

PHYSICS GRADE 11

UNIT SIX_

ANGULAR MOTION
/ROTATIONAL MOTION/

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Objective of this lesson

♣ Upon completion of this unit we should be able to

- Gain knowledge and understanding of the fundamental concepts rotational motion.
- Define basic terms in rotational motion.
- state law of conservation of angular momentum.
- Develop skills and techniques of solving theoretical problems concerning rotational motion.

What is angular motion?

NOTE:- Angular Motion is also known as rotational motion. can be defined as *the study of rotation of a rigid body about a given axis.* Here we deal with the rotation of rigid body.

♣ **Rigid body** is a body which is non-deformable, that is the relative position between every particles on it remains constant.

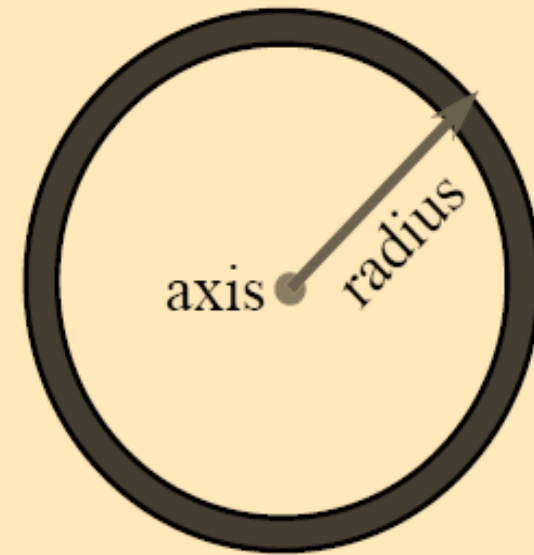
There are several examples of objects in rotational motion

Among those are

- vehicles wheel,
- swinging objects,
- planets rotating around their orbits,
- ice skaters ,
- gears ,
- connecting rods of engine.

Key points

- axis of rotation
- radius of rotation: distance from axis to point of interest



The main points we need to focus on this topic are

- Rotational kinematics
- Rotational dynamics
- Rotational work , energy and power
- Rotational / angular/ momentum

Terms Used in rotational or angular kinematics

Suppose an object shown below follows a rotational motion about a given axis. All the points around the axis follow a circular rotational motion.

A compact disc rotating about a fixed axis through O perpendicular to the plane of the figure.

(a) In order to define angular position for the disc, a fixed reference line is chosen. A particle at P is located at a distance r from the rotation axis at O .

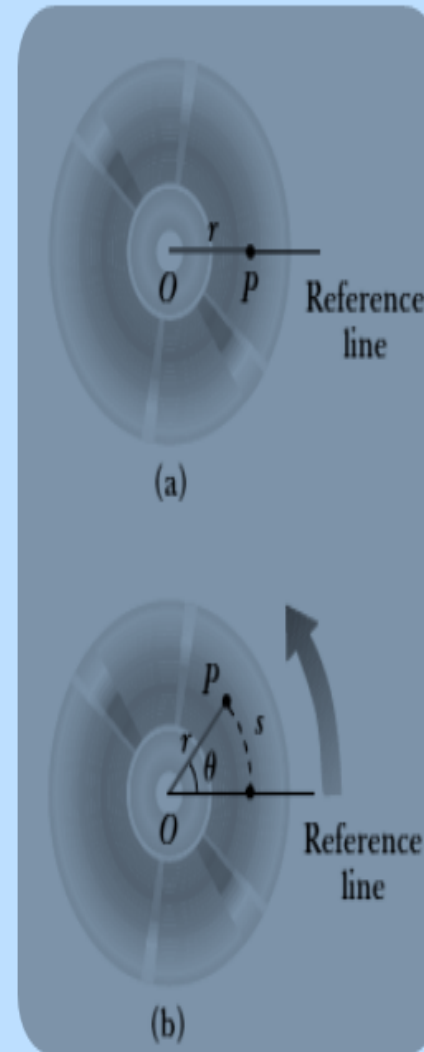
(b) As the disc rotates, point P moves through an arc length s on a circular path of radius r .

i. Angular Displacement (θ)

* It is the measure of displacement along circular path,
Mathematically it can be defined as the ratio of arc length (S) to radius (R)

$$\theta = \frac{S}{R}$$

$$[\theta] = \frac{m}{m} = \text{radian (rad)}$$



* Angular displacement is a dimension-less quantity, but it can be measured by terms degree (deg), radian (rad), and revolution (rev).

$$1 \text{ rev} = 360^\circ = 2\pi \text{ rad}$$

➤ One radian is the angle subtended by an arc length equal to the radius of the arc.

ii. Angular Velocity (ω)

* It is a rotating velocity which is the rate of change of angular displacement. i.e.

$$\omega = \frac{\Delta\theta}{\Delta t} \quad [\omega] = \frac{\text{rad}}{\text{s}}$$

iii. Angular Acceleration (α)

* It is the rate of change of angular velocity.

$$\alpha = \frac{\Delta\omega}{\Delta t} \quad [\alpha] = \frac{\text{rad}^2}{\text{s}}$$

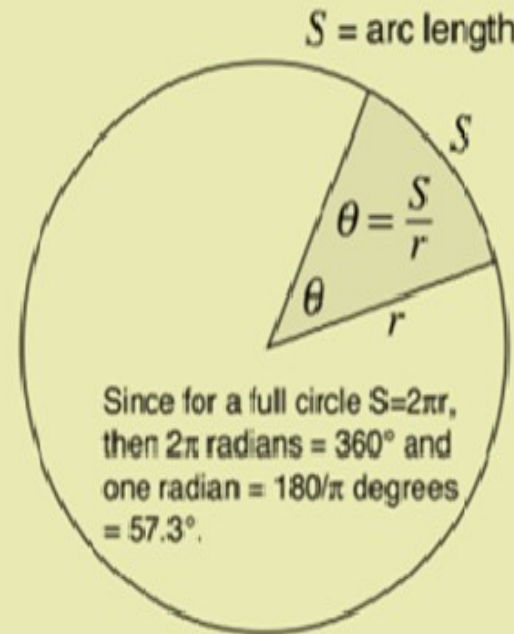
iv. Tangential velocity (linear velocity) / V_t or V

* It is a velocity tangent to the circle. $V = \frac{\Delta S}{\Delta t}$

but $\Delta S = r\Delta\theta$ $V = \frac{r\Delta\theta}{\Delta t} = r\left(\frac{\Delta\theta}{\Delta t}\right) = r\omega$

$$V = r\omega$$

Basic Rotational Quantities



The angular displacement is defined by:

$$\theta = \frac{S}{r}$$

For a circular path it follows that the angular velocity is

$$\omega = \frac{v}{r}$$

and the angular acceleration is

$$\alpha = \frac{a_t}{r}$$

In addition to any tangential acceleration, there is always the centripetal acceleration:

$$a_c = \frac{v^2}{r}$$

The standard angle of a directed quantity is taken to be counterclockwise from the positive x axis.

where the acceleration here is the tangential acceleration.

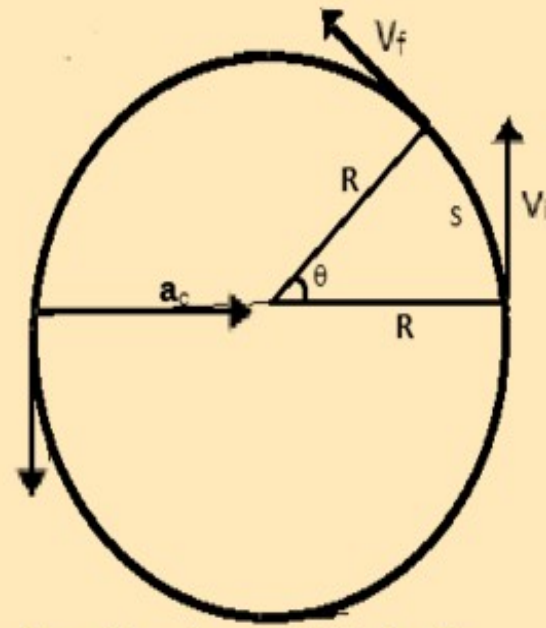
v. Tangential acceleration (linear acceleration) / a_t or a /

* It is acceleration tangent to the circle.

$$\mathbf{a} = \frac{\Delta V}{\Delta t} \text{ but } V=R\omega \Rightarrow \Delta V = R\Delta\omega$$

$$\mathbf{a} = \frac{R\Delta\omega}{\Delta t} = R \left(\frac{\Delta\omega}{\Delta t} \right) = R\alpha$$

$$a = r\alpha$$

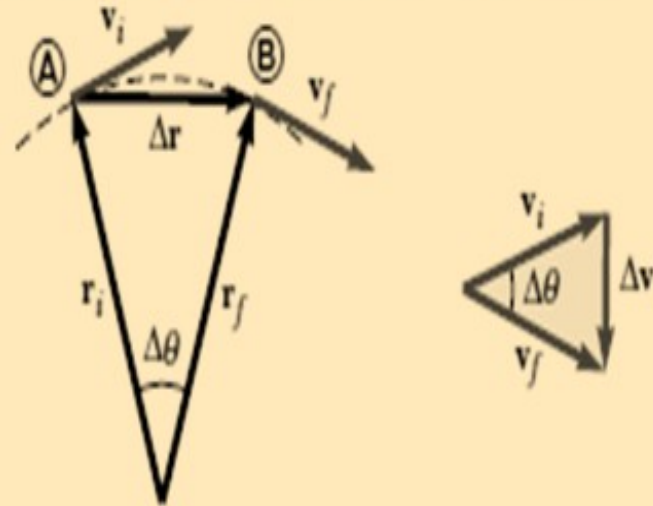


Vi. Centripetal acceleration/ radial acceleration / a_c /

* It is acceleration due to the change of direction of velocity which is directing towards the center of the circle. Centripetal acceleration is due to the change of velocity.

$$(\Delta V = V_f - V_i)$$

$$a_c = \frac{v^2}{r}$$



Vii. Centripetal Force (F_c)

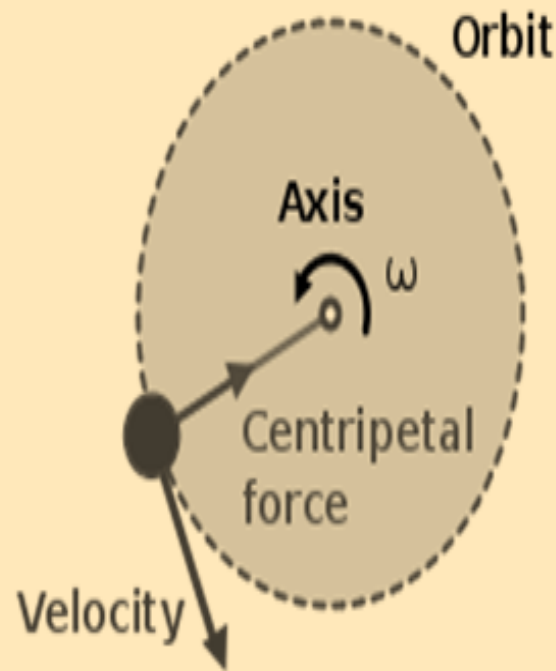
It is a force which is acting towards the center of a circle. This force is due to the centripetal acceleration.

$$F_c = ma_c$$

$$F_c = m \left(\frac{v^2}{r} \right)$$

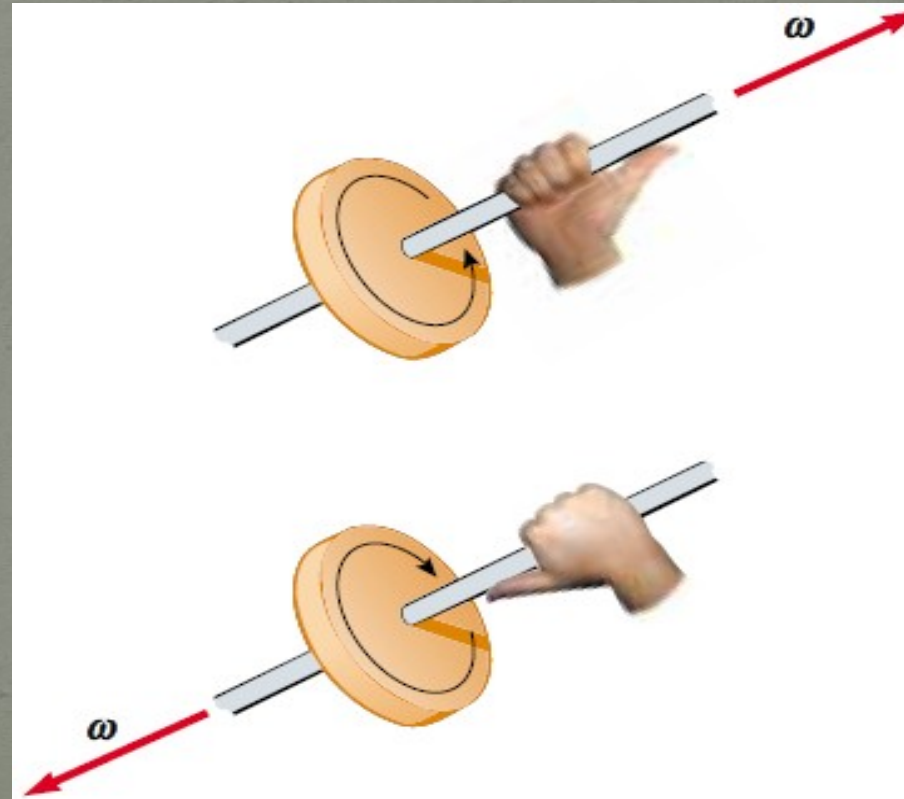
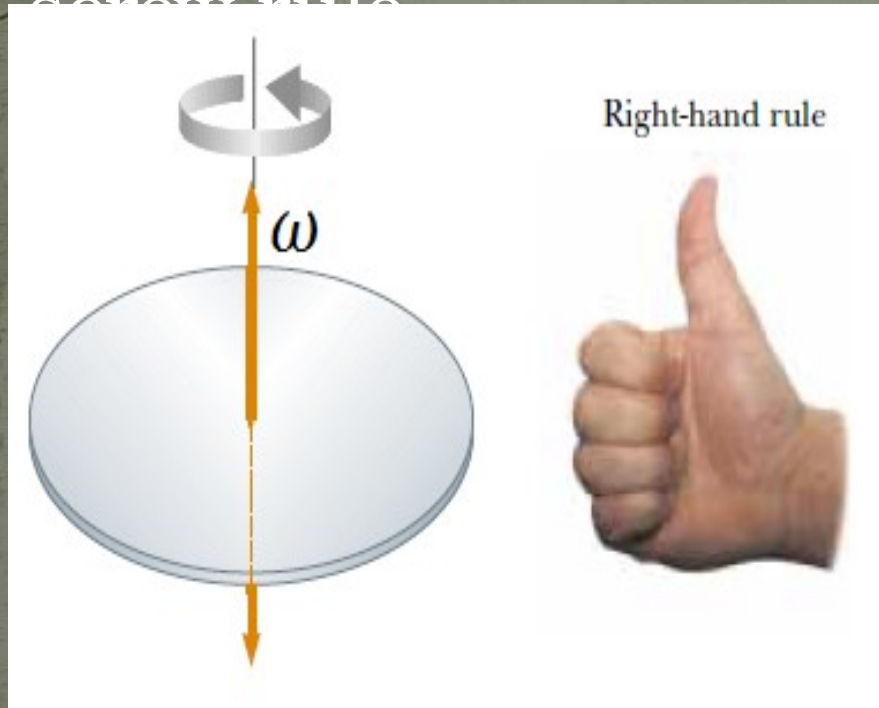
$\Rightarrow$

$$F_c = \frac{mv^2}{r}$$

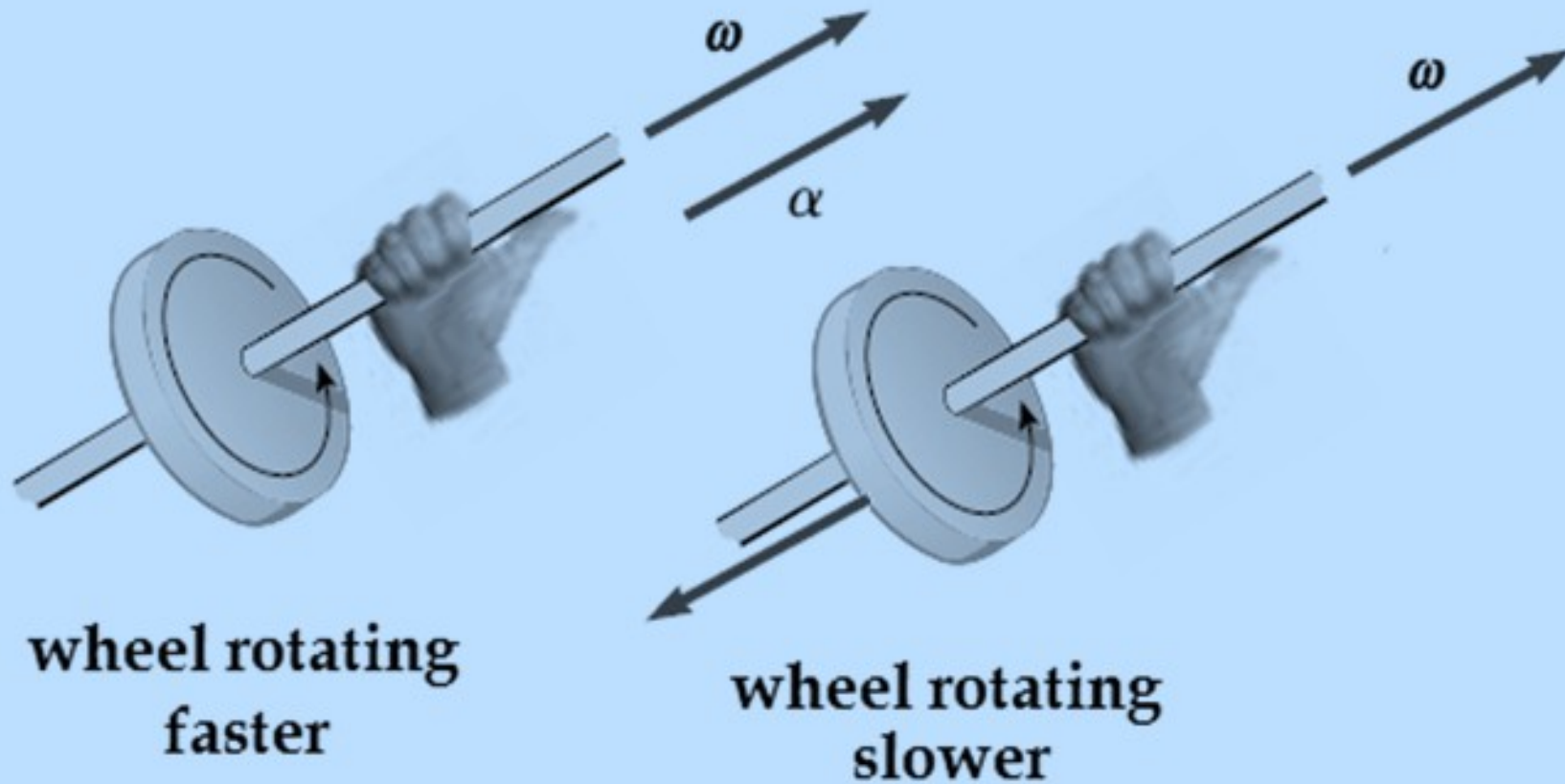


What is the direction of the angular velocity?

The direction of angular velocity and angular acceleration is always along the direction of the axis of rotation. We may use right hand rule or screw rule



Note: - for **fast** moving rotating objects angular acceleration and angular speed are acting in the **same direction** along the axis of rotation. But for **slow** moving objects the (rotating objects which decrease its angular velocity) angular acceleration and angular velocity are acting in **opposite direction** along the axis of rotation.



Summary

$$\theta = \frac{S}{R}$$

$$\alpha = \frac{\Delta\omega}{\Delta t}$$

$$a_c = \frac{v^2}{r}$$

$$\omega = \frac{\Delta\theta}{\Delta t}$$

$$\alpha = \frac{a}{r}$$

$$a_c = \omega^2 r$$

$$\omega = \frac{V}{r}$$

$$a = r\alpha$$

$$F_c = \frac{mv^2}{r}$$

$$V = r\omega$$

$$a_t = \sqrt{a_c^2 + a^2}$$

NOTE: - In translational /linear/ motion, depending on acceleration we have categorized motion in to three as

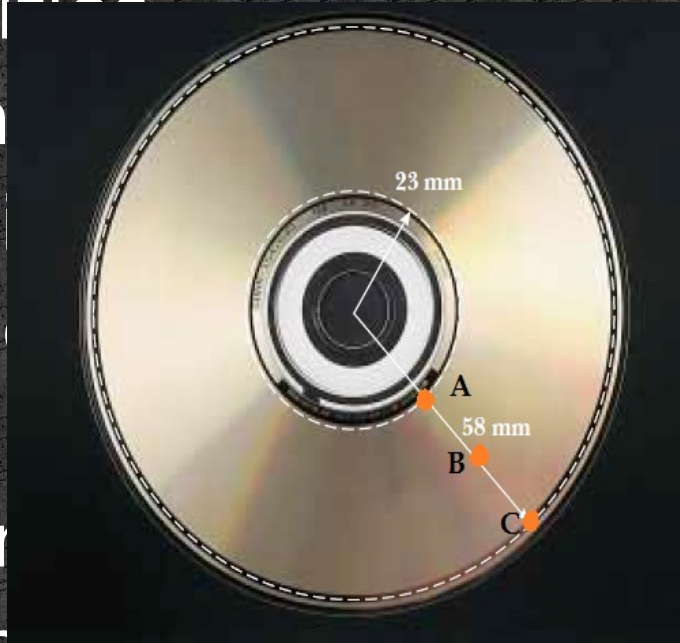
- uniform motion,
- uniformly accelerating motion and
- accelerating motion.

analogous to translational motion we may focus on rotational motion of *uniformly angular acceleration*.

rotational motion

- ♣ When a rigid object is rotating about a *fixed* axis, every particle on the object rotates through the same angle in a given time interval and has the same angular speed and the same angular acceleration. That means, the quantities θ , ω and α characterize the rotational motion of the entire rigid object as well as individual particles in the object.

- This means that the quantities θ , ω , and α **characterize** the rotational motion of the entire rigid object as well as individual particles in the object.



If we take points (A), (B) and (C) which are found at different positions on a rotating CD have the same angular quantities i.e (angular displacement, angular velocity and angular acceleration).

Average and instantaneous
angular velocity and angular
Average and instantaneous
acceleration

$$\omega = \frac{\theta_2 - \theta_1}{t_2 - t_1}$$

$$\omega_{ins} = \lim_{\Delta t \rightarrow 0} \frac{\theta_2 - \theta_1}{t_2 - t_1} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \theta}{\Delta t}$$

$$\omega_{ins} = \frac{d\theta}{dt}$$

a

$$\alpha = \frac{\omega_2 - \omega_1}{t_2 - t_1}$$

$$\alpha_{ins} = \lim_{\Delta t \rightarrow 0} \frac{\omega_2 - \omega_1}{t_2 - t_1} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \omega}{\Delta t}$$

$$\alpha_{ins} = \frac{d\omega}{dt}$$

Example-1

Suppose a toy car rotates 4 times with in 10 seconds clockwise when seen from top of a horizontal table making a radius of 20 cm from the center of the table.

Find (a) the angular displacement in rad
(b) the magnitude and direction of angular
velocity?

Answer

GIVEN:- $\theta = 4$ rev (revolution)

$t = 10$ s (seconds)

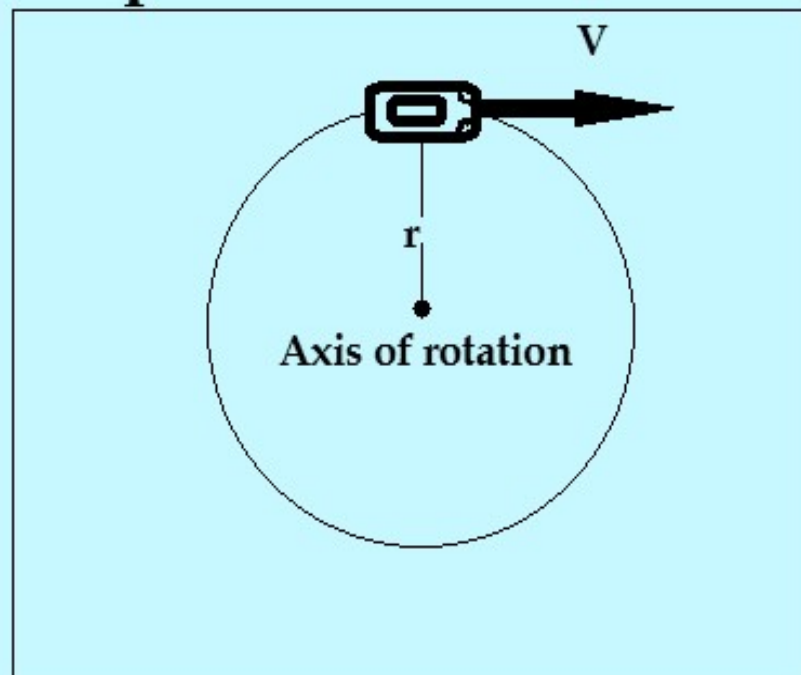
REQUIRED:- a) $\theta = ?$ b) $\omega = ?$

SOLUTION:- a) 1 rev = 2π rad, therefore 4 rev = ? (rad)

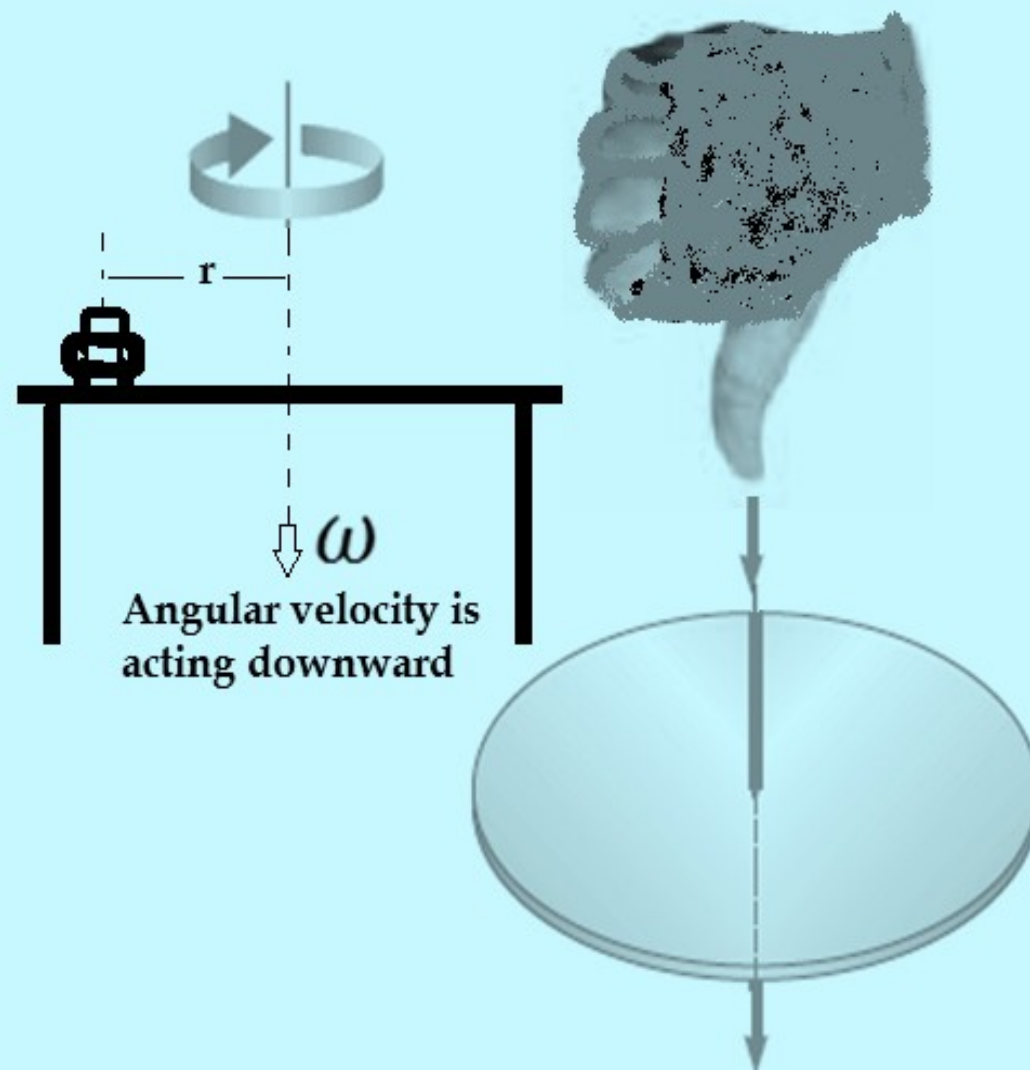
$$\theta = 4(2\pi) = 8\pi \text{ rad}$$

b) $\omega = \Delta \theta / \Delta t = \theta / t \Rightarrow \omega = 8\pi / 10 = 0.8\pi$ rad/s downward of the horizontal table since the direction of angular velocity and angular acceleration is always along the axis of rotation, it is downward !

Top view of the table



Right-hand rule



Example-2

A fly wheel under goes a rotational motion and its angular position as a function of time can be expressed as $(3t^2+4t)$ rad. find the average velocity between 2 sec and 4 sec.

Answer

Given:- $\theta(t) = (3t^2 + 4t)$ rad

Req:- ω (average angular velocity) between
 $t_1 = 2$ sec and $t_2 = 4$ sec is ---?

Solⁿ Since $t_2 = 4$ sec and $t_1 = 2$ sec the
corresponding
angular displacements are

$$\theta_2(4) = (3(4)^2 + 4(4)) = \underline{64 \text{ rad}} \text{ and}$$

$$\theta_1(t) = (3(2)^2 + 4(2)) = \underline{20}$$

rad, Therefore from our definition of
average

$$\omega = \frac{64 - 20}{4 - 2} = \frac{44}{2} \quad \omega = 22 \text{ rad/s}$$

Example-3

A fly wheel under goes a rotational motion and its angular position as a function of time can be expressed as $(3t^2+4t)$ rad. find the instantaneous angular velocity at the 5th second?

Given: - $\theta(t) = 3t^2 + 4t$

Req: - ω at $t = 5$ second

Solⁿ let $t_2 = t + \Delta t$ and $t_1 = t$ the corresponding angular displacements are $\theta(t_2)$ and $\theta(t_1)$ i.e.

$$\theta(t_2) = 3(t + \Delta t)^2 + 4(t + \Delta t)$$

$$\theta(t_1) = 3(t)^2 + 4(t) ,$$

Therefore from our definition of instantaneous angular velocity

$$\omega_{ins} = \lim_{\Delta t \rightarrow 0} \frac{\theta_2 - \theta_1}{t_2 - t_1} =$$

$$\lim_{\Delta t \rightarrow 0} \frac{3(t+\Delta t)^2 + 4(t+\Delta t) - 3(t)^2 + 4(t)}{t+\Delta t - t} \quad \text{Use limit or}$$

derivative,

$$\omega_{ins} = \frac{d\theta}{dt}$$

$$\omega(t) = \frac{d\theta}{dt} \Rightarrow \omega(t) = \frac{d}{dt}(3t^2 + 4t) = 6t + 4$$

Therefore,

$$\omega_{ins} @ t = 5 \text{ s } \text{is } \omega_{ins} = 6(5) + 4 = 34 \text{ rad/s}$$

$$\omega_{ins} = 34 \text{ rad/s}$$

Equation of kinematics

Equation of linear kinematics

$$\bar{V} = \frac{V_f + V_i}{2}$$

$$V_f = V_i + at$$

$$V_f^2 = V_i^2 + 2as$$

$$S = V_it + \frac{1}{2}at^2$$

Equation of rotational kinematics

$$\bar{\omega} = \frac{\omega + \omega_i}{2}$$

$$\omega = \omega_i + \alpha t$$

$$\omega^2 = \omega_i^2 + 2\alpha\theta$$

$$\theta = \omega_it + \frac{1}{2}\alpha t^2$$

Example-3

When a car is heading North a man see the wheel of the car at a distance, the driver stamp on the break 72 km/hr to 36 km/hr with in 2 seconds. The radius of the wheel is 40 cm. What is the

- (a) magnitude and direction of the angular velocities?
- (b) magnitude and direction of the angular acceleration?
- (c) angular displacement of the wheel in this time interval?

Answer

Given:- $V_i = 72 \text{ km/hr} \Rightarrow V_i = 72/3.6 = 20 \text{ m/s}$
 $V_f = 36 \text{ km/hr} \Rightarrow V_f = 36/3.6 = 10 \text{ m/s}$
 $r = 40 \text{ cm} = 0.4 \text{ m}$
 $t = 2 \text{ seconds}$

Req:- (a) Magnitude and direction $\omega_i = ?$, $\omega_f = ?$

(b) Magnitude and direction $\alpha = ?$

(c) angular displacement $\theta = ?$

Solⁿ:- a) $\omega_f = V_f / r \Rightarrow \omega_f = (10 \text{ m/s}) / (0.4 \text{ m}) = 25 \text{ rad/s}$

$\omega_f = 25 \text{ rad/s due west}$

$\omega_i = V_i / r \Rightarrow \omega_i = (20 \text{ m/s}) / (0.4 \text{ m}) = 50 \text{ rad/s}$

$\omega_i = 50 \text{ rad/s due west}$

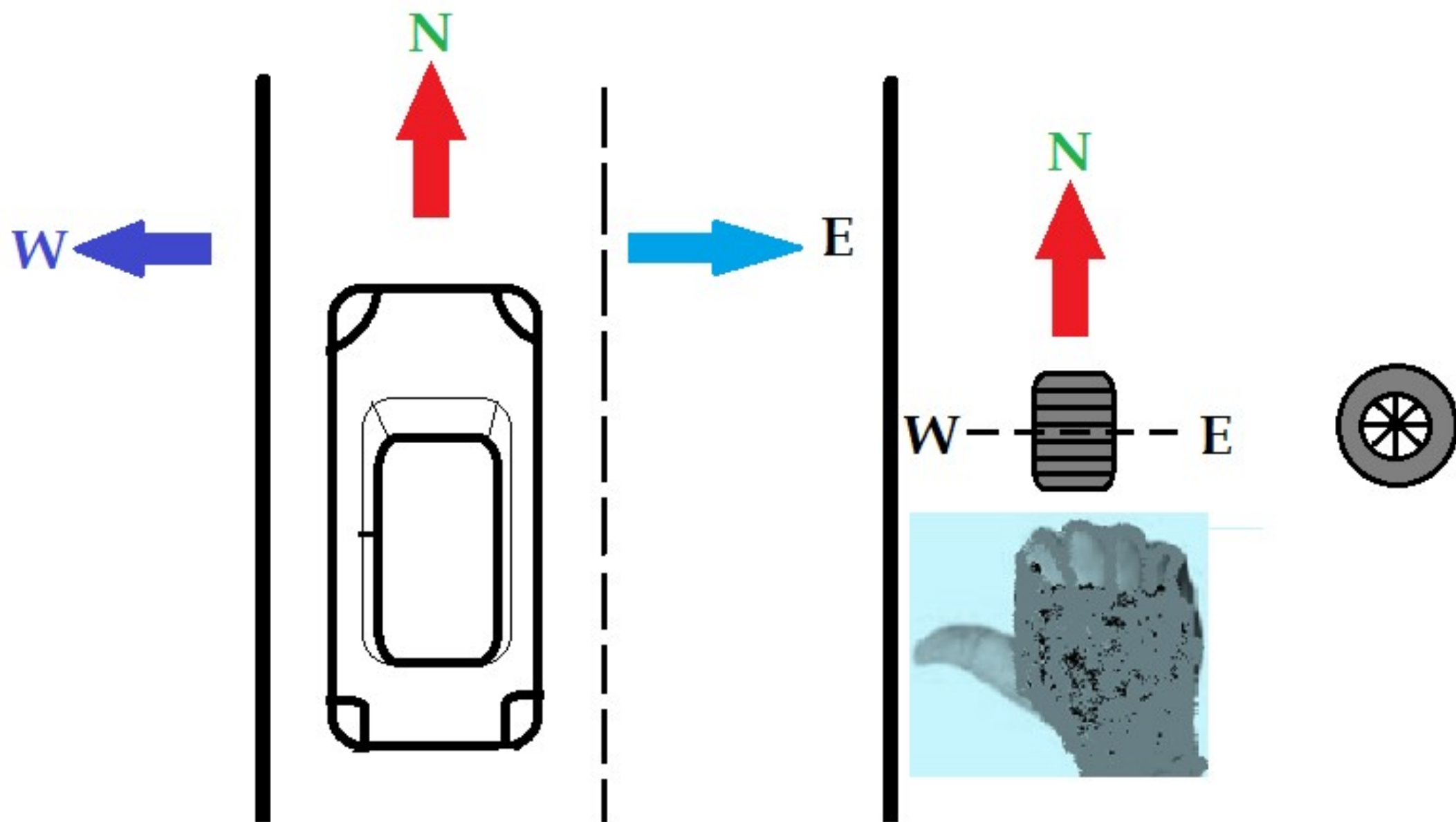
b) $\omega_f = \omega_i + \alpha t \Rightarrow \alpha = (\omega_f - \omega_i) / t \Rightarrow \alpha = (25 - 50) / 5$

$\alpha = (-25) / 5 = -5 \text{ rad/s}^2 \Rightarrow \alpha = -5 \text{ rad/s}^2$

$\alpha = 5 \text{ rad/s}^2 \text{ due east}$

C) use $\theta = \omega_i t + \frac{1}{2} \alpha t^2$ or $\omega_f^2 = \omega_i^2 + 2 \alpha \theta$ or
 $\theta = s / r$

$\theta = (50)(5) + \frac{1}{2}(-5)(5)^2 = 187.5 \text{ m} \Rightarrow \theta =$
 187.5 m



Top view of the car

Rotational dynamics

Basic terms used in rotational dynamics are moment of inertia and torque.

➤ Moment of Inertia (I) :- is the measure of distribution of mass of particles around a given axis of rotation.

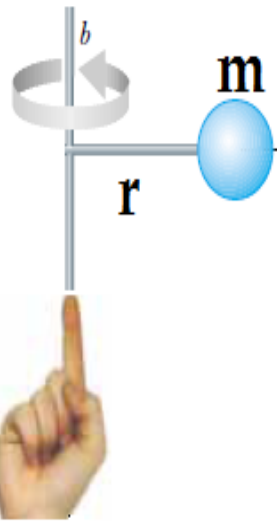
Key point:- radius (r) can be expressed in two ways

i) for point particles system it is the distance between

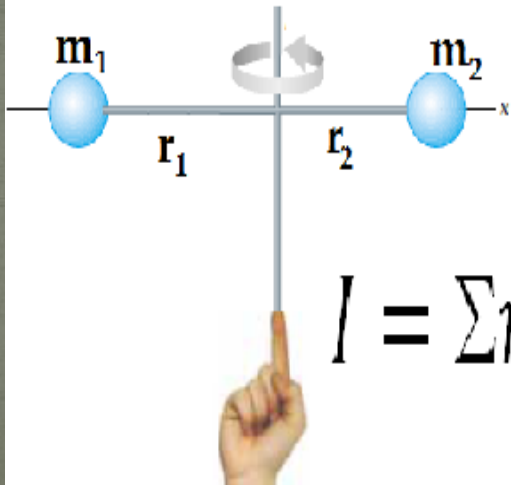
the particle and the axis.

ii) For rigid object it is the actual radius of the solid

bodies i.e sphere, cylinder, hoop, disk.



$$I = mr^2$$



$$I = \sum mr^2$$

Moment of Inertia

Moment of inertia
of particles

$$I = mr^2$$

For a particle at a distance of
(r) from a given axis.

$$I = \sum mr^2$$

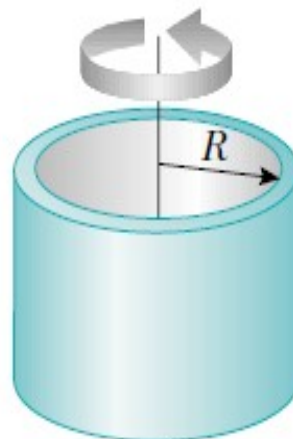
For several particles found at
different distances of (r) from
a given axis.

Moment of inertia of
solid rigid body

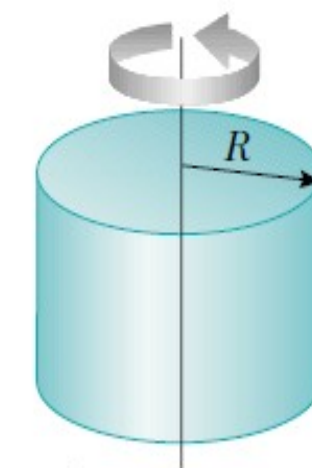
Moment of inertia of
common solid bodies as
the axis of rotation is
through their center of
mass is given as

Moment of Inertia (I) of most common solid bodies

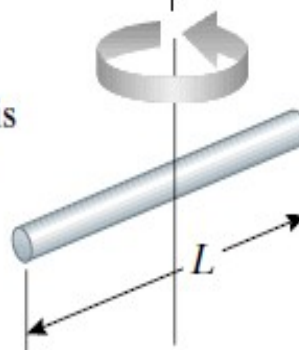
Hoop or thin cylindrical shell
 $I_{\text{CM}} = MR^2$



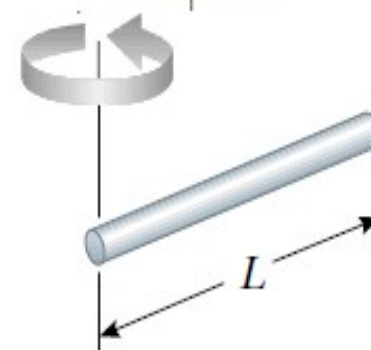
Solid cylinder or disk
 $I_{\text{CM}} = \frac{1}{2} MR^2$



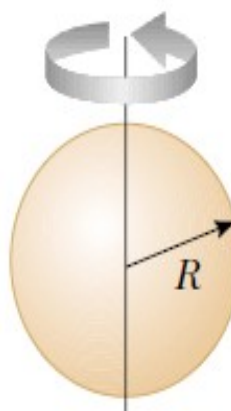
Long thin rod with rotation axis through center
 $I_{\text{CM}} = \frac{1}{12} ML^2$



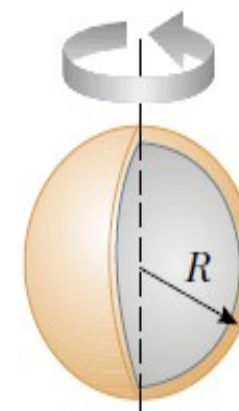
Long thin rod with rotation axis through end
 $I = \frac{1}{3} ML^2$



Solid sphere
 $I_{\text{CM}} = \frac{2}{5} MR^2$



Thin spherical shell
 $I_{\text{CM}} = \frac{2}{3} MR^2$



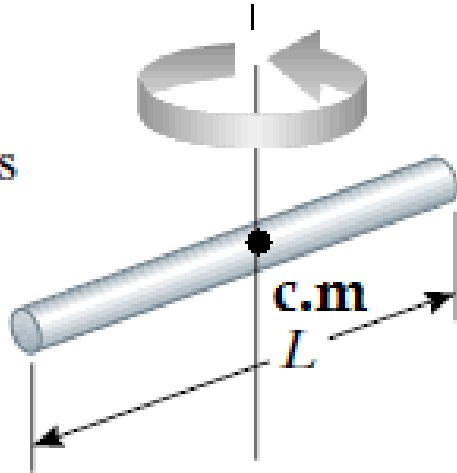
How do we determine the **moment of inertia** of a solid body?

- i) if the center of the mass is found at some point shifted from the geometrical center?
- ii) If the axis of rotation is found at different location from geometrical center ?

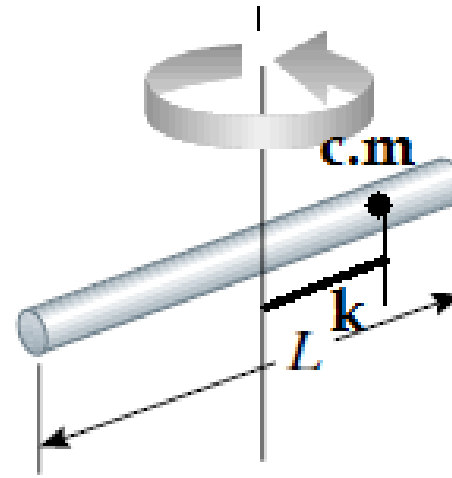
i) To determine the moment of inertia of an object if the center of the mass is found at some point, shifted from the geometrical center?

Long thin rod
with rotation axis
through center

$$I_{CM} = \frac{1}{12} ML^2$$



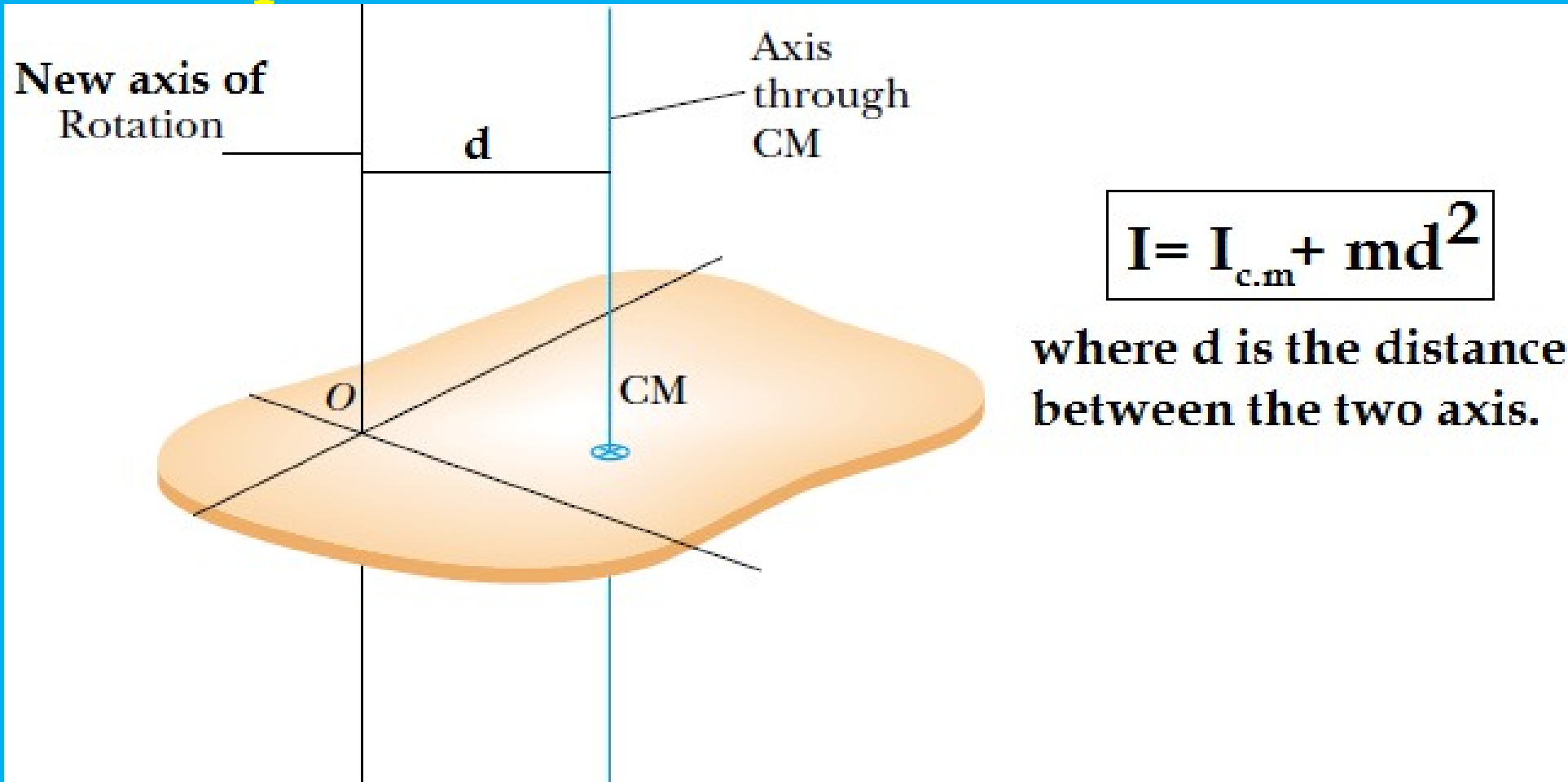
* the center of mass and the axis of rotation are along the geometrical center.



$$I = mk^2$$

where K = radius
of gyration

ii) To determine the moment of inertia of an object if the axis of rotation is found at somewhere from the geometrical center, we can use **parallel axis theorem**.



Example

What is the moment of inertia of *a uniform rod* of length (l) and mass (m), if the axis of rotation is found at $(1/3)^{\text{rd}}$ of length (l)?

Exercise

What is the moment of inertia of a *uniform sphere* of radius (r) and mass (m), if the axis of rotation is found at the rim ?

Answer

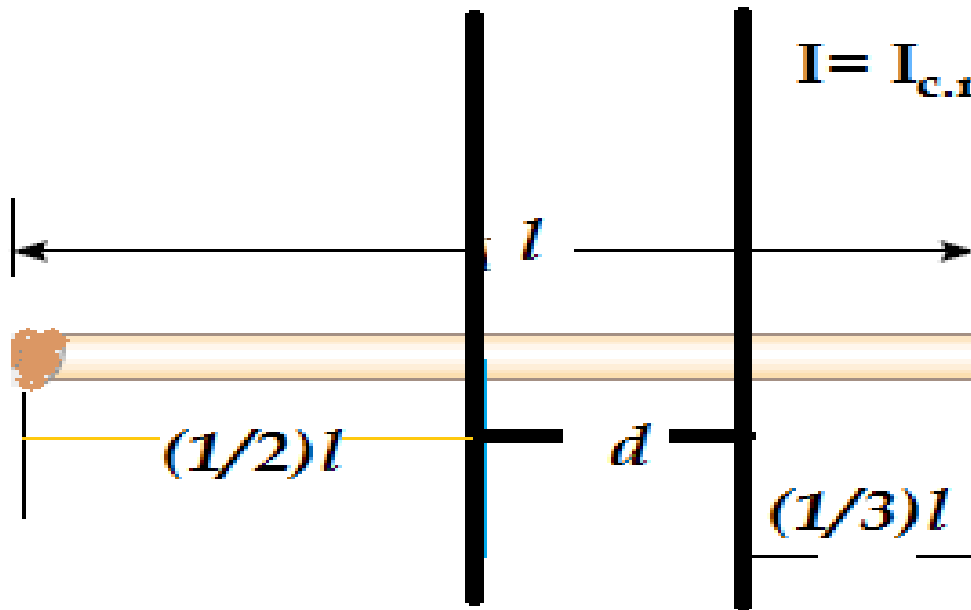
Given:- length (l) and mass (m),

Req:- I (the new moment of inertia?)

Soln:-

$$I_{c.m} = (1/12)ml^2$$

$$I = I_{c.m} + md^2$$



$$I = I_{c.m} + md^2$$

$$I = (1/12)ml^2 + m(1/6)l^2$$

$$I = (1/4)ml^2$$

$$d = (1/2)l - (1/3)l$$

$$d = (1/6)l$$

ii) Torque /moment of force/

It is a vector quantity which measure the turning effect of a given force.

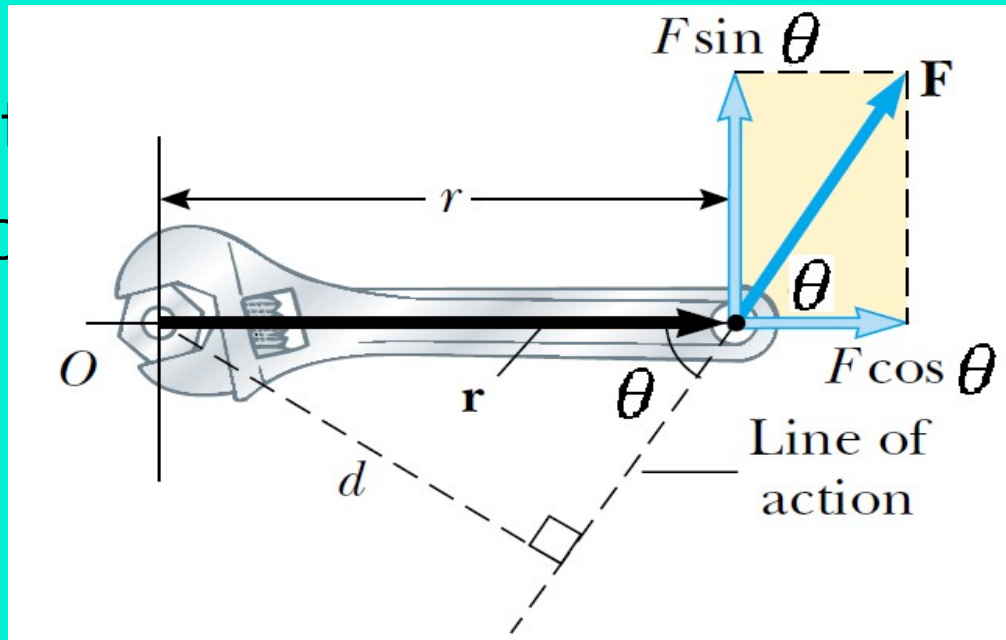
Mathematically it can be represented as

$$\vec{\tau} = \vec{r} \times \vec{F} = Fr \sin \theta$$

where $\tau = Fr \sin \theta$

r = moment arm,

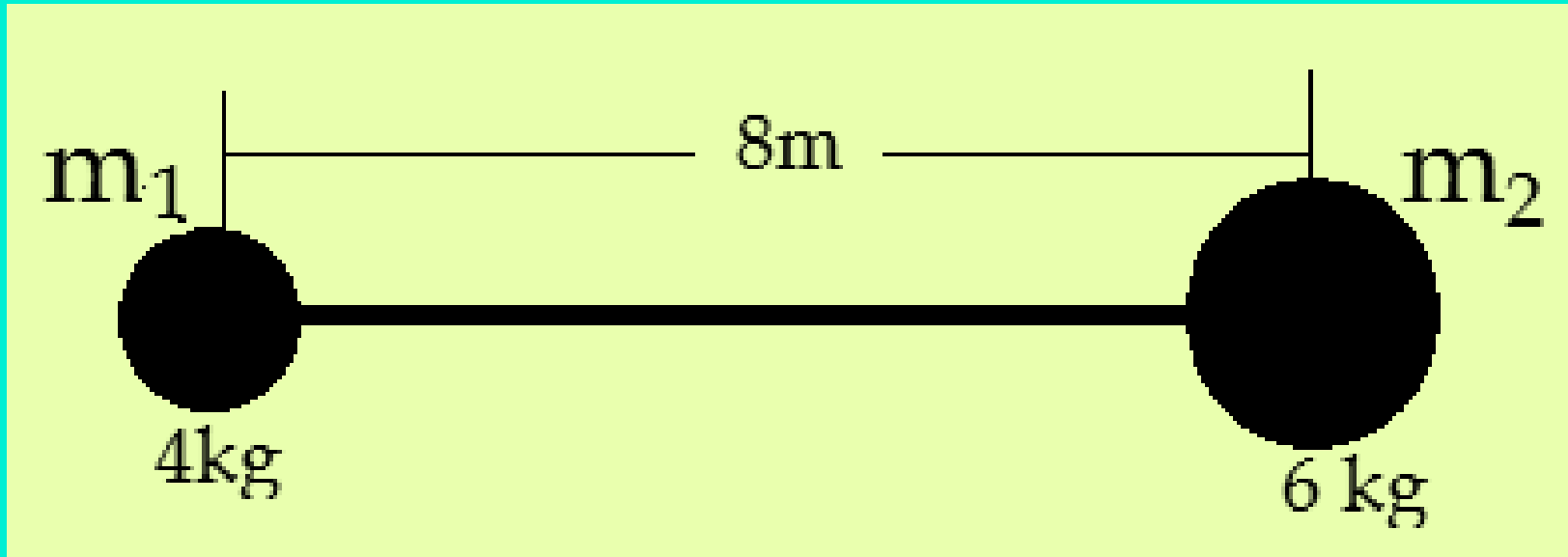
the distance between
fulcrum (pivot) and force



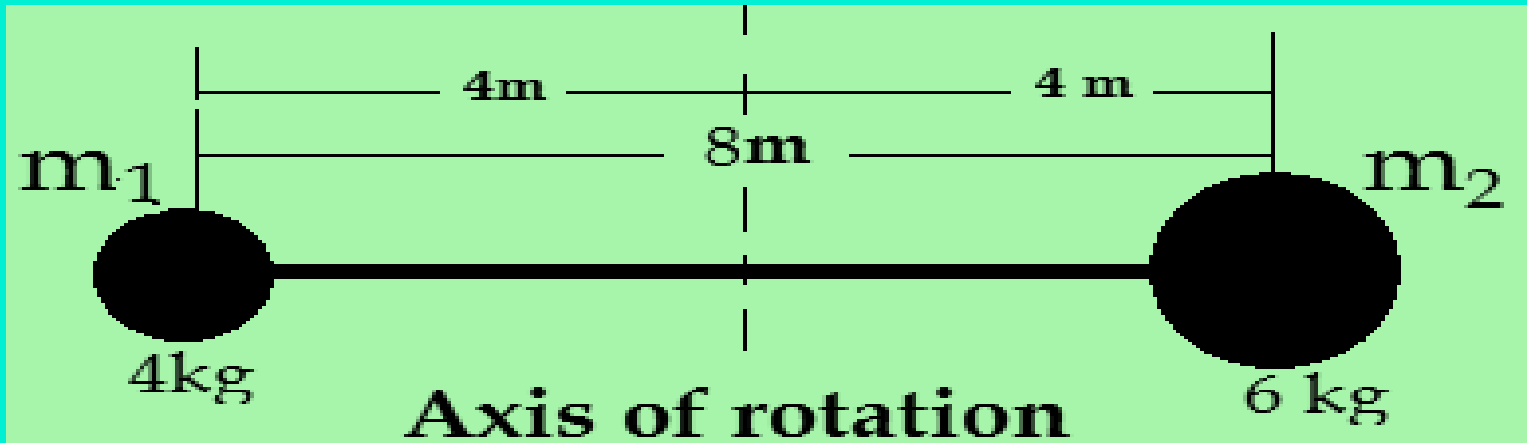
Exersice-1

What is the moment of inertia of the point-mass system shown and the rod is (a) mass-less and (b) has a mass of 2 kg; if the axis of rotation is about

- i) The center of the rod
- ii) Through m_1
- iii)



Answer



(a) Without mass on the rod

(b) with mass on the rod

Axis of rotation through center

i) $I = m_1 r_1^2 + m_2 r_2^2 \Rightarrow$

$$I = (4)(4)^2 + (6)(4)^2 = (4 + 6)(4)^2$$

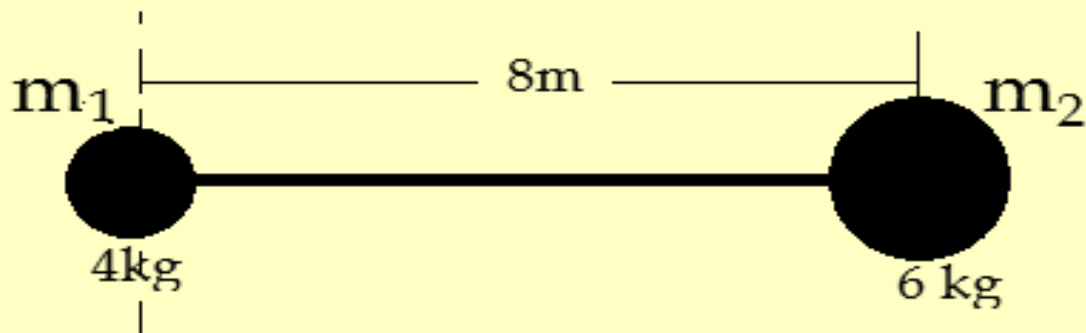
$$I = 160 \text{ kgm}^2$$

i) $I = I_{cm} + m_1 r_1^2 + m_2 r_2^2 \Rightarrow I_{cm} = \frac{1}{12} m l^2$

$$I = \frac{1}{12} m l^2 + m_1 r_1^2 + m_2 r_2^2$$

$$I = \frac{1}{12} (2)(8)^2 + (4)(4)^2 + (6)(4)^2 = 170.7$$

$$I = 170.7 \text{ kgm}^2$$



Axis of rotation through m_1

$$\text{ii) } I = m_2 r_2^2 \Rightarrow$$

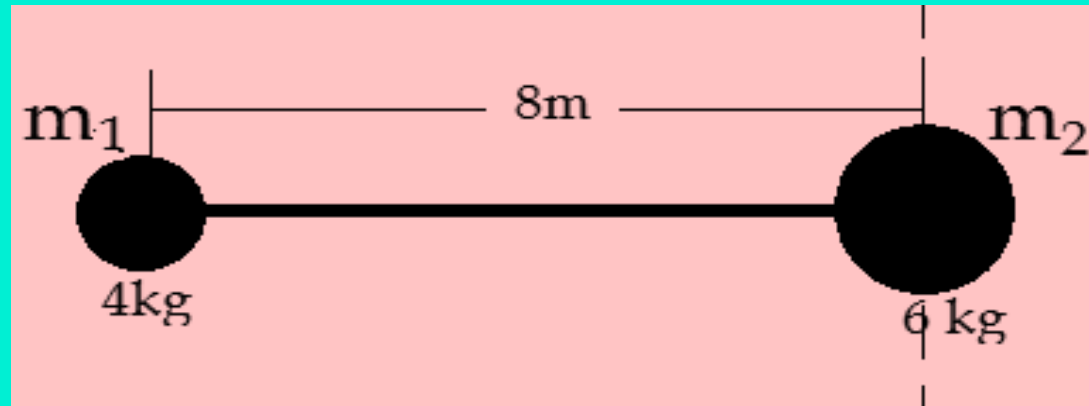
$$\text{ii) } I = I + m_1 r_1^2 + m_2 r_2^2 \Rightarrow I = \frac{1}{3} m l^2$$

$$I = (6)(4)^2 = (4 + 6)(4)^2 \quad I = \frac{1}{3} m l^2 + m_2 r^2$$

$$I = 96 \text{ kgm}^2$$

$$I = \frac{1}{3}(2)(8)^2 + (6)(8)^2 = 426.7$$

$$I = 426.7 \text{ kgm}^2$$



Axis of rotation through m_2

$$\text{iii) } I = m_1 r_1^2 \Rightarrow \quad \text{iii) } I = I + m_1 r^2 \Rightarrow I = \frac{1}{3} m l^2$$

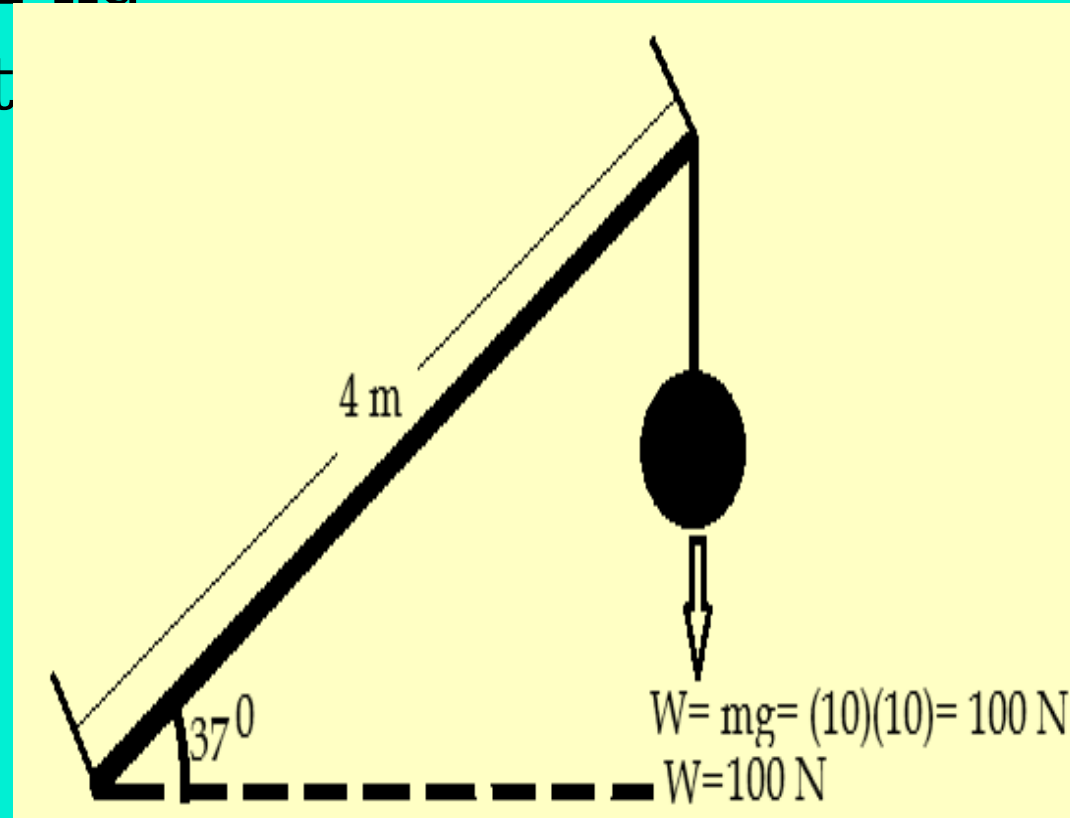
$$I = (4)(4)^2 \quad I = \frac{1}{3} m l^2 + m_1 r^2$$

$$I = 64 \text{ kgm}^2 \quad I = \frac{1}{3} (2)(8)^2 + (4)(8)^2$$

$$I = 298.7 \text{ kgm}^2$$

Example -2

What is the net moment of force (torque) exerted on the bar of mass 4 kg and length 4m pivot at the bottom?



Given: - $F=W= mg= (10) (10) =100\text{N}$

$$r = l= 4 \text{ m}$$

$$\theta= 37^\circ$$

Req:- $\tau =?$

Solution:- $\vec{\tau} = \vec{r} \times \vec{F}$

The magnitude of the torque $\tau = Fr \sin \theta \Rightarrow$

$$\tau = (100)(4) \sin 37 \Rightarrow \tau = 240 \text{ Nm}$$