

Chapter 5

Integration

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Integration is one of the two main operations in calculus, with its inverse differentiation. This chapter is devoted to integration concepts and its techniques.

5.1 Anti-derivatives; indefinite integrals

Definition 5.1.1 Let $f(x)$ be continuous on $[a, b]$. If $F(x)$ is continuous on $[a, b]$ and $F'(x) = f(x)$, for all $x \in (a, b)$, then F is called an anti-derivative of f .

The process of finding anti-derivatives is called anti-differentiation or integration. It is the reverse of differentiation.

■ **Example 5.1** Let f be defined by $f(x) = 2x + 3$ and F by $F(x) = x^2 + 3x$. Then $F(x)$ is an anti-derivatives of $f(x)$, because $F'(x) = 2x + 3 = f(x)$ for all x . ■

■ **Example 5.2** Check whether the function $F(x) = \frac{x^3}{3} + 5$ is an anti-derivative of $f(x) = x^2$ or not. Solution: From the definition $F(x)$ is an anti-derivative of $f(x)$ if and only if $F'(x) = f(x)$.

$$\frac{d}{dx}F(x) = \frac{d}{dx}\left(\frac{x^3}{3} + 5\right) = x^2 = f(x)$$

Therefore, $\frac{1}{3}x^3 + 5$ is an anti-derivative of $f(x) = x^2$. ■

Definition 5.1.2 $\int f(x)dx = F(x) + c$ is called the indefinite integral of $f(x)$ if and only if $F'(x) = f(x)$ for every x in the domain of f , c is called the constant integration. It is called indefinite integral because it has no definite numerical value. It is a function of variable x .

Integral Notation

- $\int$ an elongated of S , is the integral sign and is derived from the first letter of the latin word, 'Summa', which means sum.
- $\int f(x)dx$ denotes integration of $f(x)$ with respect to x .
- $f(x)$ is called the integrand (the function to be integrated).
- dx similar to the dx in the differentiation operator $\frac{d}{dx}$, it shows that the operator has to be performed with respect to the variables x .

Integration Rule

1. Constant Multiple Rule

$$\int cf(x)dx = c \int f(x)dx$$

where c is a constant.

■ **Example 5.3** $\int 2x^3 dx = 2 \int x^3 dx$ ■

2. Power Rule

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + c \text{ for } n \neq -1$$

■ **Example 5.4** Find $\int x^2 dx$.

Solution: $\int x^2 dx = \frac{x^{2+1}}{2+1} + c = \frac{1}{3}x^3 + c$ where c is some constant. ■

3. Sum-Difference Rule

$$\int [f(x) \pm g(x)] dx = \int f(x) dx \pm \int g(x) dx$$

■ **Example 5.5** Find $\int (3x^2 + 5x + 10) dx$.

Solution:

$$\begin{aligned} \int (3x^2 + 5x + 10) dx &= \int 3x^2 dx + \int 5x dx + \int 10 dx \\ &= 3 \int x^2 dx + 5 \int x dx + 10 \int dx \\ &\quad \text{by constant multiple rule} \\ &= \frac{3x^2}{3} + \frac{5x^2}{2} + 10x + c \\ &\quad \text{by the power rule, where } c \text{ is some constant} \\ &= x^3 + \frac{5}{2}x^2 + 10x + c \end{aligned}$$

Notice that in place of separate constants of integration for each of the three integrals, a single constant c was added at the end of (or $c = c_1 + c_2 + c_3$).

4. Exponential Rule

$$\int e^x dx = e^x + c$$

where c is any constant. The more general formula for an exponential function is

$$\int f'(x)e^{f(x)} dx = e^{f(x)} + c$$

■ **Example 5.6** Find $\int 3e^{3x} dx$.

Solution: $\int 3e^{3x} dx = e^{3x} + c$ ■

■ **Example 5.7** Find $\int e^{2x} dx$.

Solution: Here we can not directly apply the above rule. We have to rewrite the equation in order to apply the formula

$$\begin{aligned}\int e^{2x} dx &= \int \frac{2}{2} e^{2x} dx \\ &= \frac{1}{2} \int 2e^{2x} dx \\ &= \frac{1}{2} e^{2x} + c \text{ where } c \text{ is some constant}\end{aligned}$$

5. Logarithmic Rule

$$\int \frac{1}{x} dx = \ln x + c, x > 0$$

The more general formula for logarithmic functions is given as

$$\int \frac{f'(x)}{f(x)} dx = \ln f(x) + c, \text{ where } f(x) > 0$$

■ **Example 5.8** Find $\int \frac{2x}{x^2+5} dx$.

Solution: Let $f(x) = x^2 + 5$, then $f'(x) = 2x$. So, we can directly apply the logarithmic rule

$$\int \frac{2x}{x^2+5} dx = \ln(x^2 + 5) + c$$

■ **Example 5.9** Find $\int \frac{3x^2+2x+1}{x^3+x^2+x+3} dx$.

Solution: Let $f(x) = x^3 + x^2 + x + 3$, then $f'(x) = 3x^2 + 2x + 1$. Hence

$$\int \frac{3x^2+2x+1}{x^3+x^2+x+3} dx = \ln|x^3 + x^2 + x + 3| + c$$

5.2 Techniques of Integration

5.2.1 Integration by Substitution

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I , then

$$\int f(g(x)) \cdot g'(x) dx = \int f(u) du$$

We have to follow the following steps to use the substitution rule

- Step 1: Introduce the letter u to stand for some expression in x that is chosen with simplification of the integral as the goal.
- Step 2: Find $\frac{du}{dx}$.
- Step 3: Rewrite the integral in terms of u .
- Step 4: Evaluate the resulting integral and replace u by its expression in terms of x in the answer.

■ **Example 5.10** Find $\int 2x(x^2 + 1) dx$ by substitute rule.

Solution: Let $u = x^2 + 1$, then $\frac{du}{dx} = 2x$.

$$\begin{aligned}\int 2x(x^2 + 1) dx &= \int u \frac{du}{dx} dx \\ &= \int u du \text{ because } dx \text{ can be canceled} \\ &= \frac{1}{2} u^2 + c \\ &= \frac{1}{2} (x^2 + 1)^2 + c\end{aligned}$$

■ **Example 5.11** Find $\int 9(x^2 + 3x + 5)^8(2x + 3) dx$.

Solution: Since $2x + 3$ is the derivative of $x^2 + 3x + 5$, you are advised to let $u = x^2 + 3x + 5$. Then $\frac{du}{dx} = 2x + 3$.

$$\begin{aligned}\int 9(x^2 + 3x + 5)^8(2x + 3) dx &= \int 9u^8 \frac{du}{dx} dx \\ &= \int 9u^8 du \text{ because } dx \text{ can be canceled} \\ &= \frac{9}{9} u^9 + c \\ &= u^9 + c \\ &= (x^2 + 3x + 5)^9 + c\end{aligned}$$

■ **Example 5.12** Evaluate the integral $\int x e^{-x^2} dx$.

Solution: Except for a constant factor of -2 , x is the derivative of $-x^2$. Therefore, we set $u = -x^2$, $\frac{du}{dx} = -2x$, so $du = -2x dx$, or $-\frac{1}{2} du = x dx$, and

$$\begin{aligned}\int x e^{-x^2} dx &= \int e^{-x^2} x dx \\ &= \int e^u \left(-\frac{1}{2}\right) du \\ &= -\frac{1}{2} \int e^u du \\ &= -\frac{1}{2} e^u + c\end{aligned}$$

Now substituting the value of u , we get

$$\int x e^{-x^2} dx = -\frac{1}{2} e^{-x^2} + c$$

■ **Example 5.13** Find $\int \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}} dx$.

Solution: Let $u = 1 + \sqrt{x}$, so that $du = \frac{1}{\sqrt{x}} dx$. Then

$$\begin{aligned}\int \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}} dx &= \int 2\sqrt{u} du \\ &= 2 \int \sqrt{u} du \\ &= 2\left(\frac{2}{3}\right) u^{\frac{3}{2}} + c \\ &= \frac{4}{3} (1 + \sqrt{x})^{\frac{3}{2}} + c\end{aligned}$$

■ **Example 5.14** Find $\int \cos^3 x dx$.

Solution:

$$\begin{aligned}\int \cos^3 x dx &= \int \cos^2 x \cos x dx \\ &= \int (1 - \sin^2 x) \cos x dx \text{ since } \cos^2 x = 1 - \sin^2 x \\ &= \int \cos x dx - \int \sin^2 x \cos x dx \\ &= \sin x - \int \sin^2 x \cos x dx\end{aligned}$$

Let $u = \sin x$, so that $du = \cos x dx$. Then

$$\begin{aligned}\int \sin^2 x \cos x dx &= \int u^2 du \\ &= \frac{1}{3} u^3 + c \\ &= \frac{1}{3} \sin^3 x + c\end{aligned}$$

Therefore, $\int \cos^3 x dx = \sin x - \frac{1}{3} \sin^3 x + c_0$, where $c_0 = c$. ■

■ **Example 5.15** Evaluate the integral $\int x^4 \cos(x^5 + 2) dx$.

Solution: To calculate, set $u = x^5 + 2$, then $du = 5x^4 dx \Rightarrow \frac{1}{5} du = x^4 dx$.

$$\begin{aligned}\int x^4 \cos(x^5 + 2) dx &= \int \frac{1}{5} \cos u du \\ &= \frac{1}{5} \int \cos u du \\ &= \frac{1}{5} \sin u + c \\ &= \frac{1}{5} \sin(x^5 + 2) + c\end{aligned}$$

■ **Example 5.16** Evaluate the integral $\int e^{\sin x} \cos x dx$.

Solution: To calculate, set $u = \sin x$, then $du = \cos x dx$.

$$\begin{aligned}\int e^{\sin x} \cos x dx &= \int e^u du \\ &= e^u + c \\ &= e^{\sin x} + c\end{aligned}$$

■ **Example 5.17** Evaluate $\int \sec x dx$.

Solution: The integral is evaluated as

$$\begin{aligned}\int \sec x dx &= \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) dx \\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx \\ &\quad \text{let } u = \sec x + \tan x \Rightarrow du = (\sec^2 x + \sec x \tan x) du \\ &= \int \frac{1}{u} du \\ &= \ln |u| + c \\ &= \ln |\sec x + \tan x| + c\end{aligned}$$

■ **Example 5.18** Evaluate $\int \tan x dx$.

Solution:

$$\begin{aligned}\int \tan x dx &= \int \frac{\sin x}{\cos x} dx \\ &\quad \text{let } u = \cos x \Rightarrow du = -\sin x dx \\ &= -\int \frac{1}{u} du \\ &= -\ln |u| + c \\ &= -\ln |\cos x| + c \\ &= \ln |\sec x| + c\end{aligned}$$

5.2.2 Integration by Parts

This rule is a restatement of the product rule for differentiation. The rule is stated as

$$\int f(x)g(x) dx = f(x)G(x) - \int f'(x)G(x) dx$$

where G is an anti-derivative of g . Generally, to evaluate an integral of the form $\int f(x)g'(x) dx$ using integration by parts, we have to follow the following steps

- Step 1: First integrate g and multiply the result by f to get $f(x)G(x)$, where G is an anti-derivative of g .

Remark 5.2.1 The factor selected for integration should be easy to integrate, and the factor selected for differentiation should become simpler when differentiated.

- Step 2: Multiply the anti-derivative G by the derivative of f .
- Step 3: Take the integration of the result in step 2 above ($\int f'(x)G(x) dx$).
- Step 4: Subtract step 3 from step 1, we get $f(x)G(x) - \int f'(x)G(x) dx$.

In general, the formula of integration by parts become

$$\int u dv = uv - \int v du$$

■ **Example 5.19** Find $\int xe^x dx$.

Solution: The integrand xe^x can naturally be split in to two parts x and e^x . We let

$$\begin{aligned} u &= x \text{ and } dv = e^x dx, \text{ then} \\ du &= dx \text{ and } v = e^x \end{aligned}$$

Consequently integration by parts yields

$$\begin{aligned} \int xe^x dx &= xe^x - \int e^x dx \\ &= xe^x - e^x + c \\ &= e^x(x - 1) + c \end{aligned}$$

■

■ **Example 5.20** Evaluate the integral $\int 2xe^{3x} dx$.

Solution: The integrand $2xe^{3x}$ splits in to parts $2x$ and e^{3x} , we let

$$\begin{aligned} u &= 2x \text{ and } dv = e^{3x} dx, \text{ then} \\ du &= 2 dx \text{ and } v = \frac{1}{3}e^{3x} \end{aligned}$$

Consequently integration by parts yields

$$\begin{aligned} \int 2xe^{3x} dx &= \int u dv \\ &= uv - \int v du \\ &= 2x\left(\frac{1}{3}e^{3x}\right) - \int 2\left(\frac{1}{3}e^{3x}\right) dx \\ \int 2\left(\frac{1}{3}e^{3x}\right) dx &= \frac{2}{3} \int \frac{3}{3}e^{3x} dx \\ &= \frac{2}{9} \int 3e^{3x} dx \\ &= \frac{2}{9}e^{3x} + c \text{ using exponential rule} \end{aligned}$$

Therefore, $\int 2xe^{3x} dx = \frac{2}{3}xe^{3x} - \frac{2}{9}e^{3x} + c$.

■

■ **Example 5.21** Evaluate the integral $\int x \cos x dx$.

Solution: The integrand $x \cos x dx$ splits in to parts x and $\cos x dx$, we let

$$\begin{aligned} u &= x \text{ and } dv = \cos x dx, \text{ then} \\ du &= dx \text{ and } v = \sin x \end{aligned}$$

Consequently integration by parts yields

$$\begin{aligned} \int x \cos x dx &= \int u dv \\ &= uv - \int v du \\ &= x \sin x - \int \sin x dx \\ &= x \sin x + \cos x + c \end{aligned}$$

■

■ **Example 5.22** Evaluate the integral $\int e^x \cos x dx$.

Solution: The integrand $e^x \cos x$ splits in to parts e^x and $\cos x$, we let

$$\begin{aligned} u &= e^x \text{ and } dv = \cos x dx, \text{ then} \\ du &= e^x dx \text{ and } v = \sin x \end{aligned}$$

Consequently integration by parts yields

$$\begin{aligned} \int e^x \cos x dx &= \int u dv \\ &= uv - \int v du \\ &= e^x \sin x - \int e^x \sin x dx \end{aligned}$$

Next we perform integration by parts on $\int e^x \sin x dx$. Let

$$\begin{aligned} u &= e^x \text{ and } dv = \sin x dx, \text{ then} \\ du &= e^x dx \text{ and } v = -\cos x \end{aligned}$$

Consequently integration by parts yields

$$\begin{aligned} \int e^x \sin x dx &= \int u dv \\ &= uv - \int v du \\ &= e^x(-\cos x) - \int (-\cos x)e^x dx \\ &= -e^x \cos x + \int e^x \cos x dx \end{aligned}$$

Thus, by substitution

$$\begin{aligned} \int e^x \cos x dx &= e^x \sin x - \int e^x \sin x dx \\ &= e^x \sin x - (-e^x \cos x + \int e^x \cos x dx) \\ &= e^x \sin x + e^x \cos x - \int e^x \cos x dx \\ 2 \int e^x \cos x dx &= e^x \sin x + e^x \cos x \\ \int e^x \cos x dx &= \frac{1}{2}(e^x \sin x + e^x \cos x) + c \end{aligned}$$

Therefore, $\int e^x \cos x dx = \frac{1}{2}(e^x \sin x + e^x \cos x) + c$.

■

■ **Example 5.23** Evaluate the integral $\int \ln x dx$.

Solution: The integrand $\int x$ can be rewritten as $1 \cdot \int x$ and splits in to parts $\ln x$ and 1, we let

$$u = \ln x \text{ and } dv = dx, \text{ then}$$

$$du = \frac{1}{x} dx \text{ and } v = x$$

Consequently integration by parts yields

$$\begin{aligned} \int \ln x dx &= \int 1 \ln x dx \\ &= \int u dv \\ &= uv - \int v du \\ &= x \ln x - \int x \frac{1}{x} dx \\ &= x \ln x - x + c \\ &= x(\ln x - 1) + c \end{aligned}$$

■

■ **Example 5.24** Evaluate the integral $\int x \ln x dx$.

Solution: The integrand $x \ln x$ splits in to parts $\ln x$ and x , we let

$$u = \ln x \text{ and } dv = x dx, \text{ then}$$

$$du = \frac{1}{x} dx \text{ and } v = \frac{1}{2} x^2$$

Consequently integration by parts yields

$$\begin{aligned} \int x \ln x dx &= \int \ln x x dx \\ &= \int u dv \\ &= uv - \int v du \\ &= \frac{1}{2} x^2 \ln x - \int \frac{1}{2} x^2 \frac{1}{x} dx \\ &= \frac{1}{2} x^2 \ln x - \frac{1}{2} \int x dx \\ &= \frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 + c \end{aligned}$$

■

Quick Method To Find Integration by Parts

Consider the general problem $\int f(x)g(x) dx$, first we have to evaluate

$$\begin{array}{ccc} f(x) & & g(x) \\ & \searrow + & \\ f'(x) & & \int g(x) dx \\ & \searrow - & \\ f''(x) & & \int (\int g(x) dx) dx \\ & \searrow + & \\ f'''(x) & & \int (\int (\int g(x) dx) dx) dx \\ \vdots & & \vdots \end{array}$$

Now we can to evaluate $\int f(x)g(x)dx$ by

$$\begin{aligned}\int f(x)g(x)dx &= f(x)\left(\int g(x)dx\right) - (f'(x))\left(\int\left(\int g(x)dx\right)dx\right) \\ &\quad + (f''(x))\left(\int\left(\int\left(\int g(x)dx\right)dx\right)dx\right) - \dots\end{aligned}$$

■ **Example 5.25** Find $\int xe^x dx$.

Solution: Let $f(x) = x$ and $g(x) = e^x$, then

$$\begin{array}{rcl} x & & e^x \\ & \searrow + & \\ 1 & & e^x \\ & \searrow - & \\ 0 & & e^x \\ & \searrow + & \\ 0 & & e^x \\ \vdots & & \vdots \end{array}$$

Therefore

$$\begin{aligned}\int xe^x dx &= xe^x - 1e^x + 0e^x - 0e^x + c \\ &= xe^x - e^x + c \\ &= e^x(x - 1) + c\end{aligned}$$

■

5.2.3 Integration by Partial Fraction

We integrate integrals $\frac{f(x)}{g(x)}$, where f and g are polynomials, by expressing the integrand, $\frac{f(x)}{g(x)}$, as sums of simpler fraction, called partial fraction(decomposition), that we already know how to integrate. This method consists of two steps

- Step 1: Finding the partial fractions of the integrand.
- Step 2: Integrating the resulting partial fractions.

Methods of the Partial Fraction

Converting $\frac{f(x)}{g(x)}$ as a sum of relatively simpler fractions depends on two things

1. The degree of $f(x)$ must be less than the degree of $g(x)$.
2. We must know the factor of $g(x)$

To write $\frac{f(x)}{g(x)}$ as a sum of partial fractions, suppose $\frac{f(x)}{g(x)}$ is a proper fraction and all the factors of $g(x)$ are known. Then, here is how to find the partial fractions based on the following rules

- Rule 1: Linear Fraction Rule

Let $(x - a)^m$ is the highest power of $x - a$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions

$$\frac{A_1}{x - a} + \frac{A_2}{(x - a)^2} + \dots + \frac{A_m}{(x - a)^m}$$

- Rule 2: Quadratic Factor Rule

Let $(x^2 + px + q)^n$ is the highest power of a quadratic factor of $g(x)$ that can not be factored in to linear factors with real coefficients(irreducible quadratic factor). Then, to this factor, assign the sum of the n partial fractions of the form

$$\frac{B_1x + C_1}{x^2 + px + q} + \frac{B_2x + C_2}{(x^2 + px + q)^2} + \dots + \frac{B_nx + C_n}{(x^2 + px + q)^n}$$

where B_i 's and C_i 's are constants to be determined.

- Rule 3: Set the original fraction, $\frac{f(x)}{g(x)}$, equal to the sum of all these partial fractions.
- Rule 4: Equate the coefficients of the corresponding powers of x and solve the resulting equations for the undetermined coefficients
- Rule 5: Back substitute the values of the constants and integrate.

■ **Example 5.26** Evaluate $\int \frac{1}{x(x+1)} dx$.

Solution: The partial fraction has the form

$$\begin{aligned}\frac{1}{x(x+1)} &= \frac{A_1}{x} + \frac{A_2}{x+1} \\ \Rightarrow 1 &= A_1(x+1) + A_2x \\ \Rightarrow (A_1 + A_2)x + A_1 &= 1 \\ \Rightarrow A_1 = 1 \text{ and } A_2 &= -1 \\ \Rightarrow \frac{1}{x(x+1)} &= \frac{1}{x} - \frac{1}{x+1}\end{aligned}$$

Therefore

$$\begin{aligned}\int \frac{1}{x(x+1)} dx &= \int \left(\frac{1}{x} - \frac{1}{x+1} \right) dx \\ &= \int \frac{1}{x} dx - \int \frac{1}{x+1} dx \\ &= \ln|x| - \ln|x+1| + c \\ &= \ln \left| \frac{x}{x+1} \right| + c\end{aligned}$$

■

■ **Example 5.27** Find $\int \frac{2x+5}{(x+2)^3} dx$.

Solution: The partial fraction has the form

$$\begin{aligned}\frac{2x+5}{(x+2)^3} &= \frac{A_1}{x+2} + \frac{A_2}{(x+2)^2} + \frac{A_3}{(x+2)^3} \\ \Rightarrow 2x+5 &= A_1(x+2)^2 + A_2(x+2) + A_3 \\ \Rightarrow A_1(x^2+4x+4) + A_2x + 2A_2 + A_3 &= 2x+5 \\ \Rightarrow A_1x^2 + 4A_1x + 4A_1 + A_2x + 2A_2 + A_3 &= 2x+5 \\ \Rightarrow A_1 = 0, A_2 = 2 \text{ and } A_3 = 1 \\ \Rightarrow \frac{2x+5}{(x+2)^3} &= \frac{2}{(x+2)^2} + \frac{1}{(x+2)^3}\end{aligned}$$

Therefore

$$\begin{aligned}\int \frac{2x+5}{(x+2)^3} dx &= \int \left(\frac{2}{(x+2)^2} + \frac{1}{(x+2)^3} \right) dx \\ &= \int \frac{2}{(x+2)^2} dx + \int \frac{1}{(x+2)^3} dx \\ &\quad \text{put } u = x+2, \text{ so that } du = dx \\ &= 2 \int \frac{1}{u^2} du + \int \frac{1}{u^3} du \\ &= \frac{-2}{u} - \frac{1}{2u^2} + c \\ &= \frac{-2}{x+2} - \frac{1}{2(x+2)^2} + c\end{aligned}$$

■

■ **Example 5.28** Evaluate $\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)} dx$.

Solution: The partial fraction has the form

$$\begin{aligned}\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} &= \frac{A_1}{x-1} + \frac{A_2}{x+1} + \frac{A_3}{x+3} \\ \Rightarrow x^2+4x+1 &= A_1(x+1)(x+3) + A_2(x-1)(x+3) + A_3(x-1)(x+1) \\ \Rightarrow (A_1+A_2+A_3)x^3 + (4A_1+2A_2)x + (3A_1-3A_2-A_3) &= x^2+4x+1 \\ \Rightarrow A_1x^2+4A_1x+4A_1+A_2x+2A_2+1 &= 2x+5 \\ \Rightarrow A_1 = \frac{3}{4}, A_2 = \frac{1}{2} \text{ and } A_3 = \frac{-1}{4} \\ \Rightarrow \frac{x^2+4x+1}{(x-1)(x+1)(x+3)} &= \frac{\frac{3}{4}}{x-1} + \frac{\frac{1}{2}}{x+1} + \frac{\frac{-1}{4}}{x+3}\end{aligned}$$

Therefore

$$\begin{aligned}\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)} dx &= \int \left(\frac{\frac{3}{4}}{x-1} + \frac{\frac{1}{2}}{x+1} + \frac{\frac{-1}{4}}{x+3} \right) dx \\ &= \frac{3}{4} \int \frac{1}{x-1} dx + \frac{1}{2} \int \frac{1}{x+1} dx + \frac{1}{4} \int \frac{1}{x+3} dx \\ &= \frac{3}{4} \ln|x-1| + \frac{1}{2} \ln|x+1| - \frac{1}{4} \ln|x+3| + c\end{aligned}$$

■

■ **Example 5.29** Find $\int \frac{2x^2-x+4}{x^3+4x} dx$.

Solution: The partial fraction of the integrand is found as

$$\begin{aligned}\frac{2x^2-x+4}{x^3+4x} &= \frac{2x^2-x+4}{x(x^2+4)} \\ \frac{2x^2-x+4}{x^3+4x} &= \frac{A}{x} + \frac{Bx+C}{x^2+4} \\ \Rightarrow 2x^2-x+4 &= A(x^2+4) + x(Bx+C) \\ \Rightarrow (A+B)x^2+Cx+4A &= 2x^2-x+4 \\ \Rightarrow A=1, B=1 \text{ and } C &= -1 \\ \Rightarrow \frac{2x^2-x+4}{x^3+4x} &= \frac{1}{x} + \frac{x-1}{x^2+4}\end{aligned}$$

Therefore

$$\begin{aligned}\int \frac{2x^2-x+4}{x^3+4x} dx &= \int \left(\frac{1}{x} + \frac{x-1}{x^2+4} \right) dx \\ &= \int \frac{1}{x} dx + \int \frac{x-1}{x^2+4} dx \\ &= \int \frac{1}{x} dx + \int \frac{x}{x^2+4} dx - \int \frac{1}{x^2+4} dx \\ &= \ln|x| + \frac{1}{2} \ln|x^2+4| - \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + c\end{aligned}$$

Therefore, $\int \frac{2x^2-x+4}{x^3+4x} dx = \ln|x| + \frac{1}{2} \ln|x^2+4| - \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + c$.

■

5.2.4 Trigonometric Integrals

Trigonometric integrals involves algebraic combination of the six basic trigonometric functions.

Theorem 5.2.2 The Reduction Formula

The sine and cosine integrals are given by the formula

1. $\int \sin^n x dx = -\frac{1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} \int \sin^{n-2} x dx$
2. $\int \cos^n x dx = \frac{1}{n} \sin x \cos^{n-1} x + \frac{n-1}{n} \int \cos^{n-2} x dx$

■ **Example 5.30** Evaluate $\int \sin^6 x dx$.

Solution: By using reduction formula

$$\int \sin^6 x dx = -\frac{1}{6} \cos x \sin^5 x + \frac{5}{6} \int \sin^4 x dx$$

Now we perform integration on $\int \sin^4 x dx$

$$\int \sin^4 x dx = -\frac{1}{4} \cos x \sin^3 x + \frac{3}{4} \int \sin^2 x dx$$

Again we perform the integration on $\int \sin^2 x dx$

$$\int \sin^2 x dx = -\frac{1}{2} \cos x \sin x + \frac{1}{2} x + c$$

Consequently

$$\begin{aligned} \int \sin^6 x dx &= -\frac{1}{6} \cos x \sin^5 x + \frac{5}{6} \int \sin^4 x dx \\ &= -\frac{1}{6} \cos x \sin^5 x + \frac{5}{6} \left(-\frac{1}{4} \cos x \sin^3 x + \frac{3}{4} \int \sin^2 x dx \right) \\ &= -\frac{1}{6} \cos x \sin^5 x + \\ &\quad \frac{5}{6} \left(-\frac{1}{4} \cos x \sin^3 x + \frac{3}{4} \left(-\frac{1}{2} \cos x \sin x + \frac{1}{2} x \right) \right) + c_0 \\ &= -\frac{1}{6} \cos x \sin^5 x - \frac{5}{24} \cos x \sin^3 x - \frac{15}{16} \cos x \sin x + \frac{15}{48} x + c_0 \end{aligned}$$

■

■ **Example 5.31** Evaluate $\int \cos^6 x dx$.

Solution: By using reduction formula

$$\int \cos^6 x dx = \frac{1}{6} \sin x \cos^5 x + \frac{5}{6} \int \cos^4 x dx$$

Now we perform integration on $\int \cos^4 x dx$

$$\int \cos^4 x dx = \frac{1}{4} \sin x \cos^3 x + \frac{3}{4} \int \cos^2 x dx$$

Again we perform the integration on $\int \cos^2 x dx$

$$\int \cos^2 x dx = \frac{1}{2} \sin x \cos x + \frac{1}{2} x + c$$

Consequently

$$\begin{aligned} \int \cos^6 x dx &= \frac{1}{6} \sin x \cos^5 x + \frac{5}{6} \int \cos^4 x dx \\ &= \frac{1}{6} \sin x \cos^5 x + \frac{5}{6} \left(\frac{1}{4} \sin x \cos^3 x + \frac{3}{4} \int \cos^2 x dx \right) \\ &= \frac{1}{6} \sin x \cos^5 x + \frac{5}{6} \left(\frac{1}{4} \sin x \cos^3 x + \frac{3}{4} \left(\frac{1}{2} \sin x \cos x + \frac{1}{2} x \right) \right) + c \\ &= \frac{1}{6} \sin x \cos^5 x + \frac{5}{24} \sin x \cos^3 x + \frac{15}{48} \sin x \cos x + \frac{15}{48} x + c_0 \end{aligned}$$

■

Rules

1. Integrals of the type $\int \sin^m x \cos^n x dx$. To evaluate such integrals, we consider three cases.

- Case 1: When m is odd, put $m = 2k + 1$, where $k = \frac{m-1}{2}$, thus

$$\sin^m x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x$$

Now substitute $u = \cos x$ to evaluate the integral

$$\int (1 - \cos^2 x)^k \cos^n x \sin x dx$$

- Case 2: When m is even and n is odd, put $n = 2k + 1$, where $k = \frac{n-1}{2}$, thus

$$\cos^n x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x$$

Now substitute $u = \sin x$ to evaluate the integral $\int \sin^m x (1 - \sin^2 x)^k \cos x dx$.

- Case 3: If both m and n are even. We substitute $\sin^2 x = \frac{1 - \cos 2x}{2}$ and $\cos^2 x = \frac{1 + \cos 2x}{2}$ to reduce the integrand to one in the lower powers of $\cos 2x$.

2. Integrals of the type $\int \sin mx \sin nx dx$, $\int \sin mx \cos nx dx$ or $\int \cos mx \cos nx dx$. To evaluate such integrals, we consider

(a) $\sin mx \sin nx = \frac{1}{2} [\cos(m-n)x - \cos(m+n)x]$

(b) $\sin mx \cos nx = \frac{1}{2} [\sin(m-n)x + \sin(m+n)x]$

(c) $\cos mx \cos nx = \frac{1}{2} [\cos(m-n)x + \cos(m+n)x]$

3. Integrals of the type $\int \sec^m x dx$ and $\int \tan^n x dx$. To evaluate such integrals, we can use the identity $\sec^2 x = \tan^2 x + 1$. So,

(a) $\int \tan^n x dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x dx$, for $n \geq 2$

(b) $\int \sec^m x dx = \frac{\sec^{m-2} x \tan x}{m-1} - \frac{m-2}{m-1} \int \sec^{m-2} x dx$, for $m \geq 2$

4. Integrals of the form $\int \tan^m x \sec^n x dx$. To evaluate such integrals, we consider

- (a) Suppose n is even and $n > 0$, put $u = \tan x$ and $du = \sec^2 x dx$, then

$$\begin{aligned} \int \tan^m x \sec^n x dx &= \int \tan^m x \sec^{n-2} x \sec^2 x dx \\ &= \int \tan^m x (1 + \tan^2 x)^{\frac{n-2}{2}} dx \\ &= \int u^m (1 + u^2)^{\frac{n-2}{2}} du \end{aligned}$$

- (b) Suppose m is odd, try to factor out $\tan x \sec x$ and put $u = \sec x$, so that $du = \sec x \tan x dx$.

$$\begin{aligned} \int \tan^m x \sec^n x dx &= \int \tan^{m-1} x \sec^{n-1} x (\tan x \sec x) dx \\ &= \int (\sec^2 x - 1)^{\frac{m-1}{2}} \sec^{n-1} x (\tan x \sec x) dx \\ &= \int (u^2 - 1)^{\frac{m-1}{2}} u^{n-1} du \end{aligned}$$

- (c) Suppose n is odd and m is even, put $u = \tan x$, so that $du = \sec^2 x dx$.

$$\begin{aligned} \int \tan^m x \sec^n x dx &= \int \tan^m x \sec^{n-2} x \sec^2 x dx \\ &= \int \tan^m x (1 + \tan^2 x)^{\frac{n-2}{2}} dx \\ &= \int u^m (1 + u^2)^{\frac{n-2}{2}} du \end{aligned}$$

■ **Example 5.32** Find $\int \sin^4 x \cos^5 x dx$.

Solution: Put $u = \sin x$, so that $du = \cos x dx$

$$\begin{aligned}
 \int \sin^4 x \cos^5 x dx &= \int \sin^4 x \cos^4 x \cos x dx \\
 &= \int \sin^4 x (1 - \sin^2 x)^2 \cos x dx \\
 &= \int u^4 (1 - u^2)^2 du \\
 &= \int u^4 (1 - 2u^2 + u^4) du \\
 &= \int (u^4 - 2u^6 + u^8) du \\
 &= \int u^4 du - \int 2u^6 du + \int u^8 du \\
 &= \frac{1}{5}u^5 - \frac{2}{7}u^7 + \frac{1}{9}u^9 + c \\
 &= \frac{1}{5}\sin^5 x - \frac{2}{7}\sin^7 x + \frac{1}{9}\sin^9 x + c
 \end{aligned}$$

■ **Example 5.33** Find $\int \sin 5x \cos 3x dx$.

Solution: Since $\sin mx \cos nx = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x]$, we have

$$\begin{aligned}
 \int \sin 5x \cos 3x dx &= \int \frac{1}{2}[\sin(5-3)x + \sin(5+3)x] dx \\
 &= \int \frac{1}{2}[\sin 2x + \sin 8x] dx \\
 &= \frac{1}{2}\left(-\frac{1}{2}\cos 2x - \frac{1}{8}\cos 8x\right) + c
 \end{aligned}$$

■ **Example 5.34** Evaluate $\int \tan^4 x dx$.

Solution:

$$\begin{aligned}
 \int \tan^4 x dx &= \int (\tan^2 x)(\tan^2 x) dx \\
 &= \int (\tan^2 x)(\sec^2 x - 1) dx \\
 &= \int \tan^2 x \sec^2 x dx - \int \tan^2 x dx \\
 &= \int \tan^2 x \sec^2 x dx - \int (\sec^2 x - 1) dx \\
 &= \int \tan^2 x \sec^2 x dx - \int \sec^2 x dx - \int dx \\
 &= -\tan x + x + \int \tan^2 x \sec^2 x dx \\
 &\quad \text{let } u = \tan x \Rightarrow du = \sec^2 x dx \\
 &= -\tan x + x + \int u^2 dx \\
 &= -\tan x + x + \frac{1}{3}u^3 + c \\
 &= \frac{1}{3}\tan^3 x - \tan x + x + c
 \end{aligned}$$

■ **Example 5.35** Evaluate $\int \sec^3 x dx$.

Solution: We integrate $\int \sec^3 x dx$ by parts using

$$\begin{aligned} u &= \sec x \text{ and } dv = \sec^2 x dx, \text{ then} \\ du &= \sec x \tan x dx \text{ and } v = \tan x \end{aligned}$$

Then

$$\begin{aligned} \int \sec^3 x dx &= \sec x \tan x - \int \tan^2 x \sec x dx \\ &= \sec x \tan x - \int (\sec^2 x - 1) \sec x dx \\ &= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx \\ &= \frac{1}{2}(\sec x \tan x + \int \sec x dx) \\ &= \frac{1}{2}(\sec x \tan x + \ln |\tan x + \sec x|) \end{aligned}$$

■ **Example 5.36** Find $\int \tan^3 x \sec^4 x dx$.

Solution: Put $u = \tan x$, so that $du = \sec^2 x dx$

$$\begin{aligned} \int \tan^3 x \sec^4 x dx &= \int \tan^3 x \sec^2 x \sec^2 x dx \\ &= \int \tan^3 x (1 + \tan^2 x) \sec^2 x dx \\ &= \int u^3 (1 + u^2) du \\ &= \int (u^3 + u^5) du \\ &= \frac{1}{4}u^4 + \frac{1}{6}u^6 + c \\ &= \frac{1}{4}\tan^4 x + \frac{1}{6}\tan^6 x + c \end{aligned}$$

■ **Example 5.37** Find $\int \tan^5 x \sec^6 x dx$.

Solution: Put $u = \sec x$, so that $du = \tan x \sec x dx$

$$\begin{aligned} \int \tan^5 x \sec^6 x dx &= \int \tan^4 x \sec^5 x \tan x \sec x dx \\ &= \int (\sec^2 x - 1)^2 \sec^5 x (\tan x \sec x) dx \\ &= \int (u^2 - 1)^2 u^5 du \\ &= \int (u^4 - 2u^2 + 1) u^5 du \\ &= \int (u^9 - 2u^7 + u^5) du \\ &= \frac{1}{10}u^{10} - \frac{2}{8}u^8 + \frac{1}{6}u^6 + c \\ &= \frac{1}{10}\sec^{10} x - \frac{2}{8}\sec^8 x + \frac{1}{6}\sec^6 x + c \end{aligned}$$

5.2.5 Integration by Trigonometric Substitution

The following table is a table which shows an appropriate trigonometric substitution when the integrand involves any of the radicals indicated below.

Expression	Substitution	Identities Required
$\sqrt{a^2 - x^2}$	$x = a \sin \theta, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$	$1 - \sin^2 \theta = \cos^2 \theta$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2 \theta = \sec^2 \theta$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta, \pi < \theta < \frac{3\pi}{2}$	$\sec^2 \theta - 1 = \tan^2 \theta$

■ **Example 5.38** Integrate $\int \frac{1}{\sqrt{4-x^2}} dx$.

Solution: Put $x = 2 \sin \theta \Rightarrow dx = 2 \cos \theta d\theta$

$$\begin{aligned}
 \int \frac{1}{\sqrt{4-x^2}} dx &= \int \frac{2 \cos \theta}{\sqrt{4-4 \sin^2 \theta}} d\theta \\
 &= \int \frac{\cos \theta}{\cos \theta} d\theta \\
 &= \theta + c \\
 &= \sin^{-1}\left(\frac{x}{2}\right) + c
 \end{aligned}$$

■ **Example 5.39** Evaluate $\int \frac{dx}{x^2 \sqrt{4+x^2}}$.

Solution: Let $x = 2 \tan \theta$, then $d\theta = 2 \sec^2 \theta d\theta$ and hence $\sqrt{4+x^2} = 2 \sec \theta$. So,

$$\begin{aligned}
 \int \frac{dx}{x^2 \sqrt{4+x^2}} &= \int \frac{2 \sec^2 \theta d\theta}{4 \tan^2 \theta 2 \sec \theta} \\
 &= \frac{1}{4} \int \frac{\sec \theta d\theta}{\tan^2 \theta} \\
 &= \frac{1}{4} \int \frac{\cos \theta d\theta}{\sin^2 \theta} \\
 &= -\frac{1}{4 \sin \theta} + c \\
 &= -\frac{\sqrt{4+x^2}}{4x} + c
 \end{aligned}$$

since $\tan \theta = \frac{x}{2}$ and thus $\sin \theta = \frac{x}{\sqrt{4+x^2}}$.

■ **Example 5.40** Integrate $\int \frac{dx}{2x^2-12x+26}$.

Solution: The denominator does not factor with rational coefficients, so partial fraction is not a viable option. Instead, we will complete the square in the denominator to get a recognizable form for the integral. Now

$$\begin{aligned}
 2x^2 - 12x + 26 &= 2(x^2 - 6x + 13) \\
 &= 2[(x^2 - 6x + 9) + 4] \\
 &= 2[(x-3)^2 + 4]
 \end{aligned}$$

Returning to the integral,

$$\begin{aligned}
 \int \frac{dx}{2x^2 - 12x + 26} &= \int \frac{dx}{2[(x-3)^2 + 4]} \\
 &= \frac{1}{2} \int \frac{dx}{(x-3)^2 + 4} \\
 \text{Let } x-3 &= 2 \tan \theta \Rightarrow dx = 2 \sec^2 \theta d\theta \\
 &= \frac{1}{2} \int \frac{2 \sec^2 \theta d\theta}{4 \sec^2 \theta} \\
 &= \frac{1}{4} \theta + c \\
 &= \frac{1}{2} \left[\frac{1}{2} \tan^{-1}\left(\frac{x-3}{2}\right) \right] + c \\
 &= \frac{1}{4} \tan^{-1}\left(\frac{x-3}{2}\right) + c
 \end{aligned}$$

■ **Example 5.41** Integrate $\int \frac{dx}{\sqrt{21-4x-x^2}}$.

Solution: Now consider

$$\begin{aligned} 21 - 4x - x^2 &= 21 - (x^2 + 4x) \\ &= 21 + 4 - (x^2 + 4x + 4) \\ &= 25 - (x + 2)^2 \end{aligned}$$

Returning to the integral,

$$\begin{aligned} \int \frac{dx}{\sqrt{21-4x-x^2}} &= \int \frac{dx}{\sqrt{25-(x+2)^2}} \\ &\text{Let } x+2 = 5 \sin \theta \Rightarrow dx = 5 \cos \theta d\theta \\ &= \int \frac{5 \cos \theta d\theta}{\sqrt{25-25 \sin^2 \theta}} \\ &= \theta + c \\ &= \sin^{-1}\left(\frac{x+2}{5}\right) + c \end{aligned}$$

■ **Example 5.42** Evaluate the integral $\int x \sin^{-1} x dx$.

Solution: The integrand $\int x \sin^{-1} x dx$ splits in to parts $\sin^{-1} x$ and x , we let

$$\begin{aligned} u &= \sin^{-1} x \text{ and } dv = x dx, \text{ then} \\ du &= \frac{1}{\sqrt{1-x^2}} dx \text{ and } v = \frac{1}{2} x^2 \end{aligned}$$

Consequently integration by parts yields

$$\begin{aligned} \int x \sin^{-1} x dx &= \int x \sin^{-1} x dx \\ &= \int u dv \\ &= uv - \int v du \\ &= \frac{1}{2} x^2 \sin^{-1} x - \frac{1}{2} \int \frac{x^2}{\sqrt{1-x^2}} dx \end{aligned}$$

Next we perform integration by parts on $\int \frac{x^2}{\sqrt{1-x^2}} dx$. Let

$$\begin{aligned} u &= x \text{ and } dv = \frac{x}{\sqrt{1-x^2}} dx, \text{ then} \\ du &= dx \text{ and } v = -\sqrt{1-x^2} \end{aligned}$$

Consequently integration by parts yields

$$\begin{aligned} \int \frac{x^2}{\sqrt{1-x^2}} dx &= \int x \frac{x}{\sqrt{1-x^2}} dx \\ &= \int u dv \\ &= uv - \int v du \\ &= -x\sqrt{1-x^2} + \int \sqrt{1-x^2} dx \end{aligned}$$

Now we need to evaluate $\int \sqrt{1-x^2} dx$ by using trigonometric substitution by letting $x = \sin \theta \Rightarrow dx = \cos \theta d\theta$ as

$$\begin{aligned}
 \int \sqrt{1-x^2} dx &= \int \sqrt{1-\sin^2 \theta} \cos \theta d\theta \\
 &= \int \cos \theta \cos \theta d\theta \\
 &= \int \cos^2 \theta d\theta \\
 &= \int \frac{1}{2}(1 + \cos 2\theta) d\theta \\
 &= \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta \\
 &= \frac{1}{2}\theta + \frac{1}{2}\sin \theta \cos \theta \\
 &= \frac{1}{2}\theta + \frac{1}{2}\sin \theta \sqrt{1-\sin^2 \theta} \\
 &= \frac{1}{2}\sin^{-1} x + \frac{1}{2}x\sqrt{1-x^2} \text{ since } \theta = \sin^{-1} x
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \int \frac{x^2}{\sqrt{1-x^2}} dx &= -x\sqrt{1-x^2} + \int \sqrt{1-x^2} dx \\
 &= -x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x + \frac{1}{2}x\sqrt{1-x^2} \\
 &= \frac{1}{2}(\sin^{-1} x - x\sqrt{1-x^2})
 \end{aligned}$$

Thus, by substitution

$$\begin{aligned}
 \int x \sin^{-1} x dx &= \frac{1}{2}x^2 \sin^{-1} x - \frac{1}{2} \int \frac{x^2}{\sqrt{1-x^2}} dx \\
 &= \frac{1}{2}x^2 \sin^{-1} x - \frac{1}{2} \left(\frac{1}{2}(\sin^{-1} x - x\sqrt{1-x^2}) \right) + c \\
 &= \frac{1}{2}x^2 \sin^{-1} x - \frac{1}{4}\sin^{-1} x + \frac{1}{4}x\sqrt{1-x^2} + c
 \end{aligned}$$

Therefore, $\int x \sin^{-1} x dx = \frac{1}{2}x^2 \sin^{-1} x - \frac{1}{4}\sin^{-1} x + \frac{1}{4}x\sqrt{1-x^2} + c$. ■

5.3 Definite Integral; Fundamental Theorem of Calculus

Let $f(x)$ is a continuous function and if $F(x)$ is the ant-derivatives of $f(x)$, then

$$\int f(x) dx = F(x) + c$$

where c is any constant number.

If we choose two values of x in the domain, say $x = a$ and $x = b$ and substitute these values in to the right side of the equation and taking the difference we get

$$[F(b) + c] - [F(a) + c] = F(b) - F(a)$$

This value is a specific numerical value free of the variable x as well as the arbitrary constant c . It is called the definite integral of $f(x)$ from a to b . We can rewrite this as

$$\int_a^b f(x) dx$$

and read as the definite integral of f from a to b ; here

- The number a and b are called limits of integration.
- Letter a is called the lower limit of integration.
- Letter b is called the upper limit of integration.

Therefore,

$$\int_a^b f(x) dx = F(x)|_a^b = F(b) - F(a)$$

■ **Example 5.43** Evaluate $\int_0^2 x^2 dx$.

Solution: First find the anti-derivative of x^2

$$\int x^2 dx = \frac{1}{3}x^3 + c$$

Then evaluate the indefinite integral at the upper and lower limits and take the difference to get

$$\begin{aligned} \int_0^2 x^2 dx &= \left. \frac{1}{3}x^3 \right|_0^2 \\ &= \frac{1}{3}(2)^3 - \frac{1}{3}(0)^3 \\ &= \frac{8}{3} - 0 \\ &= \frac{8}{3} \end{aligned}$$

■ **Example 5.44** Evaluate $\int_1^3 (x^2 + 2x) dx$.

Solution: The indefinite integral is $\frac{1}{3}x^3 + x^2 + c$. Hence

$$\begin{aligned} \int_1^3 (x^2 + 2x) dx &= \left. \frac{1}{3}x^3 + x^2 \right|_1^3 \\ &= \frac{1}{3}(3)^3 + (3)^2 - \left(\frac{1}{3}(1)^3 + 1^2 \right) \\ &= 9 + 9 - \frac{1}{3} - 1 \\ &= \frac{50}{3} \end{aligned}$$

Properties of Definite Integral

1. Interchanging the upper and lower limits of integration will change the sign of the definite integral; that is,

$$\int_a^b f(x) dx = - \int_b^a f(x) dx$$

■ **Example 5.45** $\int_1^3 x^2 dx = - \int_3^1 x^2 dx$ ■

2. $\int_a^a f(x) dx = F(a) - F(a) = 0$, where F is the anti-derivative of f .

■ **Example 5.46** $\int_1^1 x^2 dx = \frac{1}{3}x^3 \Big|_1^1 = \frac{1}{3} - \frac{1}{3} = 0$ ■

3. Let f be continuous on an interval containing a , b , and c . Then

$$\int_a^c f(x) dx = \int_a^b f(x) dx + \int_b^c f(x) dx, \text{ where } a < b < c$$

This can be extended to n subintervals.

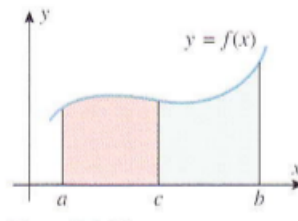


Figure 5.1: Continuity of f on the interval containing a , b and c

■ **Example 5.47** Evaluate $\int_1^3 f(x) dx$, where $f(x) = \begin{cases} x & \text{for } 1 \leq x \leq 2 \\ 2 & \text{for } 2 \leq x \leq 3 \end{cases}$
 Solution: f is continuous on $[1, 3]$. By the additive property,

$$\begin{aligned} \int_1^3 f(x) dx &= \int_1^2 f(x) dx + \int_2^3 f(x) dx \\ &= \int_1^2 x dx + \int_2^3 2 dx \\ &= \frac{1}{2}x^2 \Big|_1^2 + 2x \Big|_2^3 \\ &= \frac{1}{2}(2^2 - 1^2) + (2(3) - 2(2)) \\ &= \frac{3}{2} + 2 \end{aligned}$$

4. $\int_a^b -f(x) dx = -\int_a^b f(x) dx$
 5. $\int_a^b kf(x) dx = k \int_a^b f(x) dx$, where k is any constant. This formula means that k times an indefinite integral of f is an indefinite integral of kf .

■ **Example 5.48** $\int_1^3 3x^2 dx = 3 \int_1^3 x^2 dx = 3(\frac{7}{3}) = 7$ ■

6. If f and g are continuous on $[a, b]$, then we have the following property

$$\int_a^b (f(x) + g(x)) dx = \int_a^b f(x) dx + \int_a^b g(x) dx$$

■ **Example 5.49** Evaluate the definite integral $\int_0^1 (4x^2 + 5x^3) dx$.

Solution: First we break the integral in to its components and then integrate

$$\begin{aligned} \int_0^1 (4x^2 + 5x^3) dx &= \int_0^1 4x^2 dx + \int_0^1 5x^3 dx \\ &= 4 \int_0^1 x^2 dx + 5 \int_0^1 x^3 dx \\ &= 4 \frac{1}{3}x^3 \Big|_0^1 + 5 \frac{1}{4}x^4 \Big|_0^1 \\ &= \frac{4}{3} + \frac{5}{4} \\ &= \frac{31}{12} \end{aligned}$$

7. $\int_a^b f(x)g'(x) dx = f(x)g(x) \Big|_a^b - \int_a^b g(x)f'(x) dx$ this is integration by parts.

■ **Example 5.50** Evaluate $\int_0^1 xe^x dx$.

Solution: Using integration by parts $\int x e^x dx = e^x(x-1) + c$. So

$$\begin{aligned}\int_0^1 x e^x dx &= e^x(x-1)|_0^1 \\ &= (e^1(1-1)) - (e^0(0-1)) \\ &= 1\end{aligned}$$

■

General Comparison Property

Theorem 5.3.1 Let f and g be continuous on $[a, b]$, with $g(x) \leq f(x)$ for $a \leq x \leq b$. Then

$$\int_a^b g(x) dx \leq \int_a^b f(x) dx$$

Corollary 5.3.2 If f , g and h be continuous on $[a, b]$, with $h(x) \leq g(x) \leq f(x)$ for $a \leq x \leq b$. Then

$$\int_a^b h(x) dx \leq \int_a^b g(x) dx \leq \int_a^b f(x) dx$$

■ **Example 5.51** Show that $1 \leq \int_0^1 \sqrt{1+x^4} dx \leq \frac{4}{3}$.

Solution: We have

$$1 \leq 1+x^4 \leq 1+2x^2+x^4 = (1+x^2)^2$$

which implies that

$$1 \leq 1+x^4 \leq \sqrt{(1+x^2)^2} = 1+x^2$$

So

$$1 = \int_0^1 1 dx \leq \int_0^1 (1+x^4) dx \leq \int_0^1 (1+x^2) dx = (x + \frac{x^3}{3})|_0^1 = \frac{4}{3}$$

■

Theorem 5.3.3 If f is continuous on $[a, b]$, then

$$|\int_a^b f(x) dx| \leq \int_a^b |f(x)| dx$$

Theorem 5.3.4 Let f be continuous on $[a, b]$, and suppose $m \leq f(x) \leq M$ for all x in $[a, b]$. Then

$$m(b-a) \leq \int_a^b f(x) dx \leq M(b-a)$$

■ **Example 5.52** Using the above theorem, find lower and upper bounds for

$$\int_0^1 \sqrt{1+x^4} dx$$

Solution: If $0 \leq x \leq 1$, then $1 \leq 1+x^4 \leq 2$. So, that

$$1 \leq \sqrt{1+x^4} \leq \sqrt{2} \text{ for } 0 \leq x \leq 1$$

It follows that

$$1(1-0) \leq \int_0^1 \sqrt{1+x^4} dx \leq \sqrt{2}(1-0)$$

So that

$$1 \leq \int_0^1 \sqrt{1+x^4} dx \leq \sqrt{2}$$

Thus 1 is a lower bound and $\sqrt{2}$ is an upper bound for the integral.

■

Corollary 5.3.5 Let f be non negative and continuous on $[a, b]$. Then

$$\int_a^b f(x) dx \geq 0$$

Theorem 5.3.6 Let f be continuous on $[a, b]$. Then there is a number c in $[a, b]$ such that

$$\int_a^b f(x) dx = f(c)(b - a)$$

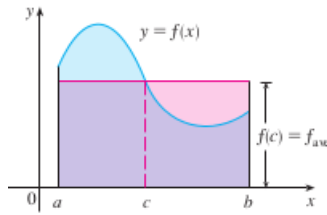


Figure 5.2: f is continuous on $[a, b]$

■ **Example 5.53** Let $f(x) = x^2$, and let A be the area of the region R between the graph f and x -axis on $[0, 2]$. Find the value of c such that A is the same as the area of the rectangle with base $[0, 2]$ and height $f(c)$.

Solution:

$$A = \int_0^2 x^2 dx = \frac{1}{3}(2^3 - 0^3) = \frac{8}{3}$$

Thus the rectangle must have area $\frac{8}{3}$ and base must have length 2. Therefore, the height $f(c)$ of the rectangle must be $\frac{4}{3}$. Consequently $c^2 = f(c) = \frac{4}{3}$, so that $c = \sqrt{\frac{4}{3}}$. ■

Remark 5.3.7 The value

$$\frac{1}{b-a} \int_a^b f(x) dx$$

that arises from the mean (or average) value of f on the interval $[a, b]$. When the interval is obvious, the mean or average value of f is denoted by f_{av} .

■ **Example 5.54** Suppose that a ball drops in to a hole 4 meter deep. Find the mean height during its fall. Solution: Measured in terms of meter, the height $h(t)$ of the ball after t seconds is given by $h(t) = -16t^2$ until the ball hits the bottom of the hole, which occurs after $\frac{1}{2}$ second. It follows that the mean value h_{av} of the height is given by

$$\begin{aligned} h_{av} &= \frac{1}{\frac{1}{2} - 0} \int_0^{\frac{1}{2}} (-16t^2) dt \\ &= 2 \int_0^{\frac{1}{2}} (-16t^2) dt \\ &= -\frac{4}{3} \end{aligned}$$

Therefore, we conclude that the mean height of the ball is $-\frac{4}{3}$ meter. ■

Theorem 5.3.8 Fundamental Theorem of Calculus part I

Suppose f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t) dt$ is continuous on $[a, b]$ and differentiable on (a, b) and the derivative is

$$F'(x) = \frac{d}{dx} \int_a^x f(t) dt = f(x)$$

■ **Example 5.55** Find the derivative of $g(x) = \int_a^x \sqrt{t^2 + 1} dt$.

Solution: Since $f(t) = \sqrt{t^2 + 1}$ is continuous, part I of the fundamental theorem of calculus gives

$$g'(x) = f(x) = \sqrt{x^2 + 1}$$

■ **Example 5.56** Find the derivative of $g(x) = \int_a^{x^4} \sec t dt$.

Solution: Here we use the chain rule in conjunction with the fundamental theorem of calculus. Let $u = x^4$

$$\begin{aligned} g'(x) &= \frac{d}{dx} \int_a^{x^4} \sec t dt \\ &= \frac{d}{du} \left[\int_a^u \sec t dt \right] \frac{du}{dx} \\ &= [\sec u] \frac{du}{dx} \\ &= [\sec x^4] (4x^3) \end{aligned}$$

■ **Example 5.57** Let $f(x) = \int_0^{x^2} t \sin t^3 dt$ for all x . Find $F'(x)$.

Solution: If we let $G(x) = \int_0^x t \sin t^3 dt$ and $g(x) = x^2$, then

$$\begin{aligned} F'(x) &= G'(g(x))g'(x) \\ &= x^2 \sin(x^2)^3 (2x) \\ &= 2x^3 \sin x^6 \end{aligned}$$

■ **Example 5.58** Find the derivative of $\int_x^{x^2} e^{-t^2} dt$.

Solution: Let $F(x) = \int_x^{x^2} e^{-t^2} dt$, then

$$\begin{aligned} \frac{d}{dx} \left(\int_x^{x^2} e^{-t^2} dt \right) &= \frac{d}{dx} \int_a^{x^2} e^{-t^2} dt + \frac{d}{dx} \int_x^a e^{-t^2} dt \\ &= \frac{d}{dx} \int_a^{x^2} e^{-t^2} dt - \frac{d}{dx} \int_a^x e^{-t^2} dt \\ &= F(x^2) - F(x) \\ &= F'(x^2)2x - F'(x) \\ &= 2xe^{-x^4} - e^{-x^2} \end{aligned}$$

Theorem 5.3.9 Fundamental Theorem of Calculus part II

If f is continuous at every point of $[a, b]$ and F is any anti-derivative f on $[a, b]$, then

$$\int_a^b f(x) dx = F(b) - F(a)$$

■ **Example 5.59** Find the definite integral $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec t dt$.

Solution:

$$\begin{aligned}
 \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec t \, dt &= \ln |\sec t + \tan t| \Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} \\
 &= \ln \left| \sec \frac{\pi}{3} + \tan \frac{\pi}{3} \right| - \ln \left| \sec \frac{\pi}{6} + \tan \frac{\pi}{6} \right| \\
 &= \ln \left| \frac{5 + \sqrt{3}}{\sqrt{3}} \right|
 \end{aligned}$$

■

5.4 Application of Integration

5.4.1 Area

Definition 5.4.1 Let f be continuous and non-negative on $[a, b]$ and let S be the area of the region that lie under the curve $y = f(x)$ from a to b . This mean that S , illustrated in the figure below, is given by the graph of continuous function f (where $f(x) \geq 0$), the vertical line $x = a$ and $x = b$, and the x -axis. The area A of the region S that lies under the graph of the continuous function defined by

$$A = \int_a^b f(x) \, dx$$

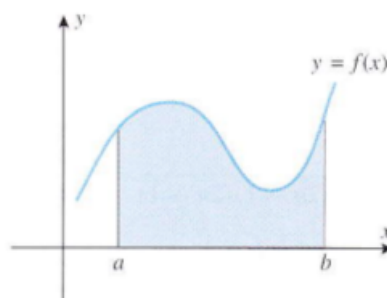


Figure 5.3: Graphs of $f(x)$

■ **Example 5.60** Find the area A of the region enclosed by the ellipse

$$\frac{x^2}{(\frac{5}{2})^2} + \frac{y^2}{5^2} = 1$$

Solution: Here $y = \sqrt{25 - 4x^2}$ on the interval $[-\frac{5}{2}, \frac{5}{2}]$.

$$\begin{aligned}
 A &= 2 \int_{-\frac{5}{2}}^{\frac{5}{2}} \sqrt{25 - 4x^2} dx \\
 &\quad \text{let } x = \frac{5}{2} \sin \theta \Rightarrow dx = \frac{5}{2} \cos \theta d\theta \\
 &= 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{25 - 25 \sin^2 \theta} \frac{5}{2} \cos \theta d\theta \\
 &= 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 5 \cos \theta \frac{5}{2} \cos \theta d\theta \\
 &= 25 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 \theta d\theta \\
 &= 25 \left[\frac{1}{2} \theta + \frac{1}{4} \sin 2\theta \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
 &= 25 \left[\left(\frac{\pi}{4} + 0 \right) - \left(-\frac{\pi}{4} + 0 \right) \right] \\
 &= 25 \left(\frac{\pi}{2} \right) \\
 &= \frac{25}{2} \pi
 \end{aligned}$$

More general, it is possible to show that the area of the region enclosed by its ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $a > 0$ and $b > 0$ is πab .

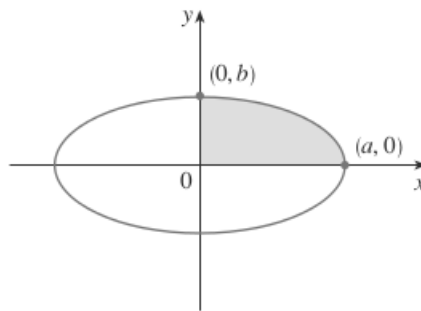


Figure 5.4: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Area between curves

Definition 5.4.2 Let f and g are continuous functions with $f(x) \geq g(x)$ for all $x \in [a, b]$. The area of the region S that lies between two curves $y = f(x)$ and $y = g(x)$ between the vertical lines $x = a$ and $x = b$ is given by

$$A = \int_a^b [f(x) - g(x)] dx$$

■ **Example 5.61** Find the area of the region bounded above by $y = e^x$, bounded below by $y = x$, and bounded on the sides by $x = 0$ and $x = 1$.

Solution: The upper boundary curve is $y = e^x$ and the lower boundary curve is $y = x$. So, we use the area formula which is formed between two curves with $f(x) = e^x$, $g(x) = x$, $a = 0$ and $b = 1$

$$A = \int_0^1 (e^x - x) dx = \left(e^x - \frac{x^2}{2} \right) \Big|_0^1 = e - 1.5$$

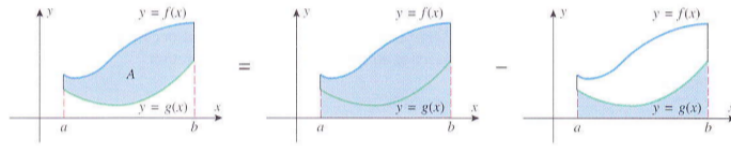


Figure 5.5: $f(x)$ and $g(x)$ are continuous on $[a, b]$ with $f(x) > g(x)$

Remark 5.4.1 • To find the area between the curves $y = f(x)$ and $y = g(x)$, where $f(x) \geq g(x)$ for some values of x but $g(x) \geq f(x)$ for other values of x , then we split the given region S into several regions $S_1, S_2, \dots$ with $A_1, A_2, \dots$. So

$$|f(x) - g(x)| = \begin{cases} f(x) - g(x) & \text{when } f(x) \geq g(x) \\ g(x) - f(x) & \text{when } g(x) \geq f(x) \end{cases}$$

Therefore, the area between the curves $y = f(x)$ and $y = g(x)$ and between $x = a$ and $x = b$ is

$$A = \int_a^b |f(x) - g(x)| dx$$

- Some regions are treated by regarding x as a function of y . If a region is bounded by curves with equations $x = f(y)$, $x = g(y)$, $y = c$ and $y = d$, where f and g are continuous and $f(y) \geq g(y)$ for $c \leq y \leq d$, then its area is

$$A = \int_c^d [f(y) - g(y)] dy$$

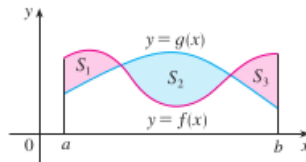


Figure 5.6: $f(x)$ and $g(x)$ are continuous on $[a, b]$

5.4.2 Volume

Volume for Regular Region

Definition 5.4.3 Let S be a solid that lies between $x = a$ and $x = b$. If the cross sectional area of S in the plane, through x and perpendicular to the x -axis, is $A(x)$, where A is a continuous function, then the volume of S is

$$V = \lim_{n \rightarrow \infty} \sum_{i=1}^n A(x_i) \Delta x = \int_a^b A(x) dx$$

■ **Example 5.62** Show that if a right pyramid has base square cross sections, then its volume is given by $V = \frac{1}{3}a^2h$, where h is the height of the pyramid and a is the length of the side base.

Solution: Let us place the y -axis along the axis of the pyramid, the origin lying at base of the pyramid, the pyramid extends from 0 to h along the y -axis. The cross section at any y in $[0, h]$ is square whose side length $s(y)$ satisfies

$$\frac{s(y)}{a} = \frac{h-y}{h} \Rightarrow s(y) = \frac{a}{h}(h-y)$$

Therefore, the cross sectional area at y is given by $A(y) = [s(y)]^2 = \frac{a^2}{h^2}(h-y)^2$.

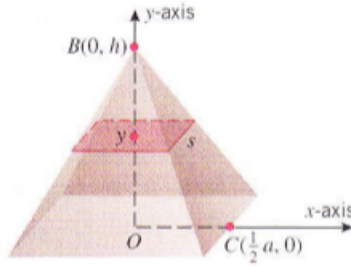


Figure 5.7: pyramid

$$\begin{aligned}
 V &= \int_0^h \left(\frac{a}{h}\right)^2 (h^2 - 2hy + y^2) dy \\
 &= \frac{a^2}{h^2} \left(h^2 y - 2h \frac{y^2}{2} + \frac{y^3}{3} \right) \Big|_0^h \\
 &= \frac{a^2}{h^2} \left(h^2 h - 2h \frac{h^2}{2} + \frac{h^3}{3} \right) \\
 &= \frac{1}{3} a^2 h
 \end{aligned}$$

■

■ **Example 5.63** Show that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.

Solution: If we place the sphere so that its center is at the origin, then the plane intersects the sphere in a circle whose radius is $y = \sqrt{r^2 - x^2}$ from the Pythagorean theorem. So, the cross sectional area is $A(x) = \pi y^2 = \pi(r^2 - x^2)$. Using the definition of the volume with $a = -r$ and $b = r$, we

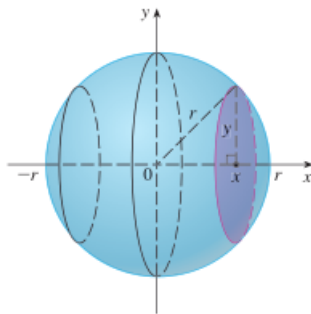


Figure 5.8: Sphere with radius r

$$\begin{aligned}
 V &= \int_{-r}^r A(x) dx \\
 &= \int_{-r}^r \pi(r^2 - x^2) dx \\
 &= 2\pi \int_0^r (r^2 - x^2) dx \text{ since the integrand is even} \\
 &= 2\pi \left[r^2x - \frac{x^3}{3} \right]_0^r \\
 &= 2\pi \left(r^3 - \frac{r^3}{3} \right) \\
 &= \frac{4\pi r^3}{3}
 \end{aligned}$$

■

Disk Method

Definition 5.4.4 When the graph of a continuous functions, non-negative function f on the interval $[a, b]$ is revolving about the x-axis, it generate a solid region having circular cross section; that is, cross section are circular discs $A(x) = \pi[f(x)]^2$. The volume V of solid that generated is given by

$$V = \int_a^b \pi[f(x)]^2 dx$$

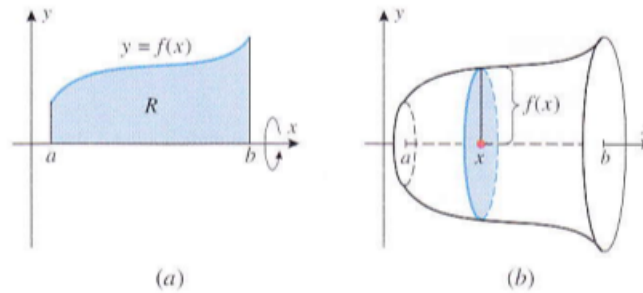


Figure 5.9: Revolving of $f(x)$ along x-axis

■ **Example 5.64** Find the volume V of the frustum of the cone.

Solution: First we have to find equation of a line through points $(0, b)$ and (h, a)

$$\frac{y-b}{x-0} = \frac{a-b}{h-0} \Rightarrow y = b + \frac{a-b}{h}x$$

since y represents the radius of the cross section of the frustum at x , then

$$\begin{aligned}
 V &= \int_0^h \pi \left(b + \frac{a-b}{h}x \right)^2 dx \\
 &= \frac{\pi h}{3(a-b)(a^3 - b^3)}
 \end{aligned}$$

If $b = 0$, the resulting region is complete cone, and the volume reduced to $V = \frac{1}{3}\pi a^2 h$.

■

Washer Method

Definition 5.4.5 Let f and g be functions such that $0 \leq g(x) \leq f(x)$ for $a \leq x \leq b$, then the plane region between the graph of f and the x-axis on $[a, b]$ is composed of the region between the graphs of f and g on $[a, b]$ and the region between the graph of g and the x-axis on $[a, b]$. Therefore, the volume of the solid generated by revolving the entire region about the x-axis is calculated by the formula

$$V = \int_a^b \pi[(f(x))^2 - (g(x))^2] dx$$

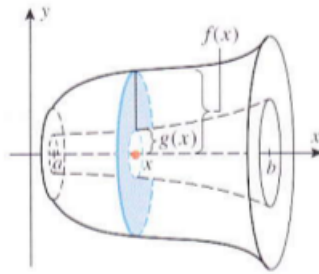


Figure 5.10: Generating of volume by $f(x)$ and $g(x)$

■ **Example 5.65** The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is resolved about the x-axis to generate a solid. Find the volume of the solid.

Solution: Let $f(x) = -x + 3$ and $g(x) = x^2 + 1$, those two graph intersect at $x = -2$ and $x = 1$. So, by washer method

$$\begin{aligned} V &= \int_{-2}^1 \pi[(f(x))^2 - (g(x))^2] dx \\ &= \int_{-2}^1 \pi[(-x + 3)^2 - (x^2 + 1)^2] dx \\ &= \int_{-2}^1 \pi[(x^2 - 6x + 9) - (x^4 + x^2 + 6x - 8)] dx \\ &= \int_{-2}^1 \pi[x^4 - x^2 - 6x - 8] dx \\ &= \left[-\frac{1}{5}x^5 - \frac{1}{3}x^3 - 3x^2 + 8x \right]_{-2}^1 \\ &= \frac{117\pi}{5} \end{aligned}$$

■

Cylindrical Shell Method

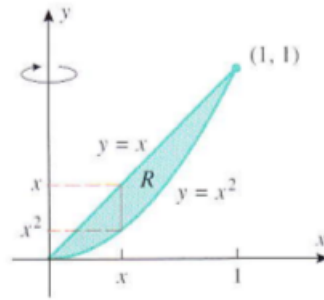
Definition 5.4.6 The volume of the solid generated by revolving the region between the x-axis and the graph of a continuous function $y = f(x) \geq 0$, $L \leq a \leq x \leq b$, about a vertical line $x = L$ is

$$V = \int_a^b 2\pi x f(x) dx$$

where x is shell radius and $f(x)$ is shell height.

■ **Example 5.66** The region R enclosed by the curve $y = x$ and $y = x^2$ is rotated about the y-axis. Find the volume of the resulting solid.

Solution: The shell formed by the region between $y = x$ and $y = x^2$ has radius x and height $x - x^2$. Then the volume is

Figure 5.11: Generating of volume by $y = x$ and $y = x^2$

$$\begin{aligned} v &= \int_0^1 2\pi x[x - x^2] dx \\ &= \frac{\pi}{6} \end{aligned}$$

■

5.4.3 Arc Length

Definition 5.4.7 If f' is continuous on $[a, b]$, then the length(arc length) of the curve $y = f(x)$ from the point $A = (a, f(a))$ to the point $B = (b, f(b))$ is the value of the integral

$$L = \int_a^b \sqrt{1 + (f'(x))^2} dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

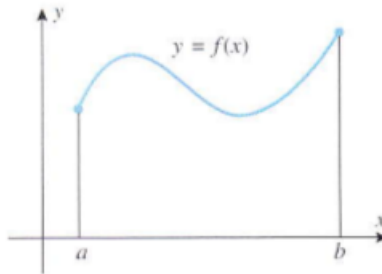


Figure 5.12: Arc Length

■ **Example 5.67** Let $f(x) = x^{\frac{3}{2}} + 2$ for $0 \leq x \leq \frac{4}{3}$. Find the length of the graph of f .

Solution: Since $f'(x) = \frac{3}{2}x^{\frac{1}{2}}$

$$\begin{aligned} L &= \int_0^{\frac{4}{3}} \sqrt{1 + \left(\frac{3}{2}x^{\frac{1}{2}}\right)^2} dx \\ &= \int_0^{\frac{4}{3}} \sqrt{1 + \frac{9}{4}x} dx \\ &= \frac{56}{27} \end{aligned}$$

■

5.4.4 Surface Area

Definition 5.4.8 If the function $f(x) \geq 0$ is continuous differentiable on $[a, b]$, the area of the surface generated by revolving the graph of $y = f(x)$ about x-axis is

$$S = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} dx = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

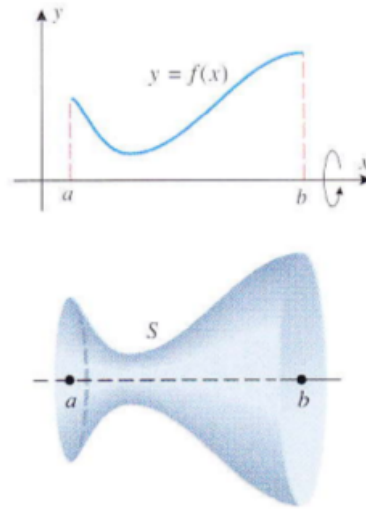


Figure 5.13: Surface Area of $f(x)$

Definition 5.4.9 If the function $g(y) \geq 0$ is continuous differentiable on $[c, d]$, the area of the surface generated by revolving the graph of $x = g(y)$ about y-axis is

$$S = \int_c^d 2\pi g(y) \sqrt{1 + (g'(y))^2} dy = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

■ **Example 5.68** The curve $y = \sqrt{4 - x^2}$, where $-1 \leq x \leq 1$ is an arc of the circle $x^2 + y^2 = 4$. Find the surface area obtained by revolving this arc about the x-axis, the surface is the portion of the sphere of radius 2 unit.

Solution: Since $\frac{dy}{dx} = \frac{-x}{\sqrt{4-x^2}}$, it follows from the formula that

$$\begin{aligned} S &= \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\ &= \int_{-1}^1 2\pi \sqrt{4 - x^2} \sqrt{1 + \left(\frac{-x}{\sqrt{4-x^2}}\right)^2} dx \\ &= 2\pi \int_{-1}^1 \sqrt{4 - x^2} \sqrt{1 + \left(\frac{-x}{\sqrt{4-x^2}}\right)^2} dx \\ &= 2\pi \int_{-1}^1 \sqrt{4 - x^2} \frac{2}{\sqrt{4 - x^2}} dx \\ &= 8\pi \end{aligned}$$

■

5.5 Improper Integral

The definition of an integral $\int_a^b f(x) dx$ requires the function $f(x)$ to be bounded on the bounded interval $[a, b]$ and all the integrals that we have performed have had finite limits and integrands with no asymptotes.

Now, there are also integrands with finite limits and integrands with vertical asymptotes and it is called improper integral. The following are some of the examples of the improper integrals

1. The integral $\int_0^1 \frac{1}{x} dx$ is improper because $\frac{1}{x}$ becomes unbounded as x approaches zero, the left end points of the interval of the integration.
2. The integral $\int_1^\infty \frac{1}{x^2} dx$ is improper because $[1, \infty)$ is an infinite interval of integration.
3. The integral $\int_{-1}^1 \frac{1}{x^2} dx$ is improper because $\frac{1}{x^2}$ becomes unbounded as x approaches zero, a point inside the interval of integration.

Infinite Integrals

Even with finite limits of integration, all integrals can still be improper if the integrand becomes infinite somewhere in the interval of integration $[a, b]$.

Definition 5.5.1 The function $f(x)$ defined on the interval $[a, b]$ where a and b are real numbers. We have two cases $f(x)$ becomes unbounded around a or unbounded around b .

$$\begin{aligned}\int_a^b f(x) dx &= \lim_{m \rightarrow a^+} \int_m^b f(x) dx \text{ unbounded around } a \\ \int_a^b f(x) dx &= \lim_{m \rightarrow b^-} \int_a^m f(x) dx \text{ unbounded around } b\end{aligned}$$

■ **Example 5.69** Evaluate $\int_0^1 \frac{1}{\sqrt{x}} dx$.

Solution: The integrand has a vertical asymptote at $x = 0$, so we must use the above definition. We have

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{x}} dx &= \lim_{m \rightarrow 0^+} \int_m^1 \frac{1}{\sqrt{x}} dx \\ &= \lim_{m \rightarrow 0^+} 2\sqrt{x} \Big|_m^1 \\ &= \lim_{m \rightarrow 0^+} (2 - 2\sqrt{m}) \\ &= 2\end{aligned}$$

■

Definition 5.5.2 Let $\int_a^b f(x) dx$ be improper integral.

1. If the limit exists and is a number, we say that the improper integral is convergent.
2. If the limit does not exist or it is infinite, then we say that the improper integral is divergent.

■ **Example 5.70** Evaluate $\int_0^1 \frac{1}{x} dx$.

Solution: This integral is infinite at lower limit of integration; that is, $\frac{1}{x} \rightarrow \infty$ as $x \rightarrow 0^+$. Therefore, we should first find the integral $\int_m^1 \frac{1}{x} dx = \ln x \Big|_m^1 = \ln 1 - \ln m = -\ln m$ for $m > 0$ and then evaluate limits as $m \rightarrow 0^+$.

$$\begin{aligned}\int_0^1 \frac{1}{x} dx &= \lim_{m \rightarrow 0^+} \int_m^1 \frac{1}{x} dx \\ &= \lim_{m \rightarrow 0^+} (-\ln m)\end{aligned}$$

since this limit does not exist as $m \rightarrow 0^+$, $\ln m \rightarrow -\infty$, then given integral is divergent.

■

■ **Example 5.71** Evaluate $\int_0^9 x^{-\frac{1}{2}} dx$.

Solution: This integral is improper because when $x \rightarrow 0^+$, the integrand $\frac{1}{\sqrt{x}}$ becomes infinite

$$\begin{aligned}\int_0^9 x^{-\frac{1}{2}} dx &= \lim_{m \rightarrow 0^+} \int_m^9 x^{-\frac{1}{2}} dx \\ &= \lim_{m \rightarrow 0^+} 2x^{\frac{1}{2}} \Big|_m^9 \\ &= \lim_{m \rightarrow 0^+} (6 - 2\sqrt{m}) \\ &= 6 - 0 \\ &= 6\end{aligned}$$

and it is convergent. ■

Infinite Limits of Integration

Definition 5.5.3 Integrals with infinite limits of integration are improper integrals

1. If $f(x)$ is continuous on $[a, \infty)$, then $\int_a^\infty f(x) dx = \lim_{m \rightarrow \infty} \int_a^m f(x) dx$
2. If $f(x)$ is continuous on $(-\infty, b]$, then $\int_{-\infty}^b f(x) dx = \lim_{m \rightarrow -\infty} \int_m^b f(x) dx$
3. If $f(x)$ is continuous on $(-\infty, \infty)$, then $\int_{-\infty}^\infty f(x) dx = \lim_{m \rightarrow -\infty} \int_m^c f(x) dx + \lim_{m \rightarrow \infty} \int_c^m f(x) dx$, where c is any constant. If both limits are finite, the improper integral is said to be convergent, otherwise the improper integral is said to diverge.

In each case, if the limit is finite we say that the improper integral converges and that the limit is the value of the improper integral. If the limit fails to exist, the improper integral diverges.

Remark 5.5.1 If the improper integral $\int_a^b f(x) dx$ is split in to a sum of improper integrals (because $f(x)$ presents more than one improper behavior on $[a, b]$), then the integral converges if and only if any single improper integral is convergent.

■ **Example 5.72** Consider the function f defined by $f(x) = e^{-x}$. Find the area of the region under the graph of f above the x -axis to the right of $x = 0$.

Solution: If m is any positive number, then the area under $f(x) = e^{-x}$ from 0 to m is certainly finite, and we know that it is equal to

$$\int_0^m e^{-x} dx$$

as $m \rightarrow \infty$, the area under $f(x)$ from 0 to m will approach the area of the region in question. Thus if

$$\lim_{m \rightarrow \infty} \int_0^m e^{-x} dx$$

then the region in question has area equal to the value of the limit. If the limit does not exist, then the area of the region is not finite. Now

$$\lim_{m \rightarrow \infty} \int_0^m e^{-x} dx = \lim_{m \rightarrow \infty} (-e^x)|_0^m \quad (5.1)$$

$$= \lim_{m \rightarrow \infty} (-e^{-m} + 1) \text{ but } \lim_{m \rightarrow \infty} -e^{-m} = 0 \quad (5.2)$$

$$= 0 + 1 = 1 \quad (5.3)$$

Therefore the shaded region in the above figure has finite area equal to 1. ■

■ **Example 5.73** Evaluate $\int_1^\infty \frac{1}{x^2} dx$.

Solution: First compute the integral from 1 to N and then let n approach infinity.

$$\begin{aligned} \int_1^\infty \frac{1}{x^2} dx &= \lim_{m \rightarrow \infty} \int_1^m \frac{1}{x^2} dx \\ &= \lim_{m \rightarrow \infty} \left(\frac{-1}{x} \right) \Big|_1^m \\ &= \lim_{m \rightarrow \infty} \left(\frac{-1}{m} + 1 \right) \\ &= 1 \end{aligned}$$

This improper integral does converge, and it has a value of 1. ■

■ **Example 5.74** Evaluate $\int_1^\infty \frac{1}{x} dx$.

Solution:

$$\begin{aligned}
 \int_1^{\infty} \frac{1}{x} dx &= \lim_{m \rightarrow \infty} \int_1^m \frac{1}{x} dx \\
 &= \lim_{m \rightarrow \infty} \ln x \Big|_1^m \\
 &= \lim_{m \rightarrow \infty} (\ln m - \ln 1) \\
 &= \lim_{m \rightarrow \infty} \ln m \\
 &= \infty
 \end{aligned}$$

Then $\lim_{m \rightarrow \infty} \ln m$ is divergent. ■

■ **Example 5.75** Evaluate the improper integral $\int_1^{\infty} \frac{\ln x}{x^2} dx$.

Solution:

$$\begin{aligned}
 \int_1^{\infty} \frac{\ln x}{x^2} dx &= \lim_{m \rightarrow \infty} \int_1^m \frac{\ln x}{x^2} dx \\
 &= \lim_{m \rightarrow \infty} \left[\frac{\ln x}{x} \right]_1^m \\
 &= \lim_{m \rightarrow \infty} \left[\frac{\ln m}{m} + 1 \right] \\
 &= \lim_{m \rightarrow \infty} \left[\frac{\ln m}{m} \right] + 1 \\
 &= 1 \text{ by using L'hospital rule } \lim_{m \rightarrow \infty} \left[\frac{\ln m}{m} \right] = \lim_{m \rightarrow \infty} \left[\frac{1}{m} \right] = 0
 \end{aligned}$$

■ **Example 5.76** Evaluate the improper integral $\int_{-\infty}^{\infty} \frac{1}{x^2+1} dx$.

Solution: We can split the improper integral as

$$\begin{aligned}
 \int_{-\infty}^{\infty} \frac{1}{x^2+1} dx &= \lim_{m \rightarrow -\infty} \int_m^0 \frac{1}{x^2+1} dx + \lim_{m \rightarrow \infty} \int_0^m \frac{1}{x^2+1} dx \\
 &= \lim_{m \rightarrow -\infty} [\tan^{-1} x]_m^0 + \lim_{m \rightarrow \infty} [\tan^{-1} x]_0^m \\
 &= \lim_{m \rightarrow -\infty} [\tan^{-1} 0 - \tan^{-1} m] + \lim_{m \rightarrow \infty} [\tan^{-1} m - \tan^{-1} 0] \\
 &= \lim_{m \rightarrow -\infty} [-\tan^{-1} m] + \lim_{m \rightarrow \infty} [\tan^{-1} m] \\
 &= -(-\frac{\pi}{2}) + (\frac{\pi}{2}) \\
 &= \pi
 \end{aligned}$$

5.6 Exercise

1. Evaluate the integral $\int_1^3 (1+2x) dx$.

Ans. 10

2. Evaluate the integral $\int_{-1}^2 |x| dx$.

Ans. $\frac{5}{2}$

3. Evaluate $\int_{\pi}^{\pi} \sin^2 x \cos^4 x dx$.

Ans. 0

4. Use the properties of integrals to verify the inequality

$$2 \leq \int_{-1}^1 \sqrt{1+x^2} dx \leq 2\sqrt{2}$$

with out evaluating the integrals.

5. Find the derivative of the function $f(x) = \int_{2x}^{3x} \frac{u^2-1}{u^2+1} du$.

$$\text{Ans. } f'(x) = -\frac{2(4x^2-1)}{4x^2+1} + \frac{3(9x^2-1)}{9x^2+1}$$

6. If $F(x) = \int_1^x f(t) dt$, where $f(t) = \int_1^{t^2} \frac{\sqrt{1+u^4}}{u} du$, find $F''(2)$.

Ans. $\sqrt{257}$

7. Show that

$$0 \leq \int_5^{10} \frac{x^2}{x^4+x^2+1} dx \leq 0.1$$

by comparing the integrand to a simpler function.

8. Find a function f and a such that $6 + \int_a^x \frac{f(t)}{t^2} dt = 2\sqrt{x}$ for all $x > 0$.

$$\text{Ans. } f(x) = x^{\frac{3}{2}}, a = 9$$

9. Evaluate the integral $\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{6}{1-x^2} dx$.

Ans. π

10. Evaluate the indefinite integral $\int \frac{\cos \sqrt{x}}{\sqrt{x}} dx$.

$$\text{Ans. } 2 \sin \sqrt{x} + c$$

11. Evaluate the indefinite integral $\int \sqrt{\cot x} \csc^2 x dx$.

$$\text{Ans. } -\frac{2}{3}(\cos x)^{\frac{3}{2}} + c$$

12. If f is continuous and $\int_0^4 f(x) dx = 10$, find $\int_0^2 f(2x) dx$.

Ans. 5

13. Find the volume of a pyramid with height h and rectangular base with dimensions b and $2b$.

$$\text{Ans. } \frac{2}{3}b^2h$$

14. Find the volume of a tetrahedron with three mutually perpendicular faces and three mutually perpendicular edges with lengths 3 cm, 4 cm, and 5 cm.

$$\text{Ans. } 10 \text{ cm}^3$$

15. Find the volume having the base of S is an elliptical region with boundary curve $9x^2 + 4y^2 = 36$. Cross-sections to the x -axis are isosceles right triangles with hypotenuse in the base.

Ans. 24

16. If f is continuous and $\int_1^3 f(x) dx = 8$, show that f takes on the value 4 at least once on the interval $[1, 3]$.

17. Find the area of the region bounded by $y = x^2$ and $y = 4x - x^2$.

Ans. $\frac{8}{3}$

18. Find the area of the region bounded by $y = 1 - 2x^2$ and $y = |x|$.

Ans. $\frac{7}{12}$

19. Find the volume of the solid obtained by rotating the region bounded by $x = 0, x = 9 - y^2$ about the axis $x = -1$.

Ans. $\frac{1656}{5}\pi$

20. The base of a solid is a circular disk with radius 3. Find the volume of the solid if parallel cross-sections perpendicular to the base are isosceles right triangles with hypotenuse lying along the base.

Ans. 36