications.

Costa, A. L. and Kallick, B. 16 Habits of Mind.  
<http://www.instituteforhabitsofmind.com/what-are-habits-mind>.

# Chapter 1

## Limits and Continuity

### 1.1 Introduction

1. **Limit.** We write  $\lim_{x \rightarrow a} f(x) = L$  and say "the limit of  $f(x)$ , as  $x$  approaches  $a$ , equals  $L$ " if it is possible to make the values of  $f(x)$  arbitrarily close to  $L$  by taking  $x$  to be sufficiently close to  $a$ .
2. **Limit -  $\varepsilon$ ,  $\delta$  Definition.** Let  $f$  be a function defined on some open interval that contains  $a$ , except possibly at  $a$  itself. Then we say that the limit of  $f(x)$  as  $x$  approaches  $a$  is  $L$ , and we write  $\lim_{x \rightarrow a} f(x) = L$  if for every number  $\varepsilon > 0$  there is a  $\delta > 0$  such that  $|f(x) - L| < \varepsilon$  whenever  $0 < |x - a| < \delta$ .
3. **Limit And Right-hand and Left-hand Limits.**  $\lim_{x \rightarrow a} f(x) = L \Leftrightarrow (\lim_{x \rightarrow a^-} f(x) = L \text{ and } \lim_{x \rightarrow a^+} f(x) = L)$
4. **Infinite Limit.** Let  $f$  be a function defined on a neighborhood of  $a$ , except possibly at  $a$  itself. Then  $\lim_{x \rightarrow a} f(x) = \infty$  means that the values of  $f(x)$  can be made arbitrarily large by taking  $x$  sufficiently close to  $a$ , but not equal to  $a$ .
5. **Vertical Asymptote.** The line  $x = a$  is called a vertical asymptote of the curve  $y = f(x)$  if at least one of the following statements is true:

$$\begin{array}{lll} \lim_{x \rightarrow a} f(x) = \infty & \lim_{x \rightarrow a^-} f(x) = \infty & \lim_{x \rightarrow a^+} f(x) = \infty \\ \lim_{x \rightarrow a} f(x) = -\infty & \lim_{x \rightarrow a^-} f(x) = -\infty & \lim_{x \rightarrow a^+} f(x) = -\infty \end{array}$$

6. **Limit At Infinity.** Let  $f$  be a function defined on  $(a, \infty)$ . Then  $\lim_{x \rightarrow \infty} f(x) = L$  means that the values of  $f(x)$  can be made arbitrarily close to  $L$  by taking  $x$  sufficiently large.

7. **Horizontal Asymptote.** The line  $y = a$  is called a horizontal asymptote of the curve  $y = f(x)$  if at least one of the following statements is true:

$$\lim_{x \rightarrow \infty} f(x) = a \text{ or } \lim_{x \rightarrow -\infty} f(x) = a.$$

8. **Limit Laws.** Let  $c$  be a constant and let the limits  $\lim_{x \rightarrow a} f(x)$  and  $\lim_{x \rightarrow a} g(x)$  exist. Then

$$(a) \lim_{x \rightarrow a} (f(x) \pm g(x)) = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x)$$

$$(b) \lim_{x \rightarrow a} (c \cdot f(x)) = c \cdot \lim_{x \rightarrow a} f(x)$$

$$(c) \lim_{x \rightarrow a} (f(x) \cdot g(x)) = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$$

$$(d) \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} \text{ if } \lim_{x \rightarrow a} g(x) \neq 0.$$

9. **Squeeze Law.** If  $f(x) \leq g(x) \leq h(x)$  when  $x$  is near  $a$  (except possibly at  $a$ ) and  $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$  then  $\lim_{x \rightarrow a} g(x) = L$ .

10. **Trigonometric Limits.**  $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$  and  $\lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\theta} = 0$ .

11. **The Number  $e$ .**  $\lim_{x \rightarrow 0} (1 + x)^{\frac{1}{x}} = e$  and  $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e$ .

12. **L'Hospital's Rule.** Suppose that  $f$  and  $g$  are differentiable and  $g'(x) \neq 0$  near  $a$  (except possibly at  $a$ ). Suppose that  $\lim_{x \rightarrow a} f(x) = 0$  and  $\lim_{x \rightarrow a} g(x) = 0$  or that  $\lim_{x \rightarrow a} f(x) = \pm\infty$  and  $\lim_{x \rightarrow a} g(x) = \pm\infty$ . Then  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$  if the limit on the right side exists (or is  $\infty$  or  $-\infty$ ).

13. **Continuity.** We say that a function  $f$  is continuous at a number  $a$  if  $\lim_{x \rightarrow a} f(x) = f(a)$ .

14. **Continuity and Limit.** If  $f$  is continuous at  $b$  and  $\lim_{x \rightarrow a} g(x) = b$  then  $\lim_{x \rightarrow a} f(g(x)) = f(\lim_{x \rightarrow a} g(x)) = f(b)$ .

15. **Intermediate Value Theorem.** Let  $f$  be continuous on the closed interval  $[a, b]$  and let  $f(a) \neq f(b)$ . For any number  $M$  between  $f(a)$  and  $f(b)$  there exists a number  $c$  in  $(a, b)$  such that  $f(c) = M$ .

## 1.2 Limits

Evaluate the following limits. Use limit theorems, not  $\varepsilon$  -  $\delta$  techniques. If any of them fail to exist, say so and say why.

1. (a)  $\lim_{x \rightarrow 10} \frac{x^2 - 100}{x - 10}$   
 (b)  $\lim_{x \rightarrow 10} \frac{x^2 - 99}{x - 10}$   
 (c)  $\lim_{x \rightarrow 10} \frac{x^2 - 100}{x - 9}$   
 (d)  $\lim_{x \rightarrow 10} f(x)$ , where  
 $f(x) = x^2$  for all  $x \neq 10$ ,  
 but  $f(10) = 99$ .  
 (e)  $\lim_{x \rightarrow 10} \sqrt{-x^2 + 20x - 100}$
2.  $\lim_{x \rightarrow -4} \frac{x^2 - 16}{x + 4} \ln |x|$
3.  $\lim_{x \rightarrow \infty} \frac{x^2}{e^{4x} - 1 - 4x}$
4.  $\lim_{x \rightarrow -\infty} \frac{3x^6 - 7x^5 + x}{5x^6 + 4x^5 - 3}$
5.  $\lim_{x \rightarrow -\infty} \frac{5x^7 - 7x^5 + 1}{2x^7 + 6x^6 - 3}$
6.  $\lim_{x \rightarrow -\infty} \frac{2x + 3x^3}{x^3 + 2x - 1}$
7.  $\lim_{x \rightarrow -\infty} \frac{5x + 2x^3}{x^3 + x - 7}$
8.  $\lim_{x \rightarrow -\infty} \frac{3x + |1 - 3x|}{1 - 5x}$
9.  $\lim_{u \rightarrow \infty} \frac{u}{\sqrt{u^2 + 1}}$
10.  $\lim_{x \rightarrow \infty} \frac{1 + 3x}{\sqrt{2x^2 + x}}$
11.  $\lim_{x \rightarrow \infty} \frac{\sqrt{4x^2 + 3x} - 7}{7 - 3x}$
12.  $\lim_{x \rightarrow 1^+} \frac{\sqrt{x - 1}}{x^2 - 1}$
13. Let  

$$f(x) = \begin{cases} \frac{x^2 - 1}{|x - 1|} & \text{if } x \neq 1, \\ 4 & \text{if } x = 1. \end{cases}$$
 Find  $\lim_{x \rightarrow 1^-} f(x)$ .
14. Let  $F(x) = \frac{2x^2 - 3x}{|2x - 3|}$ .  
 (a) Find  $\lim_{x \rightarrow 1.5^+} F(x)$ .  
 (b) Find  $\lim_{x \rightarrow 1.5^-} F(x)$ .  
 (c) Does  $\lim_{x \rightarrow 1.5} F(x)$  exist? Provide a reason.
15.  $\lim_{x \rightarrow 8} \frac{(x - 8)(x + 2)}{|x - 8|}$
16.  $\lim_{x \rightarrow 16} \frac{\sqrt{x} - 4}{x - 16}$
17.  $\lim_{x \rightarrow 8} \frac{\sqrt[3]{x} - 2}{x - 8}$
18. Find constants  $a$  and  $b$  such that  
 $\lim_{x \rightarrow 0} \frac{\sqrt{ax + b} - 2}{x} = 1$ .
19.  $\lim_{x \rightarrow 8} \frac{x^{1/3} - 2}{x - 8}$
20.  $\lim_{x \rightarrow \infty} \left( \sqrt{x^2 + x} - x \right)$
21.  $\lim_{x \rightarrow -\infty} \left( \sqrt{x^2 + 5x} - \sqrt{x^2 + 2x} \right)$
22.  $\lim_{x \rightarrow \infty} \left( \sqrt{x^2 - x + 1} - \sqrt{x^2 + 1} \right)$
23.  $\lim_{x \rightarrow \infty} \left( \sqrt{x^2 + 3x - 2} - x \right)$

24. Is there a number  $b$  such that  $\lim_{x \rightarrow -2} \frac{bx^2 + 15x + 15 + b}{x^2 + x - 2}$  exists? If so, find the value of  $b$  and the value of the limit.
25. Determine the value of  $a$  so that  $f(x) = \frac{x^2 + ax + 5}{x + 1}$  has a slant asymptote  $y = x + 3$ .
26. Prove that  $f(x) = \frac{\ln x}{x}$  has a horizontal asymptote  $y = 0$ .
27. Let  $I$  be an open interval such that  $4 \in I$  and let a function  $f$  be defined on a set  $D = I \setminus \{4\}$ . Evaluate  $\lim_{x \rightarrow 4} f(x)$ , where  $x + 2 \leq f(x) \leq x^2 - 10$  for all  $x \in D$ .
28.  $\lim_{x \rightarrow 1} f(x)$ , where  $2x - 1 \leq f(x) \leq x^2$  for all  $x$  in the interval  $(0, 2)$ .
29. Use the squeeze theorem to show that  $\lim_{x \rightarrow 0^+} (\sqrt{x} e^{\sin(1/x)}) = 0$ .
30.  $\lim_{x \rightarrow 0^+} \left[ (x^2 + x)^{1/3} \sin \left( \frac{1}{x^2} \right) \right]$
31.  $\lim_{x \rightarrow 0} x \sin \left( \frac{e}{x} \right)$
32.  $\lim_{x \rightarrow 0} x \sin \left( \frac{1}{x^2} \right)$
33.  $\lim_{x \rightarrow \pi/2^+} \frac{x}{\cot x}$
34.  $\lim_{x \rightarrow 0} \frac{1 - e^{-x}}{1 - x}$
35.  $\lim_{x \rightarrow 0} \frac{e^{-x^2} \cos(x^2)}{x^2}$
36.  $\lim_{x \rightarrow 1} \frac{x^{76} - 1}{x^{45} - 1}$
37.  $\lim_{x \rightarrow 0} \frac{(\sin x)^{100}}{x^{99} \sin 2x}$
38.  $\lim_{x \rightarrow 0} \frac{x^{100} \sin 7x}{(\sin x)^{99}}$
39.  $\lim_{x \rightarrow 0} \frac{x^{100} \sin 7x}{(\sin x)^{101}}$
40.  $\lim_{x \rightarrow 0} \frac{\arcsin 3x}{\arcsin 5x}$
41.  $\lim_{x \rightarrow 0} \frac{\sin 3x}{\sin 5x}$
42.  $\lim_{x \rightarrow 0} \frac{x^3 \sin \left( \frac{1}{x^2} \right)}{\sin x}$
43.  $\lim_{x \rightarrow 0} \frac{\sin x}{\sqrt{x \sin 4x}}$
44.  $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x \sin x}$
45.  $\lim_{x \rightarrow \infty} x \tan(1/x)$
46.  $\lim_{x \rightarrow 0} \left( \frac{1}{\sin x} - \frac{1}{x} \right)$
47.  $\lim_{x \rightarrow 0} \frac{x - \sin x}{x^3}$
48.  $\lim_{x \rightarrow 0^+} (\sin x)(\ln \sin x)$
49.  $\lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt{x}}$
50.  $\lim_{x \rightarrow \infty} \frac{\ln 3x}{x^2}$

51.  $\lim_{x \rightarrow \infty} \frac{(\ln x)^2}{x}$
52.  $\lim_{x \rightarrow 1} \frac{\ln x}{x}$
53.  $\lim_{x \rightarrow 0} \frac{\ln(2 + 2x) - \ln 2}{x}$
54.  $\lim_{x \rightarrow \infty} \frac{\ln((2x)^{1/2})}{\ln((3x)^{1/3})}$
55.  $\lim_{x \rightarrow 0} \frac{\ln(1 + 3x)}{2x}$
56.  $\lim_{x \rightarrow 1} \frac{\ln(1 + 3x)}{2x}$
57.  $\lim_{\theta \rightarrow \frac{\pi}{2}^+} \frac{\ln(\sin \theta)}{\cos \theta}$
58.  $\lim_{x \rightarrow 1} \frac{1 - x + \ln x}{1 + \cos(\pi x)}$
59.  $\lim_{x \rightarrow 0} \left( \frac{1}{x^2} - \frac{1}{\tan x} \right)$
60.  $\lim_{x \rightarrow 0} (\cosh x)^{\frac{1}{x^2}}$
61.  $\lim_{x \rightarrow 0^+} x^x$
62.  $\lim_{x \rightarrow 0^+} x^{\tan x}$
63.  $\lim_{x \rightarrow 0^+} (\sin x)^{\tan x}$
64.  $\lim_{x \rightarrow 0} (1 + \sin x)^{\frac{1}{x}}$
65.  $\lim_{x \rightarrow \infty} (x + \sin x)^{\frac{1}{x}}$
66.  $\lim_{x \rightarrow \infty} x^{\frac{1}{x}}$
67.  $\lim_{x \rightarrow \infty} \left( 1 + \sin \frac{3}{x} \right)^x$
68.  $\lim_{x \rightarrow 0^+} (x + \sin x)^{\frac{1}{x}}$
69.  $\lim_{x \rightarrow 0^+} \left( \frac{x}{x + 1} \right)^x$
70.  $\lim_{x \rightarrow e^+} (\ln x)^{\frac{1}{x-e}}$
71.  $\lim_{x \rightarrow e^+} (\ln x)^{\frac{1}{x}}$
72.  $\lim_{x \rightarrow 0} e^{x \sin(1/x)}$
73.  $\lim_{x \rightarrow 0} (1 - 2x)^{1/x}$
74.  $\lim_{x \rightarrow 0^+} (1 + 7x)^{1/5x}$
75.  $\lim_{x \rightarrow 0^+} (1 + 3x)^{1/8x}$
76.  $\lim_{x \rightarrow 0} \left( 1 + \frac{x}{2} \right)^{3/x}$
77. Let  $x_1 = 100$ , and for  $n \geq 1$ , let  $x_{n+1} = \frac{1}{2}(x_n + \frac{100}{x_n})$ . Assume that  $L = \lim_{n \rightarrow \infty} x_n$  exists, and calculate  $L$ .
78. (a) Find  $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2}$ , or show that it does not exist.  
 (b) Find  $\lim_{x \rightarrow 2\pi} \frac{1 - \cos x}{x^2}$ , or show that it does not exist.  
 (c) Find  $\lim_{x \rightarrow -1} \arcsin x$ , or show that it does not exist.
79. Compute the following limits or state why they do not exist:
  - (a)  $\lim_{h \rightarrow 0} \frac{\sqrt[4]{16 + h}}{2h}$
  - (b)  $\lim_{x \rightarrow 1} \frac{\ln x}{\sin(\pi x)}$

$$(c) \lim_{u \rightarrow \infty} \frac{u}{\sqrt{u^2 + 1}}$$

$$(d) \lim_{x \rightarrow 0} (1 - 2x)^{1/x}$$

$$(e) \lim_{x \rightarrow 0} \frac{(\sin x)^{100}}{x^{99} \sin(2x)}$$

$$(f) \lim_{x \rightarrow \infty} \frac{1.01^x}{x^{100}}$$

$$(b) \lim_{x \rightarrow 1} \frac{\sin(x-1)}{x^2 + x - 2}$$

$$(c) \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2 + 4x}}{4x + 1}$$

$$(d) \lim_{x \rightarrow \infty} (e^x + x)^{1/x}$$

80. Find the following limits. If a limit does not exist, write 'DNE'. No justification necessary.

$$(a) \lim_{x \rightarrow \infty} (\sqrt{x^2 + x} - x)$$

$$(b) \lim_{x \rightarrow 0} \cot(3x) \sin(7x)$$

$$(c) \lim_{x \rightarrow 0^+} x^x$$

$$(d) \lim_{x \rightarrow \infty} \frac{x^2}{e^x}$$

$$(e) \lim_{x \rightarrow 3} \frac{\sin x - x}{x^3}$$

81. Evaluate the following limits, if they exist.

$$(a) \lim_{x \rightarrow 0} \frac{f(x)}{|x|} \text{ given that } \lim_{x \rightarrow 0} x f(x) = 3.$$

82. Evaluate the following limits, if they exist.

$$(a) \lim_{x \rightarrow 4} \left[ \frac{1}{\sqrt{x} - 2} - \frac{4}{x - 4} \right]$$

$$(b) \lim_{x \rightarrow 1} \frac{x^2 - 1}{e^{1-x^2} - 1}$$

$$(c) \lim_{x \rightarrow 0} (\sin x)(\ln x)$$

83. Evaluate the following limits. Use " $\infty$ " or " $-\infty$ " where appropriate.

$$(a) \lim_{x \rightarrow 1^-} \frac{x + 1}{x^2 - 1}$$

$$(b) \lim_{x \rightarrow 0} \frac{\sin 6x}{2x}$$

$$(c) \lim_{x \rightarrow 0} \frac{\sinh 2x}{xe^x}$$

$$(d) \lim_{x \rightarrow 0^+} (x^{0.01} \ln x)$$

84. Use the definition of limits to prove that

$$\lim_{x \rightarrow 0} x^3 = 0.$$

85. (a) Sketch an approximate graph of  $f(x) = 2x^2$  on  $[0, 2]$ . Show on this graph the points  $P(1, 0)$  and  $Q(0, 2)$ . When using the precise definition of  $\lim_{x \rightarrow 1} f(x) = 2$ , a number  $\delta$  and another number  $\epsilon$  are used. Show points on the graph which these values determine. (Recall that the interval determined by  $\delta$  must not be greater than a particular interval determined by  $\epsilon$ .)
- (b) Use the graph to find a positive number  $\delta$  so that whenever  $|x - 1| < \delta$  it is always true that  $|2x^2 - 2| < \frac{1}{4}$ .

- (c) State exactly what has to be proved to establish this limit property of the function  $f$ .

86. If  $f'$  is continuous, use L'Hospital's rule to show that

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x-h)}{2h} = f'(x).$$

Explain the meaning of this equation with the aid of a diagram.

## 1.3 Continuity

1. Given the function

$$f(x) = \begin{cases} c - x & \text{if } x \leq \pi \\ c \sin x & \text{if } x > \pi \end{cases}$$

- (a) Find the values of the constant  $c$  so that the function  $f(x)$  is continuous.
  - (b) For the value of  $c$  found above verify whether the 3 conditions for continuity are satisfied.
  - (c) Draw a graph of  $f(x)$  from  $x = -\pi$  to  $x = 3\pi$  indicating the scaling used.
2. (a) Use the Intermediate Value Property to show that  $2^x = \frac{10}{x}$  for some  $x > 0$ .
- (b) Show that the equation  $2^x = \frac{10}{x}$  has no solution for  $x < 0$ .
3. Sketch a graph of the function

$$f(x) = \begin{cases} 2 - x^2 & \text{if } 0 \leq x < 1 \\ \frac{5}{2} & \text{if } x = 1 \\ |2 - x| & \text{if } 1 < x \leq 3 \\ \frac{1}{x-3} & \text{if } 3 < x \leq 5 \\ 2 + \sin(2\pi x) & \text{if } 5 < x \leq 6 \\ 2 & \text{if } x > 6 \end{cases}$$

Answer the following questions by TRUE or FALSE:

- (a) Is  $f$  continuous at:
  - i.  $x = 1$ ?
  - ii.  $x = 6$ ?

(b) Do the following limits exist?

i.  $\lim_{x \rightarrow 1} f(x)$

ii.  $\lim_{x \rightarrow 3^-} f(x)$

(c) Is  $f$  differentiable

i. at  $x = 1$ ?

ii. on  $(1, 3)$ ?

4. Assume that

$$f(x) = \begin{cases} 2 + \sqrt{x} & \text{if } x \geq 1 \\ \frac{x}{2} + \frac{5}{2} & \text{if } x < 1 \end{cases}$$

(a) Determine whether or not  $f$  is continuous at  $x = 1$ . Justify your answer and state your conclusion.

(b) Using the definition of the derivative, determine  $f'(1)$ .

5. Give one example of a function  $f(x)$  that is continuous for all values of  $x$  except  $x = 3$ , where it has a removable discontinuity. Explain how you know that  $f$  is discontinuous at  $x = 3$ , and how you know that the discontinuity is removable.

## 1.4 Miscellaneous

1. (a) Solve the following equation:  $\pi^{x+1} = e$ .

(b) Solve the following equation:  $2^{3^x} = 10$ .

2. Find the domain of the function  $f(x) = \frac{\ln(\ln(\ln x))}{x-3} + \sin x$ .

3. (a) What is meant by saying that  $L$  is the *limit* of  $f(x)$  as  $x$  approaches  $a$ ?

(b) What is meant by saying that the function  $f(x)$  is *continuous* at  $x = a$ ?

(c) State two properties that a continuous function  $f(x)$  can have, either of which guarantees the function is not differentiable at  $x = a$ . Draw an example for each.

# Chapter 2

## Differentiation Rules

### 2.1 Introduction

1. **Derivative.** The derivative of a function  $f$  at a number  $a$  is
$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$
if this limit exists.
2. **Tangent Line.** An equation of the tangent line to  $y = f(x)$  at  $(a, f(a))$  is given by  $y - f(a) = f'(a)(x - a)$ .
3. **Product and Quotient Rules.** If  $f$  and  $g$  are both differentiable, then
$$(fg)' = f \cdot g' + g \cdot f' \text{ and } \left(\frac{f}{g}\right)' = \frac{g \cdot f' - f \cdot g'}{g^2},$$
with  $g(x) \neq 0$ .
4. **Chain Rule** If  $f$  and  $g$  are both differentiable and  $F = f \circ g$  is the composite function defined by  $F(x) = f(g(x))$ , then  $F$  is differentiable and  $F'$  is given by  $F'(x) = f'(g(x)) \cdot g'(x)$ .
5. **Implicit Differentiation.** Let a function  $y = y(x)$  be implicitly defined by  $F(x, y) = G(x, y)$ . To find the derivative  $y'$  do the following:
  - (a) Use the chain rule to differentiate both sides of the given equation, thinking of  $x$  as the independent variable.
  - (b) Solve the resulting equation for  $\frac{dy}{dx}$ .
6. **The Method of Related Rates.** If two variables are related by an equation and both are functions of a third variable (such as time), we can find a relation

between their rates of change. We say the rates are related, and we can compute one if we know the other. We proceed as follows:

- (a) Identify the independent variable on which the other quantities depend and assign it a symbol, such as  $t$ . Also, assign symbols to the variable quantities that depend on  $t$ .
- (b) Find an equation that relates the dependent variables.
- (c) Differentiate both sides of the equation with respect to  $t$  (using the chain rule if necessary).
- (d) Substitute the given information into the related rates equation and solve for the unknown rate.

## 2.2 Derivatives

1. (a) Assume that  $f(x)$  is a real-valued function defined for all real numbers  $x$  on an open interval whose center is a certain real number  $a$ . What does it mean to say that  $f(x)$  has a derivative  $f'(a)$  at  $x = a$ , and what is the value of  $f'(a)$ ? (Give the definition of  $f'(a)$ .)  
 (b) Use the definition of  $f'(a)$  you have just given in part (a) to show that if  $f(x) = \frac{1}{2x-1}$  then  $f'(3) = -0.08$ .  
 (c) Find  $\lim_{h \rightarrow 0} \frac{\sin^7\left(\frac{\pi}{6} + \frac{h}{2}\right) - \left(\frac{1}{2}\right)^7}{h}$ .
2. Let  $I$  be a bounded function on  $\mathbb{R}$  and define  $f$  by  $f(x) = x^2 I(x)$ . Show that  $f$  is differentiable at  $x = 0$ .
3. Use the definition of derivative to find  $f'(2)$  for  $f(x) = x + \frac{1}{x}$ .
4. If  $g$  is continuous (but not differentiable) at  $x = 0$ ,  $g(0) = 8$ , and  $f(x) = xg(x)$ , find  $f'(0)$ .
5. (a) State the definition of the derivative of  $f(x)$  at  $x = a$ .  
 (b) Using the definition of the derivative of  $f(x)$  at  $x = 4$ , find the value of  $f'(4)$  if  $f(x) = \sqrt{5-x}$ .
6. Let  $f$  be a function that is continuous everywhere and let

$$F(x) = \begin{cases} \frac{f(x)\sin^2 x}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Use the definition of derivatives to evaluate  $F'(0)$ . Your answer should be in terms of  $f$ .

7. The function

$$f(x) = \begin{cases} e^x & \text{if } x \leq 1 \\ mx + b & \text{if } x > 1 \end{cases}$$

is continuous and differentiable at  $x = 1$ . Find the values for the constants  $m$  and  $b$ .

8. Suppose the functions  $F(x)$  and  $G(x)$  satisfy the following properties:

$$\begin{aligned} F(3) &= 2, & G(3) &= 4, & G(0) &= 3 \\ F'(3) &= -1, & G'(3) &= 0, & G'(0) &= 0 \end{aligned}$$

- (a) If  $S(x) = \frac{F(x)}{G(x)}$ , find  $S'(3)$ . Simplify your answer.
- (b) If  $T(x) = F(G(x))$ , find  $T'(0)$ . Simplify your answer.
- (c) If  $U(x) = \ln(F(x))$ , find  $U'(3)$ . Simplify your answer.
9. Suppose that  $f(x)$  and  $g(x)$  are differentiable functions and that  $h(x) = f(x)g(x)$ . You are given the following table of values:

|         |    |
|---------|----|
| $h(1)$  | 24 |
| $g(1)$  | 6  |
| $f'(1)$ | -2 |
| $h'(1)$ | 20 |

Using the table, find  $g'(1)$ .

10. Given  $F(x) = f^2(g(x))$ ,  $g(1) = 2$ ,  $g'(1) = 3$ ,  $f(2) = 4$ , and  $f'(2) = 5$ , find  $F'(1)$ .
11. Compute the derivative of  $f(x) = \frac{x}{x-2}$  by
- (a) using the limit definition of the derivative;
- (b) using the quotient rule.
12. (a) Write down the formula for the derivative of  $f(x) = \tan x$ . State how you could use formulas for derivatives of the sine and cosine functions to derive this formula. (DO NOT do this derivation.)
- (b) Use the formula given in part (a) to derive the formula for the derivative of the arctangent function.

- (c) Use formulas indicated in parts (a) and (b) to evaluate and simplify the derivative of  $g(x) = \tan(x^2) + \arctan(x^2)$  at  $x = \frac{\sqrt{\pi}}{2}$ . That is, you want to compute a simplified expression for  $g' \left( \frac{\sqrt{\pi}}{2} \right)$ .

13. If  $g(x) = 2x^3 + \ln x$  is the derivative of  $f(x)$ , find

$$\lim_{x \rightarrow 0} \frac{f(1+x) - f(1)}{x}.$$

14. Find

$$\lim_{x \rightarrow 0} \frac{\sqrt{1+x} + (1+x)^7 - 2}{x}.$$

15. Let  $f(x) = x^2 \sin \left( \frac{1}{x} \right)$  if  $x \neq 0$ , and  $f(0) = 0$ . Find  $f'(0)$  (or say why it doesn't exist.)

16. Let  $f(x) = 2x + \cos x$ . Say why  $f(x)$  is an increasing function for all  $x$ . Let  $g(x) = f^{-1}(x)$ , and calculate  $g'(0)$ .

17. Show that  $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$ .

18. Suppose that  $f$  is a differentiable function such that  $f(g(x)) = x$ , and  $f'(x) = 1 + (f(x))^2$ . Show that  $g'(x) = \frac{1}{1+x^2}$ .

19. If  $y = \frac{\sqrt{x^2+1} - \sqrt{x^2-1}}{\sqrt{x^2+1} + \sqrt{x^2-1}}$ , show that  $\frac{dy}{dx} = 2x - \frac{2x^3}{\sqrt{x^4-1}}$ .

20. Let  $f$  be a function differentiable on  $\mathbb{R}$  and such that for all  $x \neq 2$ ,  $f(x) = \frac{x^4 - 16}{x - 2}$ . Find  $f^{(4)}(2)$ .

21. Given  $y = \frac{1}{x} + \cos 2x$ , find  $\frac{d^5 y}{dx^5}$ . Simplify your answer.

22. Find the values of  $A$  and  $B$  that make

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x \geq 0 \\ A \sin x + B \cos x & \text{if } x < 0 \end{cases}$$

differentiable at  $x = 0$ .

23. If  $f$  and  $g$  are two functions for which  $f' = g$  and  $g' = f$  for all  $x$ , then prove that  $f^2 - g^2$  must be a constant.
24. Show that if  $f$  and  $g$  are twice differentiable functions (i.e. both have continuous second derivatives) then  $(fg)'' = f''g + 2f'g' + fg''$ .
25. Find  $y'$  when  $y = \frac{(x+2)^{3\ln x}}{(x^2+1)^{1/2}}$ .
26. Find  $y'$  when  $y = e^{4 \cosh \sqrt{x}}$ .
27. Find  $f'(0)$  for the function  $f(x) = \sin^{-1}(x^2 + x) + 5^x$ .
28. Given
- $$y = \frac{\sqrt{1+2x} \sqrt[4]{1+4x} \sqrt[6]{1+6x} \dots \sqrt[100]{1+100x}}{\sqrt[3]{1+3x} \sqrt[5]{1+5x} \sqrt[7]{1+7x} \dots \sqrt[101]{1+101x}},$$
- find  $y'$  at  $x = 0$ .

29. The following questions involve derivatives.

- (a) Evaluate  $D_t \cos^{-1}(\cosh(e^{-3t}))$ , without simplifying your answer.
- (b) Use logarithmic differentiation to find  $y'(u)$  as a function of  $u$  alone, where

$$y(u) = \left( \frac{(u+1)(u+2)}{(u^2+1)(u^2+2)} \right)^{1/3},$$

without simplifying your answer.

30. Differentiate  $y = \cosh(\arcsin(x^2 \ln x))$ .
31. Given  $y = \tan(\cos^{-1}(e^{4x}))$ , find  $\frac{dy}{dx}$ . Do not simplify your answer.
- Find the derivatives of the following functions:

- |                                                |                                     |
|------------------------------------------------|-------------------------------------|
| 32. (a) $y = e^{\cos x^2}$                     | 34. (a) $y = x^{\sinh x}$           |
| (b) $y = x^{20} \arctan x$                     | (b) $\ln(x+y) = xy - y^3$           |
| (c) $y = x^{\ln x}$                            | 35. (a) $y = \sec(\sinh x)$         |
| 33. (a) $y = e^{3 \ln(2x+1)}$                  | (b) $e^x + e^y = x^e + y^e + e^3$   |
| (b) $y = x^{2x}$                               | 36. (a) $f(x) = \frac{3x^2+1}{e^x}$ |
| (c) $y = \frac{e^{2x}}{(x^2+1)^3(1+\sin x)^5}$ | (b) $g(z) = \sin \sqrt{z^2+1}$      |
| (d) $x^2 + 2xy^2 = 3y + 4$                     | (c) $f(x) = x^x$                    |

- (d)  $h(y) = \sqrt{\frac{\cos y}{y}}$
37. (a)  $f(x) = \frac{1}{x + \frac{1}{x}}$   
 (b)  $g(x) = \ln(\sqrt{x^2 + 1} \sin^4 x)$
38. (a)  $f(x) = \arctan(\sqrt{x})$   
 (b)  $f(x) = \cosh(5 \ln x)$
39. (a)  $f(x) = 10^{3x}$   
 (b)  $f(x) = x^{10} \tanh x$   
 (c)  $f(x) = x^{\cos x}$
40. (a)  $f(x) = x^{x^2}$   
 (b)  $f(x) = \ln(\cos 3x)$
41. (a)  $f(x) = \frac{(x-1)^2}{(x+1)^3}$   
 (b)  $f(x) = 2^{2x} - (x^2 + 1)^{2/3}$   
 (c)  $f(x) = \tan^2(x^2)$   
 (d)  $f(x) = x^{\arctan x}$   
 (e) Compute  $f'''(x)$  where  $f(x) = \sinh(2x)$ .
42. (a)  $f(x) = 5x + x^5 + 5^x + \sqrt[5]{x} + \ln 5$   
 (b)  $y = x^{10} \tanh x$   
 (c)  $y = (\ln x)^{\cos x}$
43. (a)  $f(x) = \ln(\sinh x)$   
 (b)  $f(x) = e^{x \cos x}$   
 (c)  $f(x) = \frac{\sin x}{1 + \cos x}$   
 (d)  $f(x) = x^x$
44. (a)  $f(x) = g(x^3)$ , where  $g'(x) = \frac{1}{x^2}$
- (b)  $f(x) = x^2 \sin^2(2x^2)$
- (c)  $f(x) = (x+2)^x$
45. (a)  $y = \sec \sqrt{x^2 + 1}$   
 (b)  $y = x^{e^x}$
46. (a)  $y = x^3 + 3^x + x^{3x}$   
 (b)  $y = e^{-5x} \cosh 3x$   
 (c)  $\tan^{-1} \frac{y}{x} = \frac{1}{2} \ln(x^2 + y^2)$   
 (d)  $y = \frac{x^5 e^{x^3} \sqrt[3]{x^2 + 1}}{(x+1)^4}$
47. (a)  $f(x) = \frac{\ln(x^2 - 3x + 8)}{\sec(x^2 + 7x)}$   
 (b)  $f(x) = \arctan(\cosh(2x - 3))$   
 (c)  $f(x) = \cos(e^{3x-4})$   
 (d)  $f(x) = (\tan x)^{\ln x + x^2}$   
 (e)  $f(x) = (\sec^2 x - \tan^2 x)^{45}$
48. (a)  $h(t) = e^{-\tan(\frac{t}{3})}$   
 (b)  $2y^{2/3} = 4y^2 \ln x$   
 (c)  $f(y) = 3^{\log_7(\arcsin y)}$
49. (a)  $f(x) = \sin^{-1}(x^2 + x) + 5^x$   
 (b)  $g(x) = \cosh\left(\frac{\sqrt{x+1}}{x^2-3}\right)$
50. (a)  $f(x) = \frac{\sinh^{-1}(2^x)}{e^{4x} + a}$ ,  $a \in \mathbb{R}$   
 (b)  $g(x) = \frac{(2 + \cos(3x^2))e^{\pi x}}{3\sqrt{x}}$
51. (a) Find  $\frac{d^2 y}{dx^2}$  if  $y = \arctan(x^2)$ .  
 (b)  $y = x^{\sqrt{x}}$

## 2.3 Related Rates

1. A ladder 15 ft long rests against a vertical wall. Its top slides down the wall while its bottom moves away along the level ground at a speed of 2 ft/s. How fast is the angle between the top of the ladder and the wall changing when the angle is  $\pi/3$  radians?
2. A ladder 12 meters long leans against a wall. The foot of the ladder is pulled away from the wall at the rate  $\frac{1}{2}$  m/min. At what rate is the top of the ladder falling when the foot of the ladder is 4 meters from the wall?
3. A rocket  $R$  is launched vertically and its tracked from a radar station  $S$  which is 4 miles away from the launch site at the same height above sea level.

At a certain instant after launch,  $R$  is 5 miles away from  $S$  and the distance from  $R$  to  $S$  is increasing at a rate of 3600 miles per hour. Compute the vertical speed  $v$  of the rocket at this instant.

4. A boat is pulled into a dock by means of a rope attached to a pulley on the dock, Figure 2.1. The rope is attached to the bow of the boat at a point 1 m below the pulley. If the rope is pulled through the pulley at a rate of 1 m/sec, at what rate will the boat be approaching the dock when 10 m of rope is out.

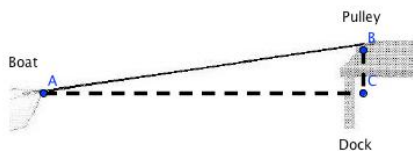
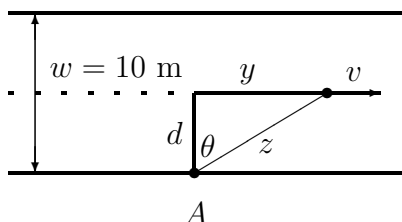


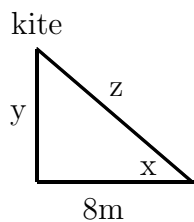
Figure 2.1: Boat, Pulley, and Dock

5. A person ( $A$ ) situated at the edge of the river observes the passage of a speed boat going downstream. The boat travels exactly through the middle of the river (at the distance  $d$  from the riverbank.) The river is 10 m wide. When the boat is at  $\theta = 60^\circ$  (see figure) the observer measures the rate of change of the angle  $\theta$  to be 2 radians/second.



What is the speed,  $v$ , of the speed boat at that instant?

6. An airplane flying horizontally at an altitude of  $y = 3 \text{ km}$  and at a speed of  $480 \text{ km/h}$  passes directly above an observer on the ground. How fast is the distance  $D$  from the observer to the airplane increasing 30 seconds later?
7. An airplane flying horizontally at a constant height of  $1000 \text{ m}$  above a fixed radar station. At a certain instant the angle of elevation  $\theta$  at the station is  $\frac{\pi}{4}$  radians and decreasing at a rate of  $0.1 \text{ rad/sec}$ . What is the speed of the aircraft at this moment.
8. A kite is rising vertically at a constant speed of  $2 \text{ m/s}$  from a location at ground level which is  $8 \text{ m}$  away from the person handling the string of the kite.



- (a) Let  $z$  be the distance from the kite to the person. Find the rate of change of  $z$  with respect to time  $t$  when  $z = 10$ .
- (b) Let  $x$  be the angle the string makes with the horizontal. Find the rate of change of  $x$  with respect to time  $t$  when the kite is  $y = 6 \text{ m}$  above ground.

9. A balloon is rising at a constant speed 4m/sec. A boy is cycling along a straight road at a speed of 8m/sec. When he passes under the balloon, it is 36 metres above him. How fast is the distance between the boy and balloon increasing 3 seconds later.
10. A helicopter takes off from a point 80 m away from an observer located on the ground, and rises vertically at 2 m/s. At what rate is elevation angle of the observer's line of sight to the helicopter changing when the helicopter is 60 m above the ground.
11. An oil slick on a lake is surrounded by a floating circular containment boom. As the boom is pulled in, the circular containment boom. As the boom is pulled in, the circular containment area shrinks (all the while maintaining the shape of a circle.) If the boom is pulled in at the rate of 5 m/min, at what rate is the containment area shrinking when it has a diameter of 100m?
12. Consider a cube of variable size. (The edge length is increasing.) Assume that the volume of the cube is increasing at the rate of  $10 \text{ cm}^3/\text{minute}$ . How fast is the surface area increasing when the edge length is 8 cm?
13. The height of a rectangular box is increasing at a rate of 2 meters per second while the volume is decreasing at a rate of 5 cubic meters per second. If the base of the box is a square, at what rate is one of the sides of the base decreasing, at the moment when the base area is 64 square meters and the height is 8 meters?
14. Sand is pouring out of a tube at 1 cubic meter per second. It forms a pile which has the shape of a cone. The height of the cone is equal to the radius of the circle at its base. How fast is the sandpile rising when it is 2 meters high?
15. A water tank is in the shape of a cone with vertical axis and vertex downward. The tank has radius 3 m and is 5 m high. At first the tank is full of water, but at time  $t = 0$  (in seconds), a small hole at the vertex is opened and the water begins to drain. When the height of water in the tank has dropped to 3 m, the water is flowing out at  $2 \text{ m}^3/\text{s}$ . At what rate, in meters per second, is the water level dropping then?
16. A boy starts walking north at a speed of 1.5 m/s, and a girl starts walking west at the same point  $P$  at the same time at a speed of 2 m/s. At what rate is the distance between the boy and the girl increasing 6 seconds later?
17. At noon of a certain day, the ship  $A$  is 60 miles due north of the ship  $B$ . If the ship  $A$  sails east at speed of 15 miles per hour and  $B$  sails north at speed

of 12.25 miles per hour, determine how rapidly the distance between them is changing 4 hours later?

18. A lighthouse is located on a small island three (3) km off-shore from the nearest point **P** on a straight shoreline. Its light makes four (4) revolutions per minute. How fast is the light beam moving along the shoreline when it is shining on a point one (1) km along the shoreline from **P**?
19. A police car, approaching right-angled intersection from the north, is chasing a speeding SUV that has turned the corner and is now moving straight east. When the police car is 0.6 km north of intersection and the SUV is 0.8 km east of intersection, the police determine with radar that the distance between them and the SUV is increasing at 20 km/hr. If the police car is moving at 60 km/hr at the instant of measurement, what is the speed of the SUV?

## 2.4 Tangent Lines and Implicit Differentiation

1. At what point on the curve  $y = \sinh x$  does the tangent line have a slope of 1?
2. Find the point(s) on the graph  $y = x^3$  where the line through the point (4, 0) is tangent to  $y$ .
3. (a) Find  $\arcsin\left(-\frac{1}{\sqrt{2}}\right)$   
(b) Find the equation of the tangent line to the graph of  $y = \arcsin x$  when  $x = -\frac{1}{\sqrt{2}}$ .
4. Find the equation of the line that is tangent to the graph of  $y = \sqrt{x} - \frac{1}{\sqrt{x}}$  at  $x = 1$ .
5. Let  $C$  be the curve  $y = (x - 1)^3$  and let  $L$  be the line  $3y + x = 0$ .
  - (a) Find the equation of all lines that are tangent to  $C$  and are also perpendicular to  $L$ .
  - (b) Draw a labeled diagram showing the curve  $C$ , the line  $L$ , and the line(s) of your solution to part (a). For each line of your solution, mark on the diagram the point where it is tangent to  $C$  and (without necessarily calculating the coordinates) the point where it is perpendicular to  $L$ .

6. Find  $\frac{dy}{dx}$  for the curve  $e^y \ln(x+y) + 1 = \cos(xy)$  at the point  $(1, 0)$ .  
Find  $\frac{dy}{dx}$  if:
7.  $x^5 + y^5 = 5xy$
8.  $x^y = y^x$
9.  $e^y - 3^x = x \sinh y$
10.  $\sinh x - \cos y = x^2 y$
11.  $\ln(x-y) = xy + y^3$
12. Find the slope of the tangent line to the curve  $y + x \ln y - 2x = 0$  at the point  $(1/2, 1)$ .
13. Use implicit differentiation to answer the following:
  - (a) Find the tangent line to the graph of  $\sin(x+y) = y^2 \cos x$  at  $(0, 0)$ .
  - (b) Show that the tangent lines to the graph of  $x^2 - xy + y^2 = 3$ , at the points where the graph crosses the  $x$ -axis, are parallel to each other.
14. The curve implicitly defined by
$$x \sin y + y \sin x = \pi$$
passes through the point  $P = P\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ .
  - (a) Find the slope of the tangent line through  $P$ .
  - (b) Write the tangent line through  $P$ .
15. Write the equation of the line tangent to the curve  $\sin(x+y) = xe^{x+y}$  at the origin  $(0, 0)$ .
16. Find the slope of the tangent line to the curve  $xy = 6e^{2x-3y}$  at the point  $(3, 2)$ .
17. (a) Find  $\frac{dy}{dx}$  for the function defined implicitly by  $x^2 y + ay^2 = b$ , where  $a$  and  $b$  are fixed constants.  
(b) For the function defined in part (a) find the values of the constants  $a$  and  $b$  if the point  $(1, 1)$  is on the graph and the tangent line at  $(1, 1)$  is  $4x + 3y = 7$ .
18. Let  $l$  be any tangent to the curve  $\sqrt{x} + \sqrt{y} = \sqrt{k}$ ,  $k > 0$ . Show that the sum of the  $x$ -intercept and the  $y$ -intercept of  $l$  is  $k$ .

19. Show that the length of the portion of any tangent line to the curve

$$x^{2/3} + y^{2/3} = 9,$$

cut off by the coordinate axis is constant. What is this length?

20. Let  $C$  denote the circle whose equation is  $(x - 5)^2 + y^2 = 25$ . Notice that the point  $(8, -4)$  lies on the circle  $C$ . Find the equation of the line that is tangent to  $C$  at the point  $(8, -4)$ .

21. The so called *devil's curve* is described by the equation

$$y^2(y^2 - 4) = x^2(x^2 - 5).$$

- (a) Compute the  $y$ -intercept of the curve.
  - (b) Use implicit differentiation to find an expression for  $\frac{dy}{dx}$  at the point  $(x, y)$ .
  - (c) Give an equation for the tangent line to curve at  $(\sqrt{5}, 0)$ .
22. The equation  $e^y + y(x - 2) = x^2 - 8$  defines  $y$  implicitly as a function of  $x$  near the point  $(3, 0)$ .
- (a) Determine the value of  $y'$  at this point.
  - (b) Use the linear approximation to estimate the value of  $y$  when  $x = 2.98$ .
23. The equation  $e^y + y(x - 3) = x^2 - 15$  defines  $y$  implicitly as a function of  $x$  near the point  $A(4, 0)$ .
- (a) Determine the values of  $y'$  and  $y''$  at this point.
  - (b) Use the tangent line approximation to estimate the value of  $y$  when  $x = 3.95$ .
  - (c) Is the true value of  $y$  greater or less than the approximation in part (b)? Make a sketch showing how the curve relates to the tangent line near the point  $A(4, 0)$ .

# Chapter 3

## Applications of Differentiation

### 3.1 Introduction

1. **Absolute Maximum and Minimum.** A function  $f$  has an absolute maximum at  $c$  if  $f(c) \geq f(x)$  for all  $x \in D$ , the domain of  $f$ . The number  $f(c)$  is called the maximum value of  $f$  on  $D$ .

A function  $f$  has an absolute minimum at  $c$  if  $f(c) \leq f(x)$  for all  $x \in D$ , the domain of  $f$ . The number  $f(c)$  is called the minimum value of  $f$  on  $D$ .

2. **Local Maximum and Minimum.** A function  $f$  has a local maximum at  $c$  if  $f(c) \geq f(x)$  for all  $x$  in an open interval containing  $c$ .

A function  $f$  has a local minimum at  $c$  if  $f(c) \leq f(x)$  for all  $x$  in an open interval containing  $c$ .

3. **Extreme Value Theorem.** If  $f$  is continuous on a closed interval  $[a, b]$ , then  $f$  attains an absolute maximum value  $f(c)$  and an absolute minimum value  $f(d)$  at some numbers  $c, d \in [a, b]$ .

4. **Fermat's Theorem.** If  $f$  has a local maximum or minimum at  $c$ , and  $f'(c)$  exists, then  $f'(c) = 0$ .

5. **Critical Number.** A critical number of a function  $f$  is a number  $c$  in the domain of  $f$  such that either  $f'(c) = 0$  or  $f'(c)$  does not exist.

6. **Closed Interval Method.** To find the absolute maximum and minimum values of a continuous function  $f$  on a closed interval  $[a, b]$ :

(a) Find the values of  $f$  at the critical numbers of  $f$  in  $(a, b)$ .

- (b) Find the values of  $f$  at the endpoints of the interval.
- (c) The largest of the values from Step 1 and Step 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

7. **Rolle's Theorem.** Let  $f$  be a function that satisfies the following three hypotheses:

- (a)  $f$  is continuous on the closed interval  $[a, b]$ .
- (b)  $f$  is differentiable on the open interval  $(a, b)$ .
- (c)  $f(a) = f(b)$ .

Then there is a number  $c$  in  $(a, b)$  such that  $f'(c) = 0$ .

8. **The Mean Value Theorem.** Let  $f$  be a function that satisfies the following hypotheses:

- (a)  $f$  is continuous on the closed interval  $[a, b]$ .
- (b)  $f$  is differentiable on the open interval  $(a, b)$ .

Then there is a number  $c$  in  $(a, b)$  such that  $f'(c) = \frac{f(b) - f(a)}{b - a}$  or, equivalently,  $f(b) - f(a) = f'(c)(b - a)$ .

9. **Increasing/Decreasing Test.**

- (a) If  $f'(x) > 0$  on an interval, then  $f$  is increasing on that interval.
- (b) If  $f'(x) < 0$  on an interval, then  $f$  is decreasing on that interval.

10. **The First Derivative Test.** Suppose that  $c$  is a critical number of a continuous function  $f$ .

- (a) If  $f'$  changes from positive to negative at  $c$ , then  $f$  has a local maximum at  $c$ .
- (b) If  $f'$  changes from negative to positive at  $c$ , then  $f$  has a local minimum at  $c$ .
- (c) If  $f'$  does not change sign at  $c$ , then  $f$  has no local minimum or maximum at  $c$ .

11. **Concavity.** If the graph of  $f$  lies above all of its tangent lines on an interval  $I$ , then it is called concave upward on  $I$ . If the graph of  $f$  lies below all of its tangents on  $I$ , it is called concave downward on  $I$ .

**12. Concavity Test.**

- (a) If  $f''(x) > 0$  for all  $x \in I$ , then the graph of  $f$  is concave upward on  $I$ .
- (b) If  $f''(x) < 0$  for all  $x \in I$ , then the graph of  $f$  is concave downward on  $I$ .

**13. Inflection Point.** A point  $P$  on a curve  $y = f(x)$  is called an inflection point if  $f$  is continuous there the curve changes from concave upward to concave downward or from concave downward to concave upward at  $P$ .

**14. The Second Derivative Test.** Suppose  $f''$  is continuous near  $c$ .

- (a) If  $f'(c) = 0$  and  $f''(c) > 0$  then  $f$  has a local minimum at  $c$ .
- (b) If  $f'(c) = 0$  and  $f''(c) < 0$  then  $f$  has a local maximum at  $c$ .

**15. Linear Approximation.** The linear function  $L(x) = f(a) + f'(a)(x - a)$  is called the linearization of  $f$  at  $a$ . For  $x$  close to  $a$  we have that  $f(x) \approx L(x) = f(a) + f'(a)(x - a)$  and this approximation is called the linear approximation of  $f$  at  $a$ .

**16. Differential.** Let  $f$  be a function differentiable at  $x \in \mathbb{R}$ . Let  $\Delta x = dx$  be a (small) given number. The differential  $dy$  is defined as  $dy = f'(x)\Delta x$ .

**17. Newton's Method.** To estimate a solution, say  $x = r$ , to the equation  $f(x) = 0$ :

- (a) Begin with an initial guess  $x_1$ .
- (b) Calculate  $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ .
- (c) If  $x_n$  is known then  $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ .
- (d) If  $x_n$  and  $x_{n+1}$  agree to  $k$  decimal places then  $x_n$  approximates the root  $r$  up to  $k$  decimal places and  $f(x_n) \approx 0$ .

**18. Antiderivative.** A function  $F$  is called an antiderivative of  $f$  on an interval  $I$  if  $F'(x) = f(x)$  for all  $x \in I$ .

**19. Natural Growth/Decay Equation.** The natural growth/decay is modeled by the initial-value problem

$$\frac{dy}{dt} = ky, \quad y(0) = y_0, \quad k \in \mathbb{R}.$$

20. **Newton's Law of Cooling and Heating** is given as

$$\frac{dT}{dt} = k(T - T_s)$$

where  $k$  is a constant,  $T = T(t)$  is the temperature of the object at time  $t$  and  $T_s$  is the temperature of surroundings.

## 3.2 Curve Sketching

1. Sketch the graph of  $f(x) = 3x^4 - 8x^3 + 10$ , after answering the following questions.
  - (a) Where is the graph increasing, and where is decreasing?
  - (b) Where is the graph concave upward, and where is it concave downward?
  - (c) Where are the local minima and local maxima? Establish conclusively that they *are* local minima and maxima.
  - (d) Where are the inflection points?
  - (e) What happens to  $f(x)$  as  $x \rightarrow \infty$  and as  $x \rightarrow -\infty$ .
2. In this question we consider the function  $f(x) = \frac{x-3}{\sqrt{x^2-9}}$ .
  - (a) Find the domain of  $f$ .
  - (b) Find the coordinates of all  $x$ - and  $y$ -intercepts, if any.
  - (c) Find all horizontal and vertical asymptotes, if any.
  - (d) Find all critical numbers, if any.
  - (e) Find all intervals on which  $f$  is increasing and those on which  $f$  is decreasing.
  - (f) Find the  $(x, y)$  coordinates of all maximum and minimum points, if any.
  - (g) Find all intervals on which  $f$  is concave up and those on which  $f$  is concave down.
  - (h) Find the  $(x, y)$  coordinates of all inflection points, if any.
  - (i) Sketch the graph of  $y = f(x)$  using all of the above information. All relevant points must be labeled.
3. Given  $f(x) = \frac{x^2-1}{x}$ :

- (a) Find the domain and  $x$ -intercepts.
  - (b) Find all asymptotes.
  - (c) Determine the intervals on which the function is increasing or decreasing. Find the local maximum and minimum, if they exist.
  - (d) Determine the intervals on which the function is concave upward or downward. Find the inflection points, if they exist.
  - (e) Sketch the graph.
4. Consider the function  $f(x) = x^3 - 2x^2 - x + 1$  on the interval  $[-1, 3]$ .
- (a) The derivative of  $f$  is:
  - (b) The critical points for  $f$  are:
  - (c) The second derivative of  $f$  is:
  - (d) The points of inflection of  $f$  are:
  - (e) The intervals on which  $f$  is increasing are:
  - (f) The intervals on which  $f$  is concave up are:
  - (g) The intervals on which  $f$  is concave down are:
  - (h)  $f$  has an absolute maximum at:
  - (i)  $f$  has an absolute minimum at:
  - (j)  $f$  has a local but not absolute maximum at:
  - (k)  $f$  has a local but not absolute minimum at:
  - (l) Sketch the graph of  $y = f(x)$  using all of the above information. All relevant points must be labeled.
5. The aim of this problem is to sketch the graph of  $f(x)$  where

$$f(x) = \frac{x^2 - 2}{x^4}, \quad f'(x) = -2\frac{x^2 - 4}{x^5}, \quad f''(x) = 2\frac{3x^2 - 20}{x^6}.$$

- (a) Find the following derivatives and say what, if anything, they tell us about asymptotes of the graph of  $y = f(x)$ .
  - i.  $\lim_{x \rightarrow 0^-} f(x)$
  - ii.  $\lim_{x \rightarrow 0^+} f(x)$
  - iii.  $\lim_{x \rightarrow -\infty} f(x)$
  - iv.  $\lim_{x \rightarrow \infty} f(x)$

- (b) Find the intervals on which  $f$  is increasing and decreasing.
- (c) Find the intervals on which  $f$  is concave up and concave down.
- (d) The the  $x$ -intercepts.
- (e) Find the coordinates of all inflection points.
- (f) Indicate the coordinates of all points where the graph of  $f$  has horizontal tangents. Are they local minima or maxima. Justify.
- (g) Sketch the graph of  $y = f(x)$  using all of the above information. All relevant points must be labeled. You may need the numerical values:

$$\sqrt{2} \approx 1.414, \sqrt{\frac{20}{3}} \approx 2.582, \text{ and } \frac{21}{200} = 0.105.$$

6. The goal of this exercise is to sketch the plot of  $f(x) = \frac{1}{(1 + e^x)^2}$ .

- (a) Find the domain of  $f$ .
- (b) Prove that the derivative of  $f$  is given by  $f'(x) = -\frac{2e^x}{(1 + e^x)^3}$ .
- (c) Prove that the second derivative of  $f$  is given by  $f''(x) = \frac{2e^x(2e^x - 1)}{(1 + e^x)^4}$ .
- (d) Find the equations of the two horizontal asymptotes by finding limits of  $f(x)$  at  $x \rightarrow +\infty$  and  $-\infty$ .
- (e) Find any  $x$ - and  $y$ -intercepts.
- (f) Prove there are no critical points.
- (g) Prove that  $\left(-\ln 2, \frac{4}{9}\right)$  is the only inflection point.
- (h) Find the intervals on which  $f$  is concave up and those on which  $f$  is concave down.
- (i) Sketch the graph of  $y = f(x)$  using all of the above information. (You may need  $\ln 2 \approx 0.7$ ,  $\frac{4}{9} \approx 0.44$ .)

7. Let

$$f(x) = \frac{x^2 - 4x}{(x + 4)^2}, \quad f'(x) = \frac{4(3x - 4)}{(x + 4)^3}, \quad f''(x) = -\frac{24(x - 4)}{(x + 4)^4}.$$

- (a) Find any  $x$ - and  $y$ -intercepts.
- (b) Determine any horizontal asymptotes of  $f(x)$  by taking appropriate limits.
- (c) Determine any vertical asymptotes of  $f(x)$  by taking appropriate limits.

- (d) Fill in the sign of each factor of  $f'(x)$  on the indicated intervals and thereby determine the sign of  $f'(x)$ . Use this to determine where  $f(x)$  is increasing/decreasing.

|             | $(-\infty, -4)$ | $(-4, -4/3)$ | $(4/3, \infty)$ |
|-------------|-----------------|--------------|-----------------|
| $(x + 4)^3$ |                 |              |                 |
| $3x - 4$    |                 |              |                 |
| $f'(x)$     |                 |              |                 |
| $f(x)$      |                 |              |                 |

- (e) Determine the location of any local extrema of  $f(x)$  and indicate whether they are minima or maxima.
- (f) Fill in the sign of each factor of  $f''(x)$  on the indicated intervals and thereby determine the sign of  $f''(x)$ . Use this to determine where  $f(x)$  is concave up/down.

|             | $(-\infty, -4)$ | $(-4, -4)$ | $(4, \infty)$ |
|-------------|-----------------|------------|---------------|
| $-24^3$     |                 |            |               |
| $(x + 4)^4$ |                 |            |               |
| $x - 4$     |                 |            |               |
| $f''(x)$    |                 |            |               |
| $f(x)$      |                 |            |               |

- (g) Determine the locations of any inflection points of  $f(x)$ .
- (h) Sketch the graph of  $y = f(x)$  using all of the above information.

8. Let

$$f(x) = \frac{4 - 4x}{x^2}, \quad f'(x) = \frac{4(x - 2)}{x^3}, \quad f''(x) = \frac{8(3 - x)}{x^4}.$$

Determine the following. Show all your work.

- The domain of  $f$ .
- The  $x$ - and  $y$ -coordinates of all intercepts.
- All asymptotes.
- The intervals on which  $f$  increases and the intervals on which  $f$  decreases.
- The  $x$ - and  $y$ -coordinates of all critical points, each classified as a local maximum, minimum or neither.
- The intervals on which  $f$  is concave up and the intervals on which  $f$  is concave down.
- The  $x$ - and  $y$ -coordinates of all inflection points.

- (h) Sketch the graph of  $f$  using all of the above information and label all pertinent points and lines.

9. Sketch the graph of

$$y = 4x^{1/3} + x^{4/3}.$$

On your graph **clearly** indicate and label all intercepts, local extrema, and inflection points.

10. Consider the function  $f(x) = x^{2/3} \left( \frac{5}{2} - x \right)$ .

- (a) Explain why  $f$  is continuous for  $x \in (-\infty, \infty)$ .
- (b) Determine the behavior of  $f$  as  $x \rightarrow \pm\infty$ .
- (c) Given that  $f'(x) = \frac{5}{3x^{1/3}}(1 - x)$ , determine the regions of increase and decrease of  $f$ .
- (d) Identify the locations of the relative extrema of  $f$  and classify them as maxima or minima.
- (e) Given that  $f''(x) = -\frac{5}{9x^{4/3}}(1 + 2x)$ , determine the concavity of  $f$ .
- (f) Identify the inflection points of  $f$ .
- (g) Sketch the graph of  $f$ .

11. Graph  $y = x^x$ .

12. Let

$$f(x) = \frac{x^2 + 2}{x^2 - 4}, \quad f'(x) = \frac{-12x}{(x^2 - 4)^2}, \quad f''(x) = \frac{12(3x^2 + 4)}{(x^2 - 4)^3}.$$

- (a) Find the horizontal and vertical asymptotes of the given function (if any).
- (b) Find the intervals where the function is increasing or decreasing, and local maximum and local minimum values of the function (if any).
- (c) Find the intervals where the function is concave upward or downward and the inflection points.
- (d) Sketch a graph of the function.

13. Let

$$f(x) = x^2 e^{-x}, \quad f'(x) = (2x - x^2)e^{-x}, \quad f''(x) = (2 - 4x + x^2)e^{-x}.$$

- (a) Does the graph  $f$  have any vertical or horizontal asymptotes. If so, what are they?

- (b) Determine the intervals of increase and decrease of this function. Also determine the extreme values.
- (c) Determine the intervals of upwards and downwards concavity of this function. Also determine any points of inflection.
- (d) Sketch a graph of this function. Clearly label the local extrema and points of inflection.

14. Let

$$f(x) = e^{1/x}, \quad f'(x) = -\frac{e^{1/x}}{x^2}, \quad f''(x) = \frac{e^{1/x}(2x+1)}{x^4}.$$

- (a) What is the domain of  $f$ .
- (b) Determine any points of intersection of the graph of  $f$  with the  $x$  and  $y$  axes.
- (c) Determine any horizontal asymptotes of  $f$ .
- (d) Determine any vertical asymptotes of  $f$ .
- (e) For each interval in the table below, indicate whether  $f$  is increasing or decreasing.

|        |                |               |
|--------|----------------|---------------|
|        | $(-\infty, 0)$ | $(0, \infty)$ |
| $f(x)$ |                |               |

- (f) Determine the  $x$  coordinates of any local extrema of  $f$ .
- (g) For each interval in the table below, indicate whether  $f$  is concave up or concave down.

|        |                   |             |               |
|--------|-------------------|-------------|---------------|
|        | $(-\infty, -1/2)$ | $(-1/2, 0)$ | $(0, \infty)$ |
| $f(x)$ |                   |             |               |

- (h) Determine the  $x$  coordinates of any inflection points on the graph of  $f$ .
- (i) Sketch the graph of  $y = f(x)$  using all of the above information. All relevant points must be labeled.

15. Let  $f(x) = e^{-2x^2}$ .

- (a) Find any horizontal and vertical asymptotes of  $f(x)$ .
- (b) Find the intervals where  $f(x)$  is increasing and the intervals where  $f(x)$  is decreasing.
- (c) Find the  $x$  values at which any local maxima or local minima occur and state whether a local maximum or a local minimum occurs.
- (d) Give the interval where  $f(x)$  is concave up and the intervals where  $f(x)$  is concave down.

- (e) Find the inflection points of  $f(x)$ .
16. Consider the function  $f(x) = x^3 e^{-x+5}$ . The first and second derivatives are:

$$f' = x^2(3-x)e^{-x+5}, \quad f'' = x(x^2 - 6x + 6)e^{-x+5}.$$

- (a) Determine the critical points of  $f$ .
- (b) Determine the intervals on which the function is increasing, and those on which the function is decreasing. Classify the critical point(s) as either local maxima, local minima, or neither.
- (c) Determine where  $f$  is concave up and where it is concave down. Identify any inflection points.
- (d) What is the end behaviour of  $f$  (i.e. what is happening as  $x \rightarrow \infty$  and  $x \rightarrow -\infty$ )?
- (e) Indicate which of graphs on Figure 3.1 is the graph of  $y = f(x)$ . Also, identify the critical points and inflection points on the graph you've chosen and write in the  $x$ -coordinates of these points.

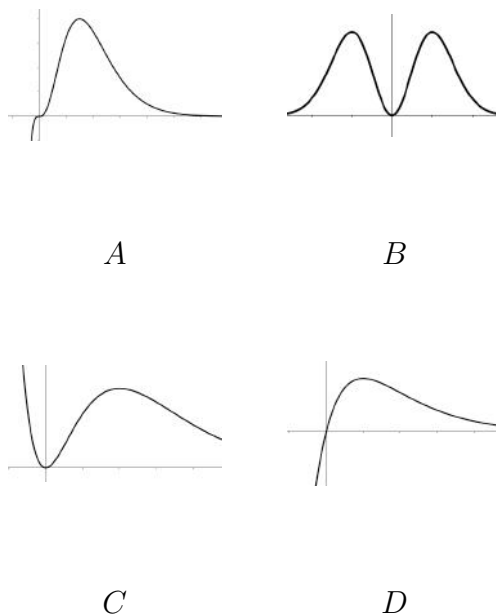


Figure 3.1: Find the graph of  $y = x^3 e^{-x+5}$

17. Let  $f(x) = \frac{18(x-1)}{x^2}$ . Then  $f'(x) = \frac{18(2-x)}{x^3}$  and  $f''(x) = \frac{36(x-3)}{x^4}$ . Give the following:

- (a) The domain of  $f$ .
- (b) The  $x$  and  $y$  coordinates of all intercepts of  $f$ .
- (c) The equations of asymptotes.
- (d) The intervals on which  $f$  is increasing and those on which  $f$  is decreasing.
- (e) The  $x$  and  $y$  coordinates of all critical points, each classified as a max, min, or neither.
- (f) The intervals on which  $f$  is concave up and those on which  $f$  is concave down.
- (g) The  $x$  and  $y$  coordinates of all inflection points of  $f$ .
- (h) Sketch the graph of  $y = f(x)$  using all of the above information. All relevant points must be labeled.

18. Let  $f(x) = \frac{x+3}{\sqrt{x^2+1}}$ . Then  $f'(x) = \frac{1-3x}{(x^2+1)^{3/2}}$  and  $f''(x) = \frac{6x^2-3x-3}{(x^2+1)^{5/2}}$ . Give the following:

- (a) The domain of  $f$ .
- (b) The  $x$  and  $y$  coordinates of all intercepts of  $f$ .
- (c) The equations of asymptotes, if any.
- (d) All critical points, if any.
- (e) The intervals on which  $f$  is increasing and those on which  $f$  is decreasing.
- (f) The classification of each critical point, if any, as a minimum or maximum, local or global, or not an extremum.
- (g) The intervals on which  $f$  is concave up and those on which  $f$  is concave down.
- (h) The  $x$  and  $y$  coordinates of all inflection points of  $f$ , if any.
- (i) Sketch the graph of  $y = f(x)$  using all of the above information. All relevant points must be labeled.

19. Consider the function

$$f(x) = (5-2x)x^{2/3}.$$

For your convenience, the first and second derivative of  $f(x)$  are

$$f'(x) = \frac{10(1-x)}{3x^{1/3}}, \quad f''(x) = -\frac{10(1+2x)}{9x^{4/3}}.$$

- (a) Locate any intercepts of  $f$ .
- (b) Locate any asymptotes of  $f$ .
- (c) Determine where  $f$  is increasing and decreasing.
- (d) Locate the local extrema of  $f$ .
- (e) Determine where  $f$  is concave upward or concave downward.
- (f) Locate all inflection points of  $f$ .
- (g) Sketch a graph of the function.

20. Consider the function

$$f(x) = \frac{x}{x^2 - 1}.$$

For your convenience, the first and second derivative of  $f(x)$  are

$$f'(x) = -\frac{x^2 + 1}{(x^2 - 1)^2}, \text{ and } f''(x) = \frac{2x(x^2 + 3)}{(x^2 - 1)^3}.$$

- (a) Determine any horizontal and vertical asymptotes of  $f$ .
- (b) Determine the open intervals on which  $f$  is increasing as well those on which  $f$  is decreasing.
- (c) Determine the open intervals on which  $f$  is concave upward as well those on which  $f$  is concave downward.
- (d) Based on the information found in Parts (a) to (c), sketch a graph of  $f$ . Indicate any relative extremum, inflection points and intercepts on your graph.

21. Given is the function

$$f(x) = \frac{x^3 - 2x}{3x^2 - 9}.$$

- (a) Give the domain of  $f$ .
- (b) Is  $f$  an even function, odd function, or neither?
- (c) Find all the  $x$  and  $y$ -intercepts of the graph  $f$ .
- (d) Find all horizontal, vertical, and slant asymptotes of the graph of  $f$ . If asymptotes of a certain kind are absent, explain why.
- (e) Find the intervals where the graph of  $f$  is increasing or decreasing and the locations and values of the local maxima and minima.
- (f) Find the intervals where the graph of  $f$  is concave upward and concave downward and the inflection points.

(g) Sketch the graph and clearly indicate the features found in parts (a)-(f).

22. Suppose  $f$  is a function satisfying the following conditions:

$$\lim_{x \rightarrow 3^+} f(x) = +\infty, \quad \lim_{x \rightarrow 3^-} f(x) = -\infty, \quad \lim_{x \rightarrow \infty} f(x) = 2, \quad \lim_{x \rightarrow -\infty} f(x) = -1,$$

$$f(0) = -3, \text{ and}$$

$$f'(x) < 0 \text{ for all } x \neq 3, \quad f''(x) > 0 \text{ for all } x > 3, \quad f''(x) < 0 \text{ for all } x < 3.$$

Draw a graph of the function  $f$  with all asymptotes and intercepts clearly labeled.

23. (a) Plot the graph of a function  $f$  which has only one point of discontinuity on its domain  $[-4, \infty)$  and that satisfies:

$$\begin{array}{ll} \lim_{x \rightarrow \infty} f(x) = -2 & f''(x) < 0 \quad \text{if } -4 < x < -1 \\ \lim_{x \rightarrow 0^-} f(x) = \infty & f''(x) > 0 \quad \text{if } -1 < x < 0 \\ f(0) = 2 & f''(x) < 0 \quad \text{if } 0 < x < 4 \\ & f''(x) > 0 \quad \text{if } 4 < x < \infty \\ & f'(x) < 0 \quad \text{if } -4 < x < -1 \\ & f'(x) > 0 \quad \text{if } -1 < x < 0 \\ & f'(x) > 0 \quad \text{if } 0 < x < 2 \\ & f'(x) < 0 \quad \text{if } 2 < x < \infty \end{array}$$

(b) Find all points of inflection for this graph, and for each point of inflection, determine if it is possible that  $f''(x) = 0$ .

24. The graphs of  $r$ ,  $s$ ,  $f'$ , and  $g'$  are shown on Figure 3.2, as labelled (these functions are all unrealized). For each question, tick the box if the corresponding function  $r$ ,  $s$ ,  $f$ , or  $g$  has the stated property. Note that you can tick more than one box.

(a) The function is increasing over the interval  $(1, 3)$ .

$$r \quad \square \quad s \quad \square \quad f \quad \square \quad g \quad \square \quad \text{None of them} \quad \square$$

(b) The function has a critical point when  $x = 3$ .

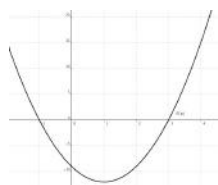
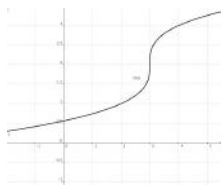
$$r \quad \square \quad s \quad \square \quad f \quad \square \quad g \quad \square \quad \text{None of them} \quad \square$$

(c) The function has an inflection point when  $x = 1$ .

$$r \quad \square \quad s \quad \square \quad f \quad \square \quad g \quad \square \quad \text{None of them} \quad \square$$

$$r(x) = (x - 1)^{1/3} + 2$$

$$f'(x) = 3x^2 - 6x - 9$$



$$s(x) = \sin[\pi(x - 1)/4] - 0.5$$

$$g'(x) = -2x^3 + 4x^2 + 6x$$

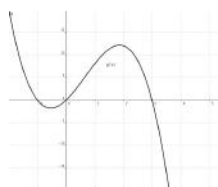
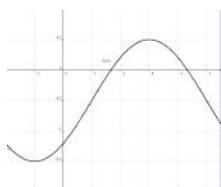


Figure 3.2: Four Graphs

- (d) The function is concave up over the interval  $(0, 2)$ .

$r$  ☐  $s$  ☐  $f$  ☐  $g$  ☐ None of them ☐

- (e) The function has a point where the second derivative does not exist.

$r$  ☐  $s$  ☐  $f$  ☐  $g$  ☐ None of them ☐

25. The graphs of four functions are shown on Figure 3.3. For each question, tick the box if the corresponding function  $r$ ,  $s$ ,  $f$ , or  $g$  has the stated property. Note that you can tick more than one box.

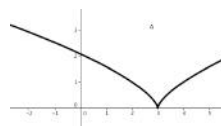
- (a) The derivative of the function is zero at  $x = -1$ .

$A$  ☐  $B$  ☐  $C$  ☐  $D$  ☐ None of them ☐

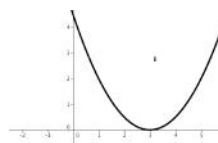
- (b) The function has a point where the second derivative does not exist.

$A$  ☐  $B$  ☐  $C$  ☐  $D$  ☐ None of them ☐

$$y = (x - 3)^{2/3}$$



$$y = 0.5(x - 3)^2$$



$$y = x^3 - 3x^2 - 9x + 1 \quad y = \sin[\pi(x - 1)/4] - 0.5$$

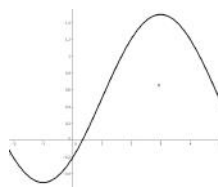
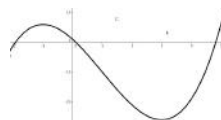


Figure 3.3: Four Graphs

(c) The derivative of the function is negative on the whole interval  $(-2, 0)$ .

$A$  ☐  $B$  ☐  $C$  ☐  $D$  ☐ None of them ☐

(d) The function has a critical point when  $x = 3$ .

$A$  ☐  $B$  ☐  $C$  ☐  $D$  ☐ None of them ☐

(e) The second derivative is positive over the whole interval  $(2, 5)$ .

$A$  ☐  $B$  ☐  $C$  ☐  $D$  ☐ None of them ☐

26. For what values of the constants  $a$  and  $b$  is  $(1, 6)$  a point of inflection of the curve  $y = x^3 + ax^2 + bx + 1$ ? Justify your answer.

### 3.3 Optimization

- Find the absolute maximum and minimum values of  $f(x) = 3x^2 - 9x$  on the interval  $[-1, 2]$ .

2. Find the absolute maximum and minimum values of  $f(x) = x^3 - 12x - 5$  on the interval  $[-4, 6]$ . Clearly explain your reasoning.
3. If  $a$  and  $b$  are positive numbers, find the maximum value of  $f(x) = x^a(1 - x)^b$ .
4. Find all critical points of the function  $f(x) = |3x - 5|$  on the interval  $[-3, 2]$ . Also find all maxima and minima of this function on  $[-3, 2]$ , both local and global.
5. The sum of two positive numbers is 12. What is the smallest possible value of the sum of their squares? Show your reasoning.
6. If  $a$  and  $b$  are positive numbers, find the  $x$  coordinate which gives the absolute maximum value of  $f(x) = x^a(1 - x)^b$  on the interval  $[0, 1]$ .
7. Find the point on the curve  $x + y^2 = 0$  that is closest to the point  $(0, -3)$ .
8. A straight piece of wire 40 cm long is cut into two pieces. One piece is bent into a circle and the other is bent into a square. How should wire be cut so that the total area of both circle and square is **minimized**?
9. A straight piece of wire of 28 cm is cut into two pieces. One piece is bent into a square (i.e., dimensions  $x$  times  $x$ .) The other piece is bent into a rectangle with aspect ratio three (i.e., dimensions  $y$  times  $3y$ .)  
  
What are dimensions, in centimeters, of the square and the rectangle such that the sum of their areas is **minimized**?
10. With a straight piece of wire 4m long, you are to create an equilateral triangle and a square, or either one only. Suppose a piece of length  $x$  meters is bent into triangle and the remainder is bent into a square. Find the value of  $x$  which maximizes the total area of both triangle and square.
11. A rectangle with sides parallel to the coordinate axes is to be inscribed in the region enclosed by the graphs of  $y = x^2$  and  $y = 4$  so that its perimeter has maximum length.
  - (a) Sketch the region under consideration.
  - (b) Supposing that the  $x$ -coordinate of the bottom right vertex of the rectangle is  $a$ , find a formula which expresses  $P$ , the length of the perimeter, in terms of  $a$ .
  - (c) Find the value of  $a$  which gives the maximum value of  $P$ , and explain why you know that this value of  $a$  gives a maximum.

- (d) What is the maximum value of  $P$ , the length of the perimeter of the rectangle?
12. Find the dimensions of the rectangle of largest area that has its base on the  $x$ -axis and its other two vertices above the  $x$ -axis and lying on the parabola  $y = 12 - x^2$ .
13. A farmer has 400 feet of fencing with which to build a rectangular pen. He will use part of an existing straight wall 100 feet long as part of one side of the perimeter of the pen. What is the maximum area that can be enclosed?
14. A  $10\sqrt{2}$  ft wall stands 5 ft from a building, see Figure 3.4. Find the length  $L$  of the shortest ladder, supported by the wall, that reaches from the ground to the building.

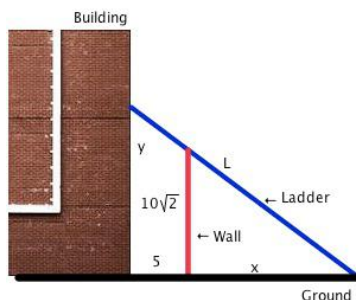


Figure 3.4: Building, Wall, and Ladder

15. In an elliptical sport field we want to design a rectangular soccer field with the maximum possible area. The sport field is given by the graph of  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ . Find the length  $2x$  and width  $2y$  of the pitch (in terms of  $a$  and  $b$ ) that maximize the area of the pitch. [Hint: express the area of the pitch as a function of  $x$  only.]
16. The top and bottom margins of a poster are each 6 cm, and the side margins are each 4 cm. If the area of the printed material on the poster (that is, the area between the margins) is fixed at  $384 \text{ cm}^2$ , find the dimensions of the poster with the smallest total area.

17. Each rectangular page of a book must contain  $30 \text{ cm}^2$  of printed text, and each page must have 2 cm margins at top and bottom, and 1 cm margin at each side. What is the minimum possible area of such a page?
18. Maya is 2 km offshore in a boat and wishes to reach a coastal village which is 6 km down a straight shoreline from the point on the shore nearest to the boat. She can row at 2 km/hr and run at 5 km/hr. Where should she land her boat to reach the village in the least amount of time?
19. A rectangular box has a square base with edge length  $x$  of at least 1 unit. The total surface area of its six sides is 150 square units.
  - (a) Express the volume  $V$  of this box as a function of  $x$ .
  - (b) Find the domain of  $V(x)$ .
  - (c) Find the dimensions of the box in part (a) with the greatest possible volume. What is this greatest possible volume?
20. An open-top box is to have a square base and a volume of  $10 \text{ m}^3$ . The cost per square meter of material is \$5 for the bottom and \$2 for the four sides. Let  $x$  and  $y$  be lengths of the box's width and height respectively. Let  $C$  be the total cost of material required to make the box.
  - (a) Express  $C$  as a function of  $x$  and find its domain.
  - (b) Find the dimensions of the box so that the cost of materials is minimized. What is this minimum cost?
21. An open-top box is to have a square base and a volume of  $13500 \text{ cm}^3$ . Find the dimensions of the box that minimize the amount of material used.
22. A water trough is to be made from a long strip of tin 6 ft wide by bending up at an the angle  $\theta$  a 2 ft strip at each side. What angle  $\theta$  would maximize the cross sectional area, and thus the volume, of the trough? (See Figure 3.5.)
23. Find the dimension of the right circular cylinder of maximum volume that can be inscribed in a right circular cone of radius  $R$  and height  $H$ .
24. A hollow plastic cylinder with a circular base and open top is to be made and  $10 \text{ m}^2$  plastic is available. Find the dimensions of the cylinder that give the maximum volume and find the value of the maximum volume.

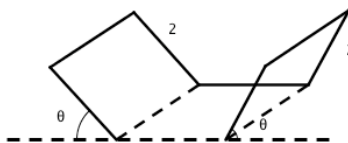


Figure 3.5: Water Trough

25. An open-topped cylindrical pot is to have volume  $250 \text{ cm}^3$ . The material for the bottom of the pot costs 4 cents per  $\text{cm}^2$ ; that for its curved side costs 2 cents per  $\text{cm}^2$ . What dimensions will minimize the total cost of this pot?
26. Cylindrical soup cans are to be manufactured to contain a given volume  $V$ . No waste is involved in cutting the material for the vertical side of each can, but each top and bottom which are circles of radius  $r$ , are cut from a square that measures  $2r$  units on each side. Thus the material used to manufacture each soup can has an area of  $A = 2\pi rh + 8r^2$  square units.
  - (a) How much material is wasted in making each soup can?
  - (b) Find the ratio of the height to diameter for the most economical can (i.e. requiring the least amount of material for manufacture.)
  - (c) Use either the first or second derivative test to verify that you have minimized the amount of material used for making each can.
27. A storage container is to be made in the form of a right circular cylinder and have a volume of  $28\pi \text{ m}^3$ . Material for the top of the container costs \$5 per square metre and material for the side and base costs \$2 per square metre. What dimensions will minimize the total cost of the container?
28. Show that the volume of the largest cone that can be inscribed inside a sphere of radius  $R$  is  $\frac{32\pi R^3}{81}$ .
29. The sound level measured in watts per square meter, varies in direct proportion to the power of the source and inversely as the square of the distance from the

source, so that is given by  $y = kPx^{-2}$ , where  $y$  is the sound level,  $P$  is the source power,  $x$  is the distance from the source, and  $k$  is a positive constant. Two beach parties, 100 meters apart, are playing loud music on their portable stereos. The second party's stereo has 64 times as much power as the first. The music approximates the white noise, so the power from the two sources arriving at a point between them adds, without any concern about whether the sources are in or out of phase. To what point on the line segment between the two parties should I go, if I wish to enjoy as much quiet as possible? Demonstrate that you have found an absolute minimum, not just a relative minimum.

### 3.4 Mean Value Theorem

1. Verify that the function

$$g(x) = \frac{3x}{x+7}$$

satisfies the hypothesis of the Mean Value Theorem on the interval  $[-1, 2]$ . Then find all numbers  $c$  that satisfy the conclusion of the Mean Value Theorem. Leave your final answer(s) exact.

2. Use the Mean Value Theorem to show that  $|\sin b - \sin a| \leq |b - a|$ , for all real numbers  $a$  and  $b$ .
3. Two horses start a race at the same time and finish in a tie. Prove that at some time during the race they have the same speed.
4. (a) Complete the following statement of the Mean Value Theorem precisely.  
Let  $f$  be a function that is continuous on the interval . . . . . and differentiable on the interval . . . . . Then there is a number  $c$  in  $(a, b)$  such that  $f(b) - f(a) = \dots\dots\dots$   
(b) Suppose that  $-1 \leq f'(x) \leq 3$  for all  $x$ . Find similar lower and upper bounds for the expression  $f(5) - f(3)$ .  
(c) Suppose  $g(x)$  is a function that is differentiable for all  $x$ . Let  $h(x)$  be a new function defined by  $h(x) = g(x) + g(2 - x)$ . Prove that  $h'(x)$  has a root in the interval  $(0, 2)$ .

### 3.5 Differential, Linear Approximation, Newton's Method

1. (a) Find the linear approximation to the function  $f(x) = \sqrt{(x+4)^3}$  at  $a = 0$ .  
(b) Use this approximation to estimate the number  $\sqrt{(3.95)^3}$ . Is your estimate an overestimate or an underestimate? (Hint: What is the concavity of the function  $f(x)$ ?)
2. Use linear approximation to estimate the value of  $\sqrt[3]{26^2}$ . Express your answer a single fraction (for example,  $\frac{16}{729}$ ).
3. Use the linear approximation to approximate  $(63)^{2/3}$ . Then use differentials to estimate the error.
4. Use linear approximation to estimate the value of  $\sqrt{80}$ .
5. Assume that  $f$  is function such that  $f(5) = 2$  and  $f'(5) = 4$ . Using a linear approximation to  $f$  near  $x = 5$ , find an approximation to  $f(4.9)$ .
6. Suppose that we don't have a formula for  $g(x)$  but we know that  $g(2) = -4$  and  $g'(x) = \sqrt{x^2 + 5}$  for all  $x$ .
  - (a) Use linear approximation to estimate  $g(2.05)$ .
  - (b) Is your estimate in part (a) larger or smaller than the actual value? Explain.
7. (a) Find a linear approximation for the function  $f(x) = \sqrt{1-x}$  valid for  $x$  close to 0.  
(b) Use your answer to find an approximate value for  $\sqrt{0.9}$ .  
(c) Find the tangent line to the graph of  $f(x) = \sqrt{1-x}$  at  $x = 0$ .  
(d) Sketch a graph to illustrate the relationship between  $f(x) = \sqrt{1-x}$  and its linear approximation near  $x = 0$ .
8. Let  $f(x) = \sqrt{1+2x}$ .
  - (a) Find the linear approximation of  $f(x)$  at  $x = 0$ .
  - (b) Use your answer to estimate the value of  $\sqrt{1.1}$ .
  - (c) Is your estimate an over- or under-estimate?
9. (a) Find a linear approximation to the function  $f(x) = \sqrt[3]{x+8}$  at  $a = 0$ .

- (b) Use this approximation to estimate the numbers  $\sqrt[3]{7.95}$  and  $\sqrt[3]{8.1}$ .
10. (a) Find the equation of the tangent line to the graph of the function  $f(x) = \sqrt[3]{27 + 3x}$  at  $x = 0$ .
- (b) Use your answer to estimate a value of  $\sqrt[3]{30}$ .
- (c) Draw a sketch to show how the graph of  $f$  and its tangent line behave around the point where  $x = 0$  and the value of  $x$  where the value in part (b) is obtained.
11. Use linear approximation to estimate the value of  $\ln 0.9$ .
12. (a) Write the linear approximation for  $f(x) = \ln x$  around 1.
- (b) Compute the approximated value for  $\exp(-0.1)$  using linear approximation.
13. Using the function  $f(x) = x^{1/3}$  and the technique of linear approximation, give an estimate for  $1001^{1/3}$ .
14. Let  $f(x) = \sqrt{x} + \sqrt[5]{x}$ .
- (a) Use linear approximation to determine which of the following is nearest the value of  $f(1.001)$ :
- |        |        |        |        |        |
|--------|--------|--------|--------|--------|
| 2.0001 | 2.0002 | 2.0003 | 2.0005 | 2.0007 |
| 2.001  | 2.002  | 2.003  | 2.005  | 2.007  |
- (b) At  $x = 1$ , is  $f(x)$  concave up or concave down?
- (c) Based on your answer above, is your estimate of  $f(1.001)$  too high or too low?
15. (a) Find the linear approximation of  $f(x) = \sin x$  about the point  $x = \pi/6$ .
- (b) Explain why  $f$  satisfies the conditions of the Mean Value Theorem. Use the theorem to prove that  $\sin x \leq \frac{1}{2} + (x - \frac{\pi}{6})$  on the interval  $[\frac{\pi}{6}, x]$  where  $x > \frac{\pi}{6}$ .
- (c) Is the differential  $df$  larger or smaller than  $\Delta f$  from  $x = \frac{\pi}{6}$  to  $x = \frac{\pi}{2}$ ? Do not perform any calculations. Use only the results in part (a) and (b) to explain your answer.
16. (a) State Newton's iterative formula that produces a sequence of approximations  $x_1, x_2, x_3, \dots$  to a root of function  $f(x)$ .

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- (b) Find the positive root of the equation  $\cos x = x^2$  using Newton's method, correct to 3 decimal points, with the first approximation  $x_1 = 1$ .
17. (a) State Newton's iterative formula that produces a sequence of approximations  $x_0, x_1, x_2, \dots$  to a solution of  $f(x) = 0$ , assuming that  $x_0$  is given.
- (b) Draw a labeled diagram showing an example of a function  $f(x)$  for which Newton's iterative formula fails to find a solution of  $f(x) = 0$ . Mark on your diagram  $x_0, x_1$ , and  $x_2$ .
18. (a) Explain how you can use Newton's Method to approximate the value of  $\sqrt{5}$ .
- (b) Explain which of the following choices is the best initial approximation when using Newton's Method as in (a):  $-1, 0$ , or  $1$ ?
- (c) Find the fourth approximation  $x_4$  to the value of  $\sqrt{5}$  using Newton's Method with the initial approximation  $x_1$  you chose in (b).
19. Apply Newton's method to  $f(x) = x^{1/3}$  with  $x_0 = 1$  and calculate  $x_1, x_2, x_3, x_4$ . Find a formula for  $|x_n|$ . What happens to  $|x_n|$  as  $n \rightarrow \infty$ ? Draw a picture that shows what is going on.
20. (a) Suppose  $k$  is a constant. Show that if we apply Newton's method to approximate the value of  $\sqrt[5]{k}$ , we get the following iterative formula:

$$x_{n+1} = \frac{x_n}{5} \left( 4 + \frac{k}{x_n^5} \right).$$

- (b) If  $x_n = \sqrt[5]{k}$ , what is the value of  $x_{n+1}$ ?
- (c) Take  $x_1 = 2$  and use the formula in part (a) to find  $x_2$ , an estimate of the value of  $\sqrt[5]{20}$  that is correct to one decimal place.
21. Use Newton's method to find the second approximation  $x_2$  of  $\sqrt[5]{31}$  starting with the initial approximation  $x_0 = 2$ .
22. (a) Suppose  $x_0$  is an initial estimate in Newton's method applied to the function  $f(x)$ . Derive Newton's formula for  $x_1$ , namely

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Support your derivation with a sketch showing a function  $f(x)$ , with  $x_0, x_1$  and the line whose slope is  $f'(x_0)$  clearly labeled.

- (b) Using one iteration of Newton's method with  $x_0 = \frac{\pi}{2}$  approximate the  $x$ -coordinate of the point where the function  $g(x) = \sin x$  crosses the line  $y = x$ .

23. The equation

$$8x^3 - 12x^2 - 22x + 25 = 0$$

has a solution near  $x_1 = 1$ . Use Newton's Method to find a better approximation  $x_2$  to this solution. Express your answer as a fraction.

24. The tangent line to the graph  $y = f(x)$  at the point  $A(2, -1)$  is given by  $y = -1 + 4(x - 2)$ . It is also known that  $f''(2) = 3$ .

- (a) Assume that Newton's Method is used to solve the equation  $f(x) = 0$  and  $x_0 = 2$  is the initial guess. Find the next approximation,  $x_1$ , to the solution.
- (b) Assume that Newton's Method is used to find a critical point for  $f$  and that  $x_0 = 2$  is the initial guess. Find the next approximation,  $x_1$ , to the critical point.

25. (a) Apply Newton's method to the equation  $\frac{1}{x} - a = 0$  to derive the following algorithm for finding reciprocals:

$$x_{n+1} = 2x_n - ax_n^2.$$

- (b) Use the algorithm from part (a) to calculate  $\frac{1}{1.128}$  correct to three decimal places, starting with the first approximation  $x_1 = 1$ .

26. You seek the approximate value of  $x$  which is near 1.8 for which  $\sin x = \frac{x}{2}$ . Your first guess is that  $x \approx x_1 = \frac{\pi}{2}$ . Use one iteration of Newton's method to find a better approximation to  $x$ . Simplify your answer as far as possible.

27. (a) For the function  $f(x) = x^3 - 3x + 5$  use the IVT, and any other tools you need to determine intervals of length 1 each of which contains a root of  $f$ .
- (b) Pick one of the intervals found in part (a). Choose the left endpoint of this interval to be  $x_0$ . Now, use this as a starting value to find two new iterations to the root of  $f$  by using Newton's method. Determine from these whether Newton's method is working. Justify your answer carefully.

28. Let  $f(x) = x^3 + 3x + 1$ .

- (a) Show that  $f$  has at least one root in the interval  $\left(-\frac{1}{2}, 0\right)$ . Explain your reasoning.
- (b) Use Newton's method to approximate the root that lies in the interval  $\left(-\frac{1}{2}, 0\right)$ . Stop when the next iteration agrees with the previous one to two decimal places.
29. In this question we investigate the solution of the equation  $\ln x = -x^2 + 3$  on the interval  $[1, 3]$ .
- (a) Explain why you know the equation has at least one solution on  $[1, 3]$ .
- (b) Show that the equation has exactly one solution on  $[1, 3]$ .
- (c) Use Newton's Method to approximate the solution of the equation by starting with  $x_1 = 1$  and finding  $x_2$ .
30. In this question we investigate the solution of the equation  $2x = \cos x$ .
- (a) Explain why you know the equation has at least one solution.
- (b) Show that the equation has exactly one solution.
- (c) Use Newton's Method to approximate the solution of the equation by starting with  $x_1 = 0$  and finding  $x_2$ .
31. In this question we investigate the solution of the equation  $2x - 1 = \sin x$ .
- (a) Explain why you know the equation has at least one solution.
- (b) Show that the equation has exactly one solution.
- (c) Use Newton's Method to approximate the solution of the equation by starting with  $x_1 = 0$  and finding  $x_2$ .

### 3.6 Antiderivatives and Differential Equations

- Find  $f$  if  $f'(x) = 2 \cos x + 8x^3 - e^x$  and  $f(0) = 7$ .
- Find  $g$  if  $g'(x) = \sin x + x^{-2} - e^x$  and  $g(\pi) = 1$ .
- Suppose  $f$  is a function such that  $f'(x) = x^2 + 2x + 3$  and  $f(5) = 1$ . What is  $f(1)$ ?

4. Suppose  $h$  is a function such that  $h'(x) = x^2 + 2e^x + 3$  and  $h(3) = 0$ . What is  $h(1)$ ?
5. Find the general form for the following antiderivative:  $\int \frac{z}{z^2 + 9} dz$
6. Find a curve  $y = f(x)$  with the following properties:
  - $\frac{d^2y}{dx^2} = 6x$
  - Its graph passes through the point  $(0, 1)$  and has a horizontal tangent there.
7. Find  $\int \frac{dx}{x + x \ln x}$ .
8. For each case compute the indefinite integral.
  - (a)  $\int (1 - x)^8 dx$
  - (b)  $\int \tan^2 x dx$
  - (c)  $\int \frac{dx}{2 + x^2}$
  - (d)  $\int e^{2x} \cosh x dx$
9. Find  $f$  if  $f''(t) = 2e^t + 3 \sin t$  and  $f(0) = 0$ ,  $f'(0) = 0$ .
10. A particle starts from rest (that is with initial velocity zero) at the point  $x = 10$  and moves along the  $x$ -axis with acceleration function  $a(t) = 12t$ . Find the resulting position function  $x(t)$ .
11. At time  $t = 0$  a car is moving at 6 m/s. and driver smoothly accelerates so that the acceleration after  $t$  seconds is  $a(t) = 3t$  m/s<sup>2</sup>.
  - (a) Write a formula for the speed  $v(t)$  of the car after  $t$  seconds.
  - (b) How far did the car travel between during the time it took to accelerate from 6 m/s to 30 m/s?
12.
  - (a) You are standing at ground level, and you throw a ball upward into the air with an initial velocity of 64 feet per second. The acceleration due to gravity is 32 feet per second squared (towards the ground). How much time is the ball in the air for? Use antiderivatives.
  - (b) What is the velocity of the ball at the time that it hits the ground?

13. In 1939, Joe Sprinz, a professional baseball player, attempted to catch a ball dropped from a blimp. This was done for the purpose of braking the record for catching a ball dropped from the greatest height set the previous year by another player on another team.
- (a) Ignoring air resistance, give an expression for the time it takes for a ball to drop  $H$  meters (from rest) under a constant gravitational acceleration  $g$ .
- (b) Give an expression for the velocity of the ball after it has fallen  $H$  meters. What is this value if  $H = 245$  m and  $g = -10$  m/s<sup>2</sup>? Would you try to catch this ball? *(Even with air resistance, the ball slammed Sprinz's glove into his face, fractured his upper jaw at 12 places, broke five teeth and knocked him unconscious. Worse still, he dropped the ball!)*
14. Find the solution  $y(x)$  to the following initial value problem:  $\frac{dy}{dx} = 2 \sin 3x + x^2 + e^{3x} + 1$ ,  $y(0) = 0$ .
15. Find the solution  $y(x)$  to the following initial value problem:  $\frac{dy}{dx} = \sqrt{1 - y^2}$ ,  $y(0) = 1$ .
16. Find the function  $y(x)$  that is the solution of the following initial value problem:  $\frac{dy}{dx} = y^2 + 1$ ,  $y(\pi/4) = 0$ .
17. Solve the initial value problem:  $\frac{dy}{dx} = 1 + y$ ,  $y(0) = 3$ .
18. Solve the initial value problem:  $\frac{dx}{dt} = \frac{36}{(4t - 7)^4}$ ,  $x(2) = 1$ .
19. Solve the initial value problem:  $\frac{dy}{dx} = e^{-y}$ ,  $y(0) = 2$ .
20. Find the solution  $y(t)$  to the following initial value problem:  $\frac{dy}{dt} = 2 - y$ ,  $y(0) = 1$ .
21. (a) Find the solution to  $\frac{dy}{dx} = 3 \cos 2x + \exp(-4x)$  such that  $y(0) = 1$ .
- (b) Use the change of variables  $u = 2x + 1$  to find the general form for the antiderivative of  $f(x) = \frac{1}{2x + 1}$ .

22. A company is manufacturing and selling towels. The marginal cost for producing towels is 0.15 CAD per towel. A market survey has shown that for every 0.10 CAD increase in the price per towel, the company will sell 50 towels less per week. Currently the company sells 1000 towels per week against the price that maximizes their profit. What is the price of one towel? (Note: As usual, even though towels are only sold in integer units, assume we can use differentiable functions to describe the relevant quantities.)

### 3.7 Exponential Growth and Decay

1. (a) An amount of  $A_0$  CAD is invested against yearly interest of  $p\%$ . Give the expression for  $A(t)$ , the value of the investment in CAD after  $t$  years if the interest is compounded continuously by writing down the differential equation that  $A$  satisfies and solving it.
- (b) Jane invests 10,000 CAD against a yearly interest  $p\%$ , compounded continuously. After 4 years the value of her investment is 15,000 CAD. What is  $p$ ?
2. The rate at which a student learns new material is proportional to the difference between a maximum,  $M$ , and the amount she already knows at time  $t$ ,  $A(t)$ . This is called a learning curve.
  - (a) Write a differential equation to model the learning curve described.
  - (b) Solve the differential equation you created in part (a).
  - (c) If took a student 100 hours to learn 50% of the material in Math 151 and she would like to know 75% in order to get a  $B$ , how much longer she should study? You may assume that the student began knowing none of the material and that the maximum she might achieve is 100%.
3. The concentration of alcohol (in %) in the blood,  $C(t)$ , obeys the decay differential equation:

$$\frac{dC}{dt} = -\frac{1}{k}C,$$

where  $k = 2.5$  hours is called the elimination time.

It is estimated that a male weighing 70 kg who drinks 3 pints of beer over a period of one hour has a concentration of 1% of alcohol in his blood. The allowed legal concentration for driving is a maximum of 0.5%.

- (a) If a person has a blood alcohol concentration of 1%, how long should she/he wait before driving in order not to disobey the law. You may need the value  $\ln 2 \approx 0.7$ .
- (b) What is the initial ( $t = 0$ ) rate of change in the concentration?

**Note:** The permissible BAC limit in the Criminal Code of Canada is .08 (80 milligrams of alcohol in 100 milliliters of blood). Some advocate a lower criminal limit of .05 (50 milligrams of alcohol in 100 milliliters of blood).

- 4. Carbon dating is used to estimate the age of an ancient human skull. Let  $f(t)$  be the proportion of original  $^{14}\text{C}$  atoms remaining in the skull after  $t$  years of radioactive decay. Since  $^{14}\text{C}$  has a half life of 5700 years we have  $f(0) = 1$  and  $f(5700) = 0.5$ .
  - (a) Sketch the graph of  $f(t)$  versus  $t$  in the domain  $0 \leq t \leq 20000$ . Label at least two points of your plot and be sure to label the axes.
  - (b) Write an expression for  $f(t)$  in terms of  $t$  and other numerical constants such as  $\ln 2$ ,  $\sin 5$ ,  $e^3$ , and  $1/5700$ . (Note: Not all of these constants need appear in your answer!)
  - (c) Suppose that only 15% of the original  $^{14}\text{C}$  is found to remain in the skull. Derive from your previous answer, an expression for the estimated age of the skull.
- 5. Plutonium-239 is part of the highly radioactive waste that nuclear power plants produce. The half-life of plutonium-239 is 24,110 years. Suppose that 10 kg of plutonium-239 has leaked into and contaminated a lake. Let  $m(t)$  denote the mass of plutonium-239 that remains in the lake after  $t$  years.
  - (a) Find an expression for  $m(t)$  based on the information given.
  - (b) How much mass remains in the lake after 1000 years.
  - (c) Suppose the lake is considered safe for use after only 1 kg of the plutonium-239 remains. How many years will this take?
- 6. On a certain day, a scientist had 1 kg of a radioactive substance  $X$  at 1:00 pm. After six hours, only 27 g of the substance remained. How much substance  $X$  was there at 3:00 pm that same day?
- 7. In a certain culture of bacteria, the number of bacteria increased tenfold in 10 hours. Assuming natural growth, how long did it take for their number to double?

8. A bacterial culture starts with 500 bacteria and after three hours there are 8000.

Assume that the culture grows at a rate proportional to its size.

- (a) Find an expression in  $t$  for the number of bacteria after  $t$  hours.
  - (b) Find the number of bacteria after six hours.
  - (c) Find an expression of the form  $m \frac{\ln a}{\ln b}$  with  $m$ ,  $a$ , and  $b$  positive integers for the number of hours it takes the number of bacteria to reach a million.
9. A bacteria culture starts with 500 bacteria and grows at a rate proportional to its size. After three hours there are 8000 bacteria. Find the number of bacteria after four hours.
10. The bacteria population  $P(t)$  quadruples every 15 minutes. The initial bacteria population is  $P(0) = 10$ . You might need the following values:  $\ln 6 \approx 1.6$ ,  $\ln 8 \approx 2.08$ ,  $(\ln 10)/(\ln 2) \approx 3.32$ ,  $\ln 2 \approx 0.69$ ,  $4^3 = 64$ ,  $4^6 = 4096$ ,  $4^9 = 262144$ ,  $4^{12} = 16777216$ .
- (a) What is the population after three hours?
  - (b) How much time does it take for the population to grow to 1 billion?
11. The population of a bacteria culture grows at a rate that is proportional to the size of the population.
- (a) Let  $P$  denote the population of the culture at time  $t$ . Express  $dP/dt$  in terms of the proportional constant  $k$  and  $P$ .
  - (b) If the population is 240 at time  $t = 1$  and is 360 at time  $t = 2$ , find a formula for the number of bacteria at time  $t$ . ( $t$  in hours.)
  - (c) How many bacteria were there at time  $t = 0$ . Your answer should be a positive integer.
  - (d) What is the value of  $dP/dt$  when  $t = 0$ .
12. Assume that Math 151 in fall of 2000 had an enrollment of 500 students and in fall 2002 had an enrollment of 750 students. Assume also that if  $P(t)$  is the enrollment at time  $t$  (let  $t$  be in years, with  $t = 0$  corresponding to year 2000), then  $P'(t) = kP(t)$  for some constant  $k$ . Calculate  $P(500)$  (the enrollment in Math 151 in fall of 2500). Simplify your answer as much as possible. The answer will be quite large.
13. A cup of coffee, cooling off in a room at temperature  $20^\circ\text{C}$ , has cooling constant  $k = 0.09\text{min}^{-1}$ .

- (a) How fast is the coffee cooling (in degrees per minute) when its temperature is  $T = 80^{\circ}\text{C}$ ?
  - (b) Use linear approximation to estimate the change in temperature over the next 6 seconds when  $T = 80^{\circ}\text{C}$ .
  - (c) The coffee is served at a temperature of  $90^{\circ}$ . How long should you wait before drinking it if the optimal temperature is  $65^{\circ}\text{C}$ ?
14. A cold drink is taken from a refrigerator and placed outside where the temperature is  $32^{\circ}\text{C}$ . After 25 minutes outside its temperature is  $14^{\circ}\text{C}$ , and after 50 minutes outside its temperature is  $20^{\circ}\text{C}$ . assuming the temperature of drink obeys Newton's Law of Heating, what was the initial temperature of the drink?

### 3.8 Miscellaneous

1. For what values of the constant  $c$  does  $\ln x = cx^2$  have solutions. Assume that  $c > 0$ .
2. Show that  $y = 3x^3 + 2x + 12$  has a unique root.
3. Show that the equation  $x^3 + 9x + 5 = 0$  has exactly one real solution.
4. For which values of  $a$  and  $b$  is  $(1, 6)$  a point of inflection of the curve  $y = x^3 + ax^2 + bx + 1$ ?
5. Prove that  $f(x) = \frac{1}{(x+1)^2} - 2x + \sin x$  has exactly one positive root.
6. A ball is thrown vertically upwards from a platform 12 feet above the ground so that after  $t$  seconds have elapsed the height  $s$  (in feet) of the ball above the ground is given by

$$s = 12 + 96t - t^2.$$

Compute the following quantities:

- (a) The initial velocity.
- (b) The time to highest point.
- (c) The maximum height attained.

7. The table below gives the values of certain functions at certain points. Calculate each of the following or write “insufficient information” if there is not enough information to compute the value.

| $x$ | $f(x)$ | $f'(x)$ | $g(x)$ | $g'(x)$ |
|-----|--------|---------|--------|---------|
| 1   | 3      | 3       | 2      | 2       |
| 2   | 4      | -1      | 4      | 0       |
| 3   | 6      | 1       | 0      | 4       |
| 4   | -1     | 0       | 1      | 1       |
| 5   | 2      | 4       | 3      | 3       |

- (a) What is  $\lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h}$ ?
- (b) What is  $\lim_{h \rightarrow 0} \frac{f(2+h)g(2+h) - f(2)g(2)}{h}$ ?
- (c) Use differentials to find the approximate value of  $f(0.98)$ .
- (d) What are the coordinates of any point on the graph of  $f$  at which there is a critical point?
8. (a) State domain and range of  $f(x) = \arcsin x$ .
- (b) Derive the differentiation formula  $\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}}$ .
- (c) Let  $g(x) = \arcsin(\sin x)$ . Graph  $g(x)$  and state its domain and range.
- (d) For the function  $g(x)$  as defined in part (c) find  $g'(x)$  using any method you like. Simplify your answer completely.
- (e) Explain carefully why the equation

$$4x - 2 + \cos\left(\frac{\pi x}{2}\right) = 0$$

has exactly one real root.

9. (a) Differentiate  $f(x) = \frac{\ln x}{x}$ , for  $x > 0$ .
- (b) Sketch the graph of  $f(x) = \frac{\ln x}{x}$ , showing all extrema.
- (c) Use what you have done to decide which is larger,  $99^{101}$  or  $101^{99}$ .
10. Answer the following questions. No justification necessary.
- (a) What is the general antiderivative of  $6x^2 + 2x + 5$ ?

- (b) What is the derivative of  $\sinh(x)$  with respect to  $x$ ?
  - (c) If  $f'(x)$  changes from negative to positive at  $c$  the  $f(x)$  has a (pick one)
    - i. local maximum at  $c$ .
    - ii. local minimum at  $c$ .
    - iii. global maximum at  $c$ .
    - iv. global minimum at  $c$ .
  - (d) If  $x^5 + y^5 = 1$ , what is  $y'$  in terms of  $x$  and  $y$ ?
  - (e) If a point has polar coordinates  $(r, \theta) = (3, 3\pi)$ , what are its Cartesian coordinates?
- 11.
- (a) State the definition of the derivative of a function  $g$  at a number  $x$ .
  - (b) State the Squeeze Theorem, clearly identifying any hypothesis and the conclusion.
  - (c) State Fermat's Theorem, clearly identifying any hypothesis and the conclusion.
  - (d) Give an example of a function with one critical point which is also an inflection point.
  - (e) Give an example of a function that satisfies  $f(-1) = 0$ ,  $f(10) = 0$ , and  $f'(x) > 0$  for all  $x$  in the domain of  $f'$ .
- 12.
- (a) State the definition of a critical number of a function  $f$ .
  - (b) State the Mean Value Theorem, clearly identifying any hypothesis and the conclusion.
  - (c) State the Extreme Value Theorem, clearly identifying any hypothesis and the conclusion.
  - (d) State the definition of a inflection point of a function  $f$ .
  - (e) Give an example of a function with a local maximum at which the second derivative is 0.
  - (f) Give an example of a quadratic function of the form  $f(x) = x^2 + bx + c$  whose tangent line is  $y = 3x + 1$  at the point  $(0, 1)$ .



# Chapter 4

## Parametric Equations and Polar Coordinates

### 4.1 Introduction

1. **Parametric Curves - Vocabulary.** Let  $I$  be an interval and let  $f$  and  $g$  be continuous on  $I$ .
  - (a) The set of points  $C = \{(f(t), g(t)) : t \in I\}$  is called a **parametric curve**.
  - (b) The variable  $t$  is called a **parameter**.
  - (c) We say that the curve  $C$  is defined by **parametric equations**  $x = f(t)$ ,  $y = g(t)$ .
  - (d) We say that  $x = f(t)$ ,  $y = g(t)$  is a **parametrization** of  $C$ .
  - (e) If  $I = [a, b]$  then  $(f(a), g(a))$  is called the **initial point** of  $C$  and  $(f(b), g(b))$  is called the **terminal point** of  $C$ .
2. **Derivative of Parametric Curves:** The derivative to the parametric curve  $x = f(t)$ ,  $y = g(t)$  is given by  $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)}$ .
3. **Polar Coordinate System.**
  - (a) Choose a point in the plane. Call it  $O$ , the **pole**.
  - (b) Choose a ray starting at  $O$ . Call it the **polar axis**.
  - (c) Take any point  $P$ , except  $O$ , in the plane. Measure the distance  $d(O, P)$  and call this distance  $r$ .

(d) Measure the angle between the polar axis and the ray starting at  $O$  and passing through  $P$  going from the polar axis in counterclockwise direction. Let  $\theta$  be this measure in radians.

(e) There is a bijection between the plane and the set

$$\mathbb{R}^+ \times [0, 2\pi) = \{(r, \theta) : r \in \mathbb{R}^+ \text{ and } \theta \in [0, 2\pi)\}.$$

This means that each point  $P$ , except  $O$ , in the plane is uniquely determined by a pair  $(r, \theta) \in \mathbb{R}^+ \times [0, 2\pi)$ .

(f)  $r$  and  $\theta$  are called **polar coordinates** of  $P$ .

4. **Polar Curves:** The graph of a polar equation  $r = f(\theta)$  consists of all points  $P$  whose polar coordinates satisfy the equation.

5. **Derivative of Polar Curves.** Suppose that  $y$  is a differentiable function of  $x$  and that  $r = f(\theta)$  is a differentiable function of  $\theta$ . Then from the parametric equations  $x = r \cos \theta$ ,  $y = r \sin \theta$  it follows that

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta} \sin \theta + r \cos \theta}{\frac{dr}{d\theta} \cos \theta - r \sin \theta}.$$

6. **Parabola.** A parabola is a set of points in the plane that are equidistant from a fixed point  $F$  (called the focus) and a fixed line called the directrix). An equation of the parabola with focus  $(0, p)$  and directrix  $y = -p$  is  $x^2 = 4py$ .

7. **Ellipse.** An ellipse is a set of point in plane the sum of whose distances from two fixed points  $F_1$  and  $F_2$  is constant. The fixed points are called foci. The ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ,  $a \geq b > 0$ , has foci  $(\pm c, 0)$ , where  $c = \sqrt{a^2 - b^2}$ , and vertices  $(\pm a, 0)$ .

8. **Hyperbola.** A hyperbola is a set of points in plane the difference of whose distances from two fixed points  $F_1$  and  $F_2$  (the foci) is constant. The hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  has foci  $(\pm c, 0)$ , where  $c = \sqrt{a^2 + b^2}$ , vertices  $(\pm a, 0)$ , and asymptotes  $y = \pm \frac{bx}{a}$ .

9. **Eccentricity.** Let  $F$  be a fixed point in the plane, let  $l$  be a fixed line in the plane, and let  $e$  be a fixed positive number (called the eccentricity). The set of all points  $P$  in the plane such that  $\frac{|PF|}{|Pl|} = e$  is a conic section. The conic is an ellipse if  $e < 1$ , a parabola if  $e = 1$ , and a hyperbola if  $e > 1$ .

## 4.2 Parametric Curves

1. The trajectory of a particle in a plane as a function of the time  $t$  in seconds is given by the parametric equations

$$x = 3t^2 + 2t - 3, \quad y = 2t^3 + 2.$$

Prove that there is exactly one time when the particle crosses the line  $y = x$ .

2. Let  $x = 2 \sin t + 1$  and  $y = 2t^3 - 3$  define a parametric curve. Find  $\frac{d^2y}{dx^2}$  as a function of  $t$ , without simplifying your answer.
3. Sketch the curve

$$x = \sin^2 \pi t, \quad y = \cos^2 \pi t, \quad 0 \leq t \leq 2.$$

Clearly label the initial and terminal points and describe the motion of the point  $(x(t), y(t))$  as  $t$  varies in the given interval.

4. The parametric equations for a curve are given by

$$x = \theta - \sin \theta, \quad y = 1 - \cos \theta.$$

- (a) Find  $\frac{dx}{dy}$  as a function of  $\theta$ .
  - (b) Find  $\frac{d^2x}{dy^2}$  as a function of  $\theta$ .
  - (c) Find the tangent line to the curve at the point of the curve obtained by setting  $\theta = \frac{\pi}{3}$ .
5. The graphs of the parametric equations  $x = f(t)$ ,  $y = g(t)$  are on Figure 4.1. Identify which is the corresponding parametric curve on Figure 4.2.
  6. Consider the parametric curve  $x(t) = -2 + 2 \cos t$ ,  $y(t) = 1 - 2 \sin t$ .
    - (a) State the Cartesian equation of the curve and sketch the curve. Determine the direction of evolution of the curve for increasing  $t$  and indicate it on the graph.
    - (b) Find the points on the curve for which the tangent line has a slope of 1.
  7. This question concerns the curve  $x = 2 \cos 3t$ ,  $y = 2 \sin 2t$  for  $0 \leq t \leq 2\pi$ .
    - (a) Find  $dy/dx$  for this curve.

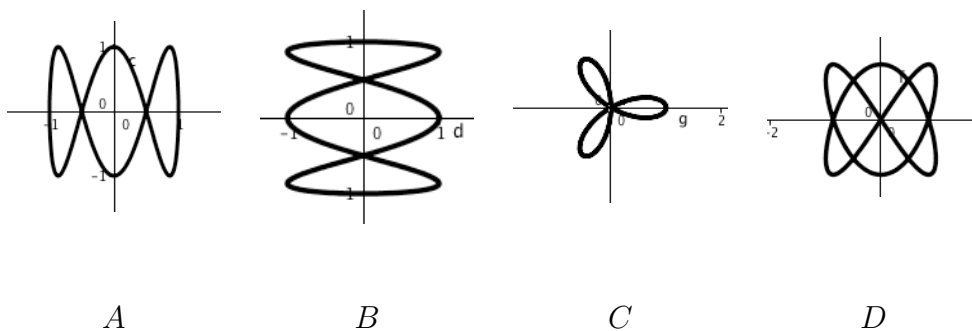
Figure 4.1:  $x = f(t), y = g(t)$ 

Figure 4.2: Only One Is Right

- (b) Find the equation of the two tangent lines at the origin.
- (c) Identify the graph on Figure 4.3 that correspond to the parametric curve.
8. For the following parametric curve  $x = t - 1$ ,  $y = t^2 - 2t + 2$ :
- Find the derivative  $\frac{dy}{dx}$  as a function of  $t$ .
  - Eliminate the parametric dependence to determine an expression of the form  $y = f(x)$  for the given curve.
  - Find an expression for  $m$  and  $b$  (as functions of  $x_1$ ) such that the equation of the line  $y = mx + b$  is tangent to the curve  $y = f(x)$  at  $(x_1, f(x_1))$ .
  - Find an expression for all tangent lines to the curve  $y = f(x)$  that pass through the point  $(2, 0)$ .

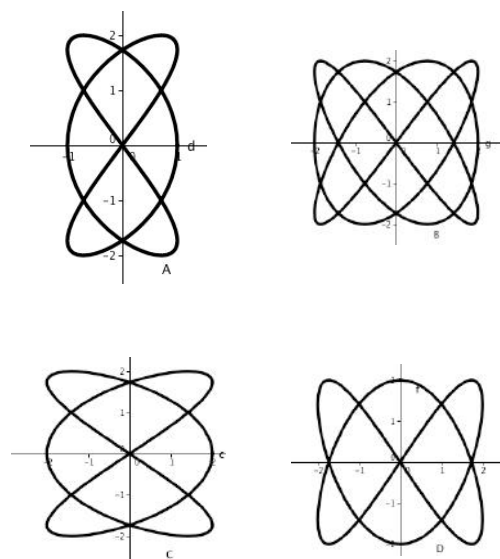


Figure 4.3: Which Graph Is Right?

9. This question concerns the parametric curve  $x = t^3 - 4t$ ,  $y = 2t^2 - 4t$ ,  $-\infty < t < \infty$ .
- Which of the two graphs on Figure 4.4 corresponds to the given parametric curve?
  - Find the  $y$ -coordinates of all points where the curve crosses the  $y$ -axis.
  - This curve crosses itself at exactly one point. Find equations of both tangent lines at that point.
10. Consider the parametric curve  $C$  described by

$$x = e^{-t}, \quad y = 2^{2t}, \quad \text{where } -\infty < t < \infty.$$

- Calculate  $\frac{dy}{dx}$  (as a function of  $t$ ) directly from these equations.
- Calculate  $\frac{d^2y}{dx^2}$  (as a function of  $t$ ) directly from these equations.
- Does the graph of  $C$  have any points of inflection?
- Eliminate  $t$  from the equations of  $C$  thus obtaining a description of the graph  $C$  of the form  $y = f(x)$ .

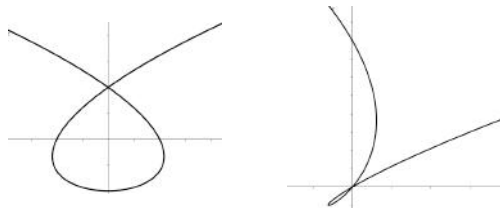


Figure 4.4: Which Graph Is Right?

- (e) Sketch  $C$ . Label the axis. Plot at least three points of the curve  $C$ , labeling the points with the corresponding values of  $t$ . Draw an arrow on your curve indicating the direction of motion as  $t$  increases toward  $+\infty$ .

11. Consider the parametric curve  $C$  described by

$$x(t) = -e^{4t}, \quad y(t) = e^{1-t}, \quad \text{where } t \in \mathbb{R}.$$

- Find  $\frac{dy}{dx}$  as a function of  $t$  directly from the above equations.
- Find  $\frac{d^2y}{dx^2}$  as a function of  $t$ . Simplify your answer.
- Determine if  $C$  is concave up or concave down at  $t = 0$ .

12. Consider the parametric curve  $C$  described by

$$x(t) = 1 + 3t, \quad y(t) = 2 - t^3, \quad \text{where } t \in \mathbb{R}.$$

- Find  $\frac{dy}{dx}$  as a function of  $t$  directly from the above equations.
- Eliminate  $t$  from the equations of  $C$  thus obtaining a description of the graph of  $C$  having the form  $y = f(x)$ .

13. For the following parametric curve:

$$\begin{cases} x = t \sin t \\ y = t \cos t \end{cases}$$

- Find the derivative  $\frac{dy}{dx}$  as a function of  $t$ .
- Find the equation of the tangent line at  $t = \frac{\pi}{2}$ .

(c) Determine if the curve is concave upwards or downwards at  $t = \frac{\pi}{2}$ .

14. A curve is defined by the parametric equations

$$x = 3(t^2 - 3), \quad y = t^3 - 3t.$$

- (a) Calculate  $\frac{dy}{dx}$  in terms of  $t$ .
- (b) Find the equation of the tangent line to the curve at the point corresponding to  $t = 2$ .

15. Consider the parametric curve

$$x = 3(t^2 - 9), \quad y = t(t^2 - 9), \quad -\infty < t < \infty.$$

Part of the graph of this curve is given in Figure 4.5)

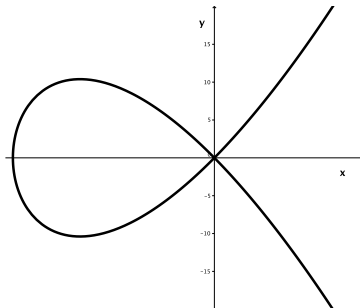


Figure 4.5:  $x = 3(t^2 - 9)$ ,  $y = t(t^2 - 9)$

- (a) Find the  $x$ -coordinate of all points where the curve crosses the  $x$ -axis.
- (b) Find the coordinate of the point on the curve corresponding to  $t = -1$ .
- (c) Find the tangent line to the curve at the point where  $t = -1$ .
- (d) Find the second derivative at the point where  $t = -1$ .

16. Consider the perimetrically defined curve  $x(t) = t - \sin 2t$ ,  $y = 4 - 3 \cos t$ ,  $0 \leq t \leq 10$ . Find the values for the parameter  $t$  where the tangent line of the curve is vertical.

17. Consider the parametric curve  $C$  described by

$$x = e^{-t}, \quad y = e^{2t}, \quad \text{where } -\infty < t < \infty.$$

- (a) Calculate  $\frac{dy}{dx}$  (as a function of  $t$ ) directly from the above equations.
- (b) Calculate  $\frac{d^2y}{dx^2}$  (as a function of  $t$ ).
- (c) Find all inflection points of  $C$  or otherwise show that  $C$  has no inflection points.
- (d) Eliminate  $t$  from the equations  $C$  to obtain a function of the form  $y = f(x)$  that describes  $C$ .

18. Given the parametric curve

$$x = \cos^3 t, \quad y = \sin^3 t.$$

- (a) Without eliminating the parameter  $t$ , show that  $\frac{dy}{dx} = -\tan t$ .
- (b) Determine the concavity of this curve when  $t = 1$ .

19. Given the parametric curve

$$x = e^t, \quad y = e^{-t}.$$

- (a) Find  $\frac{dy}{dx}$  and  $\frac{d^2y}{dx^2}$ .
- (b) Find the equation of the line tangent to the curve that is parallel to the line  $y + x = 1$ .

20. Given the parametric curve

$$x = t(t^2 - 3), \quad y = 3(t^2 - 3).$$

- (a) Find the  $y$ -intercepts of the curve.
- (b) Find the points on the curve where the tangent line is horizontal or vertical.
- (c) Sketch the curve.

## 4.3 Polar Coordinates

1. Express the polar equation  $r = \cos 2\theta$  in rectangular coordinates.
2. Sketch polar graphs of:
  - (a)  $r = 1 + \sin \theta$ .
  - (b)  $r = \cos 3\theta$ .
3. For the each of the following circles find a polar equation, i.e., an equation in  $r$  and  $\theta$ :
  - (a)  $x^2 + y^2 = 4$
  - (b)  $(x - 1)^2 + y^2 = 1$
  - (c)  $x^2 + (y - 0.5)^2 = 0.25$
4. Find the maximum height above the  $x$ -axis of the cardioid  $r = 1 + \cos \theta$ .
5. Sketch the graph of the curve whose equation in polar coordinates is  $r = 1 - 2 \cos \theta$ ,  $0 \leq \theta < 2\pi$ .
6. Sketch the graph of the curve whose equation in polar coordinates is  $r = 3 \cos 3\theta$ .
7. Sketch the curve whose polar equation is  $r = -1 + \cos \theta$ , indicating any symmetries. Mark on your sketch the polar coordinates of all points where the curve intersects the polar axis.
8. Sketch a a polar coordinate plot of:
  - (a)  $r = 2 \cos 3\theta$
  - (b)  $r^2 = -4 \sin 2\theta$
  - (c)  $r = 2 \sin \theta$
  - (d)  $r = 2 \cos \theta$
  - (e)  $r = 4 + 7 \cos \theta$
9. Consider the curve given by the polar equation  $r = 1 - 2 \sin \theta$ , for  $0 \leq \theta < 2\pi$ .
  - (a) Find the Cartesian coordinates for the point on the curve corresponding to  $\theta = \frac{3\pi}{2}$ .
  - (b) The curve intersects the  $x$ -axis at two points other than the pole. Find polar coordinates for these other points.

- (c) On Figure 4.6 identify the graphs that correspond to the following two polar curves.

$$\square \quad r = 1 - 2 \sin \theta \quad \square \quad r = 1 + 2 \sin \theta$$

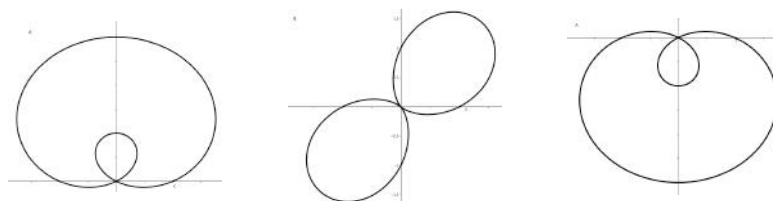


Figure 4.6: Which Graph Is Extra?

10. (a) Sketch a polar coordinate plot of

$$r = 1 + 2 \sin 3\theta, \quad 0 \leq \theta \leq 2\pi.$$

- (b) How many points lie in the intersection of the two polar graphs

$$r = 1 + 2 \sin 3\theta, \quad 0 \leq \theta \leq 2\pi$$

and

$$r = 1?$$

- (c) *Algebraically* find all values of  $\theta$  that

$$1 = 1 + 2 \sin 3\theta, \quad 0 \leq \theta \leq 2\pi.$$

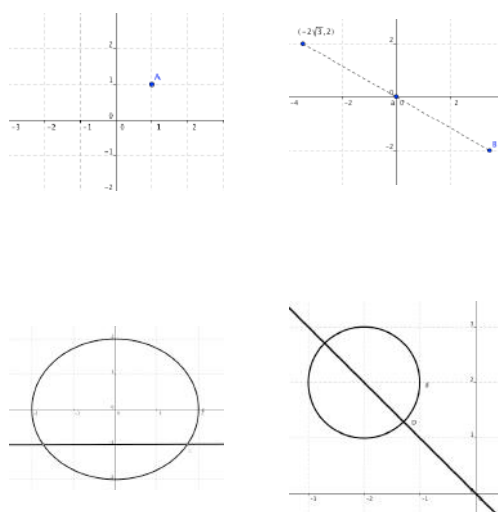
- (d) Explain in a sentence or two why the answer to part (b) differs from (or is the same as) the number of solutions you found in part (c).

11. Consider the following curve  $C$  given in polar coordinates as

$$r(\theta) = 1 + \sin \theta + e^{\sin \theta}, \quad 0 \leq \theta \leq 2\pi.$$

- (a) Calculate the value of  $r(\theta)$  for  $\theta = 0, \frac{\pi}{2}, \frac{3\pi}{2}$ .

- (b) Sketch a graph of  $C$ .
- (c) What is the minimum distance from a point on the curve  $C$  to the origin? (i.e. determine the minimum of  $|r(\theta)| = r(\theta) = 1 + \sin \theta + e^{\sin \theta}$  for  $\theta \in [0, 2\pi]$ )
12. (a) Give polar coordinates for each of the point  $A$ ,  $B$ ,  $C$  and  $D$  on Figure 4.7.

Figure 4.7: Polar Coordinates of Points  $A$ ,  $B$ ,  $C$  and  $D$ 

- (b) On Figure 4.8 identify the graphs that correspond to the following three polar curves.

$$\square r = 1 - 2 \sin \theta \quad \square r^2 \theta = 1 \quad \square r = \frac{1}{1 - 2 \sin \theta}$$

13. (a) Sketch the curve defined by  $r = 1 + 2 \sin \theta$ .
- (b) For what values of  $\theta$ ,  $\theta \in [-\pi, \pi)$ , is the radius  $r$  positive?
- (c) For what values of  $\theta$ ,  $\theta \in [-\pi, \pi)$ , is the radius  $r$  maximum and for what values is it minimum?
14. (a) Sketch the graph described in polar coordinates by the equation  $r = \theta$  where  $-\pi \leq \theta \leq 3\pi$ .

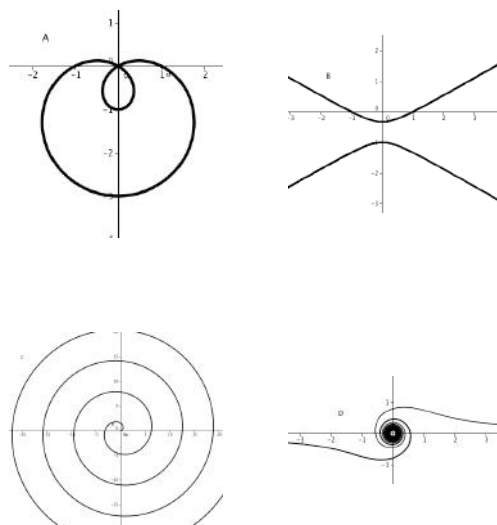


Figure 4.8: Match Polar Expressions And Graphs

- (b) Find the slope of this curve when  $\theta = \frac{5\pi}{2}$ . Simplify your answer for full credit.
- (c) Express the polar equation  $r = \theta$  in cartesian coordinates, as an equation in  $x$  and  $y$ .
15. (a) Let  $C$  denote the graph of the polar equation  $r = 5 \sin \theta$ . Find the rectangular coordinates of the point on  $C$  corresponding to  $\theta = \frac{3\pi}{2}$ .
- (b) Write a rectangular equation (i.e. using the variables  $x$  and  $y$ ) for  $C$ . (in other words, convert the equation for  $C$  into rectangular coordinates.)
- (c) Rewrite the equation of  $C$  in parametric form, i.e. express both  $x$  and  $y$  as functions of  $\theta$ .
- (d) Find an expression for  $\frac{dy}{dx}$  in terms of  $\theta$ .
- (e) Find the equation of the tangent line to  $C$  at the point corresponding to  $\theta = \frac{\pi}{6}$ .
16. Find the slope of the tangent line to the polar curve  $r = 2$  at the points where it intersects the polar curve  $r = 4 \cos \theta$ . (Hint: After you find the intersection

points, convert one of the curves to a pair of parametric equations with  $\theta$  as the perimeter.

## 4.4 Conic Sections

1. Sketch the graph of the ellipse  $\frac{x^2}{9} + \frac{y^2}{16} = 1$  and determine its foci.
2. Let  $C$  be the conic which consists of all points  $P = (x, y)$  such that the distance from  $P$  to  $F = (0, 1)$  is  $\frac{1}{2}$  the perpendicular distance from  $P$  to the line  $y = 4$ .

- (a) What is the eccentricity of the conic  $C$ ?
- (b) Show that the equation of the conic  $C$  is

$$\frac{x^2}{3} + \frac{y^2}{4} = 1.$$

- (c) What is the equation of the tangent line to the point  $A = (\sqrt{3}/2, \sqrt{3})$  on the conic  $C$ ?
3. (a) Find the equation of the tangent line to the polar curve  $r = 1 + \cos \theta$  at  $\theta = \frac{\pi}{2}$ .  
 (b) Find an equation of the ellipse with horizontal and vertical axes that passes through the points  $(2, 0)$ ,  $(-2, 0)$ ,  $(0, -1)$ , and  $(0, 3)$ .
  4. Consider the polar equation  $r(3 - k \cos \theta) = 4$ .  
 (a) For what values of the constant  $k$  is this conic section an ellipse?  
 (b) Now assume that  $k$  has the value in the middle of the interval found in part (a). Determine the location of the focus and directrix of this conic section.  
 (c) Sketch the graph of this conic section which shows the focus, directrix and vertices.
  5. (a) Assume that the earth is a sphere with radius  $s$ . ( $s$  is about 6400 kilometers, but this has nothing to do with the solution of this problem. Express everything, including your answer, in terms of  $s$ .) A satellite has an elliptical orbit with the centre of the earth at one focus. The lowest point of the orbit is  $5s$  above the surface of the earth, when the

satellite is *directly above the North Pole*. The highest point of the orbit is 11s above the surface of the earth, when the satellite is *directly above the South Pole*. What is the height of the satellite above the surface of the earth, when the satellite is *directly above the equator*? [Suggestion: Choose a coordinate system with the centre of the earth on the  $y$ -axis and the centre of the satellite's orbit at the origin.]

- (b) Let the focus  $F$  of a conic section with eccentricity  $e$  be the origin and let the directrix  $L$  be the vertical line  $x = -p$ ,  $p > 0$ . Thus the conic consists of all those points  $P = (x, y)$  such that  $|PF| = e|PL|$ . Find the polar equation of the conic section. (With the origin as the pole, and the positive  $x$ -axis as the polar axis.)

6. (a) Identify and sketch the graph of the conic section defined by

$$r = \frac{1}{2 - 4 \cos \theta}$$

in the  $x, y$ , plane without converting it to cartesian coordinates. Clearly label all foci and vertices. (Hint: You may use the equation  $r = \frac{ep}{1 - e \cos \theta}$  without deriving it.)

- (b) Convert the polar coordinates equation given above to a cartesian coordinates equation of the form  $Ax^2 + Bx + Cy^2 + Dy = E$ .
7. (a) Find an equation of the set of all points  $(x, y)$  that satisfy this condition: The distance from  $(x, y)$  to  $(5, 0)$  is exactly half the distance from  $(x, y)$  to the line  $x = -5$ .
- (b) Simplify your answer from part (a) enough to be able to tell what type of conic section it is.
8. Derive the equation of the set of all points  $P(x, y)$  that are equidistant from the point  $A(1, 0)$  and the line  $x = -5$ . Provide a diagram with your work. Simplify the equation.
9. Let  $C$  be the conic section described by the equation

$$x^2 - 2y^2 + 4x = 0.$$

- (a) Using the method of completing the square, identify the conic section  $C$ .
- (b) Sketch a graph of  $C$ . Find, foci and asymptotes (if any).

10. Given is the polar equation  $r = \frac{2}{1 - 2 \cos \theta}$ .

- (a) Which type of conic section does this polar equation represent: Parabola, ellipse, or hyperbola?
- (b) Show that the polar equation implies the following equation in Cartesian coordinates:

$$9\left(x + \frac{4}{3}\right)^2 - 3y^2 = 4.$$

- (c) Give the foci, vertices, and asymptotes of the conic if it has any.
- (d) Sketch the conic section based on the information found above. Indicate the features you found in (c) in your sketch.



## Chapter 5

### True Or False and Multiple Choice Problems

1. For each of the following ten statements answer TRUE or FALSE as appropriate:
  - (a) If  $f$  is differentiable on  $[-1, 1]$  then  $f$  is continuous at  $x = 0$ .
  - (b) If  $f'(x) < 0$  and  $f''(x) > 0$  for all  $x$  then  $f$  is concave down.
  - (c) The general antiderivative of  $f(x) = 3x^2$  is  $F(x) = x^3$ .
  - (d)  $\ln x$  exists for any  $x > 1$ .
  - (e)  $\ln x = \pi$  has a unique solution.
  - (f)  $e^{-x}$  is negative for some values of  $x$ .
  - (g)  $\ln e^{x^2} = x^2$  for all  $x$ .
  - (h)  $f(x) = |x|$  is differentiable for all  $x$ .
  - (i)  $\tan x$  is defined for all  $x$ .
  - (j) All critical points of  $f(x)$  satisfy  $f'(x) = 0$ .
2. Answer each of the following either TRUE or FALSE.
  - (a) The function  $f(x) = \begin{cases} 3 + \frac{\sin(x-2)}{x-2} & \text{if } x \neq 2 \\ 3 & \text{if } x = 2 \end{cases}$  is continuous at all real numbers  $x$ .
  - (b) If  $f'(x) = g'(x)$  for  $0 < x < 1$ , then  $f(x) = g(x)$  for  $0 < x < 1$ .
  - (c) If  $f$  is increasing and  $f(x) > 0$  on  $I$ , then  $g(x) = \frac{1}{f(x)}$  is decreasing on  $I$ .

- (d) There exists a function  $f$  such that  $f(1) = -2$ ,  $f(3) = 0$ , and  $f'(x) > 1$  for all  $x$ .
- (e) If  $f$  is differentiable, then  $\frac{d}{dx}f(\sqrt{x}) = \frac{f'(x)}{2\sqrt{x}}$ .
- (f)  $\frac{d}{dx}10^x = x10^{x-1}$
- (g) Let  $e = \exp(1)$  as usual. If  $y = e^2$  then  $y' = 2e$ .
- (h) If  $f(x)$  and  $g(x)$  are differentiable for all  $x$ , then  $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$ .
- (i) If  $g(x) = x^5$ , then  $\lim_{x \rightarrow 2} \frac{g(x) - g(2)}{x - 2} = 80$ .
- (j) An equation of the tangent line to the parabola  $y = x^2$  at  $(-2, 4)$  is  $y - 4 = 2x(x + 2)$ .
- (k)  $\frac{d}{dx} \tan^2 x = \frac{d}{dx} \sec^2 x$
- (l) For all real values of  $x$  we have that  $\frac{d}{dx}|x^2 + x| = |2x + 1|$ .
- (m) If  $f$  is one-to-one then  $f^{-1}(x) = \frac{1}{f(x)}$ .
- (n) If  $x > 0$ , then  $(\ln x)^6 = 6 \ln x$ .
- (o) If  $\lim_{x \rightarrow 5} f(x) = 0$  and  $\lim_{x \rightarrow 5} g(x) = 0$ , then  $\lim_{x \rightarrow 5} \frac{f(x)}{g(x)}$  does not exist.
- (p) If the line  $x = 1$  is a vertical asymptote of  $y = f(x)$ , then  $f$  is not defined at 1.
- (q) If  $f'(c)$  does not exist and  $f'(x)$  changes from positive to negative as  $x$  increases through  $c$ , then  $f(x)$  has a local minimum at  $x = c$ .
- (r)  $\sqrt{a^2} = a$  for all  $a > 0$ .
- (s) If  $f(c)$  exists but  $f'(c)$  does not exist, then  $x = c$  is a critical point of  $f(x)$ .
- (t) If  $f''(c)$  exists and  $f'''(c) > 0$ , then  $f(x)$  has a local minimum at  $x = c$ .

3. Are the following statements TRUE or FALSE.

- (a)  $\lim_{x \rightarrow 3} \sqrt{x - 3} = \sqrt{\lim_{x \rightarrow 3} (x - 3)}$ .
- (b)  $\frac{d}{dx} \left( \frac{\ln 2^{\sqrt{x}}}{\sqrt{x}} \right) = 0$
- (c) If  $f(x) = (1 + x)(1 + x^2)(1 + x^3)(1 + x^4)$ , then  $f'(0) = 1$ .

- (d) If  $y = f(x) = 2^{|x|}$ , then the range of  $f$  is the set of all non-negative real numbers.
- (e)  $\frac{d}{dx} \left( \frac{\log x^2}{\log x} \right) = 0$ .
- (f) If  $f'(x) = -x^3$  and  $f(4) = 3$ , then  $f(3) = 2$ .
- (g) If  $f''(c)$  exists and if  $f''(c) > 0$ , then  $f(x)$  has a local minimum at  $x = c$ .
- (h)  $\frac{d}{du} \left( \frac{1}{\csc u} \right) = \frac{1}{\sec u}$ .
- (i)  $\frac{d}{dx} (\sin^{-1}(\cos x)) = -1$  for  $0 < x < \pi$ .
- (j)  $\sinh^2 x - \cosh^2 x = 1$ .
- (k)  $\int \frac{dx}{x^2 + 1} = \ln(x^2 + 1) + C$ .
- (l)  $\int \frac{dx}{3 - 2x} = \frac{1}{2} \ln |3 - 2x| + C$ .

4. Answer each of the following either TRUE or FALSE.

- (a) For *all* functions  $f$ , if  $f$  is continuous at a certain point  $x_0$ , then  $f$  is differentiable at  $x_0$ .
- (b) For *all* functions  $f$ , if  $\lim_{x \rightarrow a^-} f(x)$  exist, and  $\lim_{x \rightarrow a^+} f(x)$  exist, then  $f$  is continuous at  $a$ .
- (c) For *all* functions  $f$ , if  $a < b$ ,  $f(a) < 0$ ,  $f(b) > 0$ , then there must be a number  $c$ , with  $a < c < b$  and  $f(c) = 0$ .
- (d) For *all* functions  $f$ , if  $f'(x)$  exists for all  $x$ , then  $f''(x)$  exists for all  $x$ .
- (e) It is impossible for a function to be discontinues at *every* number  $x$ .
- (f) If  $f, g$ , are *any* two functions which are continuous for all  $x$ , then  $\frac{f}{g}$  is continuous for all  $x$ .
- (g) It is possible that functions  $f$  and  $g$  are *not* continuous at a point  $x_0$ , but  $f + g$  is continuous at  $x_0$ .
- (h) If  $\lim_{x \rightarrow \infty} (f(x) + g(x))$  exists, then  $\lim_{x \rightarrow \infty} f(x)$  exists and  $\lim_{x \rightarrow \infty} g(x)$  exists.
- (i)  $\lim_{x \rightarrow \infty} \frac{(1.00001)^x}{x^{100000}} = 0$
- (j) Every continuous function on the interval  $(0, 1)$  has a maximum value and a minimum value on  $(0, 1)$ .

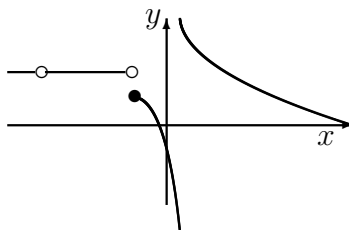
5. Answer each of the following either TRUE or FALSE.

- (a) Let  $f$  and  $g$  be *any* two functions which are continuous on  $[0, 1]$ , with  $f(0) = g(0) = 0$  and  $f(1) = g(1) = 10$ . Then there must exist  $c, d \in [0, 1]$  such that  $f'(c) = g'(d)$ .
- (b) Let  $f$  and  $g$  be *any* two functions which are continuous on  $[0, 1]$  and differentiable on  $(0, 1)$ , with  $f(0) = g(0) = 0$  and  $f(1) = g(1) = 10$ . Then there must exist  $c \in [0, 1]$  such that  $f'(c) = g'(c)$ .
- (c) For all  $x$  in the domain of  $\sec^{-1} x$ ,

$$\sec(\sec^{-1}(x)) = x.$$

6. Answer each of the following either TRUE or FALSE.

- (a) The slope of the tangent line of  $f(x)$  at the point  $(a, f(a))$  is given by  $\frac{f(a+h) - f(a)}{h}$ .
- (b) Using the Intermediate Value Theorem it can be shown that  $\lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0$ .
- (c) The graph below exhibits three types of discontinuities.



- (d) If  $w = f(x)$ ,  $x = g(y)$ ,  $y = h(z)$ , then  $\frac{dw}{dz} = \frac{dw}{dx} \cdot \frac{dx}{dy} \cdot \frac{dy}{dz}$ .
- (e) Suppose that on the open interval  $I$ ,  $f$  is a differentiable function that has an inverse function  $f^{-1}$  and  $f'(x) \neq 0$ . Then  $f^{-1}$  is differentiable and  $(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}$  for all  $x$  in the domain of  $f^{-1}$ .
- (f) Given the graph of  $f$  below to the left, the graph to the right must be that of  $f'$ .
- (g) The conclusion of the Mean Value Theorem says that the graph of  $f$  has at least one tangent line in  $(a, b)$ , whose slope is equal to the average slope on  $[a, b]$ .

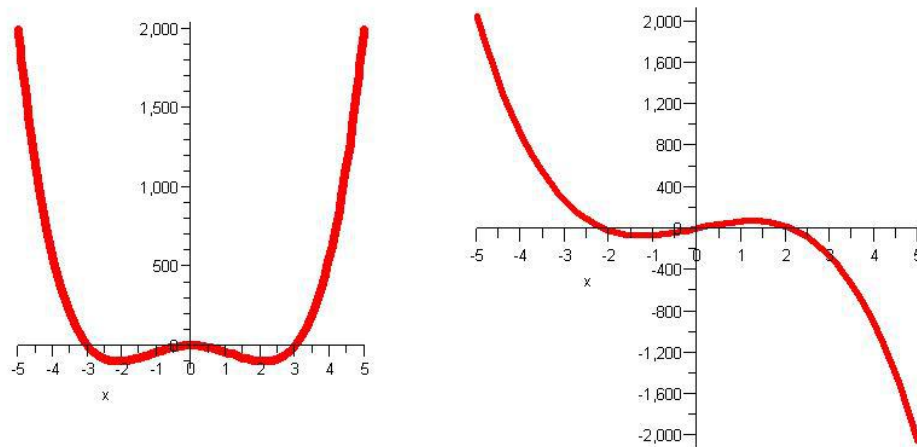
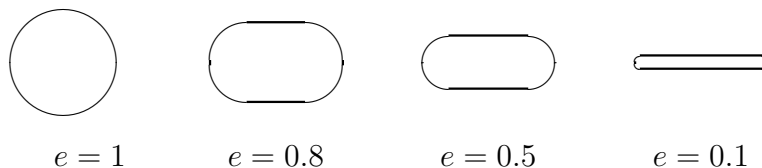


Figure 5.1: Function and Its Derivative?

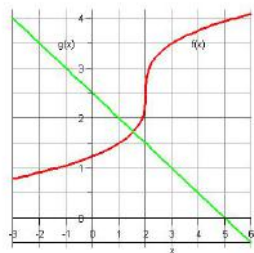
- (h) The linear approximation  $L(x)$  of a function  $f(x)$  near the point  $x = a$  is given by  $L(x) = f'(a) + f(a)(x - a)$ .
- (i) The graphs below are labeled correctly with possible eccentricities for the given conic sections:



- (j) Given  $h(x) = g(f(x))$  and the graphs of  $f$  and  $g$  below then a good estimate for  $h'(3)$  is  $-\frac{1}{4}$ .

7. Answer TRUE or FALSE to the following questions.

- (a) If  $f(x) = 7x + 8$  then  $f'(2) = f'(17.38)$ .
- (b) If  $f(x)$  is any function such that  $\lim_{x \rightarrow 2} f(x) = 6$  then  $\lim_{x \rightarrow 2^+} f(x) = 6$ .
- (c) If  $f(x) = x^2$  and  $g(x) = x + 1$  then  $f(g(x)) = x^2 + 1$ .
- (d) The average rate of change of  $f(x)$  from  $x = 3$  to  $x = 3.5$  is  $2(f(3.5) - f(3))$ .

Figure 5.2:  $h'(3) \approx -1/4$ ?

- (e) An equivalent precise definition of  $\lim_{x \rightarrow a} f(x) = L$  is: For any  $0 < \epsilon < 0.13$  there is  $\delta > 0$  such that

$$\text{if } |x - a| < \delta \text{ then } |f(x) - L| < \epsilon.$$

The last four True/False questions ALL pertain to the following function.  
Let

$$f(x) \begin{cases} x - 4 & \text{if } x < 2 \\ 23 & \text{if } x = 2 \\ x^2 + 7 & \text{if } x > 2 \end{cases}$$

- (f)  $f(3) = -1$   
 (g)  $f(2) = 11$   
 (h)  $f$  is continuous at  $x = 3$ .  
 (i)  $f$  is continuous at  $x = 2$ .

8. Answer TRUE or FALSE to the following questions.

- (a) If a particle has a constant acceleration, then its position function is a cubic polynomial.  
 (b) If  $f(x)$  is differentiable on the open interval  $(a, b)$  then by the Mean Value Theorem there is a number  $c$  in  $(a, b)$  such that  $(b - a)f'(c) = f(b) - f(a)$ .  
 (c) If  $\lim_{x \rightarrow \infty} \left( \frac{k}{f(x)} \right) = 0$  for every number  $k$ , then  $\lim_{x \rightarrow \infty} f(x) = \infty$ .  
 (d) If  $f(x)$  has an absolute minimum at  $x = c$ , then  $f'(c) = 0$ .

9. True or False. Give a brief justification for each answer.

- (a) There is a differentiable function  $f(x)$  with the property that  $f(1) = -2$  and  $f(5) = 14$  and  $f'(x) < 3$  for every real number  $x$ .
- (b) If  $f''(5) = 0$  then  $(5, f(5))$  is an inflection point of the curve  $y = f(x)$ .
- (c) If  $f'(c) = 0$  then  $f(x)$  has a local maximum or a local minimum at  $x = c$ .
- (d) If  $f(x)$  is a differentiable function and the equation  $f'(x) = 0$  has 2 solutions, then the equation  $f(x) = 0$  has no more than 3 solutions.
- (e) If  $f(x)$  is increasing on  $[0, 1]$  then  $[f(x)]^2$  is increasing on  $[0, 1]$ .

10. Answer the following questions TRUE or False.

- (a) If  $f$  has a vertical asymptote at  $x = 1$  then  $\lim_{x \rightarrow 1} f(x) = L$ , where  $L$  is a finite value.
- (b) If  $f$  has domain  $[0, \infty)$  and has no horizontal asymptotes, then  $\lim_{x \rightarrow \infty} f(x) = \pm\infty$ .
- (c) If  $g(x) = x^2$  then  $\lim_{x \rightarrow 2} \frac{g(x) - g(2)}{x - 2} = 0$ .
- (d) If  $f''(2) = 0$  then  $(2, f(2))$  is an inflection point of  $f(x)$ .
- (e) If  $f'(c) = 0$  then  $f$  has a local extremum at  $c$ .
- (f) If  $f$  has an absolute minimum at  $c$  then  $f'(c) = 0$ .
- (g) If  $f'(c)$  exists, then  $\lim_{x \rightarrow c} f(x) = f(c)$ .
- (h) If  $f(1) < 0$  and  $f(3) > 0$ , then there exists a number  $c \in (1, 3)$  such that  $f(c) = 0$ .
- (i) If  $f'(g) = \frac{1}{(3-g)^2}$ , then  $f(g)$  is differentiable on  $(-\infty, 3) \cup (3, \infty)$ .
- (j) If  $f'(g) = \frac{1}{(3-g)^2}$ , the equation of the tangent line to  $f(g)$  at  $(0, 1/3)$  is  $y = \frac{1}{9}g + \frac{1}{3}$ .

11. Are the following statements true or false?

- (a) The points described by the polar coordinates  $(2, \pi/4)$  and  $(-2, 5\pi/4)$  are the same.
- (b) If the limit  $\lim_{x \rightarrow \infty} \frac{f'(x)}{g'(x)}$  does not exist, then the limit  $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)}$  does not exist.
- (c) If  $f$  is a function for which  $f''(x) = 0$ , then  $f$  has an inflection point at  $x$ .
- (d) If  $f$  is continuous at the number  $x$ , then it is differentiable at  $x$ .

- (e) Let  $f$  be a function and  $c$  a number in its domain. The graph of the linear approximation of  $f$  at  $c$  is the tangent line to the curve  $y = f(x)$  at the point  $(c, f(c))$ .
- (f) Every function is either an odd function or an even function.
- (g) A function that is continuous on a closed interval attains an absolute maximum value and an absolute minimum value at numbers in that interval.
- (h) An ellipse is the set of all points in the plane the sum of whose distances from two fixed points is a constant.

12. For each of the following, circle only one answer.

- (a) Suppose  $y'' + y = 0$ . Which of the following is a possibility for  $y = f(x)$ .  
A.  $y = \tan x$ , B.  $y = \sin x$ , C.  $y = \sec x$ , D.  $y = 1/x$ , E.  $y = e^x$
- (b) Which of the following is  $\arcsin\left(\sin\left(\frac{3\pi}{4}\right)\right)$ ?  
A. 0, B.  $\frac{\pi}{4}$ , C.  $-\frac{\pi}{4}$ , D.  $\frac{5\pi}{4}$ , E.  $\frac{3\pi}{4}$
- (c) Let  $f(x)$  be a continuous function on  $[a, b]$  and differentiable on  $(a, b)$  such that  $f(b) = 10$ ,  $f(a) = 2$ . On which of the following intervals  $[a, b]$  would the Mean Value Theorem guarantee a  $c \in (a, b)$  such that  $f'(c) = 4$ .  
A.  $[0, 4]$ , B.  $[0, 3]$ , C.  $[2, 4]$ , D.  $[1, 10]$ , E.  $(0, \infty)$
- (d) Let  $P(t)$  be the function which gives the population as a function of time. Assuming that  $P(t)$  satisfies the natural growth equation, and that at some point in time  $t_0$ ,  $P(t_0) = 500$ ,  $P'(t_0) = 1000$ , find the growth rate constant  $k$ .  
A.  $-\frac{1}{2}$ , B.  $\ln\left(\frac{1}{2}\right)$ , C.  $\frac{1}{2}$ , D. 2, E.  $\ln 2$
- (e) Suppose that  $f$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ . If  $f'(x) > 0$  on  $(a, b)$ . Which of the following is *necessarily* true?  
A.  $f$  is decreasing on  $[a, b]$ , B.  $f$  has no local extrema on  $(a, b)$ ,  
C.  $f$  is a constant function on  $(a, b)$ , D.  $f$  is concave up on  $(a, b)$ ,  
E.  $f$  has no zero on  $(a, b)$

13. For each of the following, circle only one answer.

- (a) The equation  $x^5 + 10x + 3 = 0$  has  
A. no real roots, B. exactly one real root, C. exactly two real roots,  
D. exactly three real roots, E. exactly five real roots

- (b) The value of  $\cosh(\ln 3)$  is  
 A.  $\frac{1}{3}$ , B.  $\frac{1}{2}$ , C.  $\frac{2}{3}$ , D.  $\frac{4}{3}$ , E.  $\frac{5}{3}$
- (c) The function  $f$  has the property that  $f(3) = 2$  and  $f'(3) = 4$ . Using a linear approximation to  $f$  near  $x = 3$ , an approximation to  $f(2.9)$  is  
 A. 1.4, B. 1.6, C. 1.8, D. 1.9, E. 2.4
- (d) Suppose  $F$  is an antiderivative of  $f(x) = \sqrt[3]{x}$ . If  $F(0) = \frac{1}{4}$ , then  $F(1)$  is  
 A.  $-1$ , B.  $-\frac{3}{4}$ , C. 0, D.  $\frac{3}{4}$ , E. 1
- (e) Suppose  $f$  is a function such that  $f'(x) = 4x^3$  and  $f''(x) = 12x^2$ . Which of the following is true?  
 A.  $f$  has a local maximum at  $x = 0$  by the first derivative test  
 B.  $f$  has a local minimum at  $x = 0$  by the first derivative test  
 C.  $f$  has a local maximum at  $x = 0$  by the second derivative test  
 D.  $f$  has a local minimum at  $x = 0$  by the second derivative test  
 E.  $f$  has an inflection point at  $x = 0$
14. Circle clearly your answer to the following 10 multiple choice question.
- (a) Evaluate  $\frac{d}{dx} \sin(x^2)$   
 A.  $2x \cos(x^2)$ , B.  $2x \sin(x^2)$ , C.  $2x \cos(x)$ , D.  $2x \cos(2x)$ , E.  $2x \cos(2x)$
- (b) Evaluate  $\lim_{x \rightarrow 0^+} \frac{\ln x}{x}$   
 A. 0, B.  $\infty$ , C. 1, D.  $-1$ , E. none of the above
- (c) Evaluate  $\lim_{x \rightarrow 0^+} \frac{1 - e^x}{\sin x}$   
 A. 1, B.  $-1$ , C. 0, D.  $\infty$ , E.  $\sin e$
- (d) The circle described by the equation  $x^2 + y^2 - 2x - 4 = 0$  has center  $(h, k)$  and radius  $r$ . The values of  $h$ ,  $k$ , and  $r$  are  
 A. 0, 1, and  $\sqrt{5}$ , B. 1, 0, and 5,  
 C. 1, 0, and  $\sqrt{5}$ , D.  $-1$ , 0, and 5,  
 E.  $-1$ , 0, and  $\sqrt{5}$
- (e) The edge of the cube is increasing at a rate of 2 cm/hr. How fast is the cube's volume changing when its edge is  $\sqrt{x}$  cm in length?  
 A.  $6 \text{ cm}^3/\text{hr}$ , B.  $12 \text{ cm}^3/\text{hr}$ , C.  $3\sqrt{2} \text{ cm}^3/\text{hr}$ ,  
 D.  $6\sqrt{2} \text{ cm}^3/\text{hr}$ , E. none of the above
- (f) Given the polar equation  $r = 1$ , find  $\frac{dy}{dx}$   
 A.  $\cot \theta$ , B.  $-\tan \theta$ , C. 0, D. 1, E.  $-\cot \theta$

- (g) Let  $A(t)$  denote the amount of a certain radioactive material left after time  $t$ . Assume that  $A(0) = 16$  and  $A(1) = 12$ . How much time is left after time  $t = 3$ ?

A.  $\frac{16}{9}$ , B. 8, C.  $\frac{9}{4}$ , D.  $\frac{27}{4}$ , E. 4

- (h) Which of the following statements is always true for a function  $f(x)$ ?

1. If  $f(x)$  is concave up on the interval  $(a, b)$ , then  $f(x)$  has a local minimum  $(a, b)$ .
2. It is possible for  $y = f(x)$  to have an inflection point at  $(a, f(a))$  even if  $f'(x)$  does not exist at  $x = a$ .
3. It is possible for  $(a, f(a))$  to be both a critical point and an inflection point of  $f(x)$ .

A. 1. and 2., B. 3. only, C. 1., 2., and 3., D. 2. and 3., E. 1. only

- (i) Which of the following statements is always true for a function  $f(x)$ ?

1. If  $f(x)$  and  $g(x)$  are continuous at  $x = a$ , then  $\frac{f(x)}{g(x)}$  is continuous at  $x = a$ .
2. If  $f(x) + g(x)$  is continuous at  $x = a$  and  $f'(a) = 0$ , then  $g(x)$  is continuous at  $x = a$ .
3. If  $f(x) + g(x)$  is differentiable at  $x = a$ , then  $f(x)$  and  $g(x)$  are both differentiable at  $x = a$ .

A. 1. only, B. 2. only, C. 3. only, D. 1. and 2., E. 2. and 4.

- (j) The slant asymptote of the function  $f(x) = \frac{x^2 + 3x - 1}{x - 1}$  is

A.  $y = x + 4$ , B.  $y = x + 2$ , C.  $y = x - 2$ , D.  $y = x - 4$ , E. none of the above

15. This is a multiple choice question. No explanation is required

- (a) The derivative of  $g(x) = e^{\sqrt{x}}$  is

A.  $\sqrt{x}e^{\sqrt{x}-1}$ , B.  $2e^{\sqrt{x}}x^{-0.5}$ , C.  $\frac{0.5e^{\sqrt{x}}}{\sqrt{x}}$ , D.  $e^{\sqrt{x}}$ , E. None of these

- (b) If  $\cosh y = x + x^3y$ , then at the point  $(1, 0)$  we  $y' =$

A. 0, B.  $-1$ , C. 1, D. 3, E. Does not exist

- (c) An antiderivative of  $f(x) = x - \sin x + e^x$  is

A.  $1 - \cos x + e^x$ , B.  $x^2 + \ln x - \cos x$ , C.  $0.5x^2 + e^x - \cos x$ , D.  $\cos x + e^x + 0.5x^2$ , E. None of these

- (d) If  $h(x) = \ln(1 - x^2)$  where  $-1 < x < 1$ , then  $h'(x) =$   
 A.  $\frac{1}{1 - x^2}$ , B.  $\frac{1}{1 + x} + \frac{1}{1 - x}$ , C.  $\frac{2}{1 - x^2}$ , D.  $\frac{1}{1 + x} - \frac{1}{1 - x}$ , E. None of these
- (e) The linear approximation to  $f(x) = \sqrt[3]{x}$  at  $x = 8$  is given by  
 A. 2, B.  $\frac{x + 16}{12}$ , C.  $\frac{1}{3x^{2/3}}$ , D.  $\frac{x - 2}{3}$ , E.  $\sqrt[3]{x} - 2$
16. Answer the following questions. You need not show work for this section.
- What is the period of  $f(x) = \tan x$ ?
  - What is the derivative of  $f(x) = x \ln |x| - x$ ?
  - If  $y = \sin^2 x$  and  $\frac{dx}{dt} = 4$ , find  $\frac{dy}{dt}$  when  $x = \pi$ .
  - What is the most general antiderivative of  $f(x) = 2xe^{x^2}$ ?
  - Evaluate  $\lim_{t \rightarrow \infty} (\ln(t + 1) - \ln t)$ ?
  - Does differentiability imply continuity?
  - Convert the Cartesian equation  $x^2 + y^2 = 25$  into a polar equation.
  - Simplify  $\cosh^2 x - \sinh^2 x$ .
17. Give an example for the each of the following:
- Function  $F = f \cdot g$  so that the limits of  $F$  and  $f$  at  $a$  exist and the limit of  $g$  at  $a$  does not exist.
  - Function that is continuous but not differentiable at a point.
  - Function with a critical number but no local maximum or minimum.
  - Function with a local minimum at which its second derivative equals 0.
18.
  - State the definition of the derivative of function  $f$  at a number  $a$ .
  - State the definition of a critical number of a function.
  - State the Extreme Value Theorem.
19. Match the start of each definition/theorem with its conclusion.
- The **Mean Value Theorem** states that ...
  - The **chain rule** states that ...
  - A **critical number** is a number that ...
  - The **Extreme Value Theorem** states that ...

- (e) **Fermat's Theorem** states that
- (f) An **antiderivative** of a function  $f$  is ...
- (g) The **natural number**  $e$  is ...
- (h) An **inflection point** is a point ...
- (i) The **derivative** of a function  $f$  at a number  $a$  is ...
- (j) **L'Hospital's Rule** states that ...
- (k) The **Intermediate Value Theorem** states that ...
  - i. ... if  $f$  is continuous on the closed interval  $[a, b]$  and let  $N$  be any number between  $f(a)$  and  $f(b)$ ,  $f(a) \neq f(b)$ . Then there exists a number  $c$  in  $(a, b)$  such that  $f(c) = N$ .
  - ii. ... if  $f$  is a function that satisfies the following hypotheses:
    - A.  $f$  is continuous on the closed interval  $[a, b]$ .
    - B.  $f$  is differentiable on the open interval  $(a, b)$ .
 Then there is a number  $c$  in  $(a, b)$  such that  $f'(c) = \frac{f(b) - f(a)}{b - a}$ .
  - iii. ...  $f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$  if this limit exists.
  - iv. ... If  $f$  is continuous on a closed interval  $[a, b]$ , then  $f$  attains an absolute maximum value  $f(c)$  and an absolute minimum value  $f(d)$  at some numbers  $c, d \in [a, b]$ .
  - v. ... is in the domain of  $f$  such that either  $f'(c) = 0$  or  $f'(c)$  does not exist.
  - vi. ... if  $\lim_{x \rightarrow a} f(x) = f(a)$ .
  - vii. ... on a continuous curve where the curve changes from concave upward to concave downward or from concave downward to concave upward.
  - viii. ... the base of the exponential function which has a tangent line of slope 1 at  $(0, 1)$ .
  - ix. ... If  $f$  and  $g$  are both differentiable then  $\frac{d}{dx} [f(g(x))] = f'(g(x)) \cdot g'(x)$ .

# Chapter 6

## Answers, Hints, Solutions

### 6.1 Limits

1. (a) 20  
(b) Does not exist  
(c) 0
2.  $-8 \ln 4$
3. 0. Note the exponential function in the denominator.
4.  $\frac{3}{5}$ . Divide the numerator and denominator by the highest power.
5.  $\frac{5}{2}$
6. 3
7. 2
8. 0. What is the value of  $3x + |1 - 3x|$  if  $x < \frac{1}{3}$ ?
9. 1
10.  $\frac{3}{\sqrt{2}}$
11.  $-\frac{2}{3}$
- (d) 100
- (e) Does not exist. Consider the domain of  $g(x) = \sqrt{-x^2 + 20x - 100} = \sqrt{-(x - 10)^2}$ .
12.  $\infty$ . Note that  $x^2 - 1 = (x - 1)(x + 1)$ .
13.  $-2$ . Which statement is true for  $x < 1$ :  $|x - 1| = x - 1$  or  $|x - 1| = 1 - x$ ?
14. (a) 1.5  
(b)  $-1.5$   
(c) No. The left-hand limit and the right-hand limit are not equal.
15. Does not exist
16.  $\frac{1}{8}$ . Rationalize the numerator.
17.  $\frac{1}{12}$ . Note that  $x - 8 = (\sqrt[3]{x} - 2)(\sqrt[3]{x^2} + 2\sqrt[3]{x} + 4)$ .
18.  $a = b = 4$ . Rationalize the numerator. Choose the value of  $b$  so that  $x$  becomes

- a factor in the numerator.
19.  $\frac{1}{12}$ . Note that  
 $x - 8 = (\sqrt[3]{x} - 2)(\sqrt[3]{x^2} + 2\sqrt[3]{x} + 4)$ .
20.  $\frac{1}{2}$ . Rationalize the numerator.
21.  $-\frac{3}{2}$ . Rationalize the numerator.  
 Note that  $x \rightarrow -\infty$  and use the fact that if  $x < 0$  then  $x = -\sqrt{x^2}$ .
22.  $-\frac{1}{2}$
23.  $\frac{3}{2}$
24. Since the denominator approaches 0 as  $x \rightarrow -2$ , the necessary condition for this limit to exist is that the numerator approaches 0 as  $x \rightarrow -2$ . Thus we solve  $4b - 30 + 15 + b = 0$  to obtain  $b = 3$ .  $\lim_{x \rightarrow -2} \frac{3x^2 + 15x + 18}{x^2 + x - 2} = -1$ .
25.  $a = 4$ . Write  $f(x) = x + \frac{(a-1)x + 5}{x+1}$ .
26.  $\lim_{x \rightarrow \infty} \frac{\ln x}{x} = 0$ .
27. From  $\lim_{x \rightarrow 4} (x+2) = 6$  and  $\lim_{x \rightarrow 4} (x^2 - 10) = 6$ , by the Squeeze Theorem, it follows that  $\lim_{x \rightarrow 4} f(x) = 6$ .
28. 1
29. From the fact that  $|\sin(1/x)| \leq 1$  for all  $x \neq 0$  and the fact that the function  $y = e^x$  is increasing conclude that  $e^{-1} \leq e^{\sin(1/x)} \leq e$  for all  $x \neq 0$ . Thus  $e^{-1} \cdot \sqrt{x} \leq \sqrt{x}e^{\sin(1/x)} \leq e \cdot \sqrt{x}$  for all  $x > 0$ . By the Squeeze Theorem,  $\lim_{x \rightarrow 0^+} (\sqrt{x}e^{\sin(1/x)}) = 0$ .
30. 0. Squeeze Theorem.
31. 0. Squeeze Theorem.
32. 0. Squeeze Theorem.
33.  $-\infty$ .
34. 0.
35.  $\infty$ .
36.  $\frac{76}{45}$ . This is the case "0/0". Apply L'Hospital's rule.
37.  $\frac{1}{2}$ . Write  $\frac{1}{2} \cdot \left(\frac{\sin x}{x}\right)^{100} \cdot \frac{2x}{\sin 2x}$ .
38. 7. Write  $7 \cdot \left(\frac{x}{\sin x}\right)^{101} \cdot \frac{\sin 7x}{7x}$ .
39. 7.
40.  $\frac{3}{5}$ . This is the case "0/0". Apply L'Hospital's rule.
41.  $\frac{3}{5}$ .
42. 0. Write  $x^2 \cdot \frac{x}{\sin x} \cdot \sin\left(\frac{1}{x^2}\right)$ .
43. Does not exist.  $\frac{\sin x}{2|x|} \cdot \frac{1}{\sqrt{\frac{\sin 4x}{4x}}}$ .
44.  $\frac{1}{2}$ . Write  $\frac{1 - \cos x}{x^2} \cdot \frac{x}{\sin x}$ .
45. 1. Substitute  $t = \frac{1}{x}$ .
46. 0. This is the case " $\infty - \infty$ ". Write  $\frac{x - \sin x}{x \sin x}$  and apply L'Hospital's rule.
47.  $\frac{1}{6}$
48. 0. This is the case " $0 \cdot \infty$ ". Write  $\frac{\ln \sin x}{\frac{1}{\sin x}}$  and apply L'Hospital's rule.

49. 0. This is the case " $\infty/\infty$ ". Apply L'Hospital's rule. 54.  $\frac{3}{2}$ . Use properties of logarithms first.
50. 0. 55.  $\frac{3}{2}$ .
51. 0. 56.  $\ln 2$ . The denominator approaches 2.
52. 0. 57. 0. This is the case " $0/0$ ". Apply L'Hospital's rule.
53. 1. This is the case " $0/0$ ". Write  $\frac{\ln(1+x)}{x}$  and apply L'Hospital's rule. 58.  $-\frac{1}{\pi^2}$
59.  $\infty$ . This is the case " $\infty - \infty$ ". Write  $\frac{\sin x - x^2 \cos x}{x^2 \sin x}$  and apply L'Hospital's rule.
60.  $e^{\frac{1}{2}}$ . This is the case " $1^\infty$ ". Write  $e^{\frac{\ln \cosh x}{x^2}}$ . Apply L'Hospital's rule and use the fact that the exponential function  $f(x) = e^x$  is continuous.
61. 1. This is the case " $0^0$ ". Write  $x^x = e^{x \ln x} = e^{\frac{\ln x}{\frac{1}{x}}}$ . Apply L'Hospital's rule and use the fact that the exponential function  $f(x) = e^x$  is continuous.
62. 1. 75.  $e^{\frac{3}{8}}$ . Write  $\left((1+3x)^{\frac{1}{3x}}\right)^{\frac{3}{8}}$ .
63. 1. 76.  $e^{\frac{3}{2}}$ . Write  $\left(\left(1+\frac{x}{2}\right)^{\frac{2}{x}}\right)^{\frac{3}{2}}$ .
64.  $e$ .
65. 1. This is the case " $\infty^0$ ". 77. 10. Use the fact that  $L = \lim_{n \rightarrow \infty} x_n$  to conclude  $L^2 = 100$ . Can  $L$  be negative?
66. 1.
67.  $e^3$ .
68. 0. 78. (a)  $\frac{1}{2}$ . Write  $\frac{2 \sin^2 \frac{x}{2}}{x^2}$ , or use L'Hospital's rule.
69. 1. Write  $e^{x \ln \frac{x}{x+1}} = e^{x \ln x} \cdot e^{-x \ln(x+1)}$  and make your conclusion. (b) 0.
70.  $e^{\frac{1}{e}}$ . (c) Does not exist. Note that  $f(x) = \arcsin x$  is defined on  $[-1, 1]$ .
71. 1.
72. 1. Squeeze Theorem. 79. (a) Does not exist. Note that  $\lim_{h \rightarrow 0} \sqrt[4]{16+h} = 2$ .
73.  $e^{-2}$ . Write  $\left((1-2x)^{-\frac{1}{2x}}\right)^{-2}$ . (b)  $-\frac{1}{\pi}$ . Use L'Hospital's rule.
74.  $e^{\frac{7}{5}}$ . Write  $\left((1+7x)^{\frac{1}{7x}}\right)^{\frac{7}{5}}$ . (c) 1. Divide the numerator and denominator by  $u$ .

- (d)  $e^{-2}$ . Consider  $\lim_{x \rightarrow 0} \frac{xf(x)}{x|x|}$ ; (b)  $\frac{1}{3}$ ;  
 (e)  $\frac{1}{2}$  (c)  $-\frac{1}{4}$ . Note that  $x < 0$ ; (d)  $e$ .  
 (f)  $\infty$ . Think, exponential vs. polynomial.
80. (a)  $\frac{1}{2}$ ; (b)  $\frac{7}{3}$ ; (c) 1; (d) 0; (e)  $\frac{\sin 3 - 3}{27}$ . 82. (a)  $\frac{1}{4}$ ; (b)  $-1$ ; (c) 0.
81. (a) Does not exist. 83. (a)  $-\infty$ ; (b) 3; (c) 2; (d) 0.
84. Let  $\varepsilon > 0$  be given. We need to find  $\delta = \delta(\varepsilon) > 0$  such that  $|x-0| < \delta \Rightarrow |x^3-0| < \varepsilon$ , what is the same as  $|x| < \delta \Rightarrow |x^3| < \varepsilon$ . Clearly, we can take  $\delta = \sqrt[3]{\varepsilon}$ . Indeed. For any  $\varepsilon > 0$  we have that  $|x| < \sqrt[3]{\varepsilon} \Rightarrow |x|^3 = |x^3| < \varepsilon$  and, by definition,  $\lim_{x \rightarrow 0} x^3 = 0$ .
85. (c) For any  $\varepsilon > 0$  there exists  $\delta = \delta(\varepsilon) > 0$  such that  $|x-1| < \delta \Rightarrow |2x^2-2| < \varepsilon$ .
86.  $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x-h)}{2h} = \lim_{h \rightarrow 0} \frac{f'(x+h) + f'(x-h)}{2}$  and, since  $f'$  is continuous,  $\lim_{h \rightarrow 0} f'(x+h) = \lim_{h \rightarrow 0} f'(x-h) = f'(x)$ .

## 6.2 Continuity

1.  $c = \pi$ . Solve  $\lim_{x \rightarrow \pi^-} f(x) = \lim_{x \rightarrow \pi^-} f(x)$  for  $c$ . See Figure 6.1.

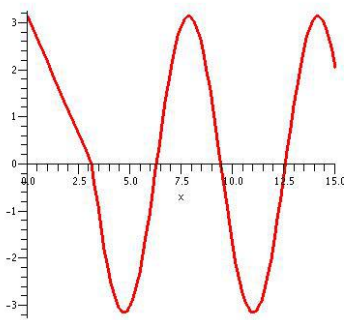


Figure 6.1:  $c = \pi$

2. Let  $f(x) = 2^x - \frac{10}{x}$ . Note that the domain of  $f$  is the set  $\mathbb{R} \setminus \{0\}$  and that on its domain, as a sum of two continuous function,  $f$  is continuous.

- (a) Since  $f$  is continuous on  $(0, \infty)$  and since  $\lim_{x \rightarrow 0^+} f(x) = -\infty$  and  $\lim_{x \rightarrow \infty} f(x) = \infty$  by the Intermediate Value Property there is  $a \in (0, \infty)$  such that  $f(a) = 0$ .
- (b) For all  $x \in (-\infty, 0)$  we have that  $\frac{10}{x} < 0$  which implies that for all  $x \in (-\infty, 0)$  we have that all  $f(x) > 0$ .

3. See Figure 6.2.

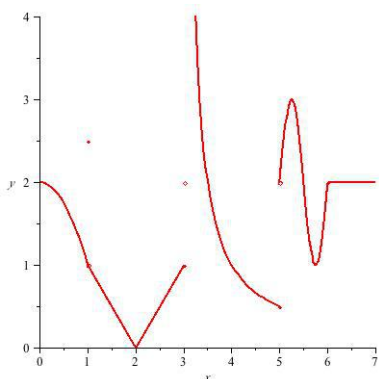


Figure 6.2: Piecewise Defined Function

- (a) (i) False; (ii) True; (b) (i) Yes; (ii) Yes; (c) (i) No; (ii) No.

4. See Figure 6.3.

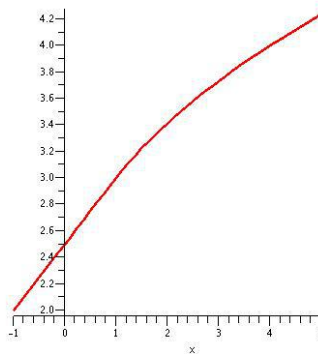


Figure 6.3: Continuous Function

- (a) Check that  $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = f(1)$ .
- (b)  $\frac{1}{2}$ . Note  $\lim_{x \rightarrow 1^-} \frac{5+x}{2} - 3 = \frac{1}{2}$  and  $\lim_{x \rightarrow 1^+} \frac{(2 + \sqrt{x}) - 3}{x - 1} = \frac{1}{2}$ .

5.  $f(x) = \frac{x^2 - 9}{x - 3}$  if  $x \neq 3$  and  $f(3) = 0$ .

## 6.3 Miscellaneous

1. (a)  $x = \frac{1 - \ln \pi}{\ln \pi}$ . (b)  $x = -\frac{\log \log 2}{\log 3}$ .  
 2.  $(e, 3) \cup (3, \infty)$ .  
 3. (a) Give a definition of the limit. (b) Give a definition of a function continuous at a point. (c) A corner or a vertical tangent;  $y = |x|$ ;  $y = x^{\frac{1}{3}}$ .

## 6.4 Derivatives

1. (b)  $f'(3) = \lim_{h \rightarrow 0} \frac{\frac{1}{2(3+h)-1} - \frac{1}{5}}{h} = \lim_{h \rightarrow 0} \frac{-2}{5(5+2h)} = -0.08$ . (c)  $\lim_{h \rightarrow 0} \frac{\sin^7(\frac{\pi}{6} + \frac{h}{2}) - (\frac{1}{2})^7}{h} = \frac{d}{dx}(\sin^7 \frac{x}{2}) \Big|_{x=\frac{\pi}{3}} = \frac{7}{2} \cdot \sin^6 \frac{\pi}{6} \cdot \cos \frac{\pi}{6} = \frac{7\sqrt{3}}{256}$ .
2. Let  $|I(x)| \leq M$  for all  $x \in \mathbb{R}$ . Then for any  $h \neq 0$ ,  $\left| \frac{h^2 I(h)}{h} \right| = |hI(h)|$ . Use the Squeeze Theorem to conclude that  $f$  is differentiable at  $x = 0$ .
3.  $f'(2) = \lim_{x \rightarrow 2} \frac{x + \frac{1}{x} - \frac{5}{2}}{x - 2} = \frac{3}{4}$ .
4. Since  $g$  is not differentiable we cannot use the product rule.  $f'(0) = \lim_{h \rightarrow 0} \frac{hg(h)}{h} = 8$ .
5. (b)  $f'(4) = \lim_{h \rightarrow 0} \frac{\sqrt{5 - (x+h)} - 1}{h} = -0.5$ .
6.  $F'(0) = \lim_{h \rightarrow 0} \frac{\frac{f(h) \sin^2 h}{h}}{h} = \lim_{h \rightarrow 0} \frac{f(h) \sin^2 h}{h^2} = f(0)$ .
7.  $m = e, b = 0$ . Solve  $\lim_{x \rightarrow 1^-} e^x = \lim_{x \rightarrow 1^+} (mx+b)$  and  $\lim_{x \rightarrow 1^-} \frac{e^x - e}{x - 1} = \lim_{x \rightarrow 1^+} \frac{mx + b - (m + b)}{x - 1}$  for  $m$  and  $b$ .
8. (a)  $S'(3) = \frac{F'(3)G(3) - F(3)G'(3)}{[G(3)]^2} = -\frac{1}{4}$ . (b)  $T'(0) = F'(G(0)) \cdot G'(0) = 0$ . (c)  $U'(3) = \frac{F'(3)}{F(3)} = -\frac{1}{2}$ .
9. From  $h(1) = f(1)g(1)$  and  $h'(1) = f'(1)g(1) + f(1)g'(1)$  it follows that  $g'(1) = 9$ .

10.  $2f(g(1)) \cdot f'(g(1)) \cdot g'(1) = 120$ .
11.  $f'(x) = -\frac{2}{(x-2)^2}$
12. (a)  $f'(x) = \sec^2 x$ . This follows from  $\tan x = \frac{\sin x}{\cos x}$  by using the quotient rule.
- (b) From  $g(x) = \arctan x$ ,  $x \in \mathbb{R}$ , and  $f'(g(x)) \cdot g'(x) = 1$ , we conclude that  $g'(x) = \cos^2(g(x))$ . Next, suppose that  $x > 0$  and consider the right triangle with the hypotenuse of the length 1 and with one angle measured  $g(x)$  radians. Then  $\tan g(x) = \tan(\arctan x) = x = \frac{\sin g(x)}{\cos g(x)} = \sqrt{\frac{1 - g'(x)}{g'(x)}}$  which implies that  $x^2 = \frac{1 - g'(x)}{g'(x)}$ . Thus  $g'(x) = \frac{1}{1 + x^2}$ .
- (c) From  $g'(x) = 2x \sec x^2 + \frac{2x}{1 + x^4}$  it follows that  $g'\left(\frac{\sqrt{\pi}}{2}\right) = 2\sqrt{\pi} + \frac{16\sqrt{\pi}}{16 + \pi^2}$ .
13.  $f'(1) = g(1) = 2$ .
14.  $\frac{d}{dx}(\sqrt{x} + x^7)\bigg|_{x=1} = \frac{15}{2}$ .
15.  $f'(0) = 0$ . Note that, for  $h \neq 0$ ,  $\left|\frac{h^2 \sin \frac{1}{h}}{h}\right| = \left|h \sin \frac{1}{h}\right| \leq |h|$ . Use the Squeeze Theorem.
16.  $f'(x) = 2 - \sin x > 0$  for all  $x \in \mathbb{R}$ . Let  $g(0) = \alpha$ . Then  $g'(0) = \frac{1}{f'(g(0))} = \frac{1}{2 - \sin \alpha}$ .
17. Let  $f(x) = \sin x$ ,  $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ . Then, for  $x \in (-1, 1)$ ,  $(f^{-1})'(x) = \frac{1}{\cos(f^{-1}(x))}$ . Suppose that  $x \in (0, 1)$  and let  $\alpha = f^{-1}(x)$ . Consider the right triangle with the hypotenuse of the length 1 and an angle measured  $\alpha$  radians. The length of the leg opposite to the angle  $\alpha$  equals  $\sin \alpha = x$  which implies  $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}$ .
18. Use the chain rule and the given property of  $f'(x)$  to get  $(1 + (f(g(x)))^2) \cdot g'(x) = 1$ .
19. Write  $y = \frac{1}{2} \cdot (2x^2 - 2\sqrt{x^4 - 1})$ .
20. 0. Note  $f(x) = (x+2)(x^2+4)$ .
21.  $y' = -\frac{5!}{x^6} - 2^5 \sin 2x$ .
22.  $A = 0$ ,  $B = 1$ .
23. Use the chain rule to differentiate  $f^2 - g^2$ .
24. Use the product rule twice.
25.  $y' = \left(\frac{3 \ln(x+2)}{x} + \frac{3 \ln x}{x+2} - \frac{x}{x^2+1}\right) \cdot \frac{(x+2)^{3 \ln x}}{(x^2+1)^{1/2}}$ . Use the logarithmic differentiation.
26.  $y = \frac{2}{\sqrt{x}} \sinh \sqrt{x} \cdot e^{4 \cosh \sqrt{x}}$ .
27. From  $f'(x) = \frac{2x+1}{\sqrt{1-(x^2+x)^2}} + 5^x \ln 5$  it follows that  $f'(0) = 1 + \ln 5$ .
28. 0. Write as a product.

$$29. \quad (a) \quad \frac{3e^{-3t} \sinh(e^{-3t})}{\sqrt{1 - \cosh^2(e^{-3t})}}$$

$$(b) \quad y'(u) = \frac{1}{3} \left( \frac{1}{u+1} + \frac{1}{u+2} - \frac{2u}{u^2+1} - \frac{2u}{u^2+2} \right) \cdot \left( \frac{(u+1)(u+2)}{(u^2+1)(u^2+2)} \right)^{1/3}.$$

$$30. \quad y' = \frac{x(2 \ln x + 1) \sinh(\arcsin(x^2 \ln x))}{\sqrt{1 - x^4 \ln^2 x}}.$$

$$31. \quad y' = -\frac{4e^{-4x}}{\sqrt{1 - e^{8x}}}.$$

$$32. \quad (a) \quad y' = -2xe^{\cos x^2} \sin x^2.$$

$$(b) \quad g'(x) = \frac{x}{x^2 + 1} + 4 \cot x.$$

$$(b) \quad y' = x^{19} \left( 20 \arctan x + \frac{x}{1+x^2} \right).$$

$$(c) \quad y' = 2x^{\ln x - 1} \ln x.$$

$$(a) \quad f'(x) = \frac{1}{2\sqrt{x}(1+x)}.$$

$$(b) \quad f'(x) = \frac{5 \sinh(5 \ln x)}{x}.$$

$$33. \quad (a) \quad y' = \frac{6e^{3 \ln(2x+1)}}{2x+1}.$$

$$(b) \quad y' = 2x^{2x}(\ln x + 1).$$

$$(c) \quad y' = \frac{e^{2x}}{(x^2+1)^3(1+\sin x)^5} \cdot \left( 2 - \frac{6x}{x^2+1} - \frac{5 \cos x}{1+\sin x} \right).$$

$$39. \quad (a) \quad f'(x) = 3 \cdot 10^{3x} \cdot \ln 10.$$

$$(b) \quad f'(x) = x^9(10 \tanh x + x \operatorname{sech}^2 x).$$

$$(c) \quad f'(x) = \left( \frac{\cos x}{x} - \sin x \ln x \right) x^{\cos x}.$$

$$(d) \quad y' = \frac{2x + 2y^2}{3 - 4xy}.$$

$$40. \quad (a) \quad f'(x) = (2 \ln x + 1)x^{x^2+1}.$$

$$(b) \quad f'(x) = -3 \tan 3x.$$

$$34. \quad (a) \quad y' = (x \cosh x + \sinh x)x^{\sinh x}.$$

$$(b) \quad y' = \frac{xy + y^2 - 1}{3xy^2 + 3y^3 - x^2 - xy - 1}.$$

$$35. \quad (a) \quad y' = \sec(\sinh x) \tan(\sinh x) \cosh x.$$

$$(b) \quad y' = \frac{ex^{e-1} - e^x}{e^y - ey^{e-1}}.$$

$$41. \quad (a) \quad f'(x) = \frac{(5-x)(x-1)}{(x+1)^4}.$$

$$(b) \quad f'(x) = 2^{2x+1} \ln 2 - \frac{4x}{3\sqrt[3]{x^2+1}}.$$

$$(c) \quad f'(x) = 4x \tan(x^2) \cdot \sec^2 x^2.$$

$$(d) \quad f'(x) = \left( \frac{\ln x}{1+x^2} + \frac{\arctan x}{x} \right) x^{\arctan x}.$$

$$36. \quad (a) \quad f'(x) = (6x - 3x^2 - 1)e^{-x}.$$

$$(b) \quad g'(z) = \frac{z \cos \sqrt{z^2+1}}{\sqrt{z^2+1}}.$$

$$(c) \quad f'(x) = x^x(\ln x + 1).$$

$$(d) \quad h'(y) = -\frac{y \tan y + 1}{2y} \sqrt{\frac{\cos y}{y}}.$$

$$(e) \quad f'''(x) = 8 \cosh(2x).$$

$$42. \quad (a) \quad f'(x) = 5 + 5x^4 + 5^x \ln 5 + \frac{1}{5\sqrt[5]{x^4}}.$$

$$(b) \quad y' = x^9(10 \tanh x + x \operatorname{sech}^2 x).$$

$$(c) \quad y' = \left( \frac{\cos x}{x \ln x} - \sin x \ln \ln x \right) (\ln x)^{\cos x}.$$

$$37. \quad (a) \quad f'(x) = \frac{1-x^2}{(x^2+1)^2}.$$

$$43. \quad (a) \quad f'(x) = \coth x.$$

- (b)  $f'(x) = (\cos x - x \sin x)e^{x \cos x}$ . 47. (a)  $f'(x) = \frac{1}{\sec(x^2 + 7x)}$ .  
 (c)  $f'(x) = \frac{1 + \cos x}{(1 + \cos x)^2}$ .  $\left( \frac{2x - 3}{x^2 - 3x + 8} - (2x + 7) \cdot \ln(x^2 - 3x + 8) \tan(x^2 + 7x) \right)$ .  
 (d)  $f'(x) = (\ln x + 1)x^x$ .  $\ln(x^2 - 3x + 8) \tan(x^2 + 7x)$ .
44. (a)  $f'(x) = \frac{3}{x^4}$ . (b)  $f'(x) = \frac{2 \sinh(2x - 3)}{1 + \cosh^2(2x - 3)}$   
 (b)  $f'(x) = \frac{2x \sin^2(2x^2)}{8x^3 \sin(2x^2) \cos(x^2)} +$  (c)  $f'(x) = -3e^{3x-4} \sin(e^{3x-4})$ .  
 (c)  $f'(x) = \left( \ln(x + 2) + \frac{x}{x + 2} \right) (x + 2)^x$ . (d)  $f'(x) = \left( (x^{-1} + 2x) \ln \tan x + \frac{\ln x + x^2}{\sin x \cos x} \right) (\tan x)^{\ln x + x^2}$ .  
 (e)  $f'(x) = 0$ .
45. (a)  $y' = \frac{x \sec \sqrt{x^2 + 1} \tan \sqrt{x^2 + 1}}{\sqrt{x^2 + 1}}$ . 48. (a)  $h'(t) = -\frac{1}{3} \sec^2 \left( \frac{t}{3} \right) \cdot e^{-\tan(\frac{t}{3})}$   
 (b)  $y' = \left( e^x \ln x + \frac{e^x}{x} \right) x^{e^x}$ . (b)  $y' = -\frac{3}{8x \ln^2 x} \left( \frac{1}{2 \ln x} \right)^{-1/4}$ .  
 Note that  $y = \left( \frac{1}{2 \ln x} \right)^{3/4}$ .
46. (a)  $y' = 3x^2 + 3^x \ln 3 + 3(\ln x + 1)x^{3x}$ . (c)  $f'(y) = \frac{\ln 3}{\ln 7 \cdot \sqrt{1 - y^2} \cdot \arcsin y}$ .  
 $3^{\log_7(\arcsin y)}$   
 (d)  $y' = \left( \frac{5}{x} + 3x^2 + \frac{2x}{3(x^2 + 1)} - \frac{449}{x + 1} \right) \cdot \frac{x^5 e^{x^3} \sqrt[3]{x^2 + 1}}{(x + 1)^4}$ . (a)  $f'(x) = \frac{2x + 1}{\sqrt{1 - (x^2 + x)^2}} + 5^x \ln 5$ .  
 (b)  $g'(x) = -\frac{3x^2 + 4x + 3}{2\sqrt{x + 1}(x^2 - 3)^2} \sinh \left( \frac{\sqrt{x + 1}}{x^2 - 3} \right)$ .
50. (a)  $f'(x) = \frac{2^x(e^{4x} + a) \ln 2 - 4e^{4x} \sinh^{-1}(2^x) \sqrt{2^{2x} + 1}}{(e^{4x} + a)^2 \sqrt{2^{2x} + 1}}$ .  
 (b)  $g'(x) = g(x) \cdot \left( -\frac{6x \sin(3x^2)}{2 + \cos(3x^2)} + \pi - \frac{3}{2} \right)$ . (Use logarithmic differentiation.)
51. (a)  $\frac{d^2 y}{dx^2} = -\frac{4x^3}{(1 + x^4)^2}$  (b)  $y' = \frac{1}{2} x^{\sqrt{x} - \frac{1}{2}} \ln(e^2 x)$

## 6.5 Related Rates

- Let  $x = x(t)$  be the distance between the bottom of the ladder and the wall. It is given that, at any time  $t$ ,  $\frac{dx}{dt} = 2$  ft/s. Let  $\theta = \theta(t)$  be the angle between the top of the ladder and the wall. Then  $\sin \theta = \frac{x(t)}{15}$ . It follows that  $\cos \theta \cdot \frac{d\theta}{dt} = \frac{1}{15} \frac{dx}{dt}$ . Thus when  $\theta = \frac{\pi}{3}$  the rate of change of  $\theta$  is given by  $\frac{d\theta}{dt} = \frac{4}{15}$  ft/s.
- Let  $x = x(t)$  be the distance between the foot of the ladder and the wall and let  $y = y(t)$  be the distance between the top of the ladder and the ground. It is given that, at any time  $t$ ,  $\frac{dx}{dt} = \frac{1}{2}$  m/min. From  $x^2 + y^2 = 144$  it follows that  $x \cdot \frac{dx}{dt} + y \frac{dy}{dt} = 0$ . Thus when  $x(t) = 4$  we have that  $y(t) = 8\sqrt{2}$  and  $4 \cdot \frac{1}{2} + 8\sqrt{2} \frac{dy}{dt} = 0$ . The top of the ladder is falling at the rate  $\frac{dy}{dt} = -\frac{\sqrt{2}}{8}$  m/min.
- Let  $x = x(t)$  be the height of the rocket at time  $t$  and let  $y = y(t)$  be the distance between the rocket and radar station. It is given that, at any time  $t$ ,  $x^2 = y^2 - 16$ . Thus, at any time  $t$ ,  $x \cdot \frac{dx}{dt} = y \frac{dy}{dt}$ . At the instant when  $y = 5$  miles and  $\frac{dy}{dt} = 3600$  mi/h we have that  $x = 3$  miles and we conclude that, at that instant,  $3 \frac{dx}{dt} = 5 \cdot 3600$ . Thus the vertical speed of the rocket is  $v = \frac{dx}{dt} = 6000$  mi/h.
- Let  $x = x(t)$  be the distance between the dock and the bow of the boat at time  $t$  and let  $y = y(t)$  be the length of the rope between the pulley and the bow at time  $t$ . It is given that  $\frac{dy}{dt} = 1$  m/sec. From  $x^2 + 1 = y^2$  it follows that  $\frac{dx}{dt} = \frac{y}{x}$  m/sec. Since  $y = 10$  implies  $x = \sqrt{99}$  we conclude that when 10 m of rope is out then the boat is approaching the dock at the rate of  $\frac{10}{\sqrt{99}}$  m/sec.
- From  $y = 5 \tan \theta$  we get that, at any time  $t$ ,  $\frac{dy}{dt} = 5 \sec^2 \theta \frac{d\theta}{dt}$ . At the instant when  $\theta = \frac{\pi}{3}$  radians we have that  $v = \frac{dy}{dt} = 5 \cdot \sec^2 \frac{\pi}{3} \cdot 2 = 40$  m/s.
- After time  $t$  (in hours) the plane is  $480t$  km away from the point directly above the observer. Thus, at time  $t$ , the distance between the observer and the plane is  $D = \sqrt{3^2 + (480t)^2}$ . We differentiate  $D^2 = 9 + 230,400t^2$  with respect to  $t$  to get  $2D \frac{dD}{dt} = 460,800t$ . Since  $30 \text{ sec} = \frac{1}{120}$  hours it follows that the distance between the observer and the plane after 30 seconds equals  $D = 5$  km. Thus, 30 seconds later the distance  $D$  from the observer to the airplane is increasing at the rate of

$$\left. \frac{dD}{dt} \right|_{t=\frac{1}{120}} = 384 \text{ km/h.}$$

7. Let  $y$  be the distance between the airplane and the radar station. Then, as the hypotenuse in a right angle triangle with the angle  $\theta$  and the opposite leg of length 1000 m,  $y = \frac{1000}{\sin \theta}$ . Since it is given that  $\frac{d\theta}{dt} = -0.1$  rad/sec, it follows that  $\frac{dy}{dt} = -\frac{1000 \cos \theta}{\sin^2 \theta} \cdot \frac{d\theta}{dt} = \frac{100 \cos \theta}{\sin^2 \theta}$  m/sec. Hence if  $\theta = \frac{\pi}{4}$ , the speed of the plane is given by  $\left. \frac{dy}{dt} \right|_{t=\frac{\pi}{4}} = 100\sqrt{2}$  m/sec.
8. (a) From  $z^2 = 64 + 4t^2$  it follows that  $2zz' = 8t$ . If  $z = 10$  then  $t = 3$  and at that instant  $z' = 1.2$  m/s. (b) Since the height of the kite after  $t$  seconds is  $2t$  meters, it follows that  $\tan x = \frac{2t}{8}$ . Thus  $\frac{x'}{\cos^2 x} = \frac{1}{4}$ . If  $y = 6$  then  $t = 3$  and  $\tan x = \frac{3}{4}$ . It follows that  $\cos x = \frac{4}{5}$  and at that instant the rate of change of  $x$  is given by  $x' = x'(3) = \frac{4}{25}$  m/s.
9. Let  $x = x(t)$  be the distance (in metres) between the boy and the balloon at time  $t$ . Then  $[x(t)]^2 = (8t)^2 + (36 + 4t)^2$ . From  $2x(t)x'(t) = 128t + 8(36 + 4t)$ . From  $x(3) = 24\sqrt{5}$  m, it follows that  $x'(3) = \frac{16}{\sqrt{5}}$  m/sec.
10. Let  $\theta = \theta(t)$  be the elevation angle. From  $\tan \theta = \frac{2t}{80}$  it follows that  $\frac{d\theta}{dt} = \frac{\cos^2 \theta}{40}$ . When  $t = 30$  we have  $\tan \theta = \frac{3}{4}$  and  $\cos \theta = \frac{4}{5}$ . Thus when the helicopter is 60 m above the ground the elevation angle of the observer's line of sight to the helicopter is changing at the rate  $\frac{1}{50}$  m/s.
11. Let  $r$  denotes the radius of the circular containment area. It is given that  $\frac{dr}{dt} = -5$  m/min. From the fact that the area at time  $t$  is given by  $A = r^2\pi$ , where  $r = r(t)$ , it follows that  $\frac{dA}{dt} = 2r\pi \frac{dr}{dt} = -10r\pi$  m<sup>2</sup>/min. Hence when  $r = 50$ m then the area shrinks at the rate of  $10 \cdot 50 \cdot \pi = 500\pi$  m<sup>2</sup>/min.
12. Let  $x = x(t)$  be the edge length. Then the volume is given by  $V = x^3$  and the surface area is given by  $S = 6x^2$ . It is given that  $\frac{dV}{dt} = 10$ . This implies that  $3x^2 \frac{dx}{dt} = 10$  at any time  $t$  and we conclude that at the instant when  $x = 8$  the edge is increasing at the rate  $\frac{5}{96}$  cm/min. This fact together with  $\frac{dS}{dt} = 12x \frac{dx}{dt}$  implies

that at the instant when  $x = 8$  the surface area is increasing at the rate  $12 \cdot 8 \cdot \frac{5}{96} = 5$   $\text{cm}^2/\text{min}$ .

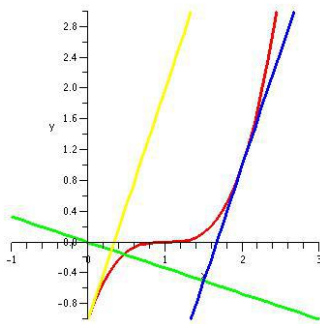
13. Let  $H = H(t)$  be the height of the box, let  $x = x(t)$  be the length of a side of the base, and  $V = V(t) = Hx^2$ . It is given that  $\frac{dH}{dt} = 2$  m/s and  $\frac{dV}{dt} = 2x\frac{dx}{dt}H + x^2\frac{dH}{dt} = -5$   $\text{m}^3/\text{s}$ . The question is to find the value of  $\frac{dx}{dt}$  at the instant when  $x^2 = 64$   $\text{m}^2$  and  $H = 8$  m. Thus, at that instant, one of the sides of the base is decreasing at the rate of  $\frac{133}{128}$  m/s.
14. Let  $H = H(t)$  be the height of the pile, let  $r = r(t)$  be the radius of the base, and let  $V = V(t)$  be the volume of the cone. It is given that  $H = r$  (which implies that  $V = \frac{H^3\pi}{3}$ ) and that  $\frac{dV}{dt} = H^2\pi\frac{dH}{dt} = 1$   $\text{m}^3/\text{sec}$ . The question is to find the value of  $\frac{dH}{dt}$  at the instant when  $H = 2$ . Thus at that instant the sandpile is rising at the rate of  $\frac{1}{4\pi}$  m/sec.
15. Let  $H = H(t)$  be the height of water, let  $r = r(t)$  be the radius of the surface of water, and let  $V = V(t)$  be the volume of water in the cone at time  $t$ . It is given that  $r = \frac{3H}{5}$  which implies that  $V = \frac{3H^3\pi}{25}$ . The question is to find the value of  $\frac{dH}{dt}$  at the instant when  $H = 3$  and  $\frac{dV}{dt} = -2$   $\text{m}^3/\text{sec}$ . Thus at that instant the water level dropping at the rate of  $\frac{50}{81\pi}$  m/sec.
16. The distance between the boy and the girl is given by  $z = \sqrt{x^2 + y^2}$  where  $x = x(t)$  and  $y = y(t)$  are the distances covered by the boy and the girl in time  $t$ , respectively. The question is to find  $z'(6)$ . We differentiate  $z^2 = x^2 + y^2$  to get  $zz' = xx' + yy'$ . From  $x(6) = 9$ ,  $y(6) = 12$ ,  $z(6) = 15$ ,  $x'(t) = 1.5$ , and  $y'(t) = 2$  it follows that  $z'(6) = 2.5$  m/s.
17. The distance between the two ships is given by  $z = \sqrt{x^2 + (60 - y)^2}$  where  $x = x(t)$  and  $y = y(t)$  are the distances covered by the ship  $A$  and the ship  $B$  in time  $t$ , respectively. The question is to find  $z'(4)$ . We differentiate  $z^2 = x^2 + (60 - y)^2$  to get  $zz' = xx' - (60 - y)y'$ . From  $x(4) = 60$ ,  $y(4) = 49$ ,  $z(4) = 61$ ,  $x'(t) = 15$ , and  $y'(t) = 12.25$  it follows that  $z'(4) = \frac{765.25}{61} \approx 12.54$  miles/hour.
18. Let the point  $L$  represents the lighthouse, let at time  $t$  the light beam shines on the point  $A = A(t)$  on the shoreline, and let  $x = x(t)$  be the distance between  $A$  and  $P$ . Let  $\theta = \theta(t)$  be the measure in radians of  $\angle PLA$ . It is given that  $x = 3 \tan \theta$  and  $\frac{d\theta}{dt} = 8\pi$  radians/minute. The question is to find  $\frac{dx}{dt}$  at the instant when  $x = 1$ .

First we note that  $\frac{dx}{dt} = 3 \sec^2 \theta \frac{d\theta}{dt}$ . Secondly, at the instant when  $x = 1$  we have that  $\tan \theta = \frac{1}{3}$  which implies that  $\cos \theta = \frac{3}{\sqrt{10}}$ . Hence, when shining on a point one kilometer away from  $P$ , the light beam moving along the shoreline at the rate of  $\frac{80\pi}{3}$  km/min.

19. Let  $x = x(t)$  be the distance between the police car and the intersection and let  $y = y(t)$  be the distance between the SUV and the intersection. The distance between the two cars is given by  $z = \sqrt{x^2 + y^2}$ . The question is to find the value of  $\frac{dy}{dt}$  at the instant when  $x = 0.6$  km,  $y = 0.8$  km,  $\frac{dz}{dt} = 20$  km/hr, and  $\frac{dx}{dt} = -60$  km/hr. We differentiate  $z^2 = x^2 + y^2$  to get  $z \frac{dz}{dt} = x \frac{dx}{dt} + y \frac{dy}{dt}$ . Since, at the given instance,  $z = 1$ , we have that  $\frac{dy}{dt} = 70$  km/hr.

## 6.6 Tangent Lines and Implicit Differentiation

1. Solve  $y' = \cosh x = 1$ . The point is  $(0, 0)$ .
2. Solutions of  $-a^3 = 3a^2(4 - a)$  are  $a = 0$  and  $a = 6$ . The points are  $(0, 0)$  and  $(6, 216)$ .
3. (a)  $-\frac{\pi}{4}$ , (b)  $y = \sqrt{2}x + 1 - \frac{\pi}{4}$ .
4.  $y = x - 1$
5. We note that  $y' = 3(x - 1)^2$ . Two lines, none of them horizontal, are perpendicular to each other if the product of their slopes equals  $-1$ . Thus to find all points on the curve  $C$  with the property that the tangent line is perpendicular to the line  $L$  we solve the equation  $-\frac{1}{3} \cdot 3(x - 1)^2 = -1$ . Hence  $x = 0$  or  $x = 2$ . The lines are  $y = 3x - 1$  and  $y = 3x - 5$ .
6. From  $e^y \cdot \left( \frac{dy}{dx} \cdot \ln(x + y) + \frac{1 + \frac{dy}{dx}}{x + y} \right) = - \left( y + \frac{dy}{dx} \right) \cdot \sin(xy)$  it follows that  $\frac{dy}{dx} \Big|_{x=1} = -1$ .

Figure 6.4:  $y = (x - 1)^3$  and  $3y = x$ 

$$7. \frac{dy}{dx} = \frac{y - x^4}{y^4 - x}$$

$$9. \frac{dy}{dx} = \frac{3^x \ln 3 + \sinh y}{e^y - x \cosh y}$$

$$8. \begin{aligned} y \ln x &= x \ln y \\ \frac{dy}{dx} \cdot \ln x + \frac{y}{x} &= \ln y + \frac{x}{y} \cdot \frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{y(x \ln y - y)}{x(y \ln x - x)} \end{aligned}$$

$$10. \frac{dy}{dx} = \frac{\cosh x - 2xy}{x^2 - \sin y}$$

$$11. \frac{dy}{dx} = \frac{1 - y(x - y)}{1 + (x - y)(x + 3y^2)}$$

12. (a)  $x + y = 0$ ; (b) The graph crosses the  $x$ -axis at the points  $(\pm\sqrt{3}, 0)$ . The claim follows from the fact that  $2x - y - xy' + 2yy' = 0$  implies that if  $x = \pm\sqrt{3}$  and  $y = 0$  then  $y' = 2$ .

$$13. x + y = \pi.$$

$$15. y'(3) = \frac{10}{21}.$$

$$14. y = 0.$$

$$16. \frac{4}{3}$$

17. (a)  $\frac{dy}{dx} = -\frac{2xy}{x^2 + 2ay}$ ; (b) We solve the system of equations  $1 + a = b$ ,  $-\frac{2}{1 + 2a} = -\frac{4}{3}$  to get  $a = \frac{1}{4}$  and  $b = \frac{5}{4}$ .

18. From  $\frac{dy}{dx} = -\sqrt{\frac{y}{x}}$  we get that the tangent line  $l$  to the curve at any of its points  $(a, b)$  is given by  $y - b = -\sqrt{\frac{b}{a}}(x - a)$ . The sum of the  $x$ -intercept and the  $y$ -intercept of  $l$  is given by  $(a + \sqrt{ab}) + (b + \sqrt{ab}) = (\sqrt{a} + \sqrt{b})^2 = k$ .

19. From  $\frac{2}{3\sqrt[3]{x}} + \frac{2y'}{3\sqrt[3]{y}} = 0$  we conclude that  $y' = -\sqrt[3]{\frac{y}{x}}$ . Thus the tangent line through the point  $(a, b)$  on the curve is given by  $y - b = -\sqrt[3]{\frac{b}{a}}(x - a)$ . Its  $x$  and  $y$  intercepts

are  $(a + \sqrt[3]{ab^2}, 0)$  and  $(0, b + \sqrt[3]{a^2b})$ . Thus the square of the portion of the tangent line cut off by the coordinate axis is  $(a + \sqrt[3]{ab^2})^2 + (b + \sqrt[3]{a^2b})^2 = a^2 + 2a\sqrt[3]{ab^2} + b\sqrt[3]{a^2b} + b^2 + 2b\sqrt[3]{a^2b} + a\sqrt[3]{ab^2} = (\sqrt[3]{a^2} + \sqrt[3]{b^2})^3 = 9^3$ . The length of the portion is  $\sqrt{9^3} = 27$ .

20.  $y + 4 = \frac{3}{4}(x - 8)$ .

21. (a)  $(0, 0)$ ,  $(0, \pm 2)$ . (b)  $y' = \frac{x(2x^2 - 5)}{2y(y^2 - 2)}$ . (c)  $x = \sqrt{5}$ .

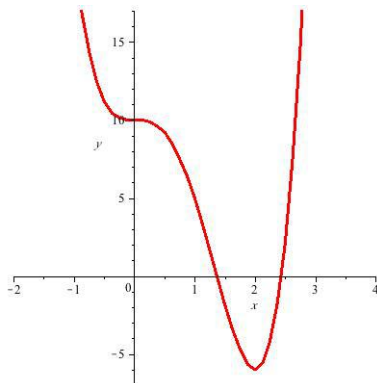
22. (a)  $y = 3x - 9$ ; (b)  $y(2.98) \approx 3 \cdot 2.98 - 9 = -0.06$

23. (a)  $y'(4) = 4$ ,  $y''(4) = -11$ ; (b)  $y(3.95) \approx -0.2$ ; (c) Since the curve is concave down, the tangent line is above the curve and the approximation is an overestimate.

## 6.7 Curve Sketching

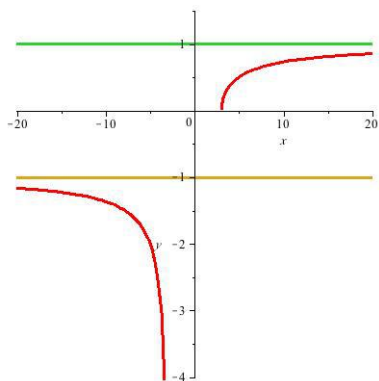
1. (a) From  $f'(x) = 12x^2(x - 2)$  we conclude that  $f'(x) > 0$  for  $x > 2$  and  $f'(x) < 0$  for  $x < 2$ . So  $f$  is increasing on  $(2, \infty)$  and decreasing on  $(-\infty, 2)$ .
- (b) From  $f''(x) = 12x(3x - 4)$  it follows that  $f''(x) > 0$  for  $x < 0$  or  $x > \frac{4}{3}$  and  $f''(x) < 0$  for  $x \in (0, \frac{4}{3})$ . Also  $f''(x) = 0$  for  $x = 0$  and  $x = \frac{4}{3}$ . Thus  $f$  is concave upward on  $(-\infty, 0)$  and on  $(\frac{4}{3}, \infty)$  and concave downward on  $(0, \frac{4}{3})$ .
- (c) Critical numbers are  $x = 0$  and  $x = 2$ . Since  $f'(x)$  does not change sign at  $x = 0$  there is no local maximum or minimum there. (Note also that  $f''(0) = 0$  and that the second derivative test is inconclusive.) Since  $f'(x)$  changes from negative to positive at  $x = 2$  there is a local minimum at  $x = 2$ . (Note also that  $f''(2) > 0$ , so second derivative test says there is a local minimum.)
- (d) Inflection points are  $(0, 10)$  and  $(\frac{4}{3}, f(\frac{4}{3}))$ .
- (e)  $\lim_{x \rightarrow \pm\infty} f(x) = \infty$ .

For the graph see Figure 6.5.

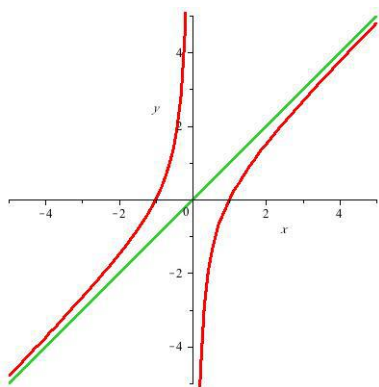
Figure 6.5:  $f(x) = 3x^4 - 8x^3 + 10$ 

2. (a) From  $x^2 - 9 > 0$  it follows that the domain of the function  $f$  is the set  $(-\infty, -3) \cup (3, \infty)$ . (b) The function is not defined at  $x = 0$ , so there is no  $y$ -intercept. Note that  $f(x) \neq 0$  for all  $x$  in the domain of  $f$ . (c) From  $\lim_{x \rightarrow \infty} f(x) = 1$  and  $\lim_{x \rightarrow -\infty} f(x) = -1$  we conclude that there are two horizontal asymptotes,  $y = 1$  (when  $x \rightarrow \infty$ ) and  $y = -1$  (when  $x \rightarrow -\infty$ ). From  $\lim_{x \rightarrow 3^+} f(x) = 0$  and  $\lim_{x \rightarrow -3^-} f(x) = -\infty$  it follows that there is a vertical asymptote at  $x = -3$ . (d) Since, for all  $x$  in the domain of  $f$ ,  $f'(x) = \frac{3(x-3)}{(x^2-9)^{3/2}} \neq 0$  we conclude that there is no critical number for the function  $f$ . (e) Note that  $f'(x) > 0$  for  $x > 3$  and  $f'(x) < 0$  for  $x < -3$ . Thus  $f$  increasing on  $(3, \infty)$  and decreasing on  $(-\infty, -3)$ . (f) Since the domain of  $f$  is the union of two open intervals and since the function is monotone on each of those intervals, it follows that the function  $f$  has neither (local or absolute) a maximum nor a minimum. (g) From  $f''(x) = -\frac{6(x-3)(x-\frac{3}{2})}{(x^2-9)^{5/2}}$  it follows that  $f''(x) < 0$  for all  $x$  in the domain of  $f$ . Therefore  $f(x)$  is concave downwards on its domain. For the graph see Figure 6.6.

3. (a) The domain of the function  $f$  is the set  $\mathbb{R} \setminus \{0\}$ . The  $x$ -intercepts are  $\pm 1$ . Since 0 not in domain of  $f$  there is no  $y$ -intercept. (b) From  $\lim_{x \rightarrow 0^-} f(x) = -\infty$  and  $\lim_{x \rightarrow 0^+} f(x) = \infty$  it follows that the vertical asymptote is the line  $x = 0$ . Since  $\lim_{x \rightarrow \pm\infty} f(x) = \lim_{x \rightarrow \pm\infty} \left(x - \frac{1}{x}\right) = \pm\infty$  we conclude that there is no horizontal asymptote. Finally, the fact  $f(x) = x - \frac{1}{x}$  implies that  $f$  has the slant (oblique) asymptote  $y = x$ . (c) For all  $x \in \mathbb{R} \setminus \{0\}$ ,  $f'(x) = \frac{x^2 + 1}{x^2} > 0$  so the function  $f$  is increasing on

Figure 6.6:  $f(x) = \frac{x-3}{\sqrt{x^2-9}}$ 

$(-\infty, 0)$  and on  $(0, \infty)$ . The function  $f$  has no critical numbers and thus cannot have a local maximum or minimum. (d) Since  $f''(x) = -\frac{2}{x^3}$  it follows that  $f''(x) > 0$  for  $x < 0$  and  $f''(x) < 0$  for  $x > 0$ . Therefore  $f$  is concave upward on  $(-\infty, 0)$  and concave downward on  $(0, \infty)$ . There are no points of inflection. (e) See Figure 6.7.

Figure 6.7:  $f(x) = \frac{x^2-1}{x}$ 

4.  $f(x) = x^3 - 2x^2 - x + 1$ ,  $f'(x) = 3x^2 - 4x - 1$ ,  $f''(x) = 6x - 4$ . See Figure 6.8.

5. See Figure 6.9.

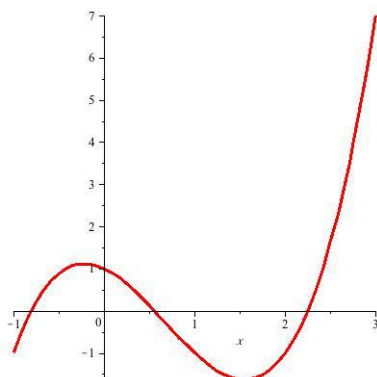


Figure 6.8:  $f(x) = x^3 - 2x^2 - x + 1$  on the interval  $[-1, 3]$

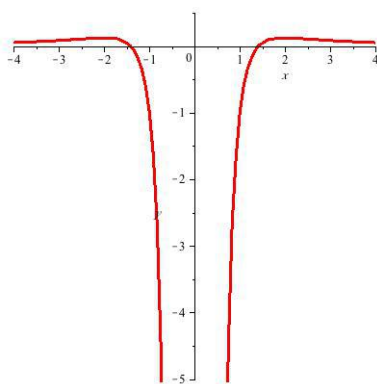
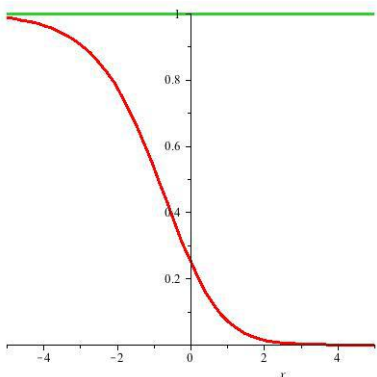
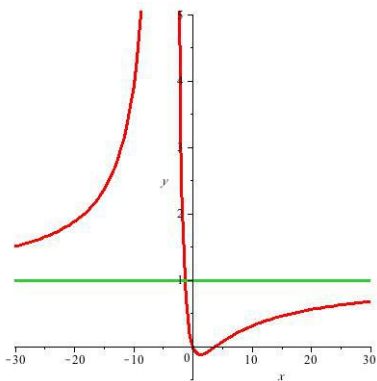


Figure 6.9:  $f(x) = \frac{x^2 - 2}{x^4}$

6. See Figure 6.10.

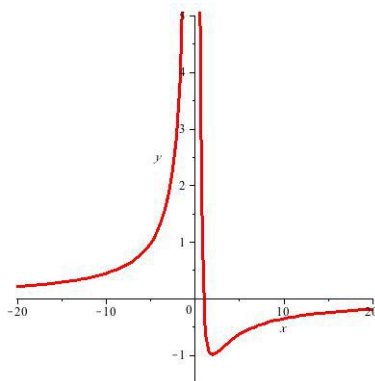
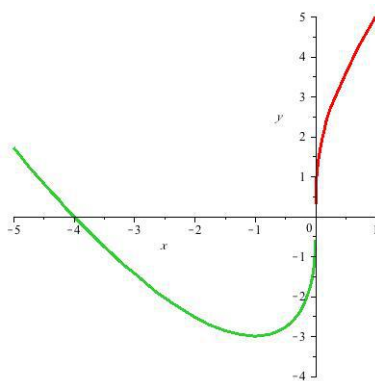
7. See Figure 6.11.

8. See Figure 6.12.

Figure 6.10:  $f(x) = \frac{1}{(1 + e^x)^2}$ Figure 6.11:  $f(x) = \frac{x^2 - 4x}{(x + 4)^2}$ 

9. Note that the domain of the given function is the set of all real numbers. The  $y$ -intercept is the point  $(0, 0)$  and the  $x$ -intercepts are  $(-4, 0)$  and  $(0, 0)$ . From  $y' = \frac{4}{3}x^{1/3} \left( \frac{1}{x} + 1 \right)$  we conclude that  $y'$  is not defined at  $x = 0$  and that  $y' = 0$  if  $x = -1$ . Thus the critical numbers are  $x = -1$  and  $x = 0$ . Also  $y' < 0$  on  $(-\infty, -1)$  and  $y' > 0$  on  $(-1, 0) \cup (0, \infty)$ . Hence the function has a local minimum at  $x = -1$ . Note that the  $y$ -axis is a vertical asymptote to the graph of the given function. From  $y'' = \frac{4}{9}x^{-5/3}(x - 2)$  it follows that  $y''(x) > 0$  on  $(-\infty, 0) \cup (2, \infty)$  and  $y''(x) < 0$  on  $(0, 2)$ . Points of inflection are  $(0, 0)$  and  $(2, 6 \cdot 2^{1/3})$ . See Figure 6.13.

10. Note that the given function is a product of a power function  $y = x^{2/3}$  and a linear function  $y = \frac{5}{2} - x$  that are both continuous on  $\mathbb{R}$ . See Figure 6.14.

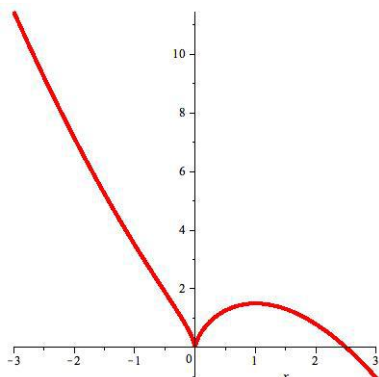
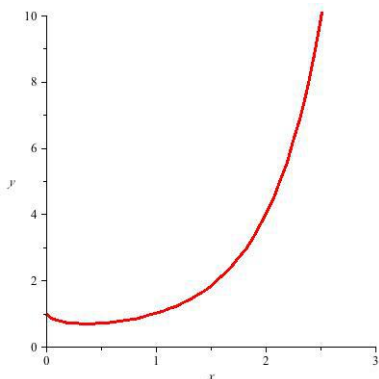
Figure 6.12:  $f(x) = \frac{4 - 4x}{x^2}$ Figure 6.13:  $y = 4x^{1/3} + x^{4/3}$ 

11. The domain is the interval  $(0, \infty)$ . Note that  $\lim_{x \rightarrow 0^+} x^x = 1$  and  $\lim_{x \rightarrow \infty} x^x = \infty$ . From

$y' = x^x(\ln x + 1)$  we get that the critical number is  $x = \frac{1}{e}$ . By the first derivative test there is a local minimum there. Also,  $y'' = x^x[(\ln x + 1)^2 + \frac{1}{x}]$ . See Figure 6.15.

12. See Figure 6.16.

13. (a)  $(0, 0)$ ,  $(3, 9e^2)$ ; (b) Increasing on  $(-\infty, 3)$  and decreasing on  $(3, \infty)$ . A local (global) maximum at  $(3, 9e^2)$ . The other critical point is neither local maximum

Figure 6.14:  $y = x^{2/3}(\frac{5}{2} - x)$ Figure 6.15:  $y = x^x$ 

nor local minimum. (c) Note that  $x^2 - 6x + 6 = (x - (3 - \sqrt{3}))(x - (3 + \sqrt{3}))$ . The function is concave up on  $(0, 3 - \sqrt{3})$  and  $(3 + \sqrt{3}, \infty)$  and concave down on  $(-\infty, 0)$  and  $(3 - \sqrt{3}, 3 + \sqrt{3})$ . The inflection points are  $(0, 0)$ ,  $(3 - \sqrt{3}, (3 - \sqrt{3})^3 e^{2+\sqrt{3}})$ , and  $(3 + \sqrt{3}, (3 + \sqrt{3})^3 e^{2-\sqrt{3}})$ . (d)  $\lim_{x \rightarrow -\infty} f(x) = -\infty$ ,  $\lim_{x \rightarrow \infty} f(x) = 0$ . (e) See Figure 6.17.

14. See Figure 6.18.

15. Note  $\lim_{x \rightarrow 0^-} f(x) = 0$ ,  $\lim_{x \rightarrow 0^+} f(x) = \infty$  and  $\lim_{x \rightarrow \pm\infty} f(x) = 1$ . See Figure 6.19.

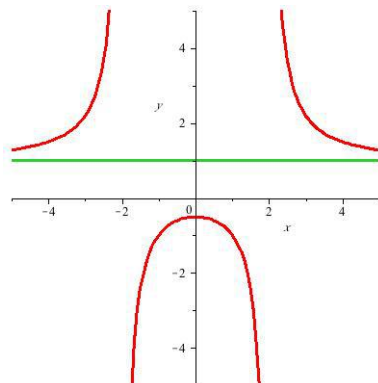


Figure 6.16:  $f(x) = \frac{x^2 + 2}{x^2 - 4}$

16. (a)  $y=0$ . (b)  $f$  is increasing on  $(-\infty, 0)$  and decreasing on  $(0, \infty)$ . (c) Local maximum at  $x = 0$ . (d) Concave up on  $(-\infty, -2)$  and  $(2, \infty)$ , concave down on  $(-2, 2)$ . (e) Inflection points at  $x = \pm 2$ .
17. See Figure 6.20.
18. See Figure 6.21.
19. Note that the function is defined on  $\mathbb{R}$ , but that the domain of its derivative is  $\mathbb{R} \setminus \{0\}$ . See Figure 6.22.
20. See Figure 6.23.
21. See Figure 6.24.
22. It is given that the  $y$ -intercept is the point  $(0, -3)$ . Note that the given function has a vertical asymptote  $x = 3$  and two horizontal asymptotes,  $y = -1$ , when  $x \rightarrow -\infty$ , and  $y = 2$ , when  $x \rightarrow \infty$ . Also, the function  $f$  is decreasing on  $(-\infty, 3)$  and  $(3, \infty)$ . Finally,  $f$  is concave upwards on  $(3, \infty)$  and concave downwards on  $(-\infty, 3)$ . See Figure 6.25.

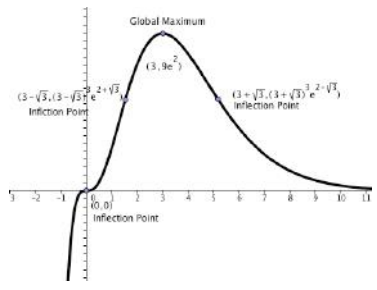


Figure 6.17:  $f(x) = x^3 e^{-x+5}$

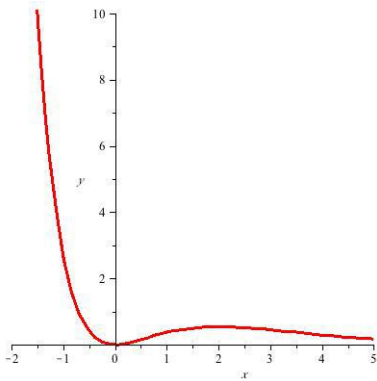
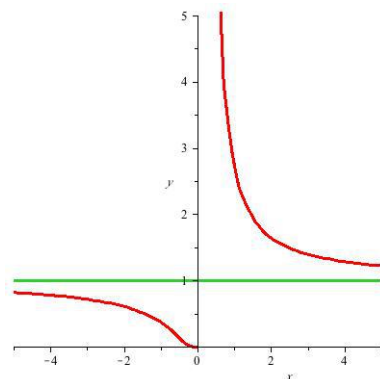
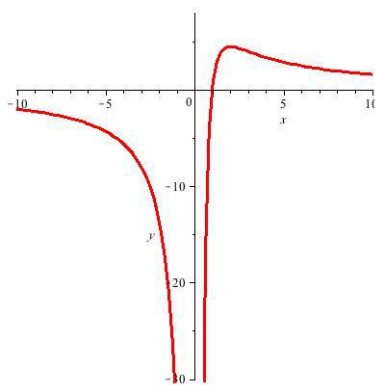


Figure 6.18:  $f(x) = x^2 e^{-x}$

23. (a) The graph has a vertical asymptote  $y = 0$  and a horizontal asymptote  $x = -2$ . The following table summarizes the rest of the given information.

| Interval     | $(-4, -1)$ | $(-1, 0)$  | $(0, 2)$   | $(2, 4)$   | $(4, \infty)$ |
|--------------|------------|------------|------------|------------|---------------|
| Monotonicity | Decreasing | Increasing | Increasing | Decreasing | Decreasing    |
| Concavity    | Downwards  | Upwards    | Downwards  | Downwards  | Upwards       |

(b) There are two points of inflection,  $x = -1$  and  $x = 4$ . We note that  $x = -1$  is also a critical number and that by the first derivative test there is a local minimum at  $x = -1$ . If  $f''(-1) = 0$ , then  $f'(-1)$  exists and  $f'(-1) = 0$ . This would imply that at this point the graph of  $f$  is **above** the tangent line at  $x = -1$  which contradicts

Figure 6.19:  $f(x) = e^{\frac{1}{x}}$ Figure 6.20:  $f(x) = \frac{18(x-1)}{x^2}$ 

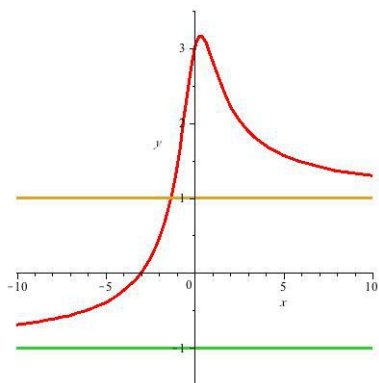
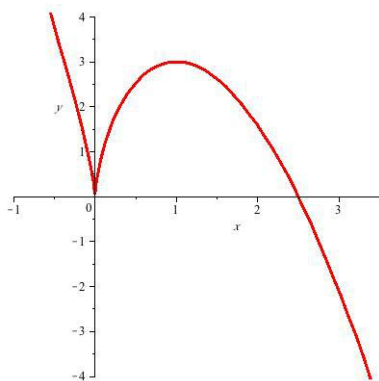
the fact that the curve **crosses** its tangent line at each inflection point. It follows that  $f'(-1)$  does not exist and therefore  $f''(-1)$  does not exist.

For a graph see Figure 6.26.

24. (a)  $\checkmark$ :  $r, s, g$ ; (b)  $\checkmark$ :  $r, s, f, g$ ; (c)  $\checkmark$ :  $r$ ; (d)  $\checkmark$ :  $g$ ; (e)  $\checkmark$ :  $r$ .

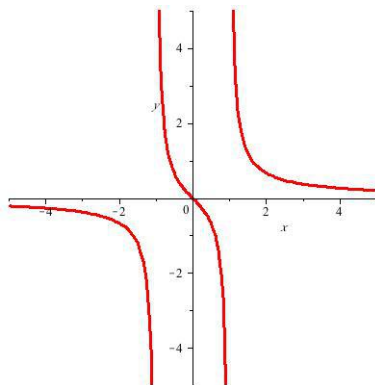
25. (a)  $\checkmark$ : C, D; (b)  $\checkmark$ : A; (c)  $\checkmark$ : A, D; (d)  $\checkmark$ : A, B, C, D; (e)  $\checkmark$ : B.

26.  $a = -3, b = 7$ . Solve the system  $y(1) = a + b + 2 = 6$  and  $y''(1) = 6 + 2a = 0$ .

Figure 6.21:  $f(x) = \frac{x+3}{\sqrt{x^2+1}}$ Figure 6.22:  $f(x) = (5-2x)x^{\frac{2}{3}}$ 

## 6.8 Optimization

1. Note that the function  $f$  is continuous on the closed interval  $[-1, 2]$ . By the Intermediate Value Theorem the function  $f$  attains its maximum and minimum values on  $[-1, 2]$ . To find those global extrema we evaluate and compare the values of  $f$  at the endpoints and critical numbers that belong to  $(-1, 2)$ . From  $f'(x) = 6x - 9 = 3(2x - 3)$  we conclude that the critical number is  $x = \frac{3}{2}$ . From  $f(-1) = 12$ ,  $f(2) = -6$ , and  $f\left(\frac{3}{2}\right) = -\frac{27}{4}$  we conclude that the maximum value is  $f(-1) = 12$  and the minimum value is  $f\left(\frac{3}{2}\right) = -\frac{27}{4}$ .
2. The global minimum value is  $f(-4) = f(2) = -21$ , and the global maximum value is  $f(6) = 139$ . Note that  $f(2) = -21$  is also a local minimum and that  $f(-2)$  is a

Figure 6.23:  $f(x) = \frac{x}{x^2 - 1}$ 

local maximum. (**Reminder:** By our definition, for  $x = c$  to be a local extremum of a function  $f$  it is necessary that  $c$  is an interior point of the domain of  $f$ . This means that there is an open interval  $I$  contained in the domain of  $f$  such that  $c \in I$ .)

3. From  $f'(x) = ax^{a-1}(1-x)^b - bx^a(1-x)^{b-1} = x^{a-1}(1-x)^{b-1}(a - (a+b)x)$  and the fact that  $a$  and  $b$  are positive conclude that  $x = \frac{a}{a+b} \in (0, 1)$  is a critical number of the function  $f$ . Since  $f(0) = f(1) = 0$  and  $f(x) > 0$  for all  $x \in (0, 1)$  it follows that the maximum value of  $f$  is  $f\left(\frac{a}{a+b}\right) = \left(\frac{a}{a+b}\right)^a \left(\frac{b}{a+b}\right)^b$ .

4. From  $f(x) = \begin{cases} 3x - 5 & \text{if } x \geq \frac{5}{3} \\ -3x + 5 & \text{if } x < \frac{5}{3} \end{cases}$  we conclude that  $f'(x) = \begin{cases} 3 & \text{if } x > \frac{5}{3} \\ -3 & \text{if } x < \frac{5}{3} \end{cases}$ .

Thus, for  $x \neq \frac{5}{3}$ ,  $f'(x) \neq 0$  and the derivative of  $f$  is not defined at  $x = \frac{5}{3}$ . We conclude that the only critical number of the function  $f$  on the interval  $[-3, 2]$  is  $x = \frac{5}{3}$ . Clearly,  $f\left(\frac{5}{3}\right) = 0$ . From  $f(-3) = 14$  and  $f(1) = 2$  it follows that the global and local minimum is  $f\left(\frac{5}{3}\right) = 0$  and that the global maximum is  $f(-3) = 14$ .

5. The question is to find the minimum value of the function  $f(x) = x^2 + (12 - x)^2$ ,  $x \in (0, 12)$ . From  $f'(x) = 4(x - 6)$  it follows that  $x = 6$  is the only critical number. From  $f''(6) = 4 > 0$ , by the second derivative test, it follows that  $f(6) = 72$  is the minimum value of the function  $f$ .
6. Note that  $f(0) = f(1) = 0$  and that  $f(x) > 0$  for  $x \in (0, 1)$ . Thus by the Intermediate Value Theorem there is  $c \in (0, 1)$  such that  $f(c)$  is the maximum value of  $f$ . Since  $f$  is differentiable on  $(0, 1)$ ,  $c$  must be a critical point. Note that  $f'(x) = x^{a-1}(1-x)^{b-1}(a - (a+b)x)$ . Since  $a$  and  $b$  are both positive we have that

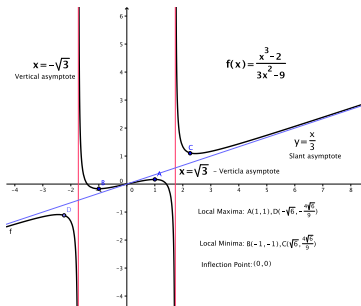


Figure 6.24:  $f(x) = \frac{x^3 - 2}{3x^2 - 9}$

$x = \frac{a}{a+b} \in (0, 1)$ . Thus  $x = \frac{a}{a+b}$  is the only critical point of the function  $f$  in the interval  $(0, 1)$  and  $f\left(\frac{a}{a+b}\right) = \frac{a^a b^b}{(a+b)^{a+b}}$  is the maximum value.

7. The distance between a point  $(x, y)$  on the curve and the point  $(0, -3)$  is  $d = \sqrt{(x-0)^2 + (y+3)^2} = \sqrt{y^4 + (y+3)^2}$ . The question is to minimize the function  $f(y) = y^4 + (y+3)^2$ ,  $y \in \mathbb{R}$ . From  $f'(y) = 2(2y^3 + y + 3) = 2(y+1)(2y^2 - 2y + 3)$  we conclude that  $y = -1$  is the only critical number of the function  $f$ . From  $f''(-1) = 10 > 0$ , by the second derivative test we conclude that  $f(-1) = 5$  is the (local and global) minimum value of  $f$ . Thus the closest point is  $(-1, -1)$ .
8. Let  $x$  be the radius of the circle. The question is to minimize the function  $f(x) = \pi x^2 + \left(\frac{40 - 2\pi x}{4}\right)^2$ ,  $x \in \left(0, \frac{20}{\pi}\right)$ . (We are given that there are TWO pieces.) The only critical number of the function  $f$  is  $x = \frac{20}{\pi + 4}$ . To minimize the total area the two pieces should be of the length  $\frac{40\pi}{\pi + 4}$  and  $\frac{160}{\pi + 4}$ .
9.  $x = 3$  and  $y = 2$ . The question is to minimize the function  $f(x) = x^2 + \frac{3(7-x)^2}{4}$ ,  $x \in (0, 7)$ . (We are given that there are TWO pieces.) The critical point is  $x = 3$ .
10. The question is to **maximize** the function  $f(x) = \frac{x^2\sqrt{3}}{36} + \frac{(4-x)^2}{16}$ ,  $x \in [0, 4]$ . Note that  $f''(x) > 0$  for  $x \in (0, 4)$  and conclude that the maximum value must occur at

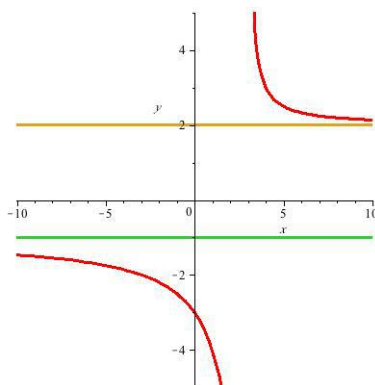
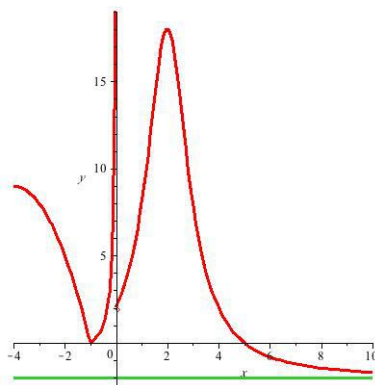


Figure 6.25: Two Horizontal Asymptotes

Figure 6.26: Discontinuity at  $x = 0$ 

$x = 0$  and/or  $x = 4$ . Since  $f(4) < f(0)$ , the maximum total area is obtained if only the square is constructed.

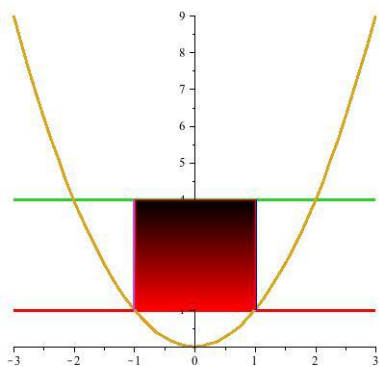
11. A rectangle with sides parallel to the coordinate axes is to be inscribed in the region enclosed by the graphs of  $y = x^2$  and  $y = 4$  so that its perimeter has maximum length.

(a) See Figure 6.27.

(b)  $P = 4a + 2(4 - a^2) = 2(4 + 2a - a^2)$ ,  $a \in (0, 1)$ .

(c) From  $\frac{dP}{da} = 4(1 - a)$  it follows that  $a = 1$  is the only critical number. The fact that  $f''(a) = -8 < 0$  for all  $a \in (0, 1)$  implies, by the second derivative test, that  $P(1)$  is the maximum value.

(d)  $P(1) = 10$ .

Figure 6.27: Rectangle inscribed in the region  $y = 4$  and  $y = x^2$ 

12. Let  $(x, 0)$  be the bottom right vertex of the rectangle. The question is to maximize  $f(x) = 2x(12 - x^2)$ ,  $x \in (0, 2\sqrt{3})$ . The only critical number is  $x = 2$ . The length of the rectangle with the largest area is 4 and its height is 8.
13. Let  $x$  be the length of one side of the fence that is perpendicular to the wall. Note that the length of the side of the fence that is parallel to the wall equals  $400 - 2x$  and that this number cannot be larger than 100. The question is to maximize the function  $f(x) = x(400 - 2x)$ ,  $x \in [150, 400)$ . The only solution of the equation  $f'(x) = 4(100 - x) = 0$  is  $x = 100$  but this value is not in the domain of the function  $f$ . Clearly  $f'(x) < 0$  for  $x \in [150, 400)$  which implies that  $f$  is decreasing on its domain. Therefore the maximum area that can be enclosed is  $f(150) = 15000$  ft<sup>2</sup>.
14.  $L = 15\sqrt{3}$ . To minimize  $L^2 = (x + 5)^2 + y^2$ , use the fact that  $\frac{x}{10\sqrt{2}} = \frac{x + 5}{y}$  and the first derivative.
15. Let  $(x, y) = \left(x, \frac{b}{a}\sqrt{a^2 - x^2}\right)$  be the upper right vertex of the rectangle. The question is to maximize the function  $f(x) = \frac{4b}{a}x\sqrt{a^2 - x^2}$ ,  $x \in (0, a)$ . From  $f'(x) = \frac{4b}{a} \frac{a^2 - 2x^2}{\sqrt{a^2 - x^2}}$  we conclude that the only critical number is  $x = \frac{a}{\sqrt{2}}$ . By the first derivative test, there is a local maximum at this critical number. Since  $\lim_{x \rightarrow 0^+} f(x) = 0$ , it follows that  $f\left(\frac{a}{\sqrt{2}}\right) = 2ab$  is the maximum value of the function  $f$ . Thus to maximize the area of the soccer field its length should be  $a\sqrt{2}$  and its width should be  $b\sqrt{2}$ .
16. Let  $a$  be the length of the printed material on the poster. Then the width of this area equals  $b = \frac{384}{a}$ . It follows that the length of the poster is  $x = a + 8$  and the

width of the poster is  $y = b + 12 = \frac{384}{a} + 12$ . The question is to minimize the function  $f(a) = xy = (a + 8) \left( \frac{384}{a} + 12 \right) = 12 \left( 40 + a + \frac{256}{a} \right)$ . It follows that the function has a local minimum at  $a = 16$ . The dimensions of the poster with the smallest area are  $x = 24$  cm and  $y = 36$  cm.

17.  $(\sqrt{15} + 2) \times (2\sqrt{15} + 4)$ .

18. Let  $P$  be the point on the shore where Maya lands her boat and let  $x$  be the distance from  $P$  to the point on the shore that is closest to her initial position. Thus to reach the village she needs to row the distance  $z = \sqrt{4 + x^2}$  and run the distance  $y = 6 - x$ . Time needed to row the distance  $z$  is given by  $T_1 = \frac{z}{2}$  and time she needs to run is  $T_2 = \frac{y}{5}$ . Therefore the question is to minimize the function  $T = T(x) = T_1 + T_2 = \frac{\sqrt{4 + x^2}}{2} + \frac{6 - x}{5}$ ,  $x \in [0, 6]$ . From  $f'(x) = \frac{x}{2\sqrt{4 + x^2}} - \frac{1}{5}$  it follows that the only critical number is  $x = \frac{4}{3}$ . From  $T(0) = \frac{11}{5} = 2.2$ ,  $T(6) = \sqrt{10}$ , and  $T\left(\frac{4}{3}\right) \approx 2.135183758$  it follows that the minimum value is  $T\left(\frac{4}{3}\right)$ . Maya should land her boat  $\frac{4}{3}$  km from the point initially nearest to the boat.

19. (a) Let  $y$  be the height of the box. Then the surface area is given by  $S = 2x^2 + 4xy$ . From  $S = 150$  it follows that  $y = \frac{1}{2} \left( \frac{75}{x} - x \right)$ . Therefore the volume of the box is given by  $V = V(x) = \frac{x}{2} (75 - x^2)$ .

(b) From the fact that  $y = \frac{1}{2} \left( \frac{75}{x} - x \right) > 0$  it follows that the domain of the function  $V = V(x)$  is the interval  $[1, 5\sqrt{3})$ .

(c) Note that  $\frac{dV}{dx} = \frac{3}{2}(25 - x^2)$  and that  $\frac{d^2V}{dx^2} = -3x < 0$  for all  $x \in (1, 5\sqrt{3})$ . Thus the maximum value is  $V(5) = 125$  cube units.

20. (a) Note that  $y = \frac{10}{x^2}$ . The cost function is given by  $C(x) = 5x^2 + 2 \cdot 4 \cdot x \cdot \frac{10}{x^2} = 5x^2 + \frac{80}{x}$ ,  $x > 0$ .

(b)  $2 \times 2 \times \frac{5}{2}$ . The minimum cost is  $C(2) = \$60$ .

21. Let  $x$  be the length and the width of the box. Then its height is given by  $y = \frac{13500}{x^2}$ . It follows that the surface area is  $S = x^2 + \frac{54000}{x}$  cm<sup>2</sup>,  $x > 0$ . The question is to

minimize  $S$ . From  $\frac{dS}{dx} = 2x - \frac{54000}{x^2}$  and  $\frac{d^2S}{dx^2} = 2 + \frac{3 \cdot 54000}{x^3} > 0$  for all  $x > 0$  it follows that the function  $S$  has a local and global minimum at  $x = 30$ .

22. We need to maximize the area of the trapezoid with parallel sides of lengths  $a = 2$  and  $c = 2 + 2 \cdot 2 \cos \theta = 2 + 4 \cos \theta$  and the height  $h = 2 \sin \theta$ . Thus we maximize the function  $A = A(\theta) = \frac{2 + (2 + 4 \cos \theta)}{2} \cdot 2 \sin \theta = 4(\sin \theta + \sin \theta \cos \theta)$ ,  $\theta \in (0, \pi)$ . From  $\frac{dA}{d\theta} = 4(\cos \theta + \cos^2 \theta - \sin^2 \theta) = 4(2 \cos^2 \theta + \cos \theta - 1) = 4(2 \cos \theta - 1)(\cos \theta + 1)$  we obtain the critical number  $\theta = \frac{\pi}{3}$ . The First Derivative Test confirms that  $\theta = \frac{\pi}{3}$  maximizes the cross sectional area of the trough.

23. Let  $r$  be the radius of the base of a cylinder inscribed in the cone and let  $h$  be its height. From  $\frac{H}{R} = \frac{h}{R-r}$  (see Figure 6.28)

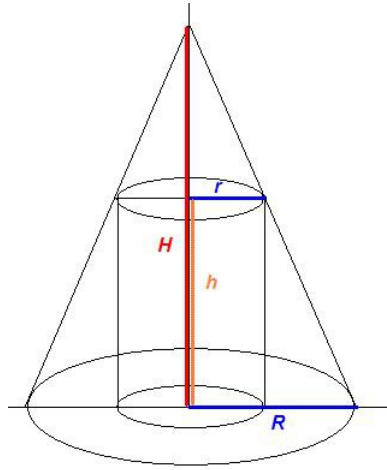


Figure 6.28: Cylinder inscribed in a cone

we conclude that  $h = \frac{H(R-r)}{R}$ . Thus the volume of the cylinder is  $V = V(r) = \frac{\pi H}{R} r^2(R-r)$ ,  $r \in (0, R)$ . From  $\frac{dV}{dr} = \frac{\pi H}{R} r(2R-3r)$  and  $\frac{d^2V}{dr^2} = \frac{2\pi H}{R}(R-3r)$  it follows that the maximum value of the volume of the cylinder is  $V\left(\frac{2R}{3}\right) = \frac{4\pi H R^2}{27}$ .

The dimensions are  $r = \frac{2R}{3}$  and  $h = \frac{H}{3}$ .

24.  $r = \sqrt{\frac{10}{3\pi}}$  m,  $h = \left(\frac{5}{\pi}\sqrt{\frac{3\pi}{10}} - \frac{1}{2}\sqrt{\frac{10}{3\pi}}\right)$  m;  $V = \frac{10}{3}\sqrt{\frac{10}{3\pi}}$  m<sup>3</sup>.

25. Let  $r$  be the radius of the base of the pot. Then the height of the pot is  $h = \frac{250}{\pi r^2}$ . The cost function is  $C(r) = 4\pi r^2 + \frac{1000}{r}$ ,  $r > 0$ . The cost function has its minimum at  $r = \frac{5}{\sqrt[3]{\pi}}$ .
26. (a) The surface area of the can is  $S = 2\pi rh + 2\pi r^2$ . The amount of material wasted is  $A - S = 2(4 - \pi)r^2$ .
- (b) From  $V = \pi r^2 h$  it follows that the amount of material needed to make a can of the given volume  $V$  is  $A = A(r) = \frac{2V}{r} + 8r^2$ . This function has its minimum at  $r = \frac{\sqrt[3]{V}}{2}$ . The ratio of the height to diameter for the most economical can is  $\frac{h}{r} = \frac{4}{\pi}$ .
- (c)  $A''(r) = \frac{4V}{r^3} + 8 > 0$  for  $r > 0$ .
27.  $r = \sqrt[3]{20}$ ,  $h = \frac{14}{\sqrt[3]{50}}$ . Minimize the cost function  $C = C(r) = 7r^2\pi + \frac{280\pi}{r}$ .
28. From Figure 6.29 conclude that  $r^2 = R^2 - (h - R)^2 = h(4R - h)$ . Then the volume of the cone as a function of  $h$  is given by  $V = \frac{\pi}{3}h^2(4R - h)$ . Maximize.

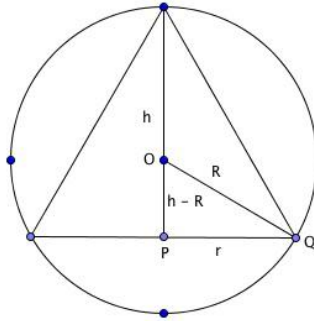


Figure 6.29: Cone inscribed in a sphere

29. Let  $P$  be the source power of the first party's stereo and let  $x$  be the distance between the person and the first party. Since the power of the second party's stereo is  $64P$ , the sound level is  $L(x) = kPx^{-2} + 64kP(100 - x)^{-2}$ ,  $x \in (0, 100)$ . From

$ds \frac{dL}{dx} = 2kP \left( \frac{64}{(100-x)^3} - \frac{1}{x^3} \right)$  it follows that  $x = 20$  is the only critical number for the function  $L$ . Since for  $x \in (0, 100)$

$$L'(x) > 0 \Leftrightarrow \frac{64}{(100-x)^3} - \frac{1}{x^3} > 0 \Leftrightarrow 64x^3 > (100-x)^3 \Leftrightarrow 4x > 100-x \Leftrightarrow x > 20$$

the function  $L$  is strictly increasing on the interval  $(20, 100)$  and strictly decreasing on the interval  $(0, 20)$ . Therefore,  $L(20)$  is the absolute minimum.

## 6.9 Mean Value Theorem

1. Since  $x + 7 \neq 0$  for all  $x \in [-1, 2]$  it follows that the function  $g$ , as a rational function, is continuous on the closed interval  $[-1, 2]$  and differentiable on the open interval  $(-1, 2)$ . Therefore the function  $g$  satisfies the hypothesis of the Mean Value Theorem on the interval  $[-1, 2]$ . By the Mean Value Theorem there is  $c \in (-1, 2)$  such that  $g'(c) = \frac{g(2) - g(-1)}{2 - (-1)}$ . Thus the question is to solve  $\frac{21}{(c+7)^2} = \frac{7}{18}$  for  $c$ . Hence  $c = -7 \pm 3\sqrt{6}$ . Clearly  $-7 - 3\sqrt{6} < -1$  and this value is rejected. From

$$-7 + 3\sqrt{6} > -1 \Leftrightarrow 3\sqrt{6} > 6 \text{ and } -7 + 3\sqrt{6} < 2 \Leftrightarrow 3\sqrt{6} < 9$$

it follows that  $c = -7 + 3\sqrt{6} \in (-1, 2)$  and it is the only value that satisfies the conclusion of the Mean Value Theorem.

2. The inequality is obviously satisfied if  $a = b$ . Let  $a, b \in \mathbb{R}$ ,  $a < b$ , and let  $f(x) = \sin x$ ,  $x \in [a, b]$ . Clearly the function  $f$  is continuous on the closed interval  $[a, b]$  and differentiable on  $(a, b)$ . Thus, by the Mean value Theorem, there is  $c \in (a, b)$  such that  $\cos c = \frac{\sin b - \sin a}{b - a}$ . Since  $|\cos c| \leq 1$  for all real numbers  $c$  it follows that  $|\sin b - \sin a| \leq |b - a|$ .
3. Let  $f(t)$  be the distance that the first horse covers from the start in time  $t$  and let  $g(t)$  be the distance that the second horse covers from the start in time  $t$ . Let  $T$  be time in which the two horses finish the race. It is given that  $f(0) = g(0)$  and  $f(T) = g(T)$ . Let  $F(t) = f(t) - g(t)$ ,  $t \in [0, T]$ . As the difference of two position functions, the function  $F$  is continuous on the closed interval  $[0, T]$  and differentiable on the open interval  $(0, T)$ . By the Mean value Theorem there is  $c \in (0, T)$  such that  $F'(c) = \frac{F(T) - F(0)}{T - 0} = 0$ . It follows that  $f'(c) = g'(c)$  which is the same as to say that at the instant  $c$  the two horse have the same speed. (Note: It is also possible to use Rolle's theorem.)

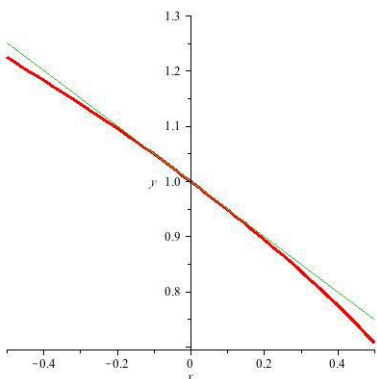
4. (a)  $[a, b]$ ;  $(a, b)$ ;  $f'(c)(b-a)$ ; (b) Note that all conditions of the Mean Value Theorem are satisfied. To get the bounds use the fact that, for some  $c \in (1, 3)$ ,  $f(5) - f(3) = 2f'(c)$ . (c) Note that  $h(2) - h(0) = 0$  and apply the Mean Value theorem for the function  $h$  on the closed interval  $[0, 2]$ .

## 6.10 Differential, Linear Approximation, Newton's Method

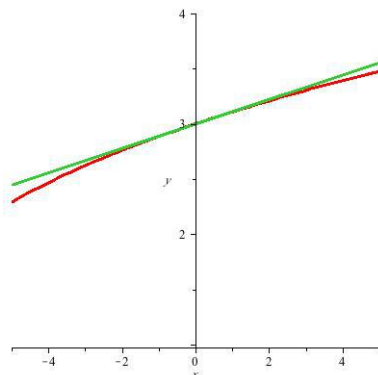
- (a) Note that  $f(0) = 8$ . From  $f'(x) = \frac{3}{2}\sqrt{x+4}$  it follows that  $f'(0) = 3$ . Thus the linearization of  $f$  at  $a = 0$  is  $L(x) = 8 + 3x$ .

(b) For  $x$  "close" to 0 we have that  $f(x) = \frac{3}{2}\sqrt{(x+4)^3} \approx L(x)$ . Thus  $\sqrt{(3.95)^3} = f(-0.05) \approx L(-0.05) = 8 - 0.15 = 7.85$ . Since  $f''(x) = \frac{3}{4\sqrt{x+4}} > 0$  we conclude that, in the neighborhood of  $x = 0$ , the graph of the function  $f$  is above the tangent line at  $x = 0$ . Thus  $L(-0.05)$  is an underestimate.
- Let  $f(x) = x^{\frac{2}{3}}$ . Then  $f(x) = \frac{2}{3}x^{-\frac{1}{3}}$ ,  $f(27) = 9$ , and  $f'(27) = \frac{2}{9}$ . Hence the linearization of the function  $f$  at  $a = 27$  is  $L(x) = 9 + \frac{2}{9}(x - 27)$ . It follows that  $\sqrt[3]{26^2} = f(26) \approx L(26) = 9 - \frac{2}{9} = \frac{79}{9}$ . (Note: MAPLE gives  $\frac{79}{9} \approx 8.777777778$  and  $\sqrt[3]{26^2} \approx 8.776382955$ .)
- Let  $f(x) = x^{\frac{2}{3}}$ . Then  $f(x) = \frac{2}{3}x^{-\frac{1}{3}}$ ,  $f(64) = 16$ , and  $f'(64) = \frac{1}{6}$ . Hence the linearization of the function  $f$  at  $a = 64$  is  $L(x) = 16 + \frac{1}{6}(x - 64)$ . It follows that  $(63)^{2/3} = f(63) \approx L(63) = 16 - \frac{1}{6} = \frac{95}{6}$ . The error is close to the absolute value of the differential  $|dy| = |f'(64)\Delta x| = \frac{1}{6}$ . (Note: MAPLE gives  $\frac{95}{6} \approx 15.83333333$  and  $\sqrt[3]{63^2} \approx 15.83289626$ .)
- Let  $f(x) = \sqrt{x}$ . Then  $f(x) = \frac{1}{2\sqrt{x}}$ ,  $f(81) = 9$ , and  $f'(81) = \frac{1}{18}$ . Hence the linearization of the function  $f$  at  $a = 81$  is  $L(x) = 9 + \frac{1}{18}(x - 81)$ . It follows that  $\sqrt{80} = f(80) \approx L(80) = 9 - \frac{1}{18} = \frac{161}{18}$ . (Note: MAPLE gives  $\frac{161}{18} \approx 8.944444444$  and  $\sqrt{80} \approx 8.944271910$ .)
- The linearization of the function  $f$  at  $a = 5$  is  $L(x) = 2 + 4(x - 5)$ . Thus  $f(4.9) \approx L(4.9) = 2 - 0.4 = 1.6$ .

6. (a) The linearization of the function  $g$  at  $a = 2$  is  $L(x) = -4 + 3(x - 2)$ . Thus  $g(2.05) \approx L(2.05) = -3.85$ . (b) From  $g''(2) = \frac{2}{3} > 0$  we conclude that the function  $g$  is concave downward at  $a = 2$ , i.e., the graph of the function lies below the tangent line. Thus, the estimate is larger than the actual value.
7. (a)  $L(x) = 1 - \frac{x}{2}$ . (b)  $\sqrt{0.9} \approx 1 - \frac{9}{20} = \frac{11}{20}$ . (c)  $y = -\frac{x}{2} + 1$ . (d) See Figure 6.30.

Figure 6.30:  $f(x) = \sqrt{1-x}$  and its tangent at  $x = 0$ 

8. (a)  $L(x) = 1 + x$ . (b)  $\sqrt{1.1} = f(0.05) \approx L(0.05) = 1.05$ . (c) An over-estimate since  $f$  is concave-down. MAPLE gives  $\sqrt{1.1} \approx 1.048808848$ .
9. (a)  $L(x) = 2 + \frac{x}{12}$ . (b)  $\sqrt[3]{7.95} \approx L(-0.05) = 2 - \frac{1}{240} = \frac{479}{240}$  and  $\sqrt[3]{8.1} \approx L(0.1) = 2 + \frac{1}{120} = \frac{241}{120}$ . (Note: MAPLE gives  $\frac{479}{240} \approx 1.995833333$  and  $\sqrt[3]{7.95} \approx 1.995824623$ . Also,  $\frac{241}{120} \approx 2.008333333$  and  $\sqrt[3]{8.1} \approx 2.008298850$ .)
10. (a)  $y = \frac{x}{9} + 3$ . (b)  $\sqrt[3]{30} \approx \frac{1}{9} + 3 = \frac{28}{9}$ . (Note: MAPLE gives  $\frac{28}{9} \approx 3.111111111$  and  $\sqrt[3]{30} \approx 3.107232506$ .) (c) See Figure 6.31.
11. The linearization of the function  $f(x) = \ln x$  at  $a = 1$  is given by  $L(x) = x - 1$ . Thus  $\ln 0.9 \approx L(0.9) = -0.1$ . (Note: MAPLE gives  $\ln 0.9 \approx -0.1053605157$ .)
12. (a)  $L(x) = x - 1$ . (b) Let  $x = \exp(-0.1)$ . Then  $\ln x = -0.1 \approx L(x) = x - 1$ . Thus  $x \approx 0.9$ . (Note: MAPLE gives  $\exp(-0.1) \approx 0.9048374180$ .)

Figure 6.31:  $f(x) = \sqrt[3]{27 - 3x}$  and its tangent at  $x = 0$ 

13.  $L(x) = 10 + \frac{1}{300}(x - 1000)$  implies  $1001^{1/3} \approx L(1001) = \frac{3001}{300}$ . (Note: MAPLE gives  $\frac{3001}{300} \approx 10.00333333$  and  $\sqrt[3]{1001} \approx 10.00333222$ .)
14. (a) The linearization of the function  $f(x) = \sqrt{x} + \sqrt[5]{x}$  at  $a = 1$  is given by  $L(x) = 2 + \frac{7}{10}(x - 1)$ . Thus  $f(1.001) \approx L(1.001) = 2 + 0.7 \cdot 0.001 = 2.0007$ . (b) Note that the domain of the function  $f$  is the interval  $[0, \infty)$ . From  $f''(x) = -\frac{1}{4}x^{-3/2} - \frac{4}{25}x^{-9/5}$  it follows that  $f$  is concave downwards on the interval  $(0, \infty)$ . (c) The graph of the function is below the tangent line at  $a = 1$ , so the estimate  $f(1.001) \approx 2.0007$  is too high.
15. (a)  $L(x) = \frac{1}{2} + \frac{\sqrt{3}}{2}(x - \frac{\pi}{6})$ . (b) By the Mean Value Theorem, for  $x > \frac{\pi}{6}$  and some  $c \in (\frac{\pi}{6}, x)$ ,  $\frac{f(x) - f(\frac{\pi}{6})}{x - \frac{\pi}{6}} = \frac{\sin x - \frac{1}{2}}{x - \frac{\pi}{6}} = f'(c) = \cos c \leq 1$ . Since  $x - \frac{\pi}{6} > 0$ , the inequality follows. (c) From (a) and (b) it follows that, for  $x > \frac{\pi}{6}$ ,  $\sin x \leq \frac{1}{2} + (x - \frac{\pi}{6}) < \frac{1}{2} + \frac{\sqrt{3}}{2}(x - \frac{\pi}{6}) = L(x)$ . Next,  $\Delta f = f(x) - f(\frac{\pi}{6}) = \sin x - \frac{1}{2} < L(x) - \frac{1}{2} = \frac{\sqrt{3}}{2}(x - \frac{\pi}{6}) = f'(\frac{\pi}{6})\Delta x = df$ .
16. (b) Let  $f(x) = \cos x - x^2$ . Then  $f'(x) = -\sin x - 2x$ . Thus  $x_2 = 1 - \frac{\cos 1 - 1}{-\sin 1 - 2} \approx 0.8382184099$ ,  $x_2 = 0.8382184099 - \frac{\cos 0.8382184099 - 0.8382184099^2}{-\sin 0.8382184099 - 2 \cdot 0.8382184099} \approx 0.8242418682$ , and  $x_3 = 0.8242418682 - \frac{\cos 0.8242418682 - 0.8242418682^2}{-\sin 0.8242418682 - 2 \cdot 0.8242418682} \approx 0.8241323190$ . (Note: MAPLE gives  $\cos 0.8241323190 - 0.8241323190^2 \approx -1.59 \cdot 10^{-8}$ .)

17. (b) Take  $f(x) = \sqrt[3]{x}$ ,  $x_0 = 1$ ,  $x_1 = -2$ ,  $x_2 = 4$ , and  $x_3 = -8$ . See Figure 6.32.

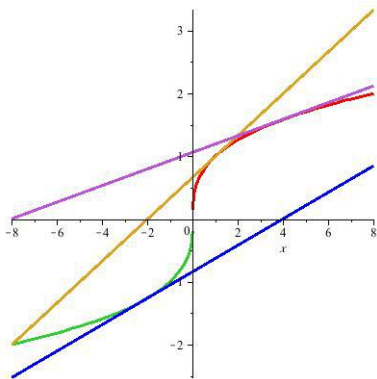


Figure 6.32: Newton's Method fails:  $f(x) = \sqrt[3]{x}$  and  $x_0 = 1$

18. (a) We use Newton's Method to solve the equation  $x^2 - 5 = 0$ ,  $x > 0$ . From  $f(x) = x^2 - 5$  and  $f'(x) = 2x$ , Newton's Method gives  $x_{n+1} = x_n - \frac{x_n^2 - 5}{2x_n} = \frac{1}{2} \left( x_n + \frac{5}{x_n} \right)$ .
- (b) A rough estimate of  $\sqrt{5}$  gives a value that is a bit bigger than 2. Thus, take  $x_1 = 1$ .
- (c)  $x_2 = 3$ ,  $x_3 = \frac{7}{3}$ ,  $x_4 = \frac{47}{21} \approx 2.23809$ . (Note: MAPLE gives  $\sqrt{5} \approx 2.23606$ .)
19. Let  $f(x) = x^{\frac{1}{3}}$ . Then Newton's method gives  $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^{\frac{1}{3}}}{\frac{1}{3}x_n^{-\frac{2}{3}}} = -2x_n$ . So  $|x_{n+1}| = 2|x_n|$ . This implies that if  $x_0 \neq 0$ ,  $|x_n| = 2^n|x_0| \rightarrow \infty$  as  $n \rightarrow \infty$ ; Newton's Method does not work in this case! See Figure 6.32.
20. (a) Take  $f(x) = x^5 - k$ . Then  $f'(x) = 5x^4$  and  $x_{n+1} = x_n - \frac{x_n^5 - k}{5x_n^4} = \frac{4x_n^5 + k}{5x_n^4} = \frac{x_n}{5} \left( 4 + \frac{k}{x_n^5} \right)$ . (b)  $x_{n+1} = \sqrt[5]{k}$ . (c)  $x_2 = 1.85$ . [MAPLE gives  $\sqrt[5]{20} \approx 1.820564203$ .]
21. From  $f(x) = x^5 - 31$  and  $f'(x) = 5x^4$  it follows that  $x_1 = \frac{159}{80}$  and  $x_2 = \frac{159}{80} - \frac{\left(\frac{159}{80}\right)^5 - 31}{5 \cdot \left(\frac{159}{80}\right)^4} \approx 1.987340780$ . (Note: MAPLE gives  $\sqrt[5]{31} = 1.987340755$ .)

22. (b) The question is approximate the solution of the equation  $F(x) = \sin x - x = 0$  with  $x_0 = \frac{\pi}{2}$ . Thus  $x_1 = \frac{\pi}{2} - \frac{\sin \frac{\pi}{2} - \frac{\pi}{2}}{\cos \frac{\pi}{2} - 1} = 1$ . (Note: Clearly the solution of the given equation is  $x = 0$ . Newton's method with  $x_0 = \frac{\pi}{2}$  gives  $x_7 = 0.08518323251$ .)
23.  $x_2 = 1 - \frac{-1}{-22} = \frac{21}{22}$ . (Note: MAPLE gives  $\frac{21}{22} \approx 0.9545454545$  and approximates the solution of the equation as 0.9555894038)
24. (a)  $x_1 = 2 - \frac{-1}{4} = \frac{9}{4}$ . (b) The question is to approximate a solution of the equation  $f'(x) = 0$  with the initial guess  $x_0 = 2$ ,  $f'(2) = 4$ , and  $f''(2) = 3$  given. Hence  $x_1 = 2 - \frac{4}{3} = \frac{2}{3}$ .
25. (a) From  $f(x) = \frac{1}{x} - a$  and  $f'(x) = -\frac{1}{x^2}$  it follows that  $x_{n+1} = x_n - \frac{\frac{1}{x_n} - a}{-\frac{1}{x_n^2}} = 2x_n - ax_n^2$ . (b) Note that  $\frac{1}{1.128}$  is the solution of the equation  $\frac{1}{x} - 1.128 = 0$ . Thus  $x_2 = 2 - 1.128 = 0.872$ ,  $x_3 = 2 \cdot 0.872 - 1.128 \cdot 0.872^2 = 0.886286848$ , and  $x_4 = 0.8865247589$ . (Note: MAPLE gives  $\frac{1}{1.128} \approx 0.8865248227$ .)
26.  $x_2 = \frac{\pi}{2} - \frac{1 - \frac{\pi}{4}}{-\frac{1}{2}} = 2$ . (Note: MAPLE estimates the positive solution of the equation  $\sin x = \frac{x}{2}$  as 1.895494267. Newton's method with the initial guess  $x_1 = \frac{\pi}{2}$  gives  $x_3 \approx 1.900995594$ .)
27. (a) From  $f'(x) = 3(x^2 - 1)$  it follows that the critical numbers are  $x = \pm 1$ . From  $f(1) = 3$ ,  $f(-1) = 7$ ,  $\lim_{x \rightarrow -\infty} f(x) = -\infty$ , and  $\lim_{x \rightarrow \infty} f(x) = \infty$  it follows that  $f$  has only one root and that root belongs to the interval  $(-\infty, -1)$ . From  $f(-2) = 3 > 0$  and  $f(-3) = -13 < 0$ , by the Intermediate Value Theorem, we conclude that the root belongs to the interval  $(-3, -2)$ . (b) Let  $x_0 = -3$ . Then  $x_1 = -3 - \frac{-13}{504} = -\frac{1499}{504} \approx -2.974206349$  and  $x_3 \approx -2.447947724$ . It seems that Newton's method is working, the new iterations are inside the interval  $(-3, -2)$  where we know that the root is. (Note: MAPLE estimates the solution of the equation  $x^3 - 3x + 5 = 0$  as  $x = -2.279018786$ .)
28. (a) The function  $f$  is continuous on the closed interval  $\left[-\frac{1}{2}, 0\right]$  and  $f\left(-\frac{1}{2}\right) = -\frac{5}{8} < 0$  and  $f(0) = 1 > 0$ . By the Intermediate Value Theorem, the function  $f$  has at least one root in the interval  $\left(-\frac{1}{2}, 0\right)$ . (b) Take  $x_1 = -\frac{1}{3}$ . Then  $x_2 =$

- $-\frac{1}{3} - \frac{-\frac{1}{27} - 3 \cdot (-\frac{1}{3}) + 1}{3(\frac{1}{9} + 1)} = -\frac{29}{90} \approx -.3222222222$  and  $x_3 \approx -.3221853550$ . (Note: MAPLE estimates the solution of the equation  $x^3 + 3x + 1 = 0$  as  $x = -.3221853546$ .)
29. (a) Take  $f(x) = \ln x + x^2 - 3$ , evaluate  $f(1)$  and  $f(3)$ , and then use the Intermediate Value Theorem. (b) Note that  $f'(x) = \frac{1}{x} + 2x > 0$  for  $x \in (1, 3)$ . (c) From  $f(1) = -2$  and  $f'(1) = 3$  it follows that  $x_2 = \frac{5}{3} \approx 1.66$ . [MAPLE gives 1.592142937 as the solution.]
30. (a) Take  $f(x) = 2x - \cos x$ , evaluate  $\lim_{x \rightarrow -\infty} f(x)$  and  $\lim_{x \rightarrow \infty} f(x)$ , and then use the Intermediate Value Theorem. (b) Note that  $f'(x) = 2 + \sin x > 0$  for  $x \in \mathbb{R}$ . (c) From  $f(0) = -1$  and  $f'(0) = 2$  it follows that  $x_2 = \frac{1}{2}$ . [MAPLE gives 0.4501836113 as the solution.]
31. (a) Take  $f(x) = 2x - 1 - \sin x$ , evaluate  $\lim_{x \rightarrow -\infty} f(x)$  and  $\lim_{x \rightarrow \infty} f(x)$ , and then use the Intermediate Value Theorem. (b) Note that  $f'(x) = 2 - \cos x > 0$  for  $x \in \mathbb{R}$ . (c) From  $f(0) = -1$  and  $f'(0) = 1$  it follows that  $x_2 = 1$ . [MAPLE gives 0.8878622116 as the solution.]

## 6.11 Antiderivatives and Differential Equations

- $f(x) = 2 \sin x + 2x^4 - e^x + 8$ .
- $g(x) = -\cos x - x^{-1} - e^x + \pi^{-1} + e^\pi$
- $f(x) = \frac{1}{3}x^3 + x^2 + 3x - \frac{242}{3}$  and  $f(1) = -\frac{229}{3}$ .
- $h(1) = 2e(1 - e^2) - \frac{44}{3}$ .
- $F(z) = \frac{1}{2} \ln(z^2 + 9)$ .
- It is given that  $f(0) = 1$  and  $f'(0) = 0$ . Thus  $f(x) = x^3 + 1$ .
- $f(t) = 2e^t - 3 \sin t + t - 2$ .
- It is given that  $x(0) = 10$ ,  $x'(0) = v(0) = 0$  and  $x''(t) = 12t$ . Hence  $x(t) = 2t^3 + 10$ .
- $\int \frac{dx}{x(1 + \ln x)} = \ln(1 + \ln x) + C$ .
- For each case compute the indefinite integral.
  - $F(x) = -\frac{1}{9}(1 - x)^9$
  - $\int \tan^2 x dx = \int (\sec^2 x - 1) dx = \tan x - x + C$
  - $F(x) = \frac{1}{\sqrt{2}} \arctan \frac{x}{\sqrt{2}} + C$
  - $F(x) = \frac{1}{6}e^{3x} + \frac{1}{2}e^x + C$

11. (a)  $v(t) = \frac{3}{2}t^2 + 6$ . (b) 4 seconds. Solve  $v(t) = 30$ .
12. (a) Let  $s(t)$  be the height of the ball after  $t$  seconds. It is given that  $s(0) = 0$ ,  $s'(0) = v(0) = 64$  ft/sec and  $s''(0) = v'(0) = a(0) = -32$  ft/sec<sup>2</sup>. Thus  $s(t) = -16t^2 + 64t = 16t(4 - t)$ . From  $s(4) = 0$  it follows that the ball is in the air for 4 seconds. (b)  $v(4) = s'(4) = -64$  ft/sec<sup>2</sup>.
13. (a) From the fact that the velocity of an falling object is approximated by  $v(t) = -gt + v(0)$  and the fact that, in the given case,  $v(t) = 0$ , we conclude that the distance  $y = y(t)$  between the ball and the surface of the Earth at time  $t$  is given by  $\frac{dy}{dt} = -gt$ . Hence,  $y = -\frac{gt^2}{2} + H$ , where  $H$  is the height of the blimp at the moment when the ball was dropped. At the moment when the ball hits surface we have that  $0 = -\frac{gt^2}{2} + H$  which implies that it takes  $t = \sqrt{\frac{2H}{g}}$  seconds for a ball to drop  $H$  meters. (b)  $v = -g \cdot \sqrt{\frac{2H}{g}} = -10 \cdot 7 = -70$  m/sec.
14.  $y = \frac{-2 \cos 3x + x^3 + e^{3x} + 1}{3} + x$ . 17.  $y = 4e^x - 1$ .
15.  $y = \sin\left(x + \frac{\pi}{2}\right)$ . 18.  $x(t) = -3(4t - 7)^{-3} + 4$
- Solve  $\frac{dy}{\sqrt{1-y^2}} = dx$ . 19.  $y = \ln(x + e^2)$ .
16.  $y = \tan\left(x - \frac{\pi}{4}\right)$ . 20.  $y = 2 - e^{-t}$ .
21. (a)  $y = \frac{3}{2} \sin 2x - \frac{1}{4} \exp(-4x) + \frac{5}{4}$ . (b)  $F(x) = \frac{1}{2} \ln(2x + 1) + C$ .
22. Let  $x$  be the number of towels sold per week at the price  $p = p(x)$ . Let  $C = C(x)$  be the cost of manufacturing  $x$  towels. It is given that  $\frac{dC}{dx} = 0.15$  CAN/towel and  $\frac{dp}{dx} = -\frac{0.10}{50}$  CAN/towel. Hence  $C(x) = 0.15x + a$  and  $p(x) = -\frac{0.10x}{50} + b$ , for some constants  $a$  and  $b$  (in CAN). Then the profit is given by  $P = P(x) = \text{Revenue} - \text{Cost} = x \cdot p(x) - C(x) = -\frac{0.10x^2}{50} + bx - 0.15x - a$ . The quantity that maximizes revenue is  $x = 1000$  towels and it must be a solution of the equation  $\frac{dP}{dx} = -\frac{0.10x}{25} + b - 0.15 = 0$ . Hence  $-\frac{0.10 \cdot 1000}{25} + b - 0.15 = 0$  and  $b = 4.15$  CAN. The price that maximizes the profit is  $p = -\frac{0.10 \cdot 1000}{50} + 4.15 = 2.15$  CAN.

## 6.12 Exponential Growth and Decay

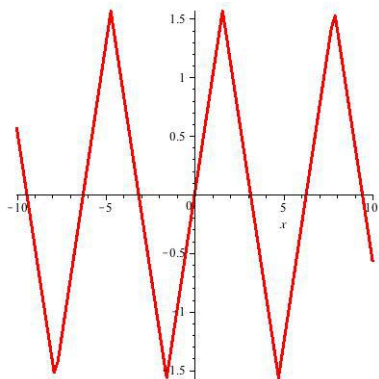
1. (a)  $\frac{dA}{dt} = pA$ ,  $A(0) = A_0$ ;  $A = A_0 e^{pt}$ . (b) Solve  $15,000 = 10,000 \cdot e^{4p}$
2. (a)  $\frac{dA}{dt} = k(M - A(t))$ . (b)  $A(t) = M - ce^{kt}$ . (c) It is given that  $M = 100$ ,  $A(0) = 0$ , and  $A(100) = 50$ . Hence  $A(t) = 100(1 - e^{-\frac{t \ln 2}{100}})$ . The question is to solve  $75 = 100(1 - e^{-\frac{t \ln 2}{100}})$  for  $t$ . It follows that the student needs to study another 100 hours.
3. (a) The model is  $C(t) = C_0 e^{-\frac{t}{2.5}}$  where  $C_0 = 1$  and the question is to solve  $0.5 = e^{-\frac{t}{2.5}}$  for  $t$ . Hence  $t = 2.5 \ln 2 \approx 1.75$  hours. (b)  $C'(0) = -\frac{1}{2.5} = -\frac{2}{7}$  hours.
4. (b)  $f(t) = e^{-\frac{t \ln 2}{5700}}$ . (c) The question is to solve  $0.15 = e^{-\frac{t \ln 2}{5700}}$  for  $t$ . Hence the age of the skull is  $t = -\frac{5700 \ln 0.15}{\ln 2} \approx 15600$  years.
5. (a) The model is  $m(t) = 10e^{-kt}$  where  $t$  is in years,  $m(t)$  is in kilograms, and  $k$  is a constant that should be determined from the fact that  $m(24110) = 5$ . Hence  $k = -\frac{\ln 2}{24110}$  and  $m(t) = 10e^{-\frac{t \ln 2}{24110}}$ . (b)  $m(1000) = 10e^{-\frac{\ln 2}{24.11}} \approx 9.716$  kilograms. (c) We solve  $1 = 10e^{-\frac{t \ln 2}{24110}}$  to get  $t = 24110 \frac{\ln 10}{\ln 2} \approx 80091.68$  years.
6. The model is  $A = A(0)e^{-kt}$ . It is given that  $A(0) = 1$  kg and  $A(6) = 0.027$  kg. Hence  $A(t) = e^{\frac{t \ln 0.0027}{6}}$ . It follows that at 3:00 there are  $A(2) = e^{\frac{\ln 0.0027}{3}} \approx 0.1392476650$  kg of substance  $X$ .
7. The model is  $P = P(t) = P_0 e^{kt}$  where  $k$  is a constant,  $P_0$  is the initial population and  $t$  is the time elapsed. It is given that  $10P_0 = P_0 e^{10k}$  which implies that  $k = \frac{\ln 10}{10}$ . The question is to solve  $2 = e^{\frac{t \ln 10}{10}}$  for  $t$ . Hence  $t = \frac{10 \ln 2}{\ln 10} \approx 3.01$  hours.
8. (a) The model is  $P = 500e^{kt}$ . From  $8000 = 500e^{3k}$  it follows that  $k = \frac{4 \ln 2}{3}$ . Thus the model is  $P = 500e^{\frac{4t \ln 2}{3}}$ . (b) 128000 bacteria. (c) Solve  $10^6 = 500e^{\frac{4t \ln 2}{3}}$  for  $t$ . It follows that  $t = \frac{3(4 \ln 2 + 3 \ln 5)}{4 \ln 2} \approx 4.7414$  hours.
9. 20158
10. (a) The model is  $P(t) = 10e^{kt}$  where  $t$  is time in hours. From  $40 = 10e^{\frac{k}{4}}$  it follows that  $k = 4 \ln 4$ . Hence  $P(3) = 10e^{12 \ln 4} = 167,772,160$  bacteria. (b) 3.32 hours.
11. (a)  $\frac{dP}{dt} = kP$ ; (b)  $P = 160e^{t \ln \frac{3}{2}}$ ; (c)  $P(0) = 160$  bacteria; (d)  $\frac{dP}{dt}(0) = 160 \ln \frac{3}{2}$

12. The model is  $P(t) = 500e^{kt}$  where  $t$  is time in years. From  $P(2) = 750$  it follows that  $k = \frac{\ln 3 - \ln 2}{2}$ . Thus  $P(500) = 500e^{250(\ln 3 - \ln 2)} = 5.27 \cdot 10^{46}$ .
13. (a)  $\left. \frac{dT}{dt} \right|_{T=80^0} = -0.09 \cdot (80 - 20) = -5.4^0\text{C/min}$ . (b) Note that 6 seconds should be used as 0.1 minutes. From  $T \approx 80 - 5.4\Delta T = 80 - 5.4 \cdot 0.1$  it follows that the change of temperature will be  $T - 80 \approx -0.54^0\text{C}$ . (c)  $t = -\frac{100}{9} \ln \frac{9}{16} \approx 6.4$  minutes. Find the function  $T = T(t)$  that is the solution of the initial value problem  $\frac{dT}{dt} = -0.09(T - 20)$ ,  $T(0) = 90$ , and then solve the equation  $T(t) = 65$  for  $t$ .
14. The model is  $\frac{dT}{dt} = k(T - 32)$  where  $T = T(t)$  is the temperature after  $t$  minutes and  $k$  is a constant. Hence  $T = 32 + (T_0 - 32)e^{kt}$  where  $T_0$  is the initial temperature of the drink. From  $14 = 32 + (T_0 - 32)e^{25k}$  and  $20 = 32 + (T_0 - 32)e^{50k}$  it follows  $\frac{3}{2} = e^{-25k}$  and  $k = -\frac{1}{25} \ln \frac{3}{2}$ . Answer:  $T_0 = 5^0\text{C}$ .

## 6.13 Miscellaneous

- Let  $f(x) = \ln x - cx^2$ . Note that the domain of  $f$  is the interval  $(0, \infty)$  and that  $\lim_{x \rightarrow \pm\infty} f(x) = -\infty$ . From  $f'(x) = \frac{1}{x} - 2cx$  it follows that there is a local (and absolute) maximum at  $x = \frac{1}{\sqrt{2c}}$ . Since the function  $f$  is continuous on its domain, by the Intermediate Value Theorem, it will have a root if and only if  $f\left(\frac{1}{\sqrt{2c}}\right) > 0$ . Thus
 
$$\ln \frac{1}{\sqrt{2c}} - c \cdot \frac{1}{2c} > 0 \Leftrightarrow \ln \frac{1}{\sqrt{2c}} > \frac{1}{2} \Leftrightarrow c < \frac{1}{2e}.$$
- Note that the function  $y = 3x^3 + 2x + 12$ , as a polynomial, is continuous on the set of real numbers. Also,  $\lim_{x \rightarrow -\infty} (3x^3 + 2x + 12) = -\infty$  and  $\lim_{x \rightarrow \infty} (3x^3 + 2x + 12) = \infty$ . By the Intermediate Value Theorem, the function has at least one root. Next, note that  $y' = 6x^2 + 2 > 0$  for all  $x \in \mathbb{R}$ . This means that the function is increasing on its domain and therefore has only one root. (If there is another root, by Rolle's theorem there will be a critical number.)
- Take  $y = x^3 + 9x + 5$ .
- $a = -3$ ,  $b = 1$ . Solve  $f(10) = 6$  and  $f''(1) = 0$ .

5. Note that the function  $f$  is continuous on its domain  $(-\infty, -1) \cup (-1, \infty)$ . Since  $\lim_{x \rightarrow \infty} \frac{1}{(x+1)^2} = 0$  and  $|\sin x| \leq 1$ , for all  $x \in \mathbb{R}$ , it follows that  $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} (-2x) = -\infty$ . Also,  $f(0) = 1 > 0$ . By the Intermediate Value Theorem, the function has at least one root in the interval  $(0, \infty)$ . Next, note that  $f'(x) = -\frac{2}{(x+1)^3} - 2 + \cos x < 0$  for all  $x \in (0, \infty)$ . This means that the function is decreasing on  $(0, \infty)$  and therefore has only one root.
6. (a) From  $s'(t) = v(t) = 96 - 2t$  it follows that the initial velocity is  $v(0) = 96$  ft/sec. (b) The only critical number is  $t = 48$ . By the second derivative test there is a local (and absolute) maximum there. (c)  $s(48) = 2316$  feet.
7. (a)  $f'(3) = 1$ . (b)  $(fg)'(2) = f'(2)g(2) + f(2)g'(2) = -4$ . (c)  $f(0.98) \approx f(1) + f'(1)(0.98 - 1) = 3 + 3(-0.02) = 2.94$ . (d) There is a critical point at  $(4, -1)$ .
8. (a) The domain of  $f(x) = \arcsin x$  is the interval  $[-1, 1]$  and its range is  $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ . (b) For  $x \in (-1, 1)$  let  $y = \arcsin x$ . Then  $\sin y = x$  and  $y' \cos y = 1$ . Since  $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$  it follows that  $\cos y > 0$ . Thus  $\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$ . (c) The domain of the function  $g$  is the set of all real numbers and its range is the set  $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ . See Figure 6.33. (d)  $g'(x) \begin{cases} 1 & \text{if } x \in (4n-1, 4n+1), n \in \mathbb{Z} \\ -1 & \text{if } x \in (4n+1, 4n+3), n \in \mathbb{Z} \end{cases}$ . The derivative of  $g$  is not defined if  $x = 2n+1, n \in \mathbb{Z}$ . (e) The function  $F(x) = 4x - 2 + \cos\left(\frac{\pi x}{2}\right)$  is continuous on the set of real numbers. From  $\lim_{x \rightarrow \pm\infty} F(x) = \pm\infty$ , by the Intermediate Value Theorem, the function  $F$  has at least one root. From  $F'(x) = 1 - \frac{2}{\pi} \sin\left(\frac{\pi x}{2}\right) > 0$  we conclude that  $F$  is monotone.

Figure 6.33:  $f(x) = \arcsin(\sin x)$

9. (a)  $f'(x) = \frac{1 - \ln x}{x^2}$ . (b) See Figure 6.34. (c) Since  $99 > e$  we have that  $\frac{\ln 99}{99} < \frac{\ln 101}{101}$ . This is the same as  $101 \ln 99 < 99 \ln 101$ . Hence  $\ln 99^{101} < \ln 101^{99}$  and  $99^{101} < 101^{99}$ .

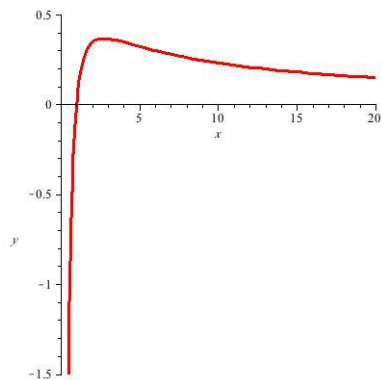
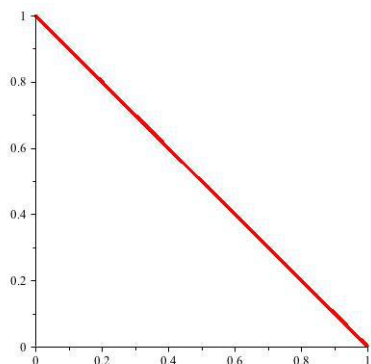


Figure 6.34:  $f(x) = \frac{\ln x}{x}$

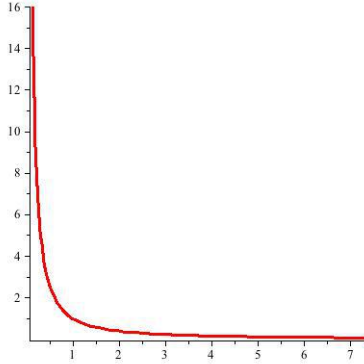
10. (a)  $2x^3 + x^2 + 5x + c, c \in \mathbb{R}$ ; (b)  $\cosh(x)$ ; (c) local minimum; (d)  $y' = -x^4 y^{-4}$ ; (e)  $(-3, 0)$ .
11. (d) Take  $f(x) = x^3$  and the point  $(0, 0)$ . (e) For example,  $f(x) = x - \frac{1}{x}$ .
12. (e) Take  $f(x) = -x^4$  and the point  $(0, 0)$ . (f)  $f(x) = x^2 + 3x + 1$ .

## 6.14 Parametric Curves

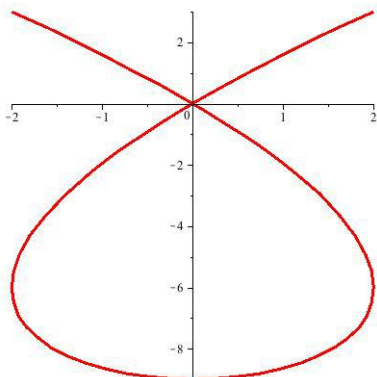
1. From  $3t^3 + 2t - 3 = 2t^3 + 2$  obtain  $t^3 + 2t - 5 = 0$ . What can you tell about the function  $f(t) = t^3 + 2t - 5$ ?
2.  $\frac{d^2 y}{dx^2} = \frac{\frac{d}{dt} \left( \frac{dy}{dx} \right)}{\frac{dx}{dt}} = \frac{3t(2 \cos t + t \sin t)}{2 \cos^3 t}$ .
3. Note that  $x + y = 1$  and that  $x, y \in [0, 1]$ . See Figure 6.35.
4. (a)  $\frac{dx}{dy} = \frac{1 - \cos \theta}{\sin \theta}$ . (b)  $\frac{d^2 x}{dy^2} = \frac{1 - \cos \theta}{\sin^3 \theta}$ . (c)  $y = \sqrt{3}x - \frac{\pi\sqrt{3}}{3} + 2$ .

Figure 6.35:  $x = \sin^2 \pi t$ ,  $y = \cos^2 \pi t$ 

5. A. Follow the curve as  $t$  increases.
6. (a) Express  $\cos t$  and  $\sin t$  in terms of  $x$  and  $y$  to get the circle  $(x+2)^2 + (y-1)^2 = 4$ . Check which points correspond to  $t = 0$  and  $t = \frac{\pi}{2}$  to get the orientation. (b) Solve  $\frac{dy}{dx} = \cot t = 1$  for  $t \in (0, 2\pi)$ .
7. (a)  $\frac{dy}{dx} = -\frac{2 \cos 2t}{3 \sin 3t}$ ; (b) Note that for both  $t = \frac{\pi}{2}$  and  $t = \frac{3\pi}{2}$  the curve passes through the origin. Thus,  $y = \pm \frac{2x}{3}$ . (c) C.
8. (a)  $\frac{dy}{dx} = 2t - 2$ . (b)  $y = x^2 + 1$ . (c)  $m = 2x_1$ ,  $b = 1 - x_1^2$ . (d) Note that the point  $(2, 0)$  does not belong to the curve. Since all tangent lines to the curve are given by  $y = 2x_1x + 1 - x_1^2$ ,  $x_1 \in \mathbb{R}$ , we need to solve  $0 = 4x_1 + 1 - x_1^2$  for  $x_1$ . Hence  $x_1 = 2 \pm \sqrt{5}$ . The tangent lines are given by  $y = 2(2 \pm \sqrt{5})x - 8 \mp 4\sqrt{5}$ .
9. (a) Note that if  $t = 0$  then  $x = y = 0$ . (b) Solve  $x = t^3 - 4t = t(t^2 - 4) = 0$ . Next,  $y(-2) = 16$ ,  $y(0) = y(2) = 0$ . (c) From  $\frac{dy}{dx} = \frac{4(t-1)}{3t^2-4}$  it follows that  $\frac{dy}{dx}\Big|_{t=0} = 1$  and  $\frac{dy}{dx}\Big|_{t=2} = \frac{1}{2}$ . The tangent lines are  $y = x$  and  $y = \frac{x}{2}$ .
10. (a)  $\frac{dy}{dx} = -e^t \cdot 2^{2t+1} \ln 2$ . (b)  $\frac{d^2y}{dx^2} = e^{2t} \cdot 2^{2t+1} \cdot (1 + 2 \ln 2) \ln 2$ . (c) No, the second derivative is never zero. (d)  $y = 2^{-2 \ln x}$ . (e) See Figure 6.36.
11. a)  $\frac{dy}{dx} = \frac{1}{4}e^{1-5t}$ . (b)  $\frac{d^2y}{dx^2} = \frac{5}{16}e^{1-9t}$ . (c) Note that  $\frac{d^2y}{dx^2}\Big|_{t=0} = \frac{5}{16}e > 0$ .
12. a)  $\frac{dy}{dx} = -t^2$ . (b)  $y = 2 - \frac{(x-1)^2}{3}$ .

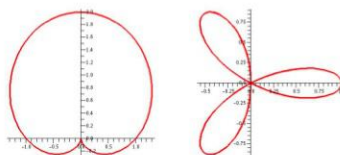
Figure 6.36:  $x = e^{-t}$ ,  $y = 2^{2t}$ 

13. (a)  $\frac{dy}{dx} = \frac{\cos t - t \sin t}{\sin t + t \cos t}$ . (b)  $y = -\frac{\pi}{2} \left( x - \frac{\pi}{2} \right)$ . (c)  $\frac{d^2y}{dx^2} \Big|_{t=\frac{\pi}{2}} = -2 - \frac{\pi^2}{4} < 0$ .
14. (a)  $\frac{dy}{dx} = \frac{t^2 - 1}{2t}$ . (b)  $y - 2 = \frac{3}{4}(x - 3)$ .
15. (a)  $-27, 0$ . Solve  $y = 0$  for  $t$ . (b)  $(-24, 8)$ . (c)  $y = x + 32$ . (d)  $\frac{d^2y}{dx^2} = \frac{t^2 + 3}{12t^3}$ .
16. From  $\frac{dy}{dx} = \frac{3 \sin t}{1 - 2 \cos 2t}$  we conclude that  $\frac{dy}{dx}$  is not defined if  $1 - 2 \cos 2t = 0$ ,  $0 \leq t \leq 10$ . Thus,  $\theta \in \left\{ \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6} \right\}$ .
17. (a)  $\frac{dy}{dx} = -2e^{3t}$ . (b)  $\frac{d^2y}{dx^2} = 6e^{4t}$ . (c)  $\frac{d^2y}{dx^2} > 0$ . (d)  $y = x^{-2}$ .
18. (a)  $\frac{dy}{dx} = \frac{3 \sin^2 t \cos t}{-3 \cos^2 t \sin t} = -\tan t$ . (b) From  $\frac{d^2y}{dx^2} = -\frac{\sec^2 t}{-3 \cos^2 t \sin t} = \frac{1}{3 \cos^4 t \sin t}$  we get  $\frac{d^2y}{dx^2} \Big|_{t=1} = \frac{1}{3 \cos^4 1 \sin 1} > 0$ .
19. (a)  $\frac{dy}{dx} = -e^{-2t}$ ,  $\frac{d^2y}{dx^2} = e^{-3t}$ . (b) From  $\frac{dy}{dx} = -e^{-2t} = -1$  we get that  $t = 0$ . Thus the tangent line is  $y = -x$ .
20. (a)  $(0, 0), (0, -9)$ . (b) From  $\frac{dy}{dx} = \frac{2t}{t^2 - 1}$  we get that the tangent line is horizontal at the point  $(0, -9)$  and vertical at the points  $(-2, -6)$  and  $(2, -6)$ . (c) See Figure 6.37.

Figure 6.37:  $x = t(t^2 - 3)$ ,  $y = 3(t^2 - 3)$ 

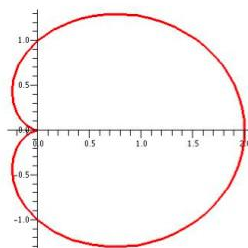
## 6.15 Polar Coordinates

1.  $(x^2 + y^2)^3 = (y^2 - x^2)$ . Multiply by  $r^2$  and use the fact that  $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$
2. See Figure 6.38.

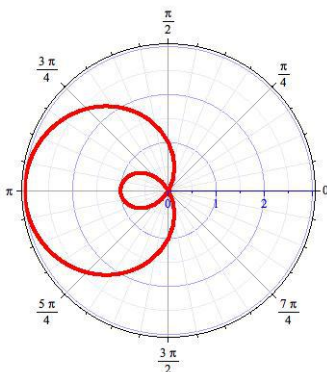
Figure 6.38:  $r = 1 + \sin \theta$  and  $r = \cos 3\theta$ 

3. (a)  $r = 2$ , (b)  $r = 2 \cos \theta$ , (c)  $r = \sin \theta$ .
4. On the given cardioid,  $x = (1 + \cos \theta) \cos \theta$  and  $y = (1 + \cos \theta) \sin \theta$ . The question is to find the maximum value of  $y$ . Note that  $y > 0$  is equivalent to  $\sin \theta > 0$ . From  $\frac{dy}{d\theta} = 2 \cos^2 \theta + \cos \theta - 1$  we get that the critical numbers of the function  $y = y(\theta)$  are the values of  $\theta$  for which  $\cos \theta = \frac{-1 \pm \sqrt{5}}{4}$ . Since  $\frac{-1 - \sqrt{5}}{4} < -1$  it follows that the critical numbers are the values of  $\theta$  for which  $\cos \theta = \frac{-1 + \sqrt{5}}{4}$ . Since  $y_{\max} > 0$  it follows that  $\sin \theta = \sqrt{1 - \left(\frac{-1 + \sqrt{5}}{4}\right)^2} = \frac{1}{2} \sqrt{\frac{5 - \sqrt{5}}{2}}$  and the maximum height equals  $y = \frac{3 + \sqrt{5}}{4} \sqrt{\frac{5 - \sqrt{5}}{2}}$ .

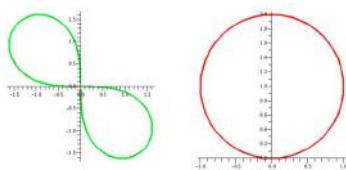
5. See the graph  $r = \cos 3\theta$  and make the appropriate stretching.
6. See Figure 6.39.

Figure 6.39:  $r = -1 + \cos \theta$ 

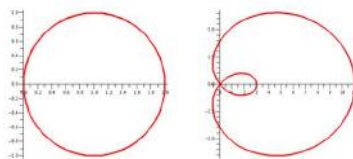
7. See Figure 6.40.

Figure 6.40:  $r = 1 - 2 \cos \theta$ 

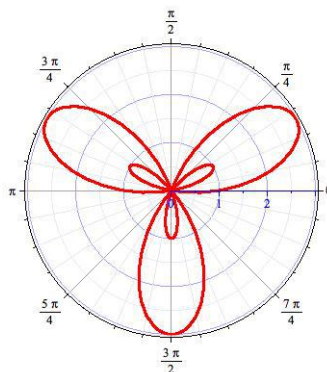
8. For (a) see the graph of  $r = \cos 3\theta$  above. For (b) and (c) see Figure 6.41 and for (d) and (e) see Figure 6.42.

Figure 6.41:  $r^2 = -4 \sin 2\theta$  and  $r = 2 \sin \theta$ 

9. (a)  $(0, -3)$ . (b) Solve  $y = (1 - 2 \sin \theta) \sin \theta = 0$ .  $(\pm 1, 0)$ . (c) The middle graph corresponds to  $r = 1 + \sin 2\theta$  and the right graph corresponds to  $r = 1 - 2 \sin \theta$ .

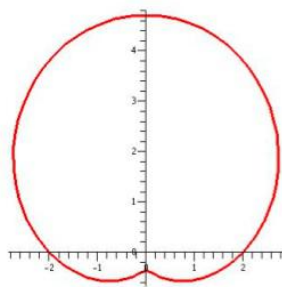
Figure 6.42:  $r = 2 \cos \theta$  and  $r = 4 + 7 \cos \theta$ 

10. (a) See Figure 6.43.

Figure 6.43:  $r = 1 + 2 \sin 3\theta$ ,  $0 \leq \theta \leq 2\pi$ 

(b) 9. (c)  $\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi$ . (d) The remaining points of intersection are obtained by solving  $-1 = 1 + 2 \sin 3\theta$ .

11. (a)  $r(0) = 2$ ,  $r\left(\frac{\pi}{2}\right) = 2 + e$ ,  $r\left(\frac{3\pi}{2}\right) = e^{-1}$ . (b) See Figure 6.44.

Figure 6.44:  $r(\theta) = 1 + \sin \theta + e^{\sin \theta}$ 

(c) From  $\frac{dr}{d\theta} = \cos \theta(1 + e^{\sin \theta}) = 0$  we conclude that the critical numbers are  $\frac{\pi}{2}$  and  $\frac{3\pi}{2}$ . By the Extreme Value Theorem, the minimum distance equals  $e^{-1}$ .

12. (a)  $A = \left(r = \sqrt{2}, \theta = \frac{\pi}{4}\right)$ ,  $B = \left(4, \frac{5\pi}{3}\right)$ ,  $C = \left(2, \frac{7\pi}{6}\right)$ ,  $D = \left(2\sqrt{2} - 1, \frac{3\pi}{4}\right)$ ; (b) A, B, D.

13. (a) See Figure 6.45. (b) Solve  $\sin \theta > -\frac{1}{2}$  to get  $\theta \in \left[-\pi, -\frac{5\pi}{6}\right) \cup \left(-\frac{\pi}{6}, \pi\right)$ . (c)

To find critical numbers solve  $\frac{dr}{d\theta} = 2 \cos \theta = 0$  in  $[-\pi, \pi)$ . It follows that  $\theta = -\frac{\pi}{2}$  and  $\theta = \frac{\pi}{2}$  are critical numbers. Compare  $r(-\pi) = r(\pi) = 1$ ,  $r\left(-\frac{\pi}{2}\right) = -1$ , and  $r\left(\frac{\pi}{2}\right) = 3$  to answer the question.

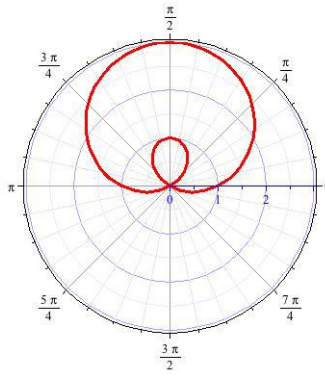


Figure 6.45:  $r(\theta) = 1 + 2 \sin \theta$

14. (a) See Figure 6.46. (b) The slope is given by  $\left.\frac{dy}{dx}\right|_{\theta=\frac{5\pi}{2}}$ . From  $x = r \cos \theta = \theta \cos \theta$

and  $y = \theta \sin \theta$  it follows that  $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\sin \theta + \theta \cos \theta}{\cos \theta - \theta \sin \theta}$ . Thus  $\left.\frac{dy}{dx}\right|_{\theta=\frac{5\pi}{2}} = -\frac{2}{5\pi}$ .

(c)  $\sqrt{x^2 + y^2} = \arctan \frac{y}{x}$ .

15. (a)  $(0, 5)$ . (b)  $x^2 + y^2 = 5y$ . (c)  $x = 5 \sin \theta \cos \theta$ ,  $y = 5 \sin^2 \theta$ . (d)  $\frac{dy}{dx} = \frac{2 \sin \theta \cos \theta}{\cos^2 \theta - \sin^2 \theta} = \tan 2\theta$ . (e)  $y - \frac{5}{4} = \sqrt{3} \left(x - \frac{5\sqrt{3}}{4}\right)$ .

16. Solve  $2 = 4 \cos \theta$  to get that curve intersect at  $(1, \sqrt{3})$ . To find the slope we note that the circle  $r = 2$  is given by parametric equations  $x = 2 \cos \theta$  and  $y = 2 \sin \theta$ . It follows that  $\frac{dy}{dx} = \frac{2 \cos \theta}{-2 \sin \theta} = -\cot \theta$ . The slope of the tangent line at the intersection point equals  $\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{3}} = -\frac{\sqrt{3}}{3}$ .



(b) If  $k = \frac{3}{2}$  then the eccentricity is given by  $e = \frac{k}{3} = \frac{1}{2}$ . The directrix is  $x = d$  where  $ed = \frac{4}{3}$ . Thus the directrix is  $x = \frac{8}{3}$ . Let  $c > 0$  be such that  $(c, 0)$  is a focus of the ellipse. Then  $\frac{c}{e}(1 - e^2) = ed$  and  $c = \frac{8}{9}$ . (c) See Figure 6.48.

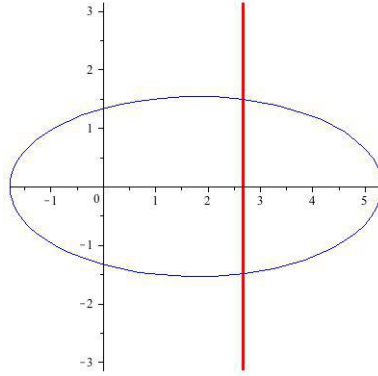
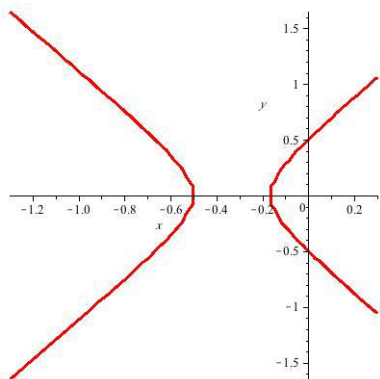


Figure 6.48:  $r(3 - \frac{3}{2} \cos \theta) = 4$  and  $x = \frac{8}{3}$

5. (a) Let the center of the earth (and a focus of the ellipse) be at the point  $(0, c)$ ,  $c > 0$ . Let the equation of the ellipse be  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ . It is given that the vertices of the ellipse on the  $y$ -axis are  $(0, (c + s) + 5s)$  and  $(0, (c - s) - 11s)$ . It follows that the length of the major axis on the  $y$ -axis is  $2b = 6s + 12s = 18s$ . Thus,  $b = 9s$  and  $c = b - 6s = 3s$ . From  $c^2 = b^2 - a^2$  we get that  $a^2 = 72s^2$ . Thus the equation of the ellipse is  $\frac{x^2}{72s^2} + \frac{y^2}{81s^2} = 1$ . The question is to evaluate the value of  $|x|$  when  $y = c = 3s$ . From  $\frac{x^2}{72s^2} + \frac{9s^2}{81s^2} = 1$  it follows that  $|x| = 8s$ .

(b)  $r = \frac{ep}{1 - e \cos \theta}$ .

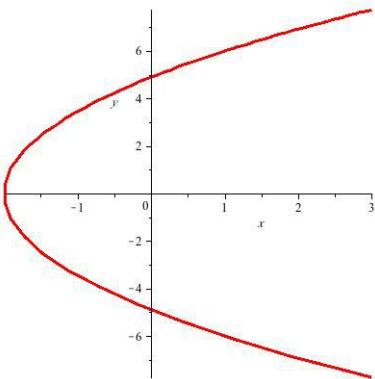
6. (a) From  $r = \frac{\frac{1}{2}}{1 - 2 \cos \theta}$  we see that the eccentricity is  $e = 2$  and the equation therefore represents a hyperbola. From  $ed = \frac{1}{2}$  we conclude that the directrix is  $x = -\frac{1}{4}$ . The vertices occur when  $\theta = 0$  and  $\theta = \pi$ . Thus the vertices are  $(-\frac{1}{2}, 0)$  and  $(-\frac{1}{6}, 0)$ . The  $y$ -intercepts occur when  $\theta = \frac{\pi}{2}$  and  $\theta = \frac{3\pi}{2}$ . Thus  $(0, \frac{1}{2})$  and  $(0, -\frac{1}{2})$ . We note that  $r \rightarrow \infty$  when  $\cos \theta \rightarrow \frac{1}{2}$ . Therefore the asymptotes are parallel to the rays  $\theta = \frac{\pi}{3}$  and  $\theta = \frac{5\pi}{3}$ . See Figure 6.49.

Figure 6.49:  $r = \frac{1}{2 - 4 \cos \theta}$ 

(b) From  $\sqrt{x^2 + y^2} = \frac{1}{2 - \frac{4x}{\sqrt{x^2 + y^2}}}$  it follows that the conic section is given by  $12x^2 - 4y^2 + 8x + 1 = 0$ .

7. (a)  $(x - 5)^2 + y^2 = \frac{(x + 5)^2}{4}$ . (b)  $3\left(x - \frac{25}{3}\right)^2 + 4y^2 = \frac{400}{3}$ . This is an ellipse.

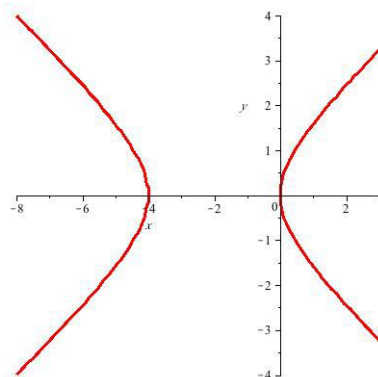
8. From  $(x - 1)^2 + y^2 = (x + 5)^2$  we get that  $y^2 = 24 + 12x$ . See Figure 6.50.

Figure 6.50:  $y^2 = 24 - 12x$ 

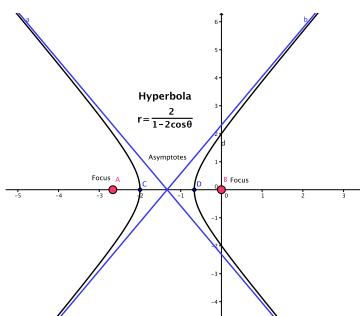
9. (a)  $\frac{(x + 2)^2}{4} - \frac{y^2}{2} = 1$ . This is a hyperbola. Foci are  $(-2 - \sqrt{6}, 0)$  and  $(-2 + \sqrt{6}, 0)$ .

The asymptotes are  $y = \pm \frac{\sqrt{2}}{2}(x + 2)$ . (b) See Figure 6.51.

10. (a) A hyperbola, since the eccentricity is  $e = 2 > 1$ . (b) From  $r(1 - 2 \cos \theta) = 2$  conclude that  $\sqrt{x^2 + y^2} = 2(x + 1)$ . Square both sides, rearrange the expression,

Figure 6.51:  $x^2 - 2y^2 + 4x = 0$ 

and complete the square. (c) From  $\frac{(x + \frac{4}{3})^2}{(\frac{2}{3})^2} - \frac{y^2}{(\frac{2}{\sqrt{3}})^2} = 1$  it follows that the foci are given by  $(-\frac{4}{3} \pm \frac{4}{3}, 0)$ , the vertices are given by  $(-\frac{4}{3} \pm \frac{2}{3}, 0)$ , and the asymptotes are given by  $y = \pm\sqrt{3}(x + \frac{4}{3})$ . (d) See Figure 6.52.

Figure 6.52:  $r = \frac{2}{1 - 2 \cos \theta}$ 

## 6.17 True Or False and Multiple Choice Problems

1.

- (a) True. (e) True. (i) False.  
 (b) False. (f) False. (j) False. Take  $f(x) = |x|$ .  
 (c) False. (g) True.  
 (d) True. (h) False.
2. (a) False. Find  $\lim_{x \rightarrow 2} f(x)$ . (l) False.  $y = |x^2 + x|$  is not differentiable for all real numbers.  
 (b) False. Think  $f(x) = 1, g(x) = 2$ . (m) False.  
 (c) True. (n) False.  
 (d) False. Apply the Mean Value Theorem. (o) False. Take  $\lim_{x \rightarrow 5} \frac{x-5}{x-5}$ .  
 (e) False. Apply the chain rule. (p) False. Take  $f(x) = \frac{1}{x-1}$  if  $x > 1$  and  $f(x) = 0$  if  $x \leq 1$ .  
 (f) False. (q) False.  
 (g) False. (r) True.  
 (h) True. (s) False.  $c$  might be an isolated point.  
 (i) True. The limit equals  $g'(2)$ . (t) False. Take  $f(x) = x^3$ .  
 (j) False.  
 (k) True.  $\tan^2 x - \sec^2 x = -1$ .
3. (a) True. (h) True  $\frac{1}{\csc u} = \sin u$  with  $\sin u \neq 0$ .  
 (b) True.  $\frac{\ln 2^{\sqrt{x}}}{\sqrt{x}} = \ln 2, x > 0$ . (i) True. Use the chain rule.  
 (c) True. (j) False.  $\sinh^2 x - \cosh^2 x = -1$ .  
 (d) False.  $f(x) \geq 1$ . (k) False.  $\int \frac{dx}{x^2+1} = \arctan x + C$ .  
 (e) True. (l) False.  $\int \frac{dx}{3-2x} = -\frac{1}{2} \ln |3-2x| + C$ .  
 (f) False  $f(x) = -\frac{x^4-256}{4} + 3$ .  
 (g) False. Take  $f(x) = x^2$  and  $c = 1$ .
4. (a) False. and  $f(x) = 0$  if  $x$  is irrational.  
 (b) False. Take  $f(x) = \frac{x^2}{x}$  and  $a = 0$ . (f) False. Take  $g(x) = 0$ .  
 (c) False. Take  $f(x) = \frac{x^2}{x}, a = -1$ , and  $b = 1$ . (g) True. Take  $f(x) = \frac{1}{x}$  if  $x \neq 0$ ,  $f(0) = 0$ , and  $g(x) = -f(x)$ .  
 (d) False. Take  $f(x) = x|x|$ . (h) False. Take  $f(x) = \sin x$  and  $g(x) = -\sin x$ .  
 (e) False. Take  $f(x) = 1$  if  $x$  is rational (i) False. The numerator is an expo-

- nential function with a base greater than 1 and the denominator is a polynomial.
- (j) False. Take  $f(x) = \tan \frac{\pi x}{2}$ .
5. (a) False. Take  $f(x) = 10x$  and  $g(x) = 20x$  if  $x \in [0, 0.5]$  and  $g(x) = 10x$  if  $x \in (0.5, 1]$ . (b) True. Take  $F(x) = f(x) - g(x)$  and apply Rolle's Theorem. (c) True.
6. (a) False. The limit is missing. (g) True. (b) False. One should use the Squeeze Theorem. (h) False. It should be  $L(x) = f(a) + f'(a)(x - a)$ . (c) True. (i) False. The eccentricity of a circle is  $e = 0$ . (d) True. (j) True. Note that  $g'(x) = -0.5$  and  $f'(3) \approx 0.5$ . (e) True. (f) False.  $f(3) = 16$ .
7. (a) True. (g) False. (b) True. (h) True. (c) False.  $f(g(x)) = (x + 1)^2$ . (i) False. (d) True. (f) False. Take  $f(x) = -x$ . (e) True.
8. (a) False. It is a quadratic polynomial. (c) False. Take  $f(x) = -x$ . (b) False. The function should be also continuous on  $[a, b]$ . (d) False. Take  $f(x) = -|x|$ . (e)
9. (a) False. Use the Mean Value Theorem. (d) True. Since  $f$  is differentiable, by Rolle's Theorem there is a local extremum between any two isolated solutions of  $f(x) = 0$ . (b) False. Take  $y = (x - 5)^4$ . (e) False. Take  $f(x) = x - 1$ . (c) False. Take  $f(x) = x^3$ ,  $c = 0$ .
10. (a) False. (e) False. (b) False. Take  $f(x) = \sin x$ . (f) False. Take  $f(x) = |x|$ . (c) False.  $g'(2) = 4$ . (g) True. If  $f$  is differentiable at  $c$  then  $f$  is continuous at  $c$ . (d) False.

- (h) False. It is not given that  $f$  is continuous.
11. (a) True.
- (b) False. Take functions  $f(x) = xe^{-1/x^2} \sin(x^{-4})$  and  $g(x) = e^{-1/x^2}$ .
- (c) False. Take  $f(x) = x^4$ .
- (d) False. Take  $f(x) = |x|$  and  $x = 0$ .
- (i) True.
- (j) True.
- (e) True.
- (f) False. Take  $f(x) = e^x$ .
- (g) True.
- (h) True.
12. (a) B.
- (c) C.  $\frac{10-2}{4-2} = 4$ .
- (d) C. Use  $\frac{dP}{dt} = kP$ .
- (e) B.  $f$  is increasing.
13. (a) B. Consider  $f(x) = x^5 + 10x + 3$  and its first derivative.
- (b) E.  $\cosh(\ln 3) = \frac{3 + \frac{1}{3}}{2}$ .
- (c) B.  $f(2.9) \approx 2 + 4(2.9 - 3)$ .
- (d) E.  $F(x) = \frac{3}{4}x^{\frac{4}{3}} + \frac{1}{4}$ .
- (e) B.
14. (a) A.
- (b) E.  $\lim_{x \rightarrow 0^+} \frac{\ln x}{x} = -\infty$ .
- (c) B. Use L'Hospital's rule.
- (d) C.  $(x-1)^2 + y^2 = 5$ .
- (e) B.  $\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$ .
- (f) E.  $\frac{dy}{dt} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$ .
- (g) D.  $A(t) = 16 \left(\frac{3}{4}\right)^t$ .
- (h) D. For (1) take  $f(x) = x^3$  on  $(0, 1)$ . For (2) take  $f(x) = \sqrt[3]{x}$ . For (3) take  $f(x) = x^4$ .
- (i) B. For (1) take  $g(x) = 0$ . For (3) take  $f(x) = |x|$ ,  $g(x) = -|x|$ , and  $a = 0$ .
- (j) A.
15. (a) C.
- (b) B. Note  $y' \sinh y = 1 + 3x^2y + x^3y'$ .
- (c) D.
- (d) D.
- (e) B.
16. (a)  $\pi$ .
- (b)  $f'(x) = \ln|x|$ .
- (c) 0.  $\frac{dy}{dt} = 2 \sin x \cdot \cos x \cdot \frac{dx}{dt}$ .
- (d)  $F(x) = e^{x^2} + C$ .
- (e) 0.  $\lim_{t \rightarrow \infty} \ln \frac{t+1}{t}$ .
- (f) Yes.
- (g)  $r = 5$ .
- (h) 1.

17. (a)  $F = x \cdot \sin \frac{1}{x}$  and  $a = 0$ . (c)  $f(x) = x^3$ .  
(b)  $f(x) = |x|$ . (d)  $f(x) = x^3$ .
18. (a) The derivative of function  $f$  at a number  $a$ , denoted by  $f'(a)$ , is  $f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$  if this limit exists.  
(b) A critical number of a function  $f$  is a number  $c$  in the domain of  $f$  such that  $f'(c) = 0$  or  $f'(c)$  does not exist.  
(c) If  $f$  is continuous on a closed interval  $[a, b]$ , the  $f$  attains an absolute maximum value  $f(c)$  and an absolute minimum value  $f(d)$  at some numbers  $c$  and  $d$  in  $[a, b]$ .
19. (a) ii (g) no match  
(b) ix (h) viii  
(c) v (i) vii  
(d) vi (j) iii  
(e) iv (k) no match  
(f) no match (l) i



# Bibliography

- [1] J. Stewart. **Calculus, Early Transcendentals**, 6th Edition, Thomson, 2008.